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Review

Bursting Oscillations in General Coupled Systems: A Review

Danjin Zhang and Youhua Qian *

School of Mathematical Sciences, Zhejiang Normal University, Jinhua 321004, China; zhangdj@zjnu.edu.cn
* Correspondence: qyh2004@zjnu.edu.cn

Abstract: In this paper, the bursting oscillation phenomenon in coupled systems with two time scales is introduced. Firstly, several types of bifurcation are briefly introduced: fold bifurcation, Hopf bifurcation, fold limit cycle bifurcation, homoclinic bifurcation, etc. The bursting oscillations of the system with two excitation terms and the bifurcation delay in the bursting oscillations are considered. Secondly, some simple bursting oscillations are introduced, such as fold/fold bursting, fold/supHopf bursting, subHopf/subHopf bursting, fold/LPC bursting, Hopf/LPC bursting, fold/homoclinic bursting, Hopf/homoclinic bursting, etc. At the same time, the system also has some complex bursting oscillations, such as asymmetric bursting, delayed bursting, bursting with hysteresis loop, etc. Finally, the practical applications of bursting oscillations, such as dynamic vibration absorbers and nonlinear vibration energy harvesting technology, are introduced.

Keywords: bursting oscillation; coupled system; bifurcation delay; fast–slow dynamic analysis method; vibration energy harvesting

MSC: 34F10; 34F99

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1. Introduction

As a type of system with special structures, coupled systems at different time scales [1] have a wide-ranging background in practical engineering involving various fields of science as well as engineering technology. At the same time, due to the coupling of different time scales, the interaction of behaviors on different time scales will lead to many special nonlinear phenomena. Its associated work has been highly attached by scholars all over the world and has become one of the hot topics in nonlinear research. Coupling at different time scales means that in the established mathematical model, state variables or some combination of them can be divided into several different groups, among which there are obvious differences in magnitude in the rate of change with time. In terms of dimensionless mathematical models, coupled systems of different time scales can be divided into two types. One is in the time domain; that is, there are vector field coupling of different orders of magnitude. The other is in the frequency domain; that is, there is an order of magnitude difference between the different frequencies of the coupled system. However, these two types of systems are essentially the same [2]. The factors leading to the coupling of different time scales can not only come from the fast–slow effect of the real time [3] but also from the scale effect of geometry [4], as well as internal physical effects, system structure effects, and so on. Thus, in the established mathematical model, state variables are divided into several different groups, and the rate of change of state variables with time among each group is significantly different in magnitude [5]. A two-time-scale coupled dynamic system is an important part of a multi-time-scale coupled dynamic system. So far, a lot of research has been conducted, and abundant results have been achieved on the fast–slow two-time-scale coupled system, but due to the particularity of this kind of system, its related research is still in the development stage, and many problems need to be further studied.

Many coupled systems at different time scales can produce complex behavioral modes, and bursting oscillation is one of the typical dynamic behaviors. The fast–slow structure plays an important role in the generation mechanism of the bursting oscillation. Kuehn [1] defined the bursting oscillation as follows: if the system has a periodic orbit, and the time series of the periodic orbit (at least one state variable or some coordinate scales) alternates between fast oscillation and near-stable behavior, then the system is said to have a bursting oscillation.

The study of the dynamic behavior of coupled systems at different time scales can be traced back to the relaxation oscillation found in the Van der Pol oscillation equation [6], but it did not receive much attention at that time. It was not until Nobel Prize winners Hodgkin and Huxley established a two-time-scale neuron model (H–H model) and successfully reproduced the firing behavior of neurons observed in the experiment that the coupled problem at different time scales gradually attracted scholars' attention [7]. Since then, how to deal with the complex dynamic behavior and the inducing mechanism of coupled systems at different time scales has been one of the topics puzzling scholars.

Due to the lack of effective analytical methods, most of the early relevant work focused on approximate solutions, such as the quasi-static method [8] and the singular perturbation method [9]. Although these methods could well approximate the exact solution of the system, they could not explain the interaction between different scales. Therefore, geometric singular perturbation theory [10] was proposed, whose main idea is to decompose the system into two coupled fast and slow subsystems by means of invariant manifold theory and obtain the dynamic behaviors and properties of the original system by analyzing the dynamic behaviors and properties of the limiting fast and slow subsystems. However, this method cannot deal with other types of attractors (such as limit cycles and chaotic) bifurcations in coupled systems of different time scales.

It was not until Rinzel [11] proposed the fast–slow dynamic analysis method, which revealed the induction mechanism of bursting oscillation in coupled systems with different time scales, and pointed out that the essential cause of bursting oscillation is the mutual transition between the system's quiescent state and spiking state. The method provides an effective tool for understanding the nature and induction mechanism of bursting oscillations. The core idea of the fast–slow dynamic analysis method [12] is to take the slow variable as the bifurcation parameter of a fast subsystem, use classical bifurcation theory to analyze the bifurcation mechanism of the attractor of the fast subsystem changing with the slow variable, and then analyze the interaction between different scales. With the change in the slow variable, the fast subsystem has different motion modes, such as the equilibrium point or the small amplitude limit cycle corresponding to the quiescent state and the large amplitude limit cycle corresponding to the spiking state. When the slow variable changes slowly and passes through the parameter domain of different vibration modes of the fast subsystem, it will also cause the fast subsystem to transform back and forth between different motion modes, resulting in the bursting oscillation of the coupled system. Therefore, in the bursting phenomenon, with the slow variable as the bifurcation parameter, the fast subsystem will generate two important bifurcation behaviors, namely the bifurcation from the quiescent state to the spiking state and the bifurcation from the spiking state back to the quiescent state (Figure 1). The introduction of the fast–slow dynamic analysis method can explain the bifurcation connection between the spiking state and quiescent state and reveal the generation mechanism of bursting oscillation. Based on this method, various oscillation modes of bursting are obtained, such as fold/fold bursting, fold/Hopf bursting, etc.

