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Feynman Integrals for the Harmonic Oscillator in an Exponentially Growing Potential

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Abstract: We construct the Feynman integral for the Schrödinger propagator with combinations of exponentially growing and harmonic oscillator potentials as well-defined white noise functionals.

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1. Introduction

Feynman integrals are undoubtedly of central importance in the most varied fields of quantum physics. It is remarkable how they have been defying attempts at a direct mathematical interpretation since the times of Gelfand and Yaglom [1]. There has been an abundance of indirect definitions of this infinite-dimensional "integral" through discretizations or via analytic continuation of better-defined integrals such as in the Feynman–Kac formula, which is the analytic continuation of the Wiener integral to a purely imaginary time (see, e.g., [2]), which together with complex scaling was a method used by Doss [3]. Another one is the work of DeWitt-Morette on promeasures (also called cylindrical measures) in relation to the Feynman path integrals [4]. Albeverio and Høegh-Krohn also developed a general theory of oscillating integrals to give a mathematical foundation to Feynman path integrals [2]. A rather fruitful alternative is to generalize the notion of such (infinite-dimensional) integrals as one does in the theory of generalized functions or "distributions", for which white noise analysis provides a natural framework [5–8]. This approach, whose construction of Feynman integrals was first proposed in [9,10], and by Khandekar and Streit [11], turned out to be useful for a large and growing class of potentials [10–16]. The case of potentials that are Laplace transforms of measures has already been explored in [12]. In the present paper, we extend this to combinations of the harmonic oscillator with potentials that are Laplace transforms of rapidly decreasing measures, such as the Morse potential. On the other hand, white noise analysis proved to be a useful approach to phase space Feynman integrals [17,18], and we shall show similar results obtained with this method in Appendix A.

This paper is organized as follows. In Section 2, we collect the necessary tools from white noise analysis that are needed later on. In Section 3, we describe how to realize the Feynman integrals in white noise analysis as well as the classes of potentials we are interested in. The main results of this paper are included in Section 4, namely Theorem 3. Finally, in Appendix A, we sketch how to realize the Feynman integrands in the phase space in white noise analysis.

2. Some Facts on White Noise Analysis

From [5–8], we collected the concepts and theorems of white noise analysis used in this paper.

Let $d \in \mathbb{N}$ be given and L^2_d be the Hilbert space of vector-valued square integrable measurable functions:

$$L^2_d := L^2(\mathbb{R}) \otimes \mathbb{R}^d.$$

The space L^2_d is unitary isomorphic to a direct sum of d identical copies of $L^2 := L^2(\mathbb{R})$, the space of real-valued square integrable measurable functions with respect to the Lebesgue measure. Any element $f \in L^2_d$ may be written in the form

$$f = (f_1 \otimes e_1, \dots, f_d \otimes e_d), \tag{1}$$

where $f_i \in L^2(\mathbb{R})$, $i = 1, \dots, d$ and $\{e_1, \dots, e_d\}$ denotes the canonical basis of $\mathbb{R}^d$. The norm of $f \in L^2_d$ is given by

$$|f|_0^2 := \sum_{k=1}^{d} |f_k|_{L^2}^2 = \sum_{k=1}^{d} \int_{\mathbb{R}} f_k^2(x)\,\mathrm{d}x. \tag{2}$$

As a densely embedded nuclear space in L^2_d, we choose $S_d := S(\mathbb{R}) \otimes \mathbb{R}^d$, where $S(\mathbb{R})$ is the Schwartz test function space. With $(|\cdot|_p)_{p\in\mathbb{N}_0}$, we denote the sequence of Hilbert norms which topologizes S_d. An element $\varphi \in S_d$ has the form

$$\varphi = (\varphi_1 \otimes e_1, \dots, \varphi_d \otimes e_d), \tag{3}$$

where $\varphi_i \in S(\mathbb{R})$, $i = 1, \dots, d$. Together with the basic nuclear space of tempered distributions $S'_d := S'(\mathbb{R}) \otimes \mathbb{R}^d$ (the dual space of S_d), we obtain the triple

$$S_d \subset L^2_d \subset S'_d. \tag{4}$$

The dual pairing between S'_d and S_d is given as an extension of the scalar product in L^2_d by

$$\langle w, \varphi \rangle = \sum_{k=1}^{d} \langle w_k, \varphi_k \rangle, \tag{5}$$

for any $w = (w_1 \otimes e_1, \dots, w_d \otimes e_d) \in S'_d$ with $w_i \in S'(\mathbb{R})$, $i = 1, \dots, d$ and φ, as in Equation (3). The space S'_d is provided with the σ algebra $\mathscr{B}$ generated by the cylinder sets.

Using Minlos' theorem, we construct a measure space $(S'_d, \mathscr{B}, \mu)$, called the white noise space, by fixing the characteristic functional in the following way:

$$C(\varphi) = \int_{S'_d} \exp(\mathrm{i}\langle w, \varphi \rangle)\mathrm{d}\mu(w) = \exp\left(-\frac{1}{2}|\varphi|_0^2\right), \quad \varphi \in S_d. \tag{6}$$

Within this formalism, a version of Wiener's Brownian motion B is given by

$$\begin{aligned} B(t, w) &:= (\langle w, \mathbb{1}_{[0,t)} \otimes e_1 \rangle, \dots, \langle w, \mathbb{1}_{[0,t)} \otimes e_d \rangle) \\ &= \left(\int_0^t w_1(s)\,\mathrm{d}s, \dots, \int_0^t w_d(s)\,\mathrm{d}s \right), \qquad w \in S'_d. \end{aligned} \tag{7}$$

