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Review

A Review of q-Difference Equations for Al-Salam–Carlitz Polynomials and Applications to $U(n+1)$ Type Generating Functions and Ramanujan's Integrals

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Abstract: In this review paper, our aim is to study the current research progress of q-difference equations for generalized Al-Salam–Carlitz polynomials related to theta functions and to give an extension of q-difference equations for q-exponential operators and q-difference equations for Rogers–Szegö polynomials. Then, we continue to generalize certain generating functions for Al-Salam–Carlitz polynomials via q-difference equations. We provide a proof of Rogers formula for general Al-Salam–Carlitz polynomials and obtain transformational identities using q-difference equations. In addition, we gain $U(n+1)$-type generating functions and Ramanujan's integrals involving general Al-Salam–Carlitz polynomials via q-difference equations. Finally, we derive two extensions of the Andrews–Askey integral via q-difference equations.

Keywords: q-difference equation; q-exponential operator; Al-Salam–Carlitz polynomials; generating functions; Ramanujan's integral

MSC: 05A30; 11B65; 33D15; 33D45; 33D60; 39A13; 39B32

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1. Introduction

Let us review some notations and definitions for basic hypergeometric series [1]. Throughout the rest of this paper, we assume that $0 < q < 1$. For complex numbers a, the q-shifted factorials are defined by:

$$(a;q)_0 := 1,\ (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \text{ and } (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), \tag{1}$$

where $n \in \mathbb{N} := \{1,2,3,\cdots\}$. For $m \in \mathbb{N} := \{1,2,3,\cdots\}$, we have:

$$(a_1,a_2,\ldots,a_r;q)_m = (a_1;q)_m(a_2;q)_m\ldots(a_r;q)_m$$

and

$$\left(\frac{q}{a};q\right)_n = (-a)^{-n}\, q^{\binom{n+1}{2}}\frac{(aq^{-n};q)_\infty}{(a;q)_\infty} = (-a)^{-n}\, q^{\binom{n+1}{2}}(aq^{-n};q)_n. \tag{2}$$

Also, for m large, we have:

$$(a_1,a_2,\ldots,a_r;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\ldots(a_r;q)_\infty.$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \frac{(q^{-n};q)_k}{(q;q)_k}(-1)^k q^{nk-\binom{k}{2}} \qquad (0 \leqq k \leqq n). \tag{3}$$

The basic hypergeometric function is defined as follows ([2] Chapter 3), ([3] p. 347, Equation (272)), [4,5]:

$$ {}_{\mathfrak{r}}\Phi_{\mathfrak{s}}\left[\begin{matrix} a_1, a_2, \ldots, a_{\mathfrak{r}}; \\ \\ b_1, b_2, \ldots, b_{\mathfrak{s}}; \end{matrix} \; q; z\right] := \sum_{n=0}^{\infty} \left[(-1)^n \, q^{\binom{n}{2}}\right]^{1+\mathfrak{s}-\mathfrak{r}} \frac{(a_1, a_2, \ldots, a_{\mathfrak{r}}; q)_n}{(b_1, b_2, \ldots, b_{\mathfrak{s}}; q)_n} \frac{z^n}{(q;q)_n}, \tag{4} $$

where the infinite series in Equation (4) is convergent for either $|q| < 1$ and $|z| < \infty$ when $\mathfrak{r} \leq \mathfrak{s}$ or $|q| < 1$ and $|z| < 1$ when $\mathfrak{r} = \mathfrak{s} + 1$, provided that no zero appears in the denominator.

The usual q-differential operator D_x and its dual θ_x, are defined by [6,7]

$$ D_x\{h(x)\} = \frac{h(x) - h(xq)}{x}, \quad \theta_x\{h(x)\} = \frac{h(xq^{-1}) - h(x)}{q^{-1}x}. \tag{5} $$

In the limit case, when the parameter is $q \to 1-$, we have:

$$ \lim_{q \to 1-} \left\{(1-q)^{-1} D_x\{h(x)\}\right\} = h'(x) \quad \text{and} \quad \lim_{q \to 1-} \left\{(1-q)^{-1} \theta_x\{h(x)\}\right\} = h'(x), $$

which implies that the derivative $h'(x)$ exists.

The Leibniz rule for D_x or θ_x is the following identity [6–8]:

$$ D_x^n\{f(x)g(x)\} = \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q D_x^k\{f(x)\} D_x^{n-k}\left\{g(xq^k)\right\}, \tag{6} $$

or

$$ \theta_x^n\{f(x)g(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \theta_x^k\{f(x)\} \theta_x^{n-k}\left\{g(xq^{-k})\right\}, \tag{7} $$

where D_x^0 and θ_x^0 are understood as the identities.

According to the action of operators D_x and θ_x, Chen and Liu [6,7] have developed the clever method name "*Parameter Augmentation*" to derive several q-series identities by introducing two suitable q-exponential operators, $\mathbb{T}(yD_x)$ and $\mathbb{E}(y\theta_x)$,

$$ \mathbb{T}(yD_x) = \sum_{n=0}^{\infty} \frac{(yD_x)^n}{(q;q)_n}, \quad \mathbb{E}(y\theta_x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (y\theta_x)^n}{(q;q)_n}, \tag{8} $$

which are rich and powerful tools for basic hypergeometric series. The definition of $(a;q)_n$ will be given in the next section. By the means of the two q-operators defined in (8), Liu [9,10] deduced several results involving Bailey's ${}_6\psi_6$, q-Mehler formulas for Rogers–Szegö polynomials and a q-integral of Sears' transformation using q-difference equations. For more information on q-exponential operators, see the details in [6,7,9–15].

In ([9] Theorems 1 and 2), Liu gave the proof of the following proposition.

Proposition 1. *Let $f(x,y)$ be a two-variable analytic function in the neighborhood of $(x,y) = (0,0) \in \mathbb{C}^2$.*

(I.1) If the function $f(x,y)$ satisfies the q-difference equation

$$ bf(xq, y) - xf(x, yq) = (y - x)f(x, y), $$

then the function $f(x,y)$ has the following form:

$$ f(x,y) = \mathbb{T}(yD_x)\{f(x,0)\}. \tag{9} $$

(I.2) If the function $f(a,b)$ satisfies the q-difference

$$ xf(xq, y) - yf(x, yq) = (x - y)f(xq, yq), $$

then the function $f(x,y)$ has the following form:

$$f(x,y) = \mathbb{E}(y\theta_x)\{f(x,0)\}. \tag{10}$$

Proposition 2 ([16] Equations (2.2) and (2.3))**.** *Let $f(x,y)$ be a two-variable analytic function at $(0,0) \in \mathbb{C}^2$. Then,*

(A) The function f can be expanded in terms of $h_n(x,y|q)$ if and only if f satisfies the functional equation

$$yf(xq,y) - xf(x,yq) = (y-x)f(x,y). \tag{11}$$

(B) The function f can be expanded in terms of $g_n(x,y|q)$ if and only if f satisfies the functional equation

$$xf(xq,y) - yf(x,yq) = (x-y)f(xq,yq), \tag{12}$$

where $h_n(b,c|q)$ and $g_n(b,c|q)$ are the Rogers–Szegö polynomials defined as [17]

$$h_n(b,c|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^k c^{n-k},$$

and

$$g_n(b,c|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} b^k c^{n-k}.$$

Liu [9,18] and Liu and Zeng [16] have systematically researched the methods of q-difference equations and q-partial difference equations through a q-series theory. This theory is called *Liu's calculus* by Aslan and Ismail [19]. For further information about q-difference equations and q-partial difference equations, see the details in [9,16,18–28].

In 1965, Al-Salam and Carlitz ([29] Equations (1.11) and (1.15)) introduced the following polynomials:

$$\phi_n^{(a)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (a;q)_k x^k,$$

and

$$\psi_n^{(a)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} (aq^{1-k};q)_k x^k.$$

Since then, these polynomials have been called *Al-Salam–Carlitz polynomials*. These polynomials play important roles in the theory of q-orthogonal polynomials. For further information about the Al-Salam–Carlitz polynomials, see the details in [4,23,24,30,31].

Askey and Suslov [32] came to the realization that the eigenfunctions of the q-oscillator are precisely proportional to the Al-Salam–Carlitz polynomials. Kim [33] gained a new combinatorial interpretation of the moments of Al-Salam–Carlitz polynomials. Wang [34] obtained a q-integral representation of the Al-Salam–Carlitz polynomials. Chen, Saad and Sun [35] deduced several properties of the Al-Salam–Carlitz polynomials. Fang [11] deduced several multilinear generating functions of the homogeneous Al-Salam–Carlitz polynomials via q-operators. Srivastava and Arjika [36] obtained bilinear generating functions involving the generalized Al-Salam–Carlitz polynomials. For further information about Al-Salam–Carlitz polynomials, see the details in [11,32–40].

In this paper, we use Liu's calculus method to research and find the q-difference equations of the general Al-Salam–Carlitz polynomials and to generalize the related results of Al-Salam–Carlitz polynomials in the above references [11,34–36].