Izhikevich improved Rinzel's classification method based on the geometric shape of the bursting and proposed a classification method based on the switching bifurcation mode between the quiescent state and the spiking state. According to the difference in bifurcation modes switching between quiescent state and spiking state, Izhikevich classified all possible bursting oscillations in which only codimension one bifurcation occurred in the fast subsystem [13,14], and pointed out six types of codimension one bifurcation in the plane: fold bifurcation, saddle node bifurcation on the invariant torus, supercritical Hopf bifurcation, subcritical Hopf bifurcation, fold limit cycle bifurcation and homoclinic bifurcation. Different categories of bursting oscillations and their inducing mechanism are given. It is confirmed that the same category of bursting behavior is not only the same bifurcation mechanism but also the same topology of bursting shape [15].

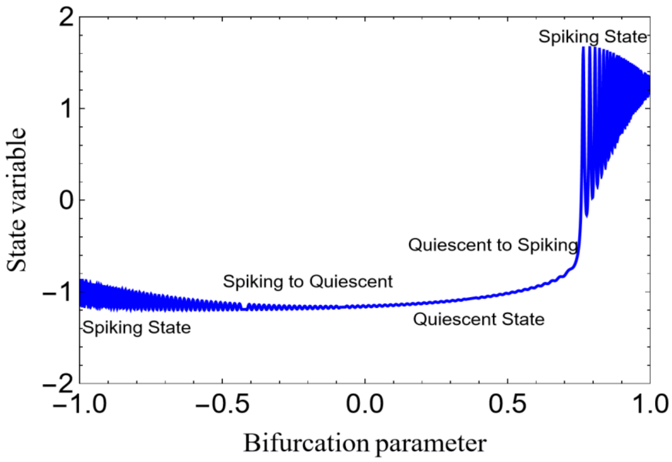

Figure 1. Two important bifurcation behaviors: the bifurcation from the quiescent state to the spiking state and the bifurcation from the spiking state back to the quiescent state.

At present, Rinzel's fast–slow dynamic analysis method and Izhikevich's classification criteria have become the classical tools for analyzing multi-time-scale coupled systems at home and abroad and have successfully solved a large number of problems of a variety of bursting oscillation mechanisms of two time scale systems.

The remainder of this paper is organized as follows. In Section 2, the equilibrium and bifurcation analysis, the bursting oscillations with two-frequency slow excitation system, the bursting oscillations with the bifurcation delay, and the bursting oscillations of the coupled system are discussed. In Section 3, learn more about the bursting oscillations from four parts, namely the simple bursting, the asymmetric bursting, the delayed bursting, and the bursting with a hysteresis loop. In Section 4, the practical application of the bursting oscillations is introduced from vibration reduction and vibration energy harvesting. In Section 5, we make conclusions and outlooks.

2. Dynamic Analysis

2.1. Bifurcation Analysis

2.1.1. Equilibrium Stability

Consider the n-dimensional nonlinear system

$$\dot{x} = f(x), x \in U \subset R^n, f : U \subset R^n \to R^n. \tag{1}$$

Then the point x_0 satisfying $f(x_0) = 0$ is an equilibrium point of System (1). Suppose that $x_0 = 0$, the linear approximation system of System (1) is

$$\dot{x} = Ax, \tag{2}$$

where $A = \frac{\partial f(x)}{\partial x}\Big|_{x=0}$ is the Jacobian matrix of $f(x)$ at $x = 0$. The stability of System (1) at the equilibrium point can be determined by the eigenvalues of the matrix A, as stated in Theorem 1.

Theorem 1 ([16])**.** *If the eigenvalues of matrix A are $\lambda_i (i = 1, 2, \cdots, n)$, then the stability of System (1) can be divided into the following three cases:*

(1) If $Re(\lambda_i) < 0 (i = 1, 2, \cdots, n)$, System (1) is asymptotically stable at the equilibrium point;
(2) If there exists at least one λ_i with $Re(\lambda_i) > 0$, System (1) is unstable at the equilibrium point;
(3) If there is no λ_i with $Re(\lambda_i) > 0$, but there is at least one $\lambda_i$with $Re(\lambda_i) = 0$, the stability of System (1) at the equilibrium point needs to be determined by the higher order term.

Let the characteristic equation of the matrix A be

$$det(A - \lambda I) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0. \tag{3}$$

We can judge the stability of the equilibrium point of System (1) by the stability of the linear approximation in System (2).