We now introduce the complex Hilbert space

$$L^2(\mu) := L^2(S'_d, \mathscr{B}, \mathrm{d}\mu) \tag{8}$$

with the inner product given by

$$((F, G)) := \int_{S'_d} \bar{F}(w) G(w)\,\mathrm{d}\mu(w), \quad F, G \in L^2(\mu). \tag{9}$$

In applications, the space $L^2(\mu)$ is often too small. We enlarge it by first choosing a nuclear subspace $(S_d)^1$ of the test functionals. Then, the corresponding Gel'fand triple is

$$(S_d)^1 \subset L^2(\mu) \subset (S_d)^{-1}. \tag{10}$$

Elements of the space $(S_d)^{-1}$ are called Kondratiev distributions, and its explicit construction is given in [19]. The dual pairing between $(S_d)^1$ and $(S_d)^{-1}$ is denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and is a bilinear extension of the inner product of $L^2(\mu)$. More precisely, if $F, G \in L^2(\mu)$, then we have

$$\langle\!\langle F, G \rangle\!\rangle = ((\bar{F}, G)) = \int_{S'_d} F(w)G(w)\,\mathrm{d}\mu(w).$$

Instead of reproducing its construction here, we shall completely characterize the Kondratiev distributions $\Phi \in (S_d)^{-1}$ by their "T-transforms".

We make use of the fact that in many physics applications, Φ will be given in terms of a "source functional". More precisely, for every $\Phi \in (S_d)^{-1}$, there exist $p, q \in \mathbb{N}_0$ such that for any $\varphi \in U_{p,q}$ with $U_{p,q} := \{\varphi \in S_d \mid |\varphi|_p^2 < 2^{-q}\}$, the T-transform of Φ is given by

$$T\Phi(\varphi) = \mathbb{E}_\mu[\Phi \exp(\mathrm{i}\langle \cdot, \varphi \rangle)] = \langle\!\langle \Phi, \exp(\mathrm{i}\langle \cdot, \varphi \rangle) \rangle\!\rangle, \tag{11}$$

where we introduce the convenient notation of the expectation $\mathbb{E}_\mu$ of a μ-integrable function. Fortunately, these expressions provide a complete characterization. Generalized functionals in the Kondratiev space are characterized by the local analyticity and local boundedness of these source functionals. To formulate precisely the characterization theorem, we need the motion of holomorphy in a nuclear space. See [20] for more details:

Definition 1. *Let $U \subseteq S_{d,\mathbb{C}}$ be an open set and $F : U \longrightarrow \mathbb{C}$ be a given function. Then, F is holomorphic on U if and only if, for all $\varphi_0 \in U$, the following apply:*

1. *For any $\varphi \in S_{d,\mathbb{C}}$, the map $\mathbb{C} \ni z \mapsto F(\varphi_0 + z\varphi) \in \mathbb{C}$ is holomorphic in a neighborhood of zero in $\mathbb{C}$;*
2. *There exists an open neighborhood U' of φ_0 such that F is bounded on U'.*

F is holomorphic at zero if and only if F is holomorphic in a neighborhood of zero.

Now, we are ready to state the announced characterization theorem of Kondratiev distributions (see [19]):

Theorem 1. *Let $U \subseteq S_{d,\mathbb{C}}$ be an open set and $F : U \to \mathbb{C}$ be holomorphic at zero. Then, there exists a unique $\Phi \in (S_d)^{-1}$ such that $T\Phi = F$. Conversely, given $\Phi \in (S_d)^{-1}$, then $T\Phi$ is holomorphic at zero. The correspondence between F and Φ is bijective if we identify holomorphic functions which coincide on an open neighborhood of zero.*

In applications, we have to handle the convergence of sequences of distributions from $(S_d)^{-1}$ as well as integrals of elements in $(S_d)^{-1}$, depending on a parameter. The following two corollaries are a consequence of Theorem 1 and will be applied in what follows:

Corollary 1. *Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in $(S_d)^{-1}$ such that there exists $U_{p,q} \subset S_d$, $p, q \in \mathbb{N}_0$ so that the following are true:*

1. *All $T\Phi_n$ are holomorphic on $U_{p,q}$;*
2. *There exists a $C > 0$ such that $|T\Phi_n(\varphi)| \le C$ for all $\varphi \in U_{p,q}$ and all $n \in \mathbb{N}$;*
3. *$(T\Phi_n(\varphi))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for all $\varphi \in U_{p,q}$.*

Then, $(\Phi_n)_{n\in\mathbb{N}}$ converges strongly in $(S_d)^{-1}$.

Corollary 2. *Let $(\Lambda, \mathscr{A}, \nu)$ be a measure space and $\lambda \mapsto \Phi_\lambda$ be a mapping from Λ to $(S_d)^{-1}$. We assume there exists $U_{p,q} \subset S_d$, $p, q \in \mathbb{N}_0$ such that the following are true:*

1. *$T\Phi_\lambda$ is holomorphic on $U_{p,q}$ for every $\lambda \in \Lambda$;*

2. *The mapping* $\lambda \mapsto T\Phi_\lambda(\varphi)$ *is measurable for every* $\varphi \in U_{p,q}$*;*
3. *There exists* $C \in L^1(\Lambda, \mathscr{A}, \nu)$ *such that*

$$|T\Phi_\lambda(\varphi)| \leq C(\lambda) \tag{12}$$

for all $\varphi \in U_{p,q}$ *and for* ν*-almost all* $\lambda \in \Lambda$.