In a dedication on the occasion of Srinivasa Ramanujan's 125th birthday, Cao ([41] Equation (4.7)) introduced the so-called *generalized* Al-Salam–Carlitz polynomials:

$$\phi_n^{(a,b,c)}(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(a,b;q)_k}{(c;q)_k} x^k y^{n-k},$$

and

$$\psi_n^{(a,b,c)}(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1)^k q^{\binom{k+1}{2}-nk}(a,b;q)_k}{(c;q)_k} x^k y^{n-k},$$

whose generating functions are ([41] Equations (4.10) and (4.11))

$$\sum_{n=0}^{\infty} \phi_n^{(a,b,c)}(x,y|q) \frac{t^n}{(q;q)_n} = \frac{1}{(yt;q)_\infty} {}_2\Phi_1 \left[\begin{matrix} a, b; & \\ & q; xt \\ c; & \end{matrix} \right], \ \left(\max\{|yt|, |xt|\} < 1\right), \tag{13}$$

and

$$\sum_{n=0}^{\infty} \psi_n^{(a,b,c)}(x,y|q) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} = (yt;q)_\infty {}_2\Phi_1 \left[\begin{matrix} a, b; & \\ & q; xt \\ c; & \end{matrix} \right], \quad \left(|xt| < 1\right).$$

Recently, the authors of [25] generalized Arjika's results [20] using q-difference equations. Saad and Abdlhusein [42] utilized the Cauchy operator in proving some identities involving the homogeneous Rogers–Szegö polynomials. Jia, Khan, Hu and Niu [43] deduced several interesting types of generating functions for q-polynomials with six variables using q-difference equations. For further results of general q-polynomials with more variables, see, for example, [20,25,26,42,43].

It is natural to ask whether some general q-polynomials exist, which are solutions of certain q-difference equations. The purpose of this paper is to search these q-difference equations that are satisfied by some of the general q-polynomials, which we have investigated in this paper. In our present investigation, we have made use of a general family of basic (or q-) polynomials, that is, the general Al-Salam–Carlitz polynomials with nine variables, as well as two q-exponential operators, with a view to constructing several q-difference equations involving nine variables. We have derived the Rogers formula for the general Al-Salam–Carlitz polynomials considered in this paper, which generalize the generating functions for the Al-Salam–Carlitz polynomials. We have also derived a class of $U(n+1)$-type generating functions and Ramanujan's integrals involving general Al-Salam–Carlitz polynomials by means of the above-mentioned q-difference equations.

In this review, our goal is to extend Arjika's [20] and Cao et al.'s [25] works using q-difference equations. We first introduce new q-polynomials, $\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$ and $\psi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$, respectively,

$$\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q) \triangleq \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(a,b,c,d;q)_k}{(e,f,g;q)_k} x^{n-k} y^k, \tag{14}$$

and

$$\psi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q) \triangleq \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1)^k q^{k(k-n)}(a,b,c,d;q)_k}{(e,f,g;q)_k} x^{n-k} y^k. \tag{15}$$

We obtain the following results for q-polynomials defined in (14) and (15).

Theorem 1. *Let $f(a,b,c,d,e,f,g,x,y)$ be a nine-variable analytic function in the neighborhood of $(a,b,c,d,e,f,g,x,y) = (0,0,0,0,0,0,0,0,0) \in \mathbb{C}^9$.*

(I) If $f(a,b,c,d,e,f,g,x,y)$ can be expanded in terms of $\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$ if and only if

$$
\begin{aligned}
&x\{f(a,b,c,d,e,f,g,x,y)-f(a,b,c,d,e,f,g,x,yq)-(e+f+g)q^{-1}[f(a,b,c,d,e,f,g,x,yq)\\
&-f(a,b,c,d,e,f,g,x,yq^2)]+(ef+eg+fg)q^{-2}[f(a,b,c,d,e,f,g,x,yq^2)\\
&-f(a,b,c,d,ev,xq,yq^3)]-efgq^{-3}[f(a,b,c,d,e,x,yq^3)-f(a,b,c,d,e,f,g,x,yq^4)]\}\\
&=y\{[f(a,b,c,d,e,f,g,x,y)-f(a,b,c,d,e,f,g,xq,y)]\\
&-(a+b+c+d)[f(a,b,c,d,e,f,g,x,yq)-f(a,b,c,d,e,f,g,xq,yq)]\\
&+(ab+ac+ad+bc+bd+cd)[f(a,b,c,d,e,f,g,x,yq^2)-f(a,b,c,d,e,f,g,xq,yq^2)]\\
&-(abc+abd+acd+bcd)[f(a,b,c,d,e,f,g,x,yq^3)-f(a,b,c,d,e,f,g,xq,yq^3)]\\
&-abcd[f(a,b,c,d,e,f,g,x,yq^4)-f(a,b,c,d,e,f,g,xq,yq^4)]\}. \qquad (16)
\end{aligned}
$$

(II) If $f(a,b,c,d,e,f,g,x,y)$ can be expanded in terms of $\psi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$ if and only if

$$
\begin{aligned}
&x\{f(a,b,c,d,e,f,g,xq,y)-f(a,b,c,d,e,f,g,xq,yq)\\
&\quad-(e+f+g)q^{-1}[f(a,b,c,d,e,f,g,xq,yq)-f(a,b,c,d,e,f,g,xq,yq^2)]\\
&\quad+(ef+eg+fg)q^{-2}[f(a,b,c,d,e,f,g,xq,yq^2)-f(a,b,c,d,e,f,g,xq,yq^3)]\\
&\quad+efgq^{-3}[f(a,b,c,d,e,f,g,xq,yq^3)-f(a,b,c,d,e,f,g,xq,yq^4)]\}\\
&=y\{[f(a,b,c,d,e,f,g,xq,yq)-f(a,b,c,d,e,f,g,x,yq)]\\
&\quad-(a+b+c+d)[f(a,b,c,d,e,f,g,xq,yq^2)-f(a,b,c,d,e,f,g,x,yq^2)]\\
&\quad+(ab+ac+ad+bc+bd+cd)[f(a,b,c,d,e,f,g,xq,yq^3)-f(a,b,c,d,e,f,g,x,yq^3)]\\
&\quad-(abc+abd+acd+bcd)[f(a,b,c,d,e,f,g,xq,yq^4)-f(a,b,c,d,e,f,g,x,yq^4)]\\
&\quad-abcd[f(a,b,c,d,e,f,g,xq,yq^5)-f(a,b,c,d,e,f,g,x,yq^5)]\}. \qquad (17)
\end{aligned}
$$

Remark 1. *For $d=g=0$ and $f=d$ in Theorem 1, we get the concluding remarks of Cao et al. [25]. For $c=d=e=f=g=0$, and $b\to\frac{1}{b}$, $y\to yb$, $b\to 0$, then Equation (16) reduces to the concluding remarks in [20]. For $a=b=c=d=e=f=g=0$ in Theorem 1, Equations (16) and (20) reduce to Equations (11) and (12), respectively.*

In fact, the new q-polynomials (14) and (15) have the following operator representations:

$$\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)=\mathbb{T}(a,b,c,d,e,f,g,yD_x)\{x^n\}, \qquad (18)$$

and

$$\psi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)=\mathbb{E}(a,b,c,d,e,f,g,y\theta_x)\{x^n\}, \qquad (19)$$

so we also obtain the following equivalent forms of Theorem 1.

Theorem 2. *Let $G(a,b,c,d,e,f,g,x,y)$ be a nine-variable analytic function in the neighborhood of $(a,b,c,d,e,f,g,x,y)=(0,0,0,0,0,0,0,0,0)\in\mathbb{C}^9$.*

(I) If $G(a,b,c,d,e,f,g,x,y)$ satisfies the difference equation

$$
\begin{aligned}
&x\{G(a,b,c,d,e,f,g,x,y) - G(a,b,c,d,e,f,g,x,yq) \\
&\quad - (e+f+g)q^{-1}[G(a,b,c,d,e,f,g,x,yq) - G(a,b,c,d,e,f,g,x,yq^2)] \\
&\quad + (ef+eg+fg)q^{-2}[G(a,b,c,d,e,f,g,x,yq^2) - G(a,b,c,d,e,f,g,x,yq^3)] \\
&\quad - efgq^{-3}[G(a,b,c,d,e,f,g,x,yq^3) - G(a,b,c,d,e,f,g,x,yq^4)]\} \\
&\quad = y\{[G(a,b,c,d,e,f,g,x,y) - G(a,b,c,d,e,f,g,xq,y)] \\
&\quad - (a+b+c+d)[G(a,b,c,d,e,f,g,x,yq) - G(a,b,c,d,e,f,g,xq,yq)] \\
&\quad + (ab+ac+ad+bc+bd+cd)[G(a,b,c,d,e,f,g,x,yq^2) - G(a,b,c,d,e,f,g,xq,yq^2)] \\
&\quad - (abc+abd+acd+bcd)[G(a,b,c,d,e,f,g,x,yq^3) - G(a,b,c,d,e,f,g,xq,yq^3)] \\
&\quad + abcd[G(a,b,c,d,e,f,g,x,yq^4) - G(a,b,c,d,e,f,g,xq,yq^4)]\},
\end{aligned} \tag{20}
$$

then we have

$$G(a,b,c,d,e,f,g,x,y) = \mathbb{T}(a,b,c,d,e,f,g,yD_x)\{G(a,b,c,d,e,f,g,x,0)\}. \tag{21}$$