Theorem 2 (Routh–Hurwitz criterion [16])**.** *The sufficient and necessary condition for all roots of Equation (3) to have negative real parts is that all the following determinants are of the same sign*

$$\Delta_0 = a_0, \Delta_1 = a_1, \Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \cdots, \Delta_n = \begin{vmatrix} a_1 & a_0 & \cdots & 0 \\ a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{2n-1} & a_{2n-2} & \cdots & a_n \end{vmatrix}. \tag{4}$$

In the elements of the determinant above, if subscript i of a_i *greater than n, let* $a_i = 0$.

Therefore, when System (2) is asymptotically stable, the equilibrium point of System (1) is asymptotically stable. When System (2) has eigenvalues of the positive real part, the equilibrium point of System (1) is unstable.

2.1.2. Bifurcation of Smooth System

Bifurcation [17] refers to the sudden qualitative or topological changes in the behavior of a dynamic system when its parameter values (bifurcation parameters) change slightly. In the bifurcation theory of differential equations, the change in the number of singularities, the change in the stability of singularities, and the change in the number of periodic solutions when the parameter changes near a critical value are studied.

Bifurcation research can be divided into different bifurcation types according to different emphases. Bifurcation can be divided into static bifurcation and dynamic bifurcation according to different research objects. Static bifurcation refers to changes in the number and stability of equilibrium points, such as fold bifurcation, etc. Dynamic bifurcation means that the topological structure of the solution of a dynamic equation changes suddenly with the change in parameters, such as Hopf bifurcation, homoclinic bifurcation, etc.

In addition, according to the different spatial regions where the bifurcation is located, the bifurcation can be divided into local bifurcation and global bifurcation. Local bifurcation only considers the change in the topological structure of the trajectory near the equilibrium point or the closed orbit, such as fold bifurcation, Hopf bifurcation, fold limit cycle bifurcation, etc. Global bifurcation considers the behavior changes in the whole vector field of the system in the bifurcation analysis, such as homoclinic bifurcation and heteroclinic bifurcation. Sometimes the global structure of the vector field is affected by local bifurcation.

Next, we briefly introduce several bifurcation types [18–21], as shown in Table 1.

Table 1. Several simple bifurcation types and their possible simple bursting types.

Bifurcation Type	Possible Simple Bursting Type
Fold Bifurcation	Fold/fold bursting Fold/supHopf bursting Fold/LPC bursting Fold/homoclinic bursting
Hopf Bifurcation	Fold/supHopf bursting Delay subHopf/delay subHopf bursting Hopf/LPC bursting Hopf/homoclinic bursting
Fold limit cycle Bifurcation	Fold/LPC bursting Hopf/LPC bursting
Homoclinic Bifurcation	Fold/homoclinic bursting Hopf/homoclinic bursting

Fold Bifurcation

Consider the one-dimensional dynamical system

$$\dot{x} = f(x, \alpha), x \in R, \alpha \in R, \tag{5}$$

where f is a smooth function. Suppose that $f(0,0) = 0$, if the following three conditions are satisfied:

(1) Bifurcation condition: $\lambda = f_x(0,0) = 0$;
(2) Non-degeneracy condition: $f_{xx}(0,0) \neq 0$;
(3) Transversality condition: $f_\alpha(0,0) \neq 0$.

Then System (5) will have a fold bifurcation at $\alpha = 0$, and the topology normal form close to $x = 0$ is

$$\dot{x} = \alpha + sx^2, s = \pm 1. \tag{6}$$

In Reference [20], the fold bifurcation diagram of System (6) is shown for $s = 1$. With the change in parameter α, two equilibrium points disappear after collision. When $\alpha < 0$, there are two equilibrium points in the system, and the collision occurs when $\alpha = 0$. With the increase in parameter α, the equilibrium point disappears.

Hopf Bifurcation

Consider the two-dimensional dynamical system

$$\dot{x} = f(x, \alpha), x = (x_1, x_2)^T \in R^2, \alpha \in R, \tag{7}$$

where f is a smooth function. Suppose that $f(0,0) = 0$, and there are a pair of complex eigenvalues $\lambda_{1,2} = u(\alpha) \pm i\omega(\alpha)$ in the domain of $x = 0$. If the following conditions are satisfied:

(1) Bifurcation condition: when $\alpha = 0$, there exist a pair of pure virtual roots, namely $u(0) = 0, \omega(0) > 0$;
(2) Non-degeneracy condition: the first Lyapunov coefficient is not zero, namely $l_1(0) \neq 0$;
(3) Transversality condition: $u_\alpha(0) \neq 0$.

Then System (7) will have a Hopf bifurcation at $\alpha = 0$, and the topology normal form close to $x = 0$ is

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + s\left(x_1^2 + x_2^2\right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{8}$$

where $s = sgn(l_1(0)) = \pm 1$.

When $l_1(0) < 0$, the Hopf bifurcation is supercritical, the equilibrium point becomes unstable, and a stable limit cycle is generated. When $l_1(0) > 0$, the Hopf bifurcation is subcritical, the equilibrium point becomes unstable, and the unstable limit cycle disappears.

Fold limit cycle Bifurcation

With the change in parameters, the phenomenon of instability, collision, and disappearance of the limit cycle of the system is called fold limit cycle bifurcation.