Then, Φ_λ *is Bochner integrable. In particular, we have*

$$\int_\Lambda \Phi_\lambda \, \mathrm{d}\nu(\lambda) \in (S_d)^{-1} \tag{13}$$

and we may interchange the dual pairing and integration such that

$$\left\langle\!\!\left\langle \int_\Lambda \Phi_\lambda \, \mathrm{d}\nu(\lambda), \xi \right\rangle\!\!\right\rangle = \int_\Lambda \langle\!\langle \Phi_\lambda, \xi \rangle\!\rangle \, \mathrm{d}\nu(\lambda), \quad \xi \in (S_d)^1. \tag{14}$$

3. Feynman Integrals in Terms of White Noise

To realize the heuristic integral

$$\int \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S} F[x] \, \mathrm{d}^\infty x(t) = \langle\!\langle I, F \rangle\!\rangle, \quad I \in (S_d)^{-1}, \tag{15}$$

where S is the classical action, as an average over paths, the basic idea is to invoke Brownian paths from (x_0, t_0) to (x, t):

$$x(\tau) = x_0 + \left(\frac{\hbar}{m_o}\right)^{1/2} \langle w, \mathbb{1}_{[t_0,\tau)} \rangle, \tag{16}$$

where $\hbar = h/(2\pi)$ is the reduced Planck's constant h and m_o is the mass of the particle. We shall set $\hbar = m_o = 1$ from here on. It is noteworthy to mention in passing that the Wiener integral [21]

$$\int \mathrm{e}^{-\int_{t_0}^t V(\gamma(\tau)+x)\,\mathrm{d}\tau} F[\gamma(t_0) + x] \, \mathrm{d}B(\gamma), \tag{17}$$

where V is a kind of "potential" and $\gamma(\tau) \in \mathbb{R}$ is an absolutely continuous function on $[t_0, t]$, having a form of functional integration similar to that in Equation (15) except for having a purely imaginary time (i.e., $t \to -\mathrm{i}t$), is rigorously defined for the heat equation (see, for example, [2]), as Equation (15) is heuristically defined for the Schrödinger equation, which is analogous to the heat equation upon replacing t with $\mathrm{i}t$. In the expression for the free Feynman integrand

$$I_0(x, t|x_0, t_0) = \mathrm{N}\exp\left(\frac{\mathrm{i}+1}{2} \int_{\mathbb{R}} |w(\tau)|^2 \, \mathrm{d}\tau\right) \delta(x(t) - x), \tag{18}$$

where N is a normalizing constant and the imaginary part of the exponent is the free action, the real part compensates the Gaussian fall-off of the white noise measure, and the Donsker delta function pins the path at the final point (x, t). From the characteristic functional in Equation (6), it is straightforward to calculate the T-transform of I_0 using, for example, the Fourier representation of the delta function to obtain the following with $\varphi \in S_d$:

$$TI_0(\varphi) = \frac{1}{(2\pi\mathrm{i}|t - t_0|)^{\frac{d}{2}}} \exp\left[-\frac{\mathrm{i}}{2} |\varphi|_0^2 \, \mathrm{d}\tau - \frac{1}{2\mathrm{i}|t - t_0|} \left| \int_{t_0}^t \varphi(\tau) \, \mathrm{d}\tau + x - x_0 \right|_0^2 \right], \tag{19}$$

which satisfies the assumptions of the characterization in Theorem 1. Hence, we arrive at a definition of the Feynman integrand I_0 as a well-defined element in $(S_d)^{-1}$. In addition, from the physics point of view, the Feynman integral $\mathbb{E}_\mu[I_0] = TI_0(0)$ is the free particle propagator $(2\pi(t-t_0))^{-d/2}\exp\left[\frac{\mathrm{i}}{2|t-t_0|}|x-x_0|^2\right]$ and is the fundamental solution to the Schrödinger equation.

For the harmonic oscillator (for space dimension $d=1$), the potential V_h is given by

$$V_h(x) = \frac{1}{2}k^2x^2, \quad x \in \mathbb{R} \tag{20}$$

and the corresponding Feynman integrand is

$$I_h := I_0 \exp\left(-\mathrm{i}\int_{t_0}^{t} V(x(\tau))\,\mathrm{d}\tau\right).$$

For $k \geq 0$, the corresponding T-transform of I_h at $\varphi \in S_d$ is given by comparison with Equation (9) in [13] (for the results obtained using "phase space white noise analysis", see (A1) of Appendix A):

$$\begin{aligned}
TI_h(\varphi) &= \left(\frac{k}{2\pi\mathrm{i}\sin k|\Delta|}\right)^{\frac{d}{2}} \exp\left(-\frac{\mathrm{i}}{2}|\varphi_\Delta|_0^2 - \frac{1}{2}|\varphi_{\Delta^c}|_0^2\right)\exp\Bigg\{\frac{\mathrm{i}k}{2\sin k|\Delta|} \\
&\times\Bigg[\left(x_0^2+x^2\right)\cos k|\Delta| + 2x\int_{t_0}^{t}\varphi(t')\cos k(t'-t_0)\,\mathrm{d}t' - 2x_0\int_{t_0}^{t}\varphi(t')\cos k(t-t')\,\mathrm{d}t' \\
&-2x_0x + 2\int_{t_0}^{t}\int_{t_0}^{s_1}\varphi(s_1)\varphi(s_2)\cos k(t-s_1)\cos k(s_2-t_0)\,\mathrm{d}s_2\,\mathrm{d}s_1\Bigg]\Bigg\},
\end{aligned} \tag{21}$$

where we denoted $\Delta := [t_0, t]$ and φ_Δ and φ_{Δ^c} are the restrictions of φ to Δ and to $\Delta^c := \mathbb{R}\backslash\Delta$, respectively.