(II) If $G(a,b,c,d,e,f,g,x,y)$ satisfies the difference equation

$$
\begin{aligned}
&x\{G(a,b,c,d,e,f,g,xq,y) - G(a,b,c,d,e,f,g,xq,yq) \\
&\quad - (e+f+g)q^{-1}[G(a,b,c,d,e,f,g,xq,yq) - G(a,b,c,d,e,f,g,xq,yq^2)] \\
&\quad + (ef+eg+fg)q^{-2}[G(a,b,c,d,e,f,g,xq,yq^2) - G(a,b,c,d,e,f,g,xq,yq^3)] \\
&\quad - efgq^{-3}[G(a,b,c,d,e,f,g,xq,yq^3) - G(a,b,c,d,e,f,g,xq,yq^4)]\} \\
&\quad = y\{[G(a,b,c,d,e,f,g,xq,yq) - G(a,b,c,d,e,f,g,x,yq)] \\
&\quad - (a+b+c+d)[G(a,b,c,d,e,f,g,xq,yq^2) - G(a,b,c,d,e,f,g,x,yq^2)] \\
&\quad + (ab+ac+ad+bc+bd+cd)[G(a,b,c,d,e,f,g,xq,yq^3) - G(a,b,c,d,e,f,g,x,yq^3)] \\
&\quad - (abc+abd+acd+bcd)[G(a,b,c,d,e,f,g,xq,yq^4) - G(a,b,c,d,e,f,g,x,yq^4)] \\
&\quad + abcd\,[G(a,b,c,d,e,f,g,xq,yq^5) - G(a,b,c,d,e,f,g,x,yq^5)]\},
\end{aligned} \tag{22}
$$

then we have

$$G(a,b,c,d,e,f,g,x,y) = \mathbb{E}(a,b,c,d,e,f,g,y\theta_x)\{G(a,b,c,d,e,f,g,x,0)\}. \tag{23}$$

Remark 2. *For $a = b = c = d = e = f = g = 0$ in Theorem 2, Equations (21) and (23) reduce to Equations (9) and (10), respectively.*

Our investigation is organized as follows: In Section 2, we give the proofs of the main theorems. In Section 3, we provide some identities of q-exponential operators involving the q- polynomials $\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$ and $\psi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$. In Section 4, we deduce generating functions for q- polynomials $\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$ and $\psi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$. In Section 5, we obtain two bilinear generating functions for q- polynomials $\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$ and $\psi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$ using q-difference equations. In Section 6, we gain a transformational identity from q-difference equations. In Section 7, we deduce $U(n+1)$-type generating functions for generalized q-polynomials $\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$ and $\psi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$ using q-difference equations. In Section 8, we deduce generalizations of Ramanujan's integrals. In Section 9, we consider two extensions of the Andrews–Askey integral.

2. Proof of Main Results

To prove our main results, the following Hartogs's theorem will be needed. For more details, see [2,18].

Lemma 1 ([44] Hartogs's theorem). *If a complex-valued function is holomorphic (analytic) in each variable separately in an open domain $D \in \mathbb{C}^n$, then it is holomorphic (analytic) in D.*

Before proving Theorem 1, we need the following fundamental property of several complex variables.

Lemma 2 ([45] Proposition 1). *If $F(x_1, x_2, \ldots, x_k)$ is analytic at the origin $(0, 0, \ldots, 0) \in \mathbb{C}^k$, then, F can be expanded in an absolutely convergent power series,*

$$F(x_1, x_2, \ldots, x_k) = \sum_{n_1, n_2, \ldots, n_k = 0}^{\infty} \alpha_{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}.$$

In order to prove our main results, we need the following lemmas.

Lemma 3. *Let D_x be defined as in (5). Then, we have [9]*

$$D_x^k \left\{ \frac{1}{(xt;q)_\infty} \right\} = \frac{t^k}{(xt;q)_\infty}, \quad \theta_x^k \{(xt;q)_\infty\} = (-t)^k (xt;q)_\infty, \tag{24}$$

$$D_x^k\{x^n\} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_{n-k}} x^{n-k}, & 0 \le k \le n-1 \\ (q;q)_n, & k = n \\ 0, & k \ge n+1. \end{cases} \tag{25}$$

Lemma 4 ([46] Equation (13)). *Let D_x be defined as in (5). Then, for $n \in \mathbb{N}$, we have:*

$$D_x^n \left\{ \frac{(xv;q)_\infty}{(xt;q)_\infty} \right\} = t^n (v/t;q)_n \frac{(xvq^n;q)_\infty}{(xt;q)_\infty}. \tag{26}$$

Lemma 5. *Let D_x be defined as in (5). Then, for $n \in \mathbb{N}$, we have:*

$$D_x^n \left\{ \frac{(xbcd;q)_\infty}{(xc, xd;q)_\infty} \right\} = \frac{(xbcd;q)_\infty}{(xc, xd;q)_\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{c^k d^{n-k} (xd;q)_k}{(xbcd;q)_k}. \tag{27}$$

Proof of Lemma 5. With straightforward calculations using Lemma 4, we have

$$\begin{aligned} D_x^n \left\{ \frac{(xbcd;q)_\infty}{(xc, xd;q)_\infty} \right\} &= \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q D_x^k \left\{ \frac{(xbcd;q)_\infty}{(xc;q)_\infty} \right\} D_x^{n-k} \left\{ \frac{1}{(xdq^k;q)_\infty} \right\} \\ &= \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q c^n (bd;q)_\infty \frac{(xbcdq^k;q)_\infty}{(xc;q)_\infty} \frac{(dq^k)^{n-k}}{(xdq^k;q)_\infty} \\ &= \frac{(xbcd;q)_\infty}{(xc, xd;q)_\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{c^k d^{n-k} (xd;q)_k}{(xbcd;q)_k}, \end{aligned}$$

which is the right-hand side of Equation (27). The proof of Lemma 5 is complete. □

Proof of Theorem 1. (I) Using Hartogs's theorem and the theory of several complex variables, we assume there exists a sequence $A_n \in \mathbb{C}$ such that

$$F(a, b, c, d, e, f, g, x, y) = \sum_{k=0}^{\infty} A_k(a, b, c, d, e, f, g, x) y^k. \tag{28}$$

First, substituting (28) into (16) yields

$$x\sum_{k=0}^{\infty}\Big\{1-q^k-(e+f+g)q^{k-1}+(ef+eg+fg)q^{2k-2}-efgq^{3k-3}\Big\}A_k(a,b,c,d,e,f,g,x)y^k$$
$$=\sum_{k=0}^{\infty}\Big\{1-(a+b+c+d)q^k+(ab+ac+ad+bc+bd+cd)q^{2k}$$
$$-(abc+abd+acd+bcd)q^{3k}-abcdq^{4k}\Big\}$$
$$\times\Big\{A_k(a,b,c,d,e,x)-A_k(a,b,c,d,e,xq)\Big\}y^{k+1}, \tag{29}$$

which implies

$$x\sum_{k=0}^{\infty}(1-q^k)(1-eq^{k-1})(1-fq^{k-1})(1-gq^{k-1})A_k(a,b,c,d,e,f,g,x)y^k$$
$$=\sum_{k=0}^{\infty}(1-aq^k)(1-bq^k)(1-cq^k)(1-dq^k)\Big\{A_k(a,b,c,d,e,f,g,x)$$
$$-A_k(a,b,c,d,e,f,g,xq)\Big\}y^{k+1}. \tag{30}$$

Equating coefficients of y^k, $k\geq 1$ on both sides of Equation (30), we have:

$$x(1-q^k)(1-eq^{k-1})(1-fq^{k-1})(1-gq^{k-1})A_k(a,b,c,d,e,f,g,x)$$
$$=(1-aq^{k-1})(1-bq^{k-1})(1-cq^{k-1})(1-dq^{k-1})-(abc+abd+acd+bcd)q^{3k}$$
$$\times\Big\{A_{k-1}(a,b,c,d,e,f,g,x)-A_{k-1}(a,b,c,d,e,f,g,xq)\Big\}, \tag{31}$$

which is equivalent to

$$A_k(a,b,c,d,e,f,g,x)$$
$$=\frac{(1-aq^{k-1})(1-bq^{k-1})(1-cq^{k-1})(1-dq^{k-1})}{(1-q^k)(1-eq^{k-1})(1-fq^{k-1})(1-gq^{k-1})}\cdot D_x\{A_{k-1}(a,b,c,d,e,f,g,x)\}.$$

By iteration, we gain

$$A_k(a,b,c,d,e,f,g,x)=\frac{(a,b,c,d;q)_k}{(q,e,f,g;q)_k}\cdot D_x^k\{A_0(a,b,c,d,e,f,g,x)\}.$$

Letting $F(a,b,c,d,e,f,g,x,0)=A_0(a,b,c,d,e,f,g,x)=\sum_{n=0}^{\infty}\mu_n x^n$ yields

$$A_k(a,b,c,d,e,f,g,x)=\frac{(a,b,c,d;q)_k}{(q,e,f,g;q)_k}\cdot\sum_{n=0}^{\infty}\mu_n\frac{(q;q)_n}{(q;q)_{n-k}}x^{n-k}, \tag{32}$$

we have

$$F(a,b,c,d,e,f,g,x,y)=\sum_{k=0}^{\infty}\frac{(a,b,c,d;q)_k}{(q,e,f,g;q)_k}\sum_{n=0}^{\infty}\mu_n\frac{(q;q)_n}{(q;q)_{n-k}}x^{n-k}y^k$$
$$=\sum_{n=0}^{\infty}\mu_n\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(a,b,c,d;q)_k}{(e,f,g;q)_k}x^{n-k}y^k$$
$$=\sum_{n=0}^{\infty}\mu_n\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q).$$

On the other hand, if the function $F(a,b,c,d,e,f,g,x,y)$ can be expanded in terms of $\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$, we verify that $F(a,b,c,d,e,f,g,x,y)$ satisfies (16). Similarly, we deduce (II). The proof of Theorem 1 is complete. □