By using the central manifold theorem and the Poincaré mapping method, the bifurcation problem of limit cycles can be transformed into the fixed-point problem of discrete systems. In Reference [20], the fold limit cycle bifurcation diagram is shown. The trajectory L_0 is the limit cycle of the continuous system, and P_α is the Poincaré mapping corresponding to the limit cycle. At $\alpha = 0$, one of the characteristic roots of the fixed point of P_α is $u_1 = 1$, and the other characteristic roots satisfy $0 < u_2 < 1$. In this case, a fold bifurcation occurs; that is, two limit cycles in the continuous system disappear due to collision.

Homoclinic Bifurcation

Consider the two-dimensional dynamical system (7). At $\alpha = 0$, there is a saddle equilibrium point $x_0 = 0$, eigenvalues $\lambda_1(0) < 0 < \lambda_2(0)$, and homoclinic orbit Γ_0. If the following conditions are satisfied:

(1) $\sigma_0 = \lambda_1(0) + \lambda_2(0) \neq 0$;

(2) $\dot{\beta}(0) \neq 0$, where $\beta(\alpha)$ is split function.

Then for any sufficiently small $|\alpha|$, and existing domain U_0 of $\Gamma_0 \cup x_0$, the unique limit cycle L_β is bifurcated from it. When $\sigma_0 < 0$ and $\beta > 0$, the limit cycle is stable. When $\sigma_0 > 0$ and $\beta < 0$, the limit cycle is unstable.

In addition, there are many types of bifurcation, such as Pitchfork bifurcation [22], Bogdanov–Takens bifurcation [22], heteroclinic bifurcation [23], Period-doubling bifurcation [24], Bautin bifurcation [25], Cusp bifurcation [25], Neimark–Sacker bifurcation [26], and so on.

2.2. Two-Frequency Slow Excitation Analysis

For the system with two excitation terms, we cannot directly use the traditional fast–slow dynamic analysis method to analyze the bursting oscillation behavior of the system. Consider the two-dimensional dynamical system

$$\begin{cases} \dot{x} = y \\ \dot{y} = g(x,y) + f_1 cos(\omega_1 t) + f_2 cos(\omega_2 t) \end{cases}. \tag{9}$$

There are two excitation terms in the system. When one excitation frequency is an integer multiple of the other excitation frequency, the two excitation terms can be transformed into the function of one basic excitation term by using the De Moivre formula. Then we analyze the bursting oscillation phenomenon of the system.

From the De Moivre formula, we can obtain the equation

$$(cosx + isinx)^n = cos(nx) + isin(nx), \tag{10}$$

where $i = \sqrt{-1}$. By balancing the real parts of both sides of Equation (10), we can obtain the equation

$$cos(nx) = C_n^0 cos^n x + C_n^2 cos^{n-2} x (isinx)^2 + \cdots + C_n^{2m} cos^{n-2m} x (isinx)^{2m}, \tag{11}$$

where $2m \leq n$, and $2m$ is the maximum value not greater than n. Letting $cos(nx) = f_n^*(cosx)$, we can obtain

$$f_n^*(x) = C_n^0 x^n - C_n^2 x^{n-2}\left(1 - x^2\right) + \cdots + (-1)^m C_n^{2m} x^{n-2m}\left(1 - x^2\right)^m. \tag{12}$$

Let $\omega_2 = n\omega_1$ and $\delta_1 = cos(\omega_1 t)$, then $\delta_2 = cos(\omega_2 t) = f_n^*(\delta_1)$. In this case, System (9) can become

$$\begin{cases} \dot{x} = y \\ \dot{y} = g(x,y) + f_1\delta_1 + f_2 f_n^*(\delta_1) \end{cases}. \tag{13}$$

Indeed, we can analyze the bursting oscillation phenomenon of System (9) by studying System (13).

For example, Han et al. [27] obtained amplitude-modulated bursting by using the multiple-frequency slow parametric modulation method, where the excitation frequencies had great effects on amplitude-modulated bursting. Han et al. [28] studied pitchfork-hysteresis bursting with additional slow parametric excitation. When the additional excitation frequency is an integer multiple of the original excitation frequency, they found two hysteresis modes. Zhang et al. [29] studied bursting oscillations of the system with two slow-varying periodic excitation terms. Zhou et al. [30] studied the zero-crossing pinched "Hopf/Hopf"-hysteresis bursting with the additional excitation term. Zhou et al. [31] studied the bursting oscillations of the smallest chemical reaction system with multi-frequency slow parametric excitation terms and observed four novel bursting oscillations. Wang et al. [32] studied a class of multi-timescale non-autonomous dynamical systems with parametric and external excitation terms. They could observe novel multi-bifurcation cascaded periodic, quasi-periodic, and chaotic bursting oscillations. Xiao et al. [33] studied hysteresis, amplitude death, and oscillation death mechanisms of the Duffing–Van der Pol model with parametric and external excitation terms. Ma et al. [34] studied the complex bursting oscillations of the Duffing–Van der Pol system with two slow-varying periodic excitation terms and obtained four novel bursting oscillations.

When two excitation terms of the system have incommensurate frequencies, a frequency-truncation fast–slow analysis method is used to truncate the incommensurate frequencies, and the De Moivre formula is used to convert them into the function of one basic excitation term, and the system's bursting oscillation is analyzed. For example, Han et al. [35] proposed the frequency-truncation fast–slow analysis method to study the bursting oscillations in parametrically and externally excited systems with two slow incommensurate excitation frequencies.