A rather surprising class of potentials is given by the Laplace transforms of a complex measure m on the Borel sets on $\mathbb{R}^d$ (see [12]):

$$V(x) = \int_{\mathbb{R}^d} \mathrm{e}^{\alpha\cdot x}\,\mathrm{d}m(\alpha), \quad x \in \mathbb{R}^d, \tag{22}$$

where m satisfies

$$\int_{\mathbb{R}^d} \mathrm{e}^{C|y|}\,\mathrm{d}|m|(y) < \infty, \quad \forall C > 0. \tag{23}$$

Example 1. *It follows from the assumption in Equation (23) that the measure m is finite. In addition, every finite measure with compact support satisfies the condition in Equation (23):*

1. *If the measure m is chosen to be $m = g\delta_a$, where $a > 0$, then the corresponding potential is $V(x) = ge^{ax}$;*
2. *Polynomials and exponential functions also belong to the class of potentials covered by Equations (22) and (23). In particular,* $\sinh(ax)$, $\cosh(ax)$, *and the Morse potential*

$$V(x) = g\left(e^{-2ax} - 2\gamma e^{-ax}\right) \qquad \text{with} \quad g, a, x \in \mathbb{R} \text{ and } \gamma > 0,$$

are also included in the class.

Example 2. *A Gaussian measure m gives the anharmonic oscillator potential $V(x) = ge^{bx^2}$, $g, b, x \in \mathbb{R}$. Entire functions of arbitrary high orders of growth are also in this class.*

Remarkably, in each case, the construction of the Feynman integrand is *perturbative*. We have to give a meaning to the pointwise multiplication

$$I_V = I_0 \cdot \exp\left(-\mathrm{i}\int_{t_0}^{t} V(x(\tau))\,\mathrm{d}\tau\right)$$

with $x(\tau)$ given as in Equation (16). We have the following result (see Theorem 16 in [12]):

Theorem 2. *If we let V be as in Equation (22) above, then*

$$I_V := \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int_{[t_0,t]^n} \mathrm{d}^n s \int_{\mathbb{R}^{dn}} I_0 \cdot \prod_{j=1}^{n} e^{\alpha_j \cdot x(s_j)} \prod_{j=1}^{n} dm(\alpha_j) \tag{24}$$

exists as a generalized white noise functional. The series converges in the strong topology of $(S_d)^{-1}$. The integrals exist in the sense of Bochner.

The proof is based on the characterization theorem and its corollaries from Section 2.

4. The Feynman Integrand for a New Class of Potentials

Our aim is to define the Feynman integrand

$$I := I_0 \cdot \exp\left(-\mathrm{i} \int_{t_0}^{t} V(x(\tau))\, \mathrm{d}\tau\right) \tag{25}$$

for a new class of potentials V of the form $V = V_1 + V_2$, with

$$x(\tau) = x_0 + \langle w, \mathbb{1}_{[t_0,\tau)} \rangle,$$

$$V_1(x) = \int_{\mathbb{R}^d} e^{\alpha \cdot x}\, \mathrm{d}m(\alpha), \qquad V_2(x) = \frac{1}{2} k^2 |x|^2, \quad x \in \mathbb{R}^d \tag{26}$$

as before. In order to accomplish this, we will first give a mathematical meaning to the heuristic expression

$$I = I_0 \exp\left(-\mathrm{i} \int_{t_0}^{t} V(x(\tau))\, \mathrm{d}\tau\right) = I_h \exp\left(-\mathrm{i} \int_{t_0}^{t} V_1(x(\tau))\, \mathrm{d}\tau\right). \tag{27}$$

We shall start from a perturbative ansatz which we then justify by using Theorem 1 and its corollaries. In Theorem 3 below, it will be shown that I is indeed a well-defined generalized white noise functional.

We formally expand the exponential in Equation (27) into a perturbation series with respect to V_1. This leads to

$$I = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int_{[t_0,t]^n} \mathrm{d}^n s \int_{\mathbb{R}^{dn}} I_h \exp\left(\sum_{l=1}^{n} \alpha_l \cdot x(s_l)\right) \prod_{l=1}^{n} \mathrm{d}m(\alpha_l), \tag{28}$$

and will show that the rhs defines a generalized function of white noise (i.e., an element in $(S_d)^{-1}$). We accomplish this in three steps:

Step 1. For each $n \in \mathbb{N}_0$, the integrand $\Phi_n := I_h \cdot \exp(\sum_{l=1}^{n} \alpha_l \cdot x(s_l))$ is a Kondratiev distribution, compared with Proposition 1 below.

Step 2. For each $n \in \mathbb{N}_0$, the s and α integration of Φ_n is a well-defined element in $(S_d)^{-1}$, compared with Lemma 1.

Step 3. The series in Equation (28) converges in $(S_d)^{-1}$, compared with Theorem 3.

Now, we show the three steps above in order to establish that I, given in Equation (28), is a well-defined element in $(S_d)^{-1}$.

Step 1. First, we need to check that the pointwise multiplication of generalized functionals for each natural number $n \in \mathbb{N}_0$, where

$$\Phi_n = I_h \cdot \exp\left(\sum_{l=1}^{n} \alpha_l \cdot x(s_l)\right) \tag{29}$$

produces a well-defined generalized functional. Due to the characterization in Theorem 1, it is enough to define this product through its T-transform. Arguing formally, for $\varphi \in S_d$, we arrive at

$$T\Phi_n(\varphi) = \int_{S'_d} \Phi_n(w) \exp(\mathrm{i}\langle w, \varphi\rangle) \mathrm{d}\mu(w) = TI_h(\varphi + \mathrm{i}\xi_t) \exp\left(\sum_{l=1}^{n} x \cdot \alpha_l\right), \tag{30}$$

where we have denoted $\xi_t := \sum_{l=1}^{n} \alpha_l \mathbb{1}_{[s_l,t)}$. Note also that we have used the fact that from Equation (16) (with $\hbar = m_o = 1$), it follows that for each $l = 1, \dots, n$, we have $x(s_l) = x(t) - \langle w, \mathbb{1}_{[s_l,t)}\rangle$. Hence, we need only to deal with the T-transform of I_h on the function $\varphi + \mathrm{i}\xi_t$ (i.e., the first factor in Equation (30)). Explicitly, by invoking the result in Equation (21), for $k \geq 0$, we obtain