Before proving Theorem 2, we need to define the following two generalized q-exponential operators as

$$\mathbb{T}(a,b,c,d,e,f,g,yD_x) = \sum_{n=0}^{\infty} \frac{(a,b,c,d;q)_n}{(q,e,f,g;q)_n}(yD_x)^n, \tag{33}$$

and

$$\mathbb{E}(a,b,c,d,e,f,g,y\theta_x) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}(a,b,c,d;q)_n}{(q,e,f,g;q)_n}(y\theta_x)^n. \tag{34}$$

Proof of Theorem 2. Starting from the theory of several complex variables, we begin to solve for the q-difference. First, we may assume there exists a sequence $A_n \in \mathbb{C}$ such that

$$G(a,b,c,d,e,f,g,x,y) = \sum_{k=0}^{\infty} A_k(a,b,c,d,e,f,g,x)y^k. \tag{35}$$

Substituting this equation into (20) and comparing coefficients of y^k $k \geq 1$, we readily find that

$$\begin{aligned} x(1-q^k)(1-eq^{k-1})(1-fq^{k-1})(1-gq^{k-1})A_k(a,b,c,d,e,f,g,x) \\ = (1-aq^{k-1})(1-bq^{k-1})(1-cq^{k-1})(1-dq^{k-1}) \\ \times \Big\{A_{k-1}(a,b,c,d,e,f,g,x) - A_{k-1}(a,b,c,d,e,f,g,xq)\Big\}, \end{aligned} \tag{36}$$

which implies

$$\begin{aligned} &A_k(a,b,c,d,e,x) \\ &= \frac{(1-aq^{k-1})(1-bq^{k-1})(1-cq^{k-1})(1-dq^{k-1})}{(1-q^k)(1-eq^{k-1})(1-fq^{k-1})(1-gq^{k-1})} \cdot D_x\{A_{k-1}(a,b,c,d,e,f,g,x)\}. \end{aligned}$$

By iteration, we find that

$$A_k(a,b,c,d,e,f,g,x) = \frac{(a,b,c,d;q)_k}{(q,e,f,g;q)_k} \cdot D_x^k\{A_0(a,b,c,d,e,f,g,x)\}. \tag{37}$$

Now, we return to calculate $A_0(a,b,c,d,e,f,g,x)$. By taking $y=0$ in Equation (35), we immediately obtain

$$A_0(a,b,c,d,e,f,g,x) = G(a,b,c,d,e,f,g,x,0).$$

Substituting (37) into (35), we achieve (21) directly. This completes the proof of Theorem 2. □

3. Some Identities of q-Exponential Operators Involving the New q-Polynomials

In this section, we derive identities of q-exponential operators (38) and (39) below, which have some relation to the families of q-polynomials $\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$, $\psi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$.

Theorem 3. *Let the operators $\mathbb{T}(a,b,c,d,e,f,g,yD_x)$ and $\mathbb{E}(a,b,c,d,e,f,g,y\theta_x)$ be defined as in (33) and (34). For $\max\{|xs|,|x\omega|,|ys|,|y\omega|\} < 1$, we have:*

$$\begin{aligned} &\mathbb{T}(a,b,c,d,e,f,g,yD_x)\left\{\frac{1}{(xs,x\omega;q)_\infty}\right\} \\ &= \frac{1}{(xs,x\omega;q)_\infty}\sum_{n=0}^{\infty}\frac{(a,b,c,d;q)_n(ys)^n}{(q,e,f,g;q)_n}\, {}_5\Phi_4\left[\begin{matrix} aq^n,bq^n,cq^n,dq^n,xs; \\ eq^n,fq^n,gq^n,0; \end{matrix}\; q;y\omega\right]. \end{aligned} \tag{38}$$

For $\max\{|y\omega|,|x\omega sy/q|\}<1$, *we have:*

$$\begin{aligned}&\mathbb{E}(a,b,c,d,e,f,g,y\theta_x)\{(xs,x\omega;q)_\infty\}\\&\qquad=(xs,x\omega;q)_\infty\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(a,b,c,d;q)_n(y\omega)^n}{(q,e,f,g;q)_n}\\&\qquad\times\ {}_5\Phi_4\left[\begin{matrix}aq^n,bq^n,cq^n,dq^n,q/(x\omega);\\eq^n,fq^n,gq^n,0;\end{matrix}\ q;-\frac{x\omega sy}{q}\right].\end{aligned}\tag{39}$$

Proof. By applying (6), we observe that

$$\begin{aligned}&\mathbb{T}(a,b,c,d,e,f,g,yD_x)\left\{\frac{1}{(x\omega,xs;q)_\infty}\right\}\\&=\sum_{n=0}^{\infty}\frac{(a,b,c,d;q)_n\,y^n}{(q,e,f,g;q)_n}\,D_x^n\left\{\frac{1}{(x\omega,xs;q)_\infty}\right\}\\&=\sum_{n=0}^{\infty}\frac{(a,b,c,d;q)_n\,y^n}{(q,e,f,g;q)_n}\,\sum_{k=0}^{n}q^{k(k-n)}\begin{bmatrix}n\\k\end{bmatrix}_q D_x^k\left\{\frac{1}{(x\omega;q)_\infty}\right\}D_x^{n-k}\left\{\frac{1}{(xsq^k;q)_\infty}\right\}\\&=\sum_{n=0}^{\infty}\frac{(a,b,c,d;q)_n\,y^n}{(q,e,f,g;q)_n}\,\sum_{k=0}^{n}q^{k(k-n)}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{\omega^k}{(x\omega;q)_\infty}D_x^{n-k}\left\{\frac{1}{(xsq^k;q)_\infty}\right\}\\&=\sum_{n,k=0}^{\infty}\frac{(a,b,c,d;q)_{n+k}\,y^{n+k}}{(q,e,f,g;q)_{n+k}}\frac{\omega^kq^{-nk}}{(q;q)_k(q;q)_n(x\omega;q)_\infty}D_x^n\left\{\frac{1}{(xsq^k;q)_\infty}\right\}\\&=\sum_{n,k=0}^{\infty}\frac{(a,b,c,d;q)_{n+k}\,y^{n+k}}{(q,e,f,g;q)_{n+k}}\frac{\omega^kq^{-nk}}{(q;q)_k(q;q)_n(x\omega;q)_\infty}\frac{(sq^k)^n}{(xsq^k;q)_\infty}\\&=\frac{1}{(x\omega,xs;q)_\infty}\sum_{n=0}^{\infty}\frac{(a,b,c,d;q)_n\,(ys)^n}{(q,e,f,g;q)_n}\sum_{k=0}^{\infty}\frac{(aq^n,bq^n,cq^n,dq^n,xs;q)_k\,(y\omega)^k}{(q,eq^n,fq^n,gq^n;q)_k}\\&=\frac{1}{(xs,x\omega;q)_\infty}\sum_{n=0}^{\infty}\frac{(a,b,c,d;q)_n(ys)^n}{(q,e,f,g;q)_n}\,{}_5\Phi_4\left[\begin{matrix}aq^n,bq^n,cq^n,dq^n,xs;\\eq^n,fq^n,gq^n,0;\end{matrix}\ q;y\omega\right].\end{aligned}\tag{40}$$

Similarly, by applying (7), we find that:

$$\begin{aligned}&\mathbb{E}(a,b,c,d,e,f,g,y\theta_a)\{(x\omega,xs;q)_\infty\}\\&=\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}(a,b,c,d;q)_n\,y^n}{(q,e,f,g;q)_n}\,\theta_x^n\{(x\omega,xs;q)_\infty\}\\&=\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}(a,b,c,d;q)_n\,y^n}{(q,e,f,g;q)_n}\,\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\theta_x^k\{(x\omega;q)_\infty\}\,\theta_x^{n-k}\left\{(xsq^{-k};q)_\infty\right\}\\&=\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}(a,b,c,d;q)_n\,y^n}{(q,e,f,g;q)_n}\,\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(-\omega)^k(x\omega;q)_\infty\,\theta_x^{n-k}\left\{(xsq^{-k};q)_\infty\right\}\\&=\sum_{n,k=0}^{\infty}\frac{(-1)^{n+k}q^{\binom{n+k}{2}}(a,b,c,d;q)_{n+k}\,y^{n+k}}{(q,e,f,g;q)_{n+k}}\frac{(-\omega)^k(x\omega;q)_\infty}{(q;q)_k(q;q)_n}\,\theta_x^n\left\{\,(xsq^{-k};q)_\infty\right\}\\&=\sum_{n,k=0}^{\infty}\frac{(-1)^{n+k}q^{\binom{n+k}{2}}(a,b,c,d;q)_{n+k}\,y^{n+k}}{(q,e,f,g;q)_{n+k}}\frac{(-\omega)^k(x\omega;q)_\infty}{(q;q)_k(q;q)_n}\,(-sq^{-k})^n(xsq^{-k};q)_\infty\\&=(x\omega,xs;q)_\infty\sum_{n,k=0}^{\infty}\frac{(-1)^{n+k}q^{\binom{n+k}{2}}(a,b,c,d;q)_{n+k}\,y^{n+k}}{(q,e,f,g;q)_{n+k}}\frac{(-\omega)^k(xsq^{-k};q)_k}{(q;q)_k(q;q)_n}\,(-sq^{-k})^n\\&=(xs,x\omega;q)_\infty\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(a,b,c,d;q)_n(y\omega)^n}{(q,e,f,g;q)_n}\\&\times\ {}_5\Phi_4\left[\begin{matrix}aq^n,bq^n,cq^n,dq^n,q/(x\omega);\\eq^n,fq^n,gq^n,0;\end{matrix}\ q;-\frac{x\omega sy}{q}\right]\end{aligned}\tag{41}$$

as asserted by Theorem 3. $\square$

4. Generating Functions for q-Polynomials

In this section, we establish and prove generating functions for q-polynomials using q-difference equations.