2.3. Bifurcation Delay

Bursting oscillation is a typical representative of the complex dynamic behavior of the multiple time-scale systems, and bifurcation delay is a common phenomenon in bursting oscillations. When the bifurcation parameter passes through the bifurcation point, the stability changes, but the system state does not change immediately and remains in the current state for a period of time, which is the phenomenon of bifurcation delay [1,36]. Bifurcation delay is manifested in a variety of bifurcations, such as Hopf bifurcation, transcritical bifurcation, period-doubling bifurcation and so on. Two important factors play a decisive role in the generation process of bifurcation-delayed bursting: one is the parameter region of bifurcation-delayed termination, and the other is the attractor to which the trajectory switches when the bifurcation-delayed termination occurs.

Many scholars have studied the bifurcation delay phenomenon of coupled systems. On the basis of central manifold reduction and Neishtadt theory [37,38], Zheng and Wang [39] proposed a method to determine the bifurcation delay exit point for the time-delay fast–slow systems. Han et al. [40] studied the chaos bursting phenomenon of the parameter-driven Lorentz system and obtained a new "delayed pitchfork/boundary crisis" chaos bursting oscillation. Tasso and Theodore [41] studied the stability delay loss caused by the Hopf bifurcation in reaction–diffusion equations with a slow-varying parameter. Zhang et al. [42] studied the bursting oscillations of the Duffing system with multi-frequency parametric excitation terms and revealed the complex "point-point" type bursting oscillation related to delayed pitchfork bifurcation. Zheng et al. [43] studied the bursting oscillations of the parametric excited three-dimensional chaotic system. When the slow-varying excitation term periodically passed through the supercritical pitchfork bifurcation point, the system had obvious delay behavior. Li et al. [44] studied a parameter-driven Rucklidge system and discussed four bursting oscillations caused by delayed pitchfork bifurcation. Ma et al. [45] studied a parameter-driven Van der Pol–Duffing system and obtained four mixed-mode vibrations caused by the pitchfork bifurcation delay phenomenon. Li et al. [46] studied a parameter-driven Shimizu–Morioka system and revealed some new bursting oscillations caused by delayed transcritical bifurcation. Deng and Li [47] proposed a chaotic memory circuit with external periodic disturbance, explored the mechanism of bursting oscillation and delay effect caused by symmetric Hopf, and measured the delay time of Hopf-induced bursting under different external excitation. The results show that the relationship between delay time and external excitation frequency is given by a power law. Ma et al. [48] studied the bursting oscillations caused by pitchfork bifurcation delay and supHopf bifurcation delay based on a generalized parameter forced Van der Pol–Duffing system. Zhang et al. [49] studied the mechanism of some special phenomena in the bursting oscillations on an improved Van der Pol–Duffing system with periodic parameter excitation. When the excitation amplitude or frequency increases to a certain extent, the delay caused by the motion inertia becomes larger, which may cause the trajectory to pass through the parameter region of the stable attractor by bifurcation.

Based on the continuous development of bifurcation delay, there are still many problems to be further studied. For example, the termination region of the bifurcation delay is further calculated and analyzed in order to improve the type of bursting oscillations under the bifurcation delay. Further analysis is made on the bursting oscillation modes with multiple bifurcation delay co-existing, and new types of bursting oscillations are sought. The bifurcation delay phenomenon of bursting oscillations is analyzed mathematically, and the mechanism of attractor transition induced by bifurcation delay is understood.

2.4. Coupled System

When using the fast–slow analysis method for the coupled system, it is necessary to first divide the fast and slow subsystems of the system according to the actual situation and then regard the slow variable as the bifurcation parameter. By analyzing the equilibrium point and its stability of the fast subsystem, various equilibrium states and their corresponding bifurcation behaviors for a change in the slow variable are given. The different motion modes of the fast subsystem are discussed for transitions between different equilibrium states. Based on the fast–slow analysis method, many scholars have performed analyses in recent years on coupled systems with two time scales.

For the two-dimensional model of one fast and one slow, Chumakov et al. [50] established a two-dimensional model of the metal catalytic oxidation process and analyzed the bursting oscillations of the model. Kiss et al. [51] established a two-dimensional model of point chemical oscillators caused by the geometric structure. Urvolgyi et al. [52] further analyzed the bursting oscillations of the two-dimensional model. Yang et al. [53] and Wermus et al. [54] studied the bursting oscillations of a laser system.

For the three-dimensional model of two fast and one slow, Bao et al. [55] proposed a novel third-order autonomous memristive diode bridge-based oscillator and discussed the bursting oscillations and their bifurcation mechanisms. Bao et al. [56] proposed a fast–slow three-dimensional autonomous Morris–Lecar neuron model and studied the bursting oscillation behaviors. Baldemir et al. [57] studied a three-dimensional reduced IHC model and discussed the path connecting pseudo-plateau bursting and mixed-mode oscillations. Barrio et al. [58] studied the Hindmarsh–Rose neuron model and the pancreatic β-cell model. Gou et al. [59] considered the vector field with the Hopf bifurcation at the origin and observed several kinds of bursting attractors. Rakaric et al. [60] considered the time-varying asymmetric potential method to study the bursting oscillations in the system with low-frequency excitation. Zhao et al. [61] considered a hybrid Rayleigh–Van der Pol–Duffing system driven by external and parametric slow-varying excitation terms and discussed the complex bursting oscillations of the system.