$$\begin{aligned} TI_h(\varphi + \mathrm{i}\xi_t) = &\left(\frac{k}{2\pi \mathrm{i} \sin k|\Delta|}\right)^{\frac{d}{2}} \exp\left(-\frac{\mathrm{i}}{2}|(\varphi + \mathrm{i}\xi_t)_\Delta|_0^2 - \frac{1}{2}|\varphi_{\Delta^c}|_0^2\right) \exp\left\{\frac{\mathrm{i}k}{2\sin k|\Delta|}\right. \\ &\times \left[\left(|x_0|^2 + |x|^2\right) \cos k|\Delta| + 2x \cdot \int_{t_0}^{t} (\varphi + \mathrm{i}\xi_t)(t') \cos k(t' - t_0)\, \mathrm{d}t'\right. \\ &- 2x_0 \cdot \int_{t_0}^{t} (\varphi + \mathrm{i}\xi_t)(t') \cos k(t - t')\, \mathrm{d}t' \; - 2x_0 x \\ &+ 2\int_{t_0}^{t}\int_{t_0}^{s_1} (\varphi + \mathrm{i}\xi_t)(s_1) \cdot (\varphi + \mathrm{i}\xi_t)(s_2) \\ &\left.\left.\times \cos k(t - s_1) \cos k(s_2 - t_0)\, \mathrm{d}s_2\, \mathrm{d}s_1\right]\right\}. \end{aligned} \tag{31}$$

Note that the scalar products in the first exponent must be considered as a bilinear analytic continuations from the case of real-valued functions φ. As a result, we have

$$\begin{aligned} TI_h(\varphi + \mathrm{i}\xi_t) = &\, TI_h(\varphi) \exp\left((\varphi_\Delta, \xi_t)_0 + \frac{\mathrm{i}}{2}|\xi_t|_0^2\right) \exp\left\{\frac{\mathrm{i}k}{2\sin k|\Delta|}\right. \\ &\times \left[2\mathrm{i}x \cdot \int_{t_0}^{t} \xi_t(t') \cos k(t' - t_0)\, \mathrm{d}t' - 2\mathrm{i}x_0 \cdot \int_{t_0}^{t} \xi_t(t') \cos k(t - t')\, \mathrm{d}t'\right. \\ &+ 2\mathrm{i}\int_{t_0}^{t}\int_{t_0}^{s_1} (\varphi(s_1) \cdot \xi_t(s_2) + \varphi(s_2) \cdot \xi_t(s_1)) \cos k(t - s_1) \cos k(s_2 - t_0)\, \mathrm{d}s_2\, \mathrm{d}s_1 \\ &\left.\left.- 2\int_{t_0}^{t}\int_{t_0}^{s_1} \xi_t(s_1) \cdot \xi_t(s_2) \cos k(t - s_1) \cos k(s_2 - t_0) \mathrm{d}s_2\, \mathrm{d}s_1\right]\right\}. \end{aligned} \tag{32}$$

Now, we are ready to state the result of Step 1:

Proposition 1. *Let $k \geq 0$ be such that $0 < k|\Delta| < 1$, $s_l \in [t_0, t]$, and $\alpha_j \in \mathbb{R}^d$, $1 \leq l \leq n$ are given. Then, for every $n \in \mathbb{N}_0$, the product*

$$\Phi_n = I_h \cdot \exp\left(\sum_{l=1}^{n} \alpha_l \cdot x(s_l)\right) \tag{33}$$

defined for any $\varphi \in S_d$ by

$$T\Phi_n(\varphi) = TI_h\left(\varphi + \mathrm{i}\sum_{l=1}^{n} \alpha_l \mathbb{1}_{[s_l,t)}\right) \exp\left(x \cdot \sum_{l=1}^{n} \alpha_l\right)$$

is a well-defined element in the Kondratiev distribution space $(S_d)^{-1}$.

Proof. We write

$$T\Phi_n(\varphi) = TI_h(\varphi)\Psi_n(\varphi, \xi_t) \tag{34}$$

with

$$\begin{aligned}\Psi_n(\varphi,\xi_t) := & \exp\left(x\cdot\sum_{l=1}^n \alpha_l\right)\exp\left((\varphi_\Delta,\xi_t)_0 + \frac{\mathrm{i}}{2}|\xi_t|_0^2\right)\exp\left\{\frac{\mathrm{i}k}{2\sin k|\Delta|}\right.\\ & \times\left[2\mathrm{i}x\cdot\int_{t_0}^t \xi_t(t')\cos k(t'-t_0)\,\mathrm{d}t' - 2\mathrm{i}x_0\cdot\int_{t_0}^t \xi_t(t')\cos k(t-t')\,\mathrm{d}t'\right.\\ & +2\mathrm{i}\int_{t_0}^t \mathrm{d}s_1\int_{t_0}^{s_1}\mathrm{d}s_2\,(\varphi(s_1)\cdot\xi_t(s_2)+\varphi(s_2)\cdot\xi_t(s_1))\cos k(t-s_1)\cos k(s_2-t_0)\\ & \left.\left. -2\int_{t_0}^t \mathrm{d}s_1\int_{t_0}^{s_1}\mathrm{d}s_2\,\xi_t(s_1)\cdot\xi_t(s_2)\cos k(t-s_1)\cos k(s_2-t_0)\right]\right\}.\end{aligned} \tag{35}$$