Theorem 4. *Each of the following assertions hold true:*

$$\sum_{n=0}^{\infty} \phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)\frac{t^n}{(q;q)_n} = \frac{1}{(xt;q)_\infty}{}_4\Phi_3\left[\begin{matrix} a,b,c,d; \\ & q;yt \\ e,f,g; \end{matrix}\right], \ (\max\{|xt|,|yt|\}<1), \tag{42}$$

$$\sum_{n=0}^{\infty} \psi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)\frac{t^n}{(q;q)_n} = (xt;q)_\infty\, {}_4\Phi_4\left[\begin{matrix} a,b,c,d; \\ & q;-yt \\ 0,e,f,g; \end{matrix}\right], \quad (|yt|<1). \tag{43}$$

Remark 3. *For $d=g=0$ in Theorem 4, we get the concluding remarks of [25]. For $d=c=f=g=0$ in Theorem 4, Equation (42) reduces to Equation (13).*

Proof of Theorem 4. Let $F(a,b,c,d,e,f,g,x,y)$ be the right-hand side of Equation (42). Then, we can verify that $G(a,b,c,d,e,f,g,x,y)$ satisfies (16). So, we have

$$G(a,b,c,d,e,f,g,x,y) = \sum_{k=0}^{\infty} \mu_n \phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q),$$

and

$$G(a,b,c,d,e,f,g,x,0) = \sum_{k=0}^{\infty} \mu_n x^n = \frac{1}{(xt;q)_\infty} = \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}x^n.$$

So, $G(a,b,c,d,e,f,g,x,y)$ is equal to

$$G(a,b,c,d,e,f,g,x,y) = \sum_{k=0}^{\infty} \phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)\frac{t^n}{(q;q)_n},$$

which is the right-hand side of Equation (42).

Similarly, let $G(a,b,c,d,e,f,g,x,y)$ be the right-hand side of Equation (43). We can verify that $G(a,b,c,d,e,f,g,x,y)$ satisfies (17). So, we can use the same method to achieve Equation (43). This completes the proof. $\square$

Now, we will give and prove the following extended generating function for q-polynomials using q-difference equations.

Theorem 5 (Extended generating function for $\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$)**.** *Let the polynomials $\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$ be defined as (14). For $s\in\mathbb{N}_0$, we have:*

$$\sum_{n=0}^{\infty} \phi_{n+s}^{\binom{a,b,c,d}{e,f,g}}(x,y|q)\frac{t^n}{(q;q)_n}$$
$$= \frac{x^s}{(xt;q)_\infty}\sum_{n=0}^{\infty}\frac{(a,b,c,d;q)_n}{(q,e,f,g;q)_n}(yt)^n\sum_{j=0}^{n}\begin{bmatrix} n \\ s \end{bmatrix}_q \frac{(-1)^j q^{nj-\binom{j}{2}}(q^{-s},xt;q)_j}{(xt)^j} \tag{44}$$

provided that $\max\{|xt|,|yt|\}<1$.

Remark 4. *For $s=0$ in Theorem 5, Equation (44) reduces to (42).*

Proof of Theorem 5. Let $F(a,b,c,d,e,f,g,x,y)$ be the right-hand side of (44). Then, we have:

$$F(a,b,c,d,e,f,g,x,y) = \frac{x^k}{(xt;q)_\infty} \sum_{n=0}^{\infty} \frac{(a,b,c,d;q)_n}{(q,e,f,g;q)_n} (yt)^n \sum_{j=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1)^j q^{nj-\binom{j}{2}} (q^{-k},xt;q)_j}{(xt)^j}. \tag{45}$$

By a direct computation, one can check that (45) satisfies (16), so there exists a sequence $\mu_n \in \mathbb{C}$ such that

$$F(a,b,c,d,e,f,g,x,y) = \sum_{k=0}^{\infty} \mu_n \phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q). \tag{46}$$

For $y = 0$, Equation (46) becomes

$$\begin{aligned} &F(a,b,c,d,e,f,g,x,0) \\ &= \sum_{n=0}^{\infty} \mu_n x^n = \frac{x^s}{(xt;q)_\infty} = \sum_{n=0}^{\infty} x^s \frac{(xt)^n}{(q;q)_n} = \sum_{n=0}^{\infty} x^{n+s} \frac{t^n}{(q;q)_n} = \sum_{n=s}^{\infty} x^n \frac{t^{n-s}}{(q;q)_{n-s}}. \end{aligned}$$

Hence,

$$F(a,b,c,d,e,f,g,x,y) = \sum_{n=s}^{\infty} \phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q) \frac{t^{n-s}}{(q;q)_{n-s}} = \sum_{n=0}^{\infty} \phi_{n+s}^{\binom{a,b,c,d}{e,f,g}}(x,y|q) \frac{t^n}{(q;q)_n}.$$

This completes the proof of Theorem 5. □

Theorem 6 (Mixed generating function for $\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$). *We have:*

$$\sum_{n=0}^{\infty} \phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q) \frac{p_n(t,s)}{(q;q)_n} = \frac{(xs;q)_\infty}{(xt;q)_\infty} {}_5\Phi_4 \left[\begin{matrix} a,b,c,d,s/t; \\ e,f,g,xs; \end{matrix} \; q; yt \right], \tag{47}$$

with $\max\{|xt|,|yt|\} < 1$, *where* $p_n(x,y)$ *is the Cauchy polynomial*

$$\sum_{n=0}^{\infty} p_n(x,y) \frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}. \tag{48}$$

Corollary 1. *We have:*

$$\sum_{n=0}^{\infty} \phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} = (xt;q)_\infty \, {}_4\Phi_4 \left[\begin{matrix} a,b,c,d; \\ e,f,g,xt; \end{matrix} \; q; yt \right], \tag{49}$$

where $|yt| < 1$.

Remark 5. *Setting* $t = 0$, *in Theorem 6, Equation (47) reduces to (49). For* $s = 0$ *in Theorem 6, Equation (47) reduces to (42), respectively.*

Proof of Theorem 6. Let $G(a,b,c,d,e,f,g,x,y)$ be the right-hand side of (47). We can verify that $G(a,b,c,d,e,f,g,x,y)$ satisfies (16). So, there exists a sequence $\mu_n \in \mathbb{C}$ such that

$$G(a,b,c,d,e,f,g,x,y) = \sum_{n=0}^{\infty} \mu_n \phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q). \tag{50}$$

For $y = 0$, we have:

$$G(a,b,c,d,e,f,g,x,0) = \sum_{n=0}^{\infty} \mu_n x^n = \frac{(xs;q)_\infty}{(xt;q)_\infty} = \sum_{n=0}^{\infty} \frac{p_n(t,s)}{(q;q)_n} x^n.$$

By a direct computation, one can complete the proof. □

Theorem 7. *For* $\max|x_1x_2t|, |x_1y_2t|, |y_1/x_1|, |x_2y_1t| < 1$, *we have:*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\phi_n^{\binom{a_1,b_1,c_1,d_1}{e_1,f_1,g_1}}(x_1,y_1|q)\phi_n^{\binom{a_2,b_2,c_2,d_2}{e_2,f_2,g_2}}(x_2,y_2|q)\frac{t^n}{(q;q)_n}\\
&=\frac{1}{(x_1x_2t;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_2,b_2,c_2,d_2;q)_n(x_1y_2t)^n}{(q,e_2,f_2,g_2;q)_n}\\
&\times\sum_{j=0}^{\infty}\frac{(q^{n-j+1},x_1x_2t,a_1,b_1,c_1,d_1;q)_j(\frac{y_1}{x_1})^j}{(q,e_1,f_1,g_1;q)_j}{}_4\Phi_3\left[\begin{matrix}a_1q^j,b_1q^j,c_1q^j,d_1q^j;\\ e_1q^j,f_1q^j,g_1q^j;\end{matrix}\; q;x_2y_1t\right].
\end{aligned}\tag{51}$$

Remark 6. *For* $a_1=b_1=c_1=d_1=e_1=f_1=g_1=a_2=b_2=c_2=d_2=e_2=f_2=g_2=0$ *in Theorem 7, Equation (51) reduces to the following corollary.*

Corollary 2 ([47] Theorem 5)**.** *We have:*

$$\sum_{n=0}^{\infty}h_n(a,b|q)h_n(c,d|q)\frac{t^n}{(q;q)_n}=\frac{(abcdt^2;q)_\infty}{(act,adt,bct,bdt;q)_\infty},\tag{52}$$

where $\max\{|act|,|adt|,|bct|,|bdt|\}<1$.