For the four-dimensional model with two fast and two slow, Guckenheimer [62] studied the singular Hopf bifurcation condition with the two-dimensional slow subsystem and obtained the bursting oscillations. Curtu [63] further analyzed the bursting oscillation behaviors of two fast and two slow four-dimensional systems. Domestic scholars, represented by Lu et al., have carried out a lot of work on different neuron models in this field and achieved a lot of achievements [64–66].

For the four-dimensional model with three fast and one slow, Ma et al. [67] studied the bursting oscillations and its mechanism of a modified Van der Pol–Duffing circuit system with slow-varying periodic excitation. Some compound bursting oscillations, namely "delayed supHopf/fold cycle-subHopf/supHopf" bursting and "subHopf/supHopf" bursting via "delayed supHopf/supHopf" hysteresis loop, are observed. Huang et al. [68] studied a novel three-dimensional chaotic system and discussed different types of bursting oscillation behaviors. Li et al. [69] studied the fast–slow Chay model and found out that bursting oscillations exhibit period-adding bifurcations. Wu et al. [70] studied the Hindmarsh–Rose model and discussed the fold/homoclinic bursting oscillation behaviors. Chen and Chen [71] studied three types of aperiodic MMOs of a three-dimensional nonautonomous system with slow-varying parametric excitation. Lin et al. [72] designed a simple autonomous three-element-based memristive circuit and discussed the bifurcation mechanism of the symmetric periodic fold/Hopf cycle-cycle bursting oscillation. Zhang et al. [73] studied the permanent magnet synchronous motor system and discussed the influence of complex bursting oscillation behaviors.

In addition to the above types of models, many higher-dimensional models [74–80] have been studied. At the same time, when there are time delay factors in the coupled system, the system will produce a more abundant bursting oscillation phenomenon. For example, Yu et al. [81–83] studied the influence of time-invariant delay on a non-autonomous system with slow parametric excitation and discussed different types of bursting oscillations.

3. Analysis of Bursting Oscillations

Using Rinzel's fast–slow dynamic analysis method, the bursting oscillations can be classified in two ways. One is to classify bursting oscillations according to their geometric structure, such as point/point bursting [84] and cycle/cycle bursting [84]. The other is to classify according to the bifurcation mode of fast–slow conversion, such as fold/fold bursting and fold/Hopf bursting. In this section, Izhikevich's naming method for the types of bursting oscillation is adopted [14]; that is, two bifurcations that make the system enter the spiking state from the quiescent state and return to the quiescent state from the spiking state are used to name the types of bursting oscillation. At present, the classification method and the fast–slow dynamic analysis method have become the classical methods for studying the bursting oscillation mechanism of multi-time-scale systems. Saggio et al. [85] studied a model with two subsystems with different time scales.

3.1. Simple Bursting

In coupled systems, the common bifurcations are fold bifurcation, Hopf bifurcation (including supercritical Hopf bifurcation and subcritical Hopf bifurcation), fold limit cycle (that is LPC) bifurcation and homoclinic bifurcation. These bifurcations can form some simple bursting oscillations, as shown in Table 1.

Fold/fold bursting

In Reference [86], the overlap of the equilibrium curve and transformed phase portrait of fold/fold bursting oscillation is shown. The system enters the spiking state from the quiescent state after two-fold bifurcations due to symmetry.

Fold/supHopf bursting

In Reference [86], the overlap of the equilibrium curve and transformed phase portrait of fold/supHopf bursting oscillation is shown. When the fold bifurcation is passed, the system enters the spiking state from the quiescent state. When the supHopf bifurcation is passed, the system generates a limit cycle around which the orbit move.

Delay subHopf/delay subHopf bursting

In Reference [86], the overlap of the equilibrium curve and transformed phase portrait of delay subHopf/delay subHopf bursting oscillation is shown. Due to symmetry, the system enters the spiking state from the quiescent state after two subHopf bifurcations. There is a delayed phenomenon when the subHopf bifurcation is encountered.

Fold/LPC bursting

In Reference [87], the overlap of the equilibrium curve and transformed phase portrait of fold/LPC bursting oscillation is shown. When the fold bifurcation is passed, the system enters the spiking state from the quiescent state. When passing through the LPC bifurcation, the stable and unstable limit cycles collide and disappear, and the system enters the quiescent state from the spiking state.

Hopf/LPC bursting

In Reference [87], the overlap of the equilibrium curve and transformed phase portrait of Hopf/LPC bursting oscillation is shown. After passing the subcritical Hopf bifurcation point, due to the influence of the slow-varying process, the system enters the spiking state after a period of time. When passing through the LPC bifurcation, the stable and unstable limit cycles collide and disappear, and the system enters the quiescent state from the spiking state.

Fold/Homoclinic bursting

In Reference [87], the overlap of the equilibrium curve and transformed phase portrait of fold/homoclinic bursting oscillation is shown. When the fold bifurcation is passed, the system enters the spiking state from the quiescent state. When passing through homoclinic bifurcation, the stable limit cycle disappears, and the system enters the quiescent state.