The first factor on the right-hand side of Equation (34) is known to be the T-transform of a Hida distribution (which forms a subspace of the Kondratiev distribution). Hence, we only need to show that Ψ_n fulfills the conditions of Theorem 1. Obviously, $\Psi_n(\varphi_0 + z\varphi, \xi_t)$ is entire in z. To show boundedness, we perform an estimation as follows:

$$\begin{aligned} & |\Psi_n(\varphi,\xi_t)| \le \exp\left(|x|\sum_{l=1}^n|\alpha_l|\right)\exp|(\varphi_\Delta,\xi_t)_0|\exp\left\{\frac{k}{\sin k|\Delta|}\right.\\ & \quad\times\left[\left|x\cdot\int_{t_0}^t \mathrm{d}t'\xi_t(t')\cos k(t'-t_0)\right| + \left|x_0\cdot\int_{t_0}^t \mathrm{d}t'\xi_t(t')\cos k(t-t')\right|\right.\\ & \quad\left.\left.+\left|\int_{t_0}^t\mathrm{d}s_1\int_{t_0}^{s_1}\mathrm{d}s_2\,(\varphi(s_1)\cdot\xi_t(s_2)+\varphi(s_2)\cdot\xi_t(s_1))\cos k(t-s_1)\cos k(s_2-t_0)\right|\right]\right\}\\ \le\ & \exp\left((|x|+\sqrt{t-t_0}|\varphi_\Delta|_0)\sum_{l=1}^n|\alpha_l|\right)\\ & \quad\times\exp\left\{\frac{k}{\sin k|\Delta|}\left[\left((t-t_0)(|x|+|x_0|)+2(t-t_0)^{3/2}|\varphi_\Delta|_0\right)\sum_{l=1}^n|\alpha_l|\right]\right\}\\ =\ & \prod_{l=1}^n \exp(C\,|\alpha_l|) \qquad \text{with} \qquad C = C(t-t_0, |\varphi_\Delta|_0, |x|, |x_0|).\end{aligned} \tag{36}$$

In other words, it is bounded on any ball with $|\varphi|_0 < const$. Hence, Ψ_n is an element in $(S_d)^{-1}$. The result of the proposition follows the characterization in Theorem 1. □

Step 2. The next step will be to tackle the integration of $T\Phi_n(\varphi)$ with respect to s and α as in Equation (28) under the three conditions of Corollary 2.

Lemma 1. *For every $n \in \mathbb{N}_0$, the integral*

$$\int_{[t_0,t]^n}\mathrm{d}^n s\int_{\mathbb{R}^{dn}} I_h \exp\left(\sum_{l=1}^n \alpha_l\cdot x(s_l)\right)\prod_{l=1}^n \mathrm{d}m(\alpha_l)$$

defined for any $\varphi \in S_d$ by

$$\begin{aligned}&\int_{[t_0,t]^n}\mathrm{d}^n s\int_{\mathbb{R}^{dn}} T\left(I_h\cdot\exp\left(\sum_{l=1}^n\alpha_l\cdot x(s_l)\right)\right)(\varphi)\prod_{l=1}^n\mathrm{d}m(\alpha_l)\\ &\qquad = TI_h(\varphi)\int_{[t_0,t]^n}\mathrm{d}^n s\int_{\mathbb{R}^{dn}}\prod_{l=1}^n\mathrm{d}m(\alpha_l)\Psi_n(\varphi)\end{aligned} \tag{37}$$

is a Kondratiev distribution.

Proof. We have to check the conditions of Corollary 2:

1. For every $(s, \alpha) \in [t_0, t]^n \times \mathbb{R}^{dn}$, $n \in \mathbb{N}$, it follows from Equations (30) and (32) that $T\Phi_n(\varphi)$ is holomorphic in a neighborhood of zero.
2. Additionally, Equations (30) and (32) imply that

$$[t_0, t]^n \times \mathbb{R}^{nd} \ni (s, \alpha) \mapsto T\left(I_h \cdot \exp\left(\sum_{l=1}^{n} \alpha_l \cdot x(s_l)\right)\right)(\varphi) \in \mathbb{C}$$

is measurable for every φ in a neighborhood of zero.

3. The estimate in Equation (36) provides an integrable bound: it is independent of s, and the finiteness of Equation (23) ensures integrability in α:

$$\begin{aligned} & \left| \int_{[t_0,t]^n} \mathrm{d}^n s \int_{\mathbb{R}^{dn}} T\left(I_h \exp\left(\sum_{l=1}^{n} \alpha_l \cdot x(s_l)\right)\right)(\varphi) \prod_{l=1}^{n} dm(\alpha_l) \right| \\ \leq \quad & |TI_h(\varphi)| \int_{[t_0,t]^n} \mathrm{d}^n s \int_{\mathbb{R}^{dn}} \prod_{l=1}^{n} d|m|(\alpha_l) \exp(C\,|\alpha_l|) \\ = \quad & |TI_h(\varphi)| (t - t_0)^n \left(\int_{\mathbb{R}^d} d|m|(\alpha) \exp(C\,|\alpha|) \right)^n. \quad \square \end{aligned} \tag{38}$$

Step 3. To justify the expression in Equation (28), there remains the summation over n. This is performed with the help of Corollary 1: All terms are Kondratiev distributions, and hence their T-transforms are holomorphic and uniformly bounded in n for bounded φ. In addition, their series is absolutely convergent since

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{1}{n!} \left| \int_{[t_0,t]^n} \mathrm{d}^n s \int_{\mathbb{R}^{dn}} T\left(I_h \exp\left(\sum_{l=1}^{n} \alpha_l \cdot x(s_l)\right)\right)(\varphi) \prod_{l=1}^{n} dm(\alpha_l) \right| \\ < \quad & |TI_h(\varphi)| \sum_{n=0}^{\infty} \frac{1}{n!} (t - t_0)^n \left(\int_{\mathbb{R}^d} d|m|(\alpha) \exp(C\,|\alpha|) \right)^n \\ = \quad & |TI_h(\varphi)| \exp\left((t - t_0) \left(\int_{\mathbb{R}^d} d|m|(\alpha) \exp(C\,|\alpha|) \right) \right). \end{aligned} \tag{39}$$

In this way, we have established the existence of the Feynman integrand in Equation (28) as a Kondratiev distribution for the class of potentials described in Equations (25) and (26). This is our main result, which we state in the following. In Appendix A we show our "alternative" result using another approach within white noise analysis.