Proof of Theorem 7. Let $H(a_1,b_1,c_1,d_1,e_1,f_1,g_1,x_1,y_1)$ be the right-hand side of (51). We have:

$$\begin{aligned}
H(a_1,b_1,c_1,d_1,e_1,f_1,g_1,x_1,y_1)&=\frac{1}{(x_1x_2t;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_2,b_2,c_2,d_2;q)_n(x_1y_2t)^n}{(q,e_2,f_2,g_2;q)_n}\\
\times\sum_{j=0}^{\infty}\frac{(q^{n-j+1},x_1x_2t,a_1,b_1,c_1,d_1;q)_j(\frac{y_1}{x_1})^j}{(q,e_1,f_1,g_1;q)_j}&{}_4\Phi_3\left[\begin{matrix}a_1q^j,b_1q^j,c_1q^j,d_1q^j;\\ e_1q^j,f_1q^j,g_1q^j;\end{matrix}\; q;x_2y_1t\right].
\end{aligned}\tag{53}$$

Because Equation (57) satisfies (20), we have:

$$\begin{aligned}
&H(a_1,b_1,c_1,d_1,e_1,f_1,g_1,x_1,y_1)\\
&=\mathbb{T}(a_1,b_1,c_1,d_1,e_1,f_1,g_1,y_1D_{x_1})\{H(a_1,b_1,c_1,d_1,e_1,f_1,g_1,x_1,0)\}\\
&=\mathbb{T}(a_1,b_1,c_1,d_1,e_1,f_1,g_1,y_1D_{x_1})\left\{\frac{1}{(x_1x_2t;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_2,b_2,c_2,d_2;q)_n(x_1y_2t)^n}{(q,e_2,f_2,g_2;q)_n}\right\}\\
&=\mathbb{T}(a_1,b_1,c_1,d_1,e_1,f_1,g_1,y_1D_{x_1})\left\{\sum_{n=0}^{\infty}\phi_n^{\binom{a_2,b_2,c_2,d_2}{e_2,f_2,g_2}}(x_2,y_2|q)\frac{(x_1t)^n}{(q;q)_n}\right\}\\
&=\sum_{n=0}^{\infty}\phi_n^{\binom{a_2,b_2,c_2,d_2}{e_2,f_2,g_2}}(x_2,y_2|q)\frac{t^n}{(q;q)_n}\mathbb{T}(a_1,b_1,c_1,d_1,e_1,f_1,g_1,y_1D_{x_1})\{x_1^n\}\\
&=\sum_{n=0}^{\infty}\phi_n^{\binom{a_1,b_1,c_1,d_1}{e_1,f_1,g_1}}(x_1,y_1|q)\phi_n^{\binom{a_2,b_2,c_2,d_2}{e_2,f_2,g_2}}(x_2,y_2|q)\frac{t^n}{(q;q)_n}.
\end{aligned}$$

This completes the proof of Theorem 7. □

5. Bilinear Generating Functions for $\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$ and $\psi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$

We will provide a q-difference equation approach to bilinear generating functions for $\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$ and $\psi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$.

Theorem 8. *Let the polynomials* $\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$ *be defined as* (14)*; then:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\phi_{n+m}^{\binom{a,b,c,d}{e,f,g}}(x,y|q)\frac{\omega^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$=\frac{1}{(xs,x\omega;q)_\infty}\sum_{n=0}^{\infty}\frac{(a,b,c,d,xs;q)_n(y\omega)^n}{(q,e,f,g;q)_n}\,{}_4\Phi_3\left[\begin{matrix}aq^n,bq^n,cq^n,dq^n;\\ eq^n,fq^n,gq^n;\end{matrix}\;q;sy\right]$$
$$(\max\{|xs|,|x\omega|,|y\omega|,|sy|\}<1). \tag{54}$$

Theorem 9. *Let the polynomials* $\psi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q)$*, as in* (15)*. For* $\max\{|y\omega|,|x\omega sy/q|\}<1$*, then:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}\,q^{\binom{n}{2}+\binom{m}{2}}\,\psi_{n+m}^{\binom{a,b,c,d}{e,f,g}}(x,y|q)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}$$
$$=(xs,x\omega;q)_\infty\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(a,b,c,d;q)_n(y\omega)^n}{(q,e,f,g;q)_n}$$
$$\times\,{}_5\Phi_4\left[\begin{matrix}aq^n,bq^n,cq^n,dq^n,q/(x\omega);\\ eq^n,fq^n,gq^n,0;\end{matrix}\;q;-\frac{x\omega sy}{q}\right]. \tag{55}$$

Proof of Theorems 8 and 9. Let $H(a,b,c,d,e,f,g,x,y)$ be the right-hand side of (54). For $y=0$, we have

$$H(a,b,c,d,e,f,g,x,0)=\frac{1}{(xs,x\omega;q)_\infty}.$$

Because Equation (54) satisfies (20), we have

$$\begin{aligned}
H(a,b,c,d,e,f,g,x,y)&=\mathbb{T}(a,b,c,d,e,f,g,yD_x)\{H(a,b,c,d,e,f,g,x,0)\}\\
&=\mathbb{T}(a,b,c,d,e,f,g,yD_x)\left\{\frac{1}{(xs,x\omega;q)_\infty}\right\}\\
&=\mathbb{T}(a,b,c,d,e,f,g,yD_x)\left\{\sum_{n=0}^{\infty}\frac{(x\omega)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(xs)^m}{(q;q)_m}\right\}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathbb{T}(a,b,c,d,e,f,g,yD_x)\{x^{n+m}\}\frac{\omega^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\phi_{n+m}^{\binom{a,b,c,d}{e,f,g}}(x,y|q)\frac{\omega^n}{(q;q)_n}\frac{s^m}{(q;q)_m},
\end{aligned} \tag{56}$$

which is the left-hand side of (54). The proof of Theorem 8 is complete.

Similarly, let $H'(a,b,c,d,e,f,g,x,y)$ be the right-hand side of (55). For $y=0$, we have:

$$H'(a,b,c,d,e,f,g,x,0)=(xs,x\omega;q)_\infty.$$

Because Equation (55) satisfies (22), we have:

$$
\begin{aligned}
&H'(a,b,c,d,e,f,g,x,y)\\
&=\mathbb{E}(a,b,c,d,e,f,g,y\theta_x)\{H'(a,b,c,d,e,f,g,x,0)\}\\
&=\mathbb{E}(a,b,c,d,e,f,g,y\theta_x)\{(xs,x\omega;q)_\infty\}\\
&=\mathbb{E}(a,b,c,d,e,f,g,y\theta_x)\left\{\sum_{n=0}^{\infty}\frac{(-1)^n\,q^{\binom{n}{2}}(x\omega)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(-1)^m\,q^{\binom{m}{2}}(xs)^m}{(q;q)_m}\right\}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}\,q^{\binom{n}{2}+\binom{m}{2}}\,\mathbb{E}(a,b,c,d,e,f,g,y\theta_x)\{x^{n+m}\}\,\frac{\omega^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}\,q^{\binom{n}{2}+\binom{m}{2}}\,\psi_{n+m}^{\binom{a,b,c,d}{e,f,g}}(x,y|q)\,\frac{\omega^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}.
\end{aligned}
\tag{57}
$$

This completes the proof of the assertion of Theorem 9. □

6. A Transformational Identity from q-Difference Equations

This section is devoted to transformational identities involving generating functions for q-polynomials using the q-difference equation. For more details, see [4,15,48].

Theorem 10. *Let $f(k)$ and $g(k)$ be two sequences such that*

$$\sum_{k=0}^{\infty}f(k)x^k=\sum_{k=0}^{\infty}g(k)\frac{(xtq^k;q)_\infty}{(xq^k;q)_\infty} \tag{58}$$

is convergent. Then, we have:

$$\sum_{k=0}^{\infty}f(k)\phi_k^{\binom{a,b,c,d}{e,f,g}}(x,y|q)=\sum_{k=0}^{\infty}g(k)\frac{(xtq^k;q)_\infty}{(xq^k;q)_\infty}{}_5\Phi_4\left[\begin{matrix}a,b,c,d,1/t;\\ e,f,g,xtq^k;\end{matrix}\;q;yq^k\right], \tag{59}$$

and both sides of (59) are also convergent.

Corollary 3. *For $\max\{|r|,|x|,|s|,|y|\}<1$, we have:*

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\phi_k^{\binom{a,b,c,d}{e,f,g}}(x,y|q)\frac{(t,s;q)_k}{(q,r;q)_k}\\
&\qquad=\frac{(xt,s;q)_\infty}{(x,r;q)_\infty}\sum_{k=0}^{\infty}\frac{(r/s,x;q)_k s^k}{(q,xt;q)_k}{}_5\Phi_4\left[\begin{matrix}a,b,c,d,1/t;\\ e,f,g,xtq^k;\end{matrix}\;q;yq^k\right].
\end{aligned}
\tag{60}
$$

Remark 7. *Upon taking $g(k)$ and $g(k)$ as in (63), Equation (59) reduces to (60). Setting $y=0$ in (60), we obtain the Heine's q-Euler transformations ([29] Equation (1.4.1)) given in (62) below.*

Proof of Theorem 10. Let $F(a,b,c,d,e,f,g,x,y)$ be the right-hand side of (59). We can check that the function $F(a,b,c,d,e,f,g,x,y)$ satisfies (16). Then, there exists a sequence $\{\mu_n\}$, such that:

$$F(a,b,c,d,e,f,g,x,y)=\sum_{k=0}^{\infty}\mu_n\phi_n^{\binom{a,b,c,d}{e,f,g}}(x,y|q). \tag{61}$$

Taking $y = 0$, the equation (61) becomes:

$$F(a,b,c,d,e,f,g,x,0) = \sum_{k=0}^{\infty} \mu_n x^n = \sum_{k=0}^{\infty} g(k) \frac{(xtq^k;q)_\infty}{(xq^k;q)_\infty} \quad \text{by (58)}$$
$$= \sum_{k=0}^{\infty} f(k) x^k.$$

Hence,

$$F(a,b,c,d,e,f,g,x,y) = \sum_{k=0}^{\infty} f(k) \phi_k^{\binom{a,b,c,d}{e,f,g}}(x,y|q),$$

which reads the left-hand side of (59). This completes the proof of Theorem 10. □

Proof of Corollary 3. Now, let us turn back and make use of the Heine's q-Euler transformations ([29] Equation (1.4.1))

$$ {}_2\Phi_1 \left[\begin{matrix} t,s; \\ r; \end{matrix} \; q;x \right] = \frac{(s,xt;q)_\infty}{(r,x;q)_\infty} {}_2\Phi_1 \left[\begin{matrix} r/s,x; \\ xt; \end{matrix} \; q;s \right]. \tag{62}$$

Setting

$$f(k) = \frac{(t,s;q)_k}{(q,r;q)_k} \quad \textit{and} \quad g(k) = \frac{(s;q)_\infty}{(r;q)_\infty} \sum_{k=0}^{\infty} \frac{(r/s;q)_k}{(q;q)_k} s^k, \tag{63}$$

the formula (58) is valid. Next, making use of (59), we get the desired result. This completes the proof of Corollary 3. □

7. $U(n+1)$-Type Generating Functions for Generalized Al-Salam–Carlitz Polynomials

In this section, we derive $U(n+1)$-type generating functions for q-polynomials using the q-difference equation method as an application of our study.