Hopf/Homoclinic bursting

In Reference [88], the overlap of the equilibrium curve and transformed phase portrait of Hopf/homoclinic bursting oscillation is shown. When passing homoclinic bifurcation, the large amplitude limit cycle disappears, causing the system to move around the limit cycle. Through the Hopf bifurcation, the system enters the quiescent state from the spiking state.

3.2. Asymmetric Bursting

Many bursting oscillations generated in the system are symmetrical, but there are also asymmetric cases. For instance, Huang et al. [68] studied a novel three-dimensional chaotic system with multiple coexisting attractors. Four periodic bursting oscillations, namely periodic asymmetric fold/fold bursting, periodic asymmetric fold/Hopf bursting, periodic symmetric fold/fold bursting, and periodic symmetric fold/Hopf bursting, were observed. Kpomahou et al. [89] studied the mixed Rayleigh–Liénard oscillator with asymmetric double-well potential driven by parametric and external excitation terms. The model had three types of asymmetric bursting oscillations, namely asymmetric fold/Hopf bursting, asymmetric fold/fold bursting, and asymmetric Hopf/Hopf bursting via fold/fold hysteresis loop.

3.3. Delayed Bursting

There is often a delay phenomenon in the bursting oscillation. Many scholars have studied the delay phenomenon. For example, Wen et al. [90] studied the bursting oscillations and their bifurcation mechanisms of a memristor-based Shimizu–Morioka system. Some complex bursting oscillations with delayed, namely symmetric compound Fold/Fold-delayed supHopf/supHopf bursting, symmetric delayed supHopf/delayed supHopf bursting, and symmetric delayed supHopf-supHopf/supHopf bursting are revealed. Zhang et al. [91] studied the bursting oscillations and their bifurcation mechanisms in a permanent magnet synchronous motor system with external load perturbation. The system had the periodic delayed subHopf/subHopf bursting with four scrolls, the signal period delayed subHopf/periodic delayed subHopf bursting, and the periodic delayed symmetric subHopf/subHopf bursting. Ma et al. [92] studied the bursting oscillations in a Lü system with orthogonal parametric and external excitation terms. Six novel bursting oscillations with delayed, namely delayed supHopf/cascaded with PD and IPD/Fold/cascaded with PD and IPD/supHopf-Fold bursting, delayed supHopf/supHopf/Fold-Fold bursting, delayed supHopf/supHopf-Fold bursting, delayed supHopf/homoclinic connections/supHopf-Fold bursting, delayed supHopf/homoclinic connections/supHopf bursting, and delayed supHopf/cascaded with PD and IPD/supHopf-Fold bursting have been discovered.

3.4. Bursting with Hysteresis Loop

The hysteresis loop phenomenon will also appear in the bursting oscillation [93–95]. For example, Lü et al. [96] studied the bursting oscillations of the pre-Bötzinger complex inspiratory neuron single-compartment model. Five types of bursting oscillations with hysteresis loop, namely the Hopf/Hopf bursting via "fold/Hopf" hysteresis loop, the Hopf/homoclinic bursting via "fold/homoclinic" hysteresis loop, the subHopf/homoclinic bursting via "fold/homoclinic" hysteresis loop, the fold cycle/homoclinic bursting via "fold/homoclinic" hysteresis loop, and the subHopf/subHopf bursting via "fold/homoclinic" hysteresis loop were observed. Duan et al. [97] studied the bursting oscillations of the pre-Bötzinger complex under a washout filter controller. Three bursting oscillations with hysteresis loop, namely Hopf/fold limit cycle/homoclinic bursting via "fold/homoclinic" hysteresis loop, Hopf/Hopf bursting via "fold/homoclinic" hysteresis loop, and subHopf/subHopf bursting via "fold/homoclinic" hysteresis loop were found. Ma et al. [98] studied the bursting oscillations of the Rayleigh–Van der Pol–Duffing system. Six kinds of bursting oscillations with hysteresis loop, namely compound fold/homoclinic-homoclinic/Hopf bursting via "fold/Homoclinic" hysteresis loop, compound homoclinic/homoclinic bursting via "Homoclinic/Homoclinic" hysteresis loop, compound fold/homoclinic-Hopf/Hopf bursting via "fold/Homoclinic" hysteresis loop, fold/homoclinic bursting via "fold/Homoclinic" hysteresis loop, fold/Hopf bursting via "fold/fold" hysteresis loop, and Hopf/Hopf bursting via "fold/fold" hysteresis loop were studied.

4. The Application of Bursting

4.1. Vibration Reduction

A vibration system attached dynamic vibration absorber is one of the common vibration reduction methods. A dynamic vibration absorber is a device that is coupled with the system to absorb the vibration energy of the main system and suppress the large vibration of the system.

The vibration of a multi-time-scale coupled system usually has not only large vibration but also high-frequency vibration. The damage effect of these vibrations on the system cannot be ignored. Therefore, it is very necessary to study the vibration control of a multi-time-scale coupled system deeply. For example, Wan et al. [99] studied the vibration control of the two-time-scale coupled Duffing system under low frequency parametric excitation by linear vibrator. It is found that the system will change from the single vibration mode to the mixed vibration mode after the addition of dynamic vibration absorber, and the vibration amplitude is significantly reduced, especially the high frequency vibration is significantly inhibited.