Theorem 3. *For a potential V of the form*

$$V(x) = \int_{\mathbb{R}^d} e^{\alpha \cdot x} \, \mathrm{d}m(\alpha) + \frac{1}{2} k^2 |x|^2, \quad x \in \mathbb{R}^d, \tag{40}$$

where m is any complex measure with

$$\int_{\mathbb{R}^d} e^{C|\alpha|} \, \mathrm{d}|m|(\alpha) < \infty, \qquad \forall C > 0, \tag{41}$$

and $0 < k|\Delta| < \frac{\pi}{2}$, the Feynman integrand

$$\begin{aligned} I \quad &= \quad I_0 \exp\left(-\mathrm{i} \int_{t_0}^{t} V(x(\tau)) \, \mathrm{d}\tau\right) \\ &= \quad \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int_{[t_0,t]^n} \mathrm{d}^n s \int_{\mathbb{R}^{dn}} I_h \cdot \exp\left(\sum_{l=1}^{n} \alpha_l \cdot x(s_l)\right) \prod_{l=1}^{n} \mathrm{d}m(\alpha_l) \end{aligned} \tag{42}$$

exists as a generalized white noise functional. The series converges strongly in $(S_d)^{-1}$, and the integrals exist in the sense of Bochner integrals.

Remark 1. *Generalization to time-dependent potentials is straightforward (see [12,15]).*

Remark 2. *For smooth φ, the T-transform of I solves the Schrödinger equation for all x, x_0, and $t_0 < t$. The propagator $K^{(\varphi)}$ with*

$$\left(\mathrm{i}\frac{\partial}{\partial t} + \frac{1}{2}\Delta_d - gV(x) - x \cdot \dot{\varphi}(t)\right)K^{(\varphi)}(x, t \mid x_0, t_0) = 0 \tag{43}$$

and initial condition

$$\lim_{t \searrow t_0} K^{(\varphi)}(x, t \mid x_0, t_0) = \delta(x - x_0) \tag{44}$$

are given by

$$K^{(\varphi)}(x, t \mid x_0, t_0) = TI(\varphi) \cdot \exp\left(\frac{\mathrm{i}}{2}|\varphi_{\Delta^c}|_0^2 + ix \cdot \varphi(t) - ix_0 \cdot \varphi(t_0)\right). \tag{45}$$

This can be verified explicitly, for example, as shown in [12,14].

Remark 3. *Note that Theorem 3 thus implies the existence of a convergent perturbation series for the propagator, although the potentials*

$$V_1(x) = \int_{\mathbb{R}^d} \mathrm{e}^{\alpha \cdot x} \, \mathrm{d}m(\alpha), \quad x \in \mathbb{R}^d, \tag{46}$$

are singular perturbations (i.e., not Kato-small with respect to the free Hamiltonian).

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Appendix A. A White Noise Analysis Approach to Feynman Integrands in a Phase Space

Applications of white noise analysis have, through the years, improved since the 1970s, when Takeyuki Hida first developed the theory. For instance, one may choose to construct Feynman integrals for admissible potentials with white noise analysis for a coordinate space using either the "conventional" approach (for the harmonic oscillator, see, for example, [13]) or the phase space approach (for the harmonic oscillator, see, for example, [17]). One can also construct momentum space Feynman integrals as well-defined white noise functionals using the phase space approach (see, e.g., [18]).

With that, to exhaust these available white noise analysis methods, we outline here the proofs on an alternative to our main result in Theorem 3 by using the white noise analysis approach to phase space Feynman integrals.

In [17], the coordinate x (of dimension $d_x = 1$) and the momentum p (of dimension $d_p = 1$) are taken as individual dimensions. Hence, if the potential $V = V(x)$ depends on x but not on p, then it takes the one-dimensional case, while $\boldsymbol{w} := (w_x, w_p) \in S_2'$ and $\boldsymbol{\varphi} := (\varphi_x, \varphi_p) \in S_2$, having both x and p components, are therefore two-dimensional. Here, $(\cdot)_x$ denotes that of the coordinate part, while $(\cdot)_p$ denotes that of the momentum part. It has been shown in particular that the Feynman integrand for the harmonic oscillator in a phase space $I_{h,\text{phase}}$ exists as a Hida distribution, and the T-transform of $I_{h,\text{phase}}$ at $\boldsymbol{\varphi} = (\varphi_x, \varphi_p) \in S_2$, is given by the following, compared with Equation (11) in [17] ($t_0 = 0, x_0 = 0$):

$$\begin{aligned} TI_{h,\text{phase}}(\boldsymbol{\varphi}) &= \sqrt{\frac{k}{2\pi \mathrm{i} \sin kt}} \exp\left\{ \frac{\mathrm{i}k}{2\tan kt} [x - (\boldsymbol{\eta}, \boldsymbol{\varphi})_0]^2 \right\} \\ &\quad \times \exp\left[-\frac{1}{2} \left(\boldsymbol{\varphi}, \mathbb{1}_{[0,t)^c} \boldsymbol{\varphi} \right)_0 \right] \exp\left[-\frac{1}{2} (\boldsymbol{\varphi}, Q(k,t)\boldsymbol{\varphi})_0 \right], \end{aligned} \tag{A1}$$

where $\boldsymbol{\eta} = (\mathbb{1}_{[0,t)}, 0)$ and

$$Q(k,t) = \begin{pmatrix} -\mathrm{i}\mathbb{1}_{[0,t)}(k^2 A - \mathbb{1}_{[0,t)})^{-1} & -\mathrm{i}\mathbb{1}_{[0,t)}(k^2 A - \mathbb{1}_{[0,t)})^{-1} \\ -\mathrm{i}\mathbb{1}_{[0,t)}(k^2 A - \mathbb{1}_{[0,t)})^{-1} & -\mathrm{i}k\mathbb{1}_{[0,t)} A (k^2 A - \mathbb{1}_{[0,t)})^{-1} \end{pmatrix}.$$