Theorem 11. *Let b,z and $x_1,\ldots,x_n$ be indeterminate, and let $n \geq 1$. Suppose that none of the denominators in the following identity vanish, and that $0 < |q| < 1$, $|y| < 1$ and $|z| < |x_1,\ldots,x_n||x_m|^{-n}|q|^{(n-1)/2}$, for $m = 1,2,\ldots,n$. Then, the following assertion*

$$\sum_{\substack{y_n \geq 0 \\ k=1,2,\ldots,n}} \Bigg\{ \prod_{1 \leq r < s \leq n} \left[\frac{1-(x_r/x_s q^{y_r - y_s})}{1-(x_r/x_s)} \right] \prod_{r,s=1}^{n} (q\frac{x_r}{x_s};q)_{y_r}^{-1} \prod_{i=1}^{n} (x_i)^{ny_i - (y_1+\ldots+y_n)}$$
$$\times (-1)^{(n-1)(y_1+\ldots+y_n)} q^{y_2+2y_3+\ldots+(n-1)y_n + (n-1)[\binom{y_1}{2}+\ldots+\binom{y_n}{2}] - e_2(y_1,\ldots,y_n)}$$
$$\times \phi_{y_1+\ldots+y_n}^{\binom{r,s,t,\omega}{u,v,\alpha}}(z,y|q)(b;q)_{y_1+\ldots+y_n} \Bigg\} = \frac{(bz;q)_\infty}{(z;q)_\infty} {}_5\Phi_4 \left[\begin{matrix} r,s,t,\omega,b; \\ u,v,\alpha,bz; \end{matrix} \; q;y \right], \tag{64}$$

holds, where $e_2(y_1,\ldots,y_n)$ is the second elementary symmetric function of the variables $\{y_1,\ldots,y_n\}$.

For $y = 0$, in Theorem 11, Equation (64) reduces to the following nonterminating $U(n+1)$ generalizations of the q-binomial theorem.

Proposition 3 ([9] Theorem 5.42)**.** *Let b, z and $x_1, \ldots, x_n$ be indeterminate, and let $n \geq 1$. Suppose that none of the denominators in the following identity vanishes, and that $0 < |q| < 1$ and $|z| < |x_1, \ldots, x_n||x_m|^{-n}|q|^{(n-1)/2}$, for $m = 1, 2, \ldots, n$. Then*

$$\sum_{\substack{y_n \geq 0 \\ k=1,2,\ldots,n}} \left\{ \prod_{1 \leq r < s \leq n} \left[\frac{1 - (x_r/x_s q^{y_r - y_s})}{1 - (x_r/x_s)} \right] \prod_{r,s=1}^{n} (q\frac{x_r}{x_s}; q)_{y_r}^{-1} \prod_{i=1}^{n} (x_i)^{ny_i - (y_1 + \ldots + y_n)} \right.$$
$$\times (-1)^{(n-1)(y_1+\ldots+y_n)} q^{y_2 + 2y_3 + \ldots + (n-1)y_n + (n-1)[\binom{y_1}{2} + \ldots + \binom{y_n}{2}] - e_2(y_1, \ldots, y_n)}$$
$$\left. \times (b; q)_{y_1+\ldots+y_n} z^{y_1+\ldots+y_n} \right\} = \frac{(bz; q)_\infty}{(z; q)_\infty}, \tag{65}$$

where $e_2(y_1, \ldots, y_n)$ is the second elementary symmetric function of $\{y_1, \ldots, y_n\}$.

Proof of Theorem 11. Let $f(r, s, t, \omega, u, v, z, \alpha, y)$ be the right-hand side of Equation (64). On can easily verify that $f(r, s, t, \omega, u, v, z, \alpha, y)$ satisfies (20). Then, we have:

$$\begin{aligned}
&f(r, s, t, \omega, u, v, z, \alpha, y) \\
&= \mathbb{T}(r, s, t, \omega, u, v, \alpha, yD_z)\{f(r, s, t, \omega, u, v, z, \alpha, 0)\} \\
&= \sum_{\substack{y_n \geq 0 \\ k=1,2,\ldots,n}} \left\{ \prod_{1 \leq r < s \leq n} \left[\frac{1 - (x_r/x_s q^{y_r - y_s})}{1 - (x_r/x_s)} \right] \prod_{r,s=1}^{n} (q\frac{x_r}{x_s}; q)_{y_r}^{-1} \prod_{i=1}^{n} (x_i)^{ny_i - (y_1 + \ldots + y_n)} \right. \\
&\quad \times (-1)^{(n-1)(y_1+\ldots+y_n)} q^{y_2 + 2y_3 + \ldots + (n-1)y_n + (n-1)[\binom{y_1}{2} + \ldots + \binom{y_n}{2}] - e_2(y_1, \ldots, y_n)} (b; q)_{y_1+\ldots+y_n} \\
&\quad \left. \times \mathbb{T}(r, s, t, \omega, u, v, z, \alpha, yD_z)\{z^{y_1+\ldots+y_n}\} \right\} \\
&= \mathbb{T}(r, s, t, \omega, u, v, \alpha, yD_z) \left\{ \frac{(bz; q)_\infty}{(z; q)_\infty} \right\}.
\end{aligned}$$

Using (14) and (59), we get the desired result. This completes the proof of Theorem 11. □

8. A Generalization of Ramanujan's Integrals

In this section, we give the following generalization using the q-difference equation method.

Theorem 12. *For $0 < q = e^{-2k^2} < 1$, $m \in \mathbb{R}$, and supposing that $\max\{|abq|, |byq|\} < 1$, we have:*

$$\int_{\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q)_\infty} {}_4\Phi_3 \left[\begin{matrix} r, s, t, \omega; \\ u, v, z; \end{matrix} \; q; yq^{1/2}e^{2ikx} \right] \mathrm{d}_q x$$
$$= \sqrt{\pi} e^{m^2} \frac{(-aqe^{2mki}, -bqe^{-2mki}; q)_\infty}{(abq; q)_\infty} {}_5\Phi_4 \left[\begin{matrix} r, s, t, \omega, e^{2mki}/b; \\ u, v, z, -aqe^{2mki}; \end{matrix} \; q; ybq \right]. \tag{66}$$

Remark 8. *Upon taking $y = 0$ in Theorem 12, Equation (66) reduces to the well-known Ramanujan's integral ([49] Equation (2))*

$$\int_{\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q)_\infty} \mathrm{d}_q x = \sqrt{\pi} e^{m^2} \frac{(-aqe^{2mki}, -bqe^{-2mki}; q)_\infty}{(abq; q)_\infty}, \tag{67}$$

where $0 < q = e^{-2k^2} < 1$, $m \in \mathbb{R}$ and $|abq| < 1$.

Proof of Theorem 12. If we denote by $G(r,s,t,u,v,a,y)$ the right-hand side of (66), one can verify that $G(r,s,t,u,v,a,y)$ satisfies (16). Then, there exists a sequence $\{\mu_n\}_n \in \mathbb{C}$, such that:

$$G(r,s,t,\omega,u,v,z,a,y) = \sum_{n=0}^{\infty} \mu_n \phi_n^{\binom{r,s,t,\omega}{u,v,z}}(a,y|q). \tag{68}$$

Taking $y=0$ in (68), we obtain:

$$\begin{aligned} G(r,s,t,\omega,u,v,z,a,0) = \sum_{n=0}^{\infty} \mu_n a^n &= \sqrt{\pi} e^{m^2} \frac{(-aqe^{2mki}, -bqe^{-2mki};q)_\infty}{(abq;q)_\infty} \quad by \ (67) \\ &= \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx};q)_\infty} \, \mathrm{d}_q\, x \\ &= \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(bq^{1/2}e^{-2ikx};q)_\infty} \left\{ \sum_{n=0}^{\infty} \frac{(q^{1/2}e^{2ikx})^n}{(q;q)_n} a^n \right\} \mathrm{d}_q\, x. \end{aligned}$$

Using relation (14), we have:

$$G(r,s,t,\omega,u,v,z,a,y) = \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(bq^{1/2}e^{-2ikx};q)_\infty} \left\{ \sum_{n=0}^{\infty} \phi_n^{\binom{r,s,t,\omega}{u,v,z}}(a,y|q) \frac{(q^{1/2}e^{2ikx})^n}{(q;q)_n} \right\} \mathrm{d}_q\, x. \tag{69}$$

By making use of relation (38) in the right-hand side of (69), we get the left-hand side of (66). This completes the proof of Theorem 12. □

9. Two Extensions of the Andrews–Askey Integral

In 1981, Andrews and Askey [50] derived a famous formula of integration from Ramanujan's ${}_1\psi_1$ summation.