4.2. Vibration Energy Harvesting

Vibration energy harvesting technology can reduce harmful vibration to protect equipment. Nonlinear vibration energy harvesting technology has been widely applied in engineering fields in the past decade. Yang et al. [100] summarized the research progress of nonlinear vibration energy harvesting technology in the past ten years. Jiang et al. [101] studied the vibration-based bistable Duffing energy harvester, the vibration-based tristable energy harvester, and the vibration-based asymmetric bistable energy harvester. The bursting oscillations are observed in the energy-harvesting systems. Jiang et al. [102] discussed a novel bursting oscillation to collect energy and revealed the dynamical mechanism of the bursting oscillation. Ma et al. [103] studied the Mathieu–Van der Pol–Duffing energy harvester with parameter excitation. Five kinds of bursting oscillations, namely the "delayed supHopf/supHopf" bursting, the "delayed pitchfork/pitchfork" bursting, the "delayed Hopf-pitchfork/Hopf-pitchfork" bursting, the "delayed subHopf/supHopf" bursting and the "delayed subHopf/fold-cycle" bursting were found. Qian and Chen [104] analyzed the bursting oscillation behaviors of the bistable piezoelectric energy harvester. Lin et al. [105] studied the bursting oscillations and energy harvesting efficiency of the piezoelectric energy harvester in rotational motion with low-frequency excitation. Chen and Chen [106] discussed energy harvesting in the bursting oscillations of the piezoelectric buckled beam system. Chen et al. [107] studied the vibrational energy of a multistable nonlinear mechanical oscillator using bursting energy harvesting. Lin et al. [108] studied the bursting oscillations of the flow-induced vibration piezoelectric energy harvester with magnets by low-frequency excitation. Ma et al. [109] investigated five novel compound bursting oscillations of the parametrically amplified Mathieu–Duffing nonlinear energy harvesters. Qian and Chen [110] studied the multi-stable series model. It was found that the pentastable energy harvester has a richer multivalued response. Wu et al. [111] collected the vibrational energy from the shape memory oscillator using bursting energy harvesting.

5. Conclusions and Outlooks

5.1. Conclusions

In this paper, the bursting oscillations and their bifurcation mechanisms in the coupled systems with two time scales are introduced. Firstly, the equilibrium stability analysis and bifurcation analysis of the system are carried out, and several bifurcation types are briefly introduced: fold bifurcation, Hopf bifurcation, fold limit cycle bifurcation, and homoclinic bifurcation. For the system with two excitation terms, the two excitation terms are converted into a function of one basic excitation term by using the De Moivre formula, and then the bursting oscillations of the system are analyzed. Bifurcation delay often occurs in bursting oscillations. Bifurcation delay is found in many kinds of bifurcations, such as Hopf bifurcation, transcritical bifurcation, pitchfork bifurcation and period-doubling bifurcation. The fast–slow systems are divided into several categories according to the dimensions of the fast and slow subsystems, and the bursting oscillations are studied by using the fast–slow dynamic analysis method. Then, using the naming method of Izhikevich to name the type of bursting oscillation, the discovered bursting oscillation phenomenon is named. The system has some simple bursting oscillations, such as fold/fold bursting, fold/supHopf bursting, subHopf/subHopf bursting, fold/LPC bursting, Hopf/LPC bursting, fold/homoclinic bursting, Hopf/homoclinic bursting, etc. At the same time, the system also has some complex bursting oscillations, such as asymmetric bursting, delayed bursting, bursting with hysteresis loop, etc. Finally, the bursting oscillation phenomenon of the system has very important practical applications, such as dynamic vibration absorber and nonlinear vibration energy harvesting technology.

5.2. Outlooks

Some achievements have been made in the study of coupled systems at different time scales. However, many questions still need further study: Firstly, the study of bursting oscillations in high-dimensional systems. At present, the research on nonlinear coupled systems mainly focuses on low-dimensional systems, and the research results of the bursting oscillations in high-dimensional systems are few. With the development of nonlinear dynamics, the dynamics research of high-dimensional nonlinear systems needs further research. Secondly, the study of the bursting oscillations under high codimension bifurcation. At present, most of the research is on the bursting oscillations of codimension one in fast subsystems. The high codimension bifurcation model is the focus and difficulty of nonlinear multi-time-scale dynamics. The bursting oscillations under high codimension bifurcation need further study. Thirdly, the bursting oscillations of the nonlinear coupled systems with special structures are studied, such as non-smooth, time delay, etc. The bifurcation analysis of the fast subsystems of unconventional systems is more complex and may involve non-smooth bifurcations. Therefore, the nonlinear coupled systems with special structures also need to be discussed in depth. Fourthly, the bifurcation delay phenomenon of the bursting oscillation is studied. The bifurcation delay phenomenon still has many problems to be further studied. For example, the bursting oscillation mode with multiple bifurcation delays co-existing is further analyzed, and new bursting oscillation types are searched for. The bifurcation delay phenomenon of the bursting oscillations is analyzed mathematically, and the mechanism of attractor transition induced by bifurcation delay is understood. Fifthly, the study of classification standard of the bursting oscillations. According to Izhikevich's method, different bursting can be classified according to the bifurcation form of their fast and slow transition. However, the classification work is not comprehensive, so it is necessary to further study the classification criteria of the bursting oscillation and consider its essential structure.

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