Proposition 4 ([50] Equation (2.1))**.** *For* $\max\{|ac|, |ad|, |bc|, |bd|\} < 1$*, we have:*

$$\int_c^d \frac{(q\theta/c, q\theta/d;q)_\infty}{(a\theta, b\theta;q)_\infty} \, \mathrm{d}_q\, \theta = \frac{d(1-q)(q, dq/c, c/d, abcd;q)_\infty}{(ac, ad, bc, bd;q)_\infty}. \tag{70}$$

They also show that the integral (70) plays an important role in the system of q-series theory [1]. Motivated by this, our aim in this section is to derive the following two extensions of the Andrews–Askey integral using the q-difference equation method.

Theorem 13. *For* $\max\{|ac|, |ad|, |bc|, |bd|, |cy|, |dy|\} < 1$*, we have:*

$$\begin{aligned} &\int_c^d \frac{(q\theta/c, q\theta/d;q)_\infty}{(a\theta, b\theta;q)_\infty} {}_4\Phi_3 \left[\begin{matrix} r, s, t, \omega; \\ \\ u, v, z; \end{matrix} \; q; y\theta \right] \mathrm{d}_q\, \theta \\ &= \frac{d(1-q)(q, dq/c, c/d, abcd;q)_\infty}{(ac, ad, bc, bd;q)_\infty} \sum_{k=0}^{\infty} \frac{(r,s,t,w,ad;q)_k (cy)^k}{(q,u,v,z,ac,ad,abcd;q)_k} \\ &\times {}_6\Phi_5 \left[\begin{matrix} rq^k, sq^k, tq^k, wq^k, 0, 0; \\ \\ uq^k, vq^k, zq^k, acq^k, adq^k; \end{matrix} \; q; dy \right]. \end{aligned} \tag{71}$$

Theorem 14. *For* $\max\{|ac|,|ad|,|bc|,|bd|,|dy|\}<1$*, we have:*

$$\int_c^d \frac{(q\theta/c,q\theta/d;q)_\infty}{(a\theta,b\theta;q)_\infty} {}_5\Phi_4\left[\begin{matrix} r,s,t,\omega,c/\theta; \\ \\ u,v,z,ac; \end{matrix} q;y\theta\right] \mathrm{d}_q\,\theta \\ = \frac{d(1-q)(q,dq/c,c/d,abcd;q)_\infty}{(ac,ad,bc,bd;q)_\infty} {}_5\Phi_4\left[\begin{matrix} r,s,t,\omega,bc; \\ \\ u,v,z,abcd; \end{matrix} q;dy\right]. \tag{72}$$

Remark 9. *Upon setting* $y=0$ *in Theorems 13 and 14, Equations (71) and (72) reduce to the Andrews–Askey integral given in (70).*

In order to prove main theorems in Section 9, we need the following lemma.

Lemma 6. *For* $\max\{|ac|,|ad|,|cy|,|dy|\}<1$*, we have:*

$$\mathbb{T}(r,s,t,w,u,v,z,yD_a)\left\{\frac{(abcd;q)_\infty}{(ac,ad;q)_\infty}\right\} \\ = \frac{(abcd;q)_\infty}{(ac,ad;q)_\infty}\sum_{k=0}^{\infty}\frac{(r,s,t,w,ad;q)_k(cy)^k}{(q,u,v,z,ac,ad,abcd;q)_k} {}_6\Phi_5\left[\begin{matrix} rq^k,sq^k,tq^k,wq^k,0,0; \\ \\ uq^k,vq^k,zq^k,acq^k,adq^k; \end{matrix} q;dy\right]. \tag{73}$$

Proof of Lemma 6. Let D_a be defined as in (5). By the definition of q-operator $\mathbb{T}$ in Equation (14), we have:

$$\mathbb{T}(r,s,t,w,u,v,z,yD_a)\left\{\frac{(abcd;q)_\infty}{(ac,ad;q)_\infty}\right\} \\ = \sum_{n=0}^{\infty}\frac{(r,s,t,\omega;q)_n y^n}{(q,u,v,z;q)_n}D_a^n\left\{\frac{(abcd;q)_\infty}{(ac,ad;q)_\infty}\right\}. \tag{74}$$

Upon using (27), relation (74) becomes:

$$\mathbb{T}(r,s,t,w,u,v,z,yD_a)\left\{\frac{(abcd;q)_\infty}{(ac,ad;q)_\infty}\right\} \\ = \sum_{n=0}^{\infty}\frac{(r,s,t,\omega;q)_n y^n}{(q,u,v,z;q)_n}\frac{(abcd;q)_\infty}{(ac,ad;q)_\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q \frac{c^k d^{n-k}(ad;q)_k}{(abcd;q)_k},$$

which is the right-hand side of Equation (73) after simplification. This completes the proof of Lemma 6. □

Now, we are in a position to prove Theorem 13.

Proof of Theorem 13. We denote $f(r,s,t,\omega,u,v,z,y,a)$ by the right-hand side of Equation (71). By using Lemmas 4–6, we check that the right-hand side of Equation (71) satisfies Equation (16). So, we have

$$f(r,s,t,\omega,u,v,z,y,a) = \sum_{n=0}^{\infty}\mu_n\phi_n^{\binom{r,s,t,\omega}{u,v,z}}(a,y|q). \tag{75}$$

Letting $y=0$ in Equation (75) yields

$$f(r,s,t,\omega,u,v,z,0,a) = \sum_{n=0}^{\infty}\mu_n a^n = \frac{d(1-q)(q,dq/c,c/d,abcd;q)_\infty}{(ac,ad,bc,bd;q)_\infty}. \tag{76}$$

Utilizing the Andrews–Askey integral (70) and equating the coefficients of a^n in (75), we obtain:

$$\mu_n = \int_c^d \frac{(q\theta/c, q\theta/d; q)_\infty}{(b\theta; q)_\infty} \left\{ \sum_{n=0}^{\infty} \frac{\theta^n}{(q;q)_n} \right\} \mathrm{d}_q \theta. \tag{77}$$

Thus, we have

$$\begin{aligned} f(r,s,t,\omega,u,v,z,y,a) &= \int_c^d \frac{(q\theta/c, q\theta/d; q)_\infty}{(b\theta; q)_\infty} \left\{ \sum_{n=0}^{\infty} \frac{\phi_n^{\binom{r,s,t,\omega}{u,v,z}}(a,y|q)\theta^n}{(q;q)_n} \right\} \mathrm{d}_q \theta \\ &= \int_c^d \frac{(q\theta/c, q\theta/d; q)_\infty}{(b\theta; q)_\infty} \left\{ \frac{1}{(a\theta;q)_\infty} {}_4\Phi_3 \left[\begin{matrix} r,s,t,\omega; \\ u,v,z; \end{matrix} q; y\theta \right] \right\} \mathrm{d}_q \theta. \end{aligned}$$

Using (42), we obtain the left-hand side of Equation (71). This completes the proof of Theorem 13. □

Proof of Theorem 14. In the proof of Theorem 14, we need:

$$\mathbb{T}(r,s,t,w,u,v,z,yD_a) \left\{ \frac{(abcd;q)_\infty}{(ad;q)_\infty} \right\} = \frac{(abcd;q)_\infty}{(ad;q)_\infty} {}_5\Phi_4 \left[\begin{matrix} r,s,t,\omega,bc; \\ u,v,z,abcd; \end{matrix} q; yd \right], \tag{78}$$

$$\mathbb{T}(r,s,t,w,u,v,z,yD_a) \left\{ \frac{(ac;q)_\infty}{(a\theta;q)_\infty} \right\} = \frac{(ac;q)_\infty}{(a\theta;q)_\infty} {}_5\Phi_4 \left[\begin{matrix} r,s,t,\omega,c/\theta; \\ u,v,z,ac; \end{matrix} q; y\theta \right], \tag{79}$$

which are derived from Lemma 4. Because we use the same procedure as in the proof of Theorem 13, we achieve the proof of Theorem 14. □

10. Concluding Remarks

In our present investigation, motivated by methods of "parameter augmentation" and q-difference equations, it is natural to ask whether some general q-polynomials exist, which are solutions of certain q-difference equations and have expressions by q-exponential operators. We focus on new generalized Al-Salam–Carlitz polynomials and search their q-difference equations. We calculate some identities of q-exponential operators involving the new generalized Al-Salam–Carlitz polynomials, count generating functions for new generalized Al-Salam–Carlitz polynomials, deduce two bilinear generating functions for generalized Al-Salam–Carlitz polynomials using q-difference equations, obtain a transformational identity from q-difference equations, obtain $U(n+1)$-type generating functions for generalized Al-Salam–Carlitz polynomials using q-difference equations and gain generalizations of Ramanujan's integrals.

We remark in conclusion that we consider two theorems (Theorems 1 and 2) to be related to the q-difference equations. The results in the two theorems are the same, but they have completely different power in calculating q-series. It is difficult but meaningful to augment parameters and to generalize the q-difference equations for q-polynomials with more variables. Our findings not only generalize some of the results presented in [11,14–16,20,29,31,42], but also enable the production of q-difference equations with a large number of new results by augmenting the parameters involved or by exploring the q-version of the recent works by Wituła et al. (see [51]) and Hetmaniok et al. (see [52–54]), which will bring new perspectives for further exploration of the theory of q-difference equations and q-polynomials.

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