Here, the operator A, which has the properties described in [22], applies to $f \in L^2(\mathbb{R}, \mathbb{C})$ as [17]

$$Af(s) = \mathbb{1}_{[0,t)}(s) \int_s^t \int_0^\tau f(r)\, \mathrm{d}r\, \mathrm{d}\tau, \quad s \in \mathbb{R}.$$

Let $I_{V,\text{phase}}$ and $I_{0,\text{phase}}$ be the Feynman integrands in the phase space for the potential V and for the free particle, respectively. Then, using Equation (A1), and following similar procedures to those shown in the main body above, we obtain an alternative result (with respect to Theorem 3), as stated in the following:

Theorem A1. *For a potential V of the form ($d_x = 1$)*

$$V(x) = \int_{\mathbb{R}} e^{\alpha x}\, \mathrm{d}m(\alpha) + \frac{1}{2}k|x|^2, \quad x \in \mathbb{R}, \tag{A2}$$

where m is any complex measure with

$$\int_{\mathbb{R}} e^{C|\alpha|}\, \mathrm{d}|m|(\alpha) < \infty, \qquad \forall C > 0, \tag{A3}$$

and $0 < k|\Delta| < \frac{\pi}{2}$, the Feynman integrand in phase space

$$\begin{aligned} I_{V,\text{phase}} &= I_{0,\text{phase}} \cdot \exp\left(-\mathrm{i} \int_0^t V(x(\tau))\, \mathrm{d}\tau \right) \\ &= \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int_{[0,t]^n} \int_{\mathbb{R}^n} I_{h,\text{phase}} \cdot \exp\left(\sum_{l=1}^n \alpha_l x(s_l) \right) \prod_{l=1}^n \mathrm{d}m(\alpha_l)\, \mathrm{d}^n s \end{aligned} \tag{A4}$$

exists as a generalized white noise functional. The series converges strongly in $(S_2)^{-1}$, and the integrals exist in the sense of Bochner integrals.

Proof. Let $\Phi_{n,\text{phase}} := I_{h,\text{phase}} \cdot \exp(\sum_{l=1}^{n} \alpha_l x(s_l))$, $\boldsymbol{w} := (w_x, w_p)$ and $\boldsymbol{\xi}_t := \left(\sum_{l=1}^{n} \alpha_l \mathbb{1}_{[s_l,t)}, 0\right)$. Then, the T-transform of $\Phi_{n,\text{phase}}$ at $\boldsymbol{\varphi} \in S_2$ yields

$$\begin{aligned} T\Phi_{n,\text{phase}}(\boldsymbol{\varphi}) &= \int_{S_2'} \Phi_{\text{phase}}(\boldsymbol{w}) \exp(\mathrm{i}\langle \boldsymbol{w}, \boldsymbol{\varphi}\rangle) \mathrm{d}\mu(\boldsymbol{w}) \\ &= TI_{h,\text{phase}}(\boldsymbol{\varphi} + \mathrm{i}\boldsymbol{\xi}_t) \exp\left(x \sum_{l=1}^{n} \alpha_l \right), \end{aligned} \tag{A5}$$

where we explicitly have

$$\begin{aligned} TI_{h,\text{phase}}(\boldsymbol{\varphi} + \mathrm{i}\boldsymbol{\xi}_t) &= \sqrt{\frac{k}{2\pi \mathrm{i} \sin kt}} \exp\left\{ \frac{\mathrm{i}k}{2\tan kt} [x - (\boldsymbol{\eta}, \boldsymbol{\varphi} + \mathrm{i}\boldsymbol{\xi}_t)_0]^2 \right\} \\ &\quad \times \exp\left[-\frac{1}{2} (\boldsymbol{\varphi} + \mathrm{i}\boldsymbol{\xi}_t, \mathbb{1}_{[0,t)^c} (\boldsymbol{\varphi} + \mathrm{i}\boldsymbol{\xi}_t))_0 \right] \\ &\quad \times \exp\left[-\frac{1}{2} (\boldsymbol{\varphi} + \mathrm{i}\boldsymbol{\xi}_t, Q(k,t)(\boldsymbol{\varphi} + \mathrm{i}\boldsymbol{\xi}_t))_0 \right], \\ &= TI_{h,\text{phase}}(\boldsymbol{\varphi}) \Psi_{n,\text{phase}}(\boldsymbol{\varphi}, \boldsymbol{\xi}_t), \end{aligned} \tag{A6}$$

with

$$\begin{aligned} \Psi_{n,\text{phase}}(\boldsymbol{\varphi}, \boldsymbol{\xi}_t) &= \exp\left\{ \sum_{l=1}^{n} \alpha_l \left[\frac{k}{\tan kt} x(t - s_l) + \left(\frac{\mathrm{i}}{2} \sum_{j=1}^{n} \alpha_j (t - s_l \wedge s_j) \right.\right.\right. \\ &\quad \left.\left.\left. + \int_{s_l}^{t} \varphi_x(s)\, \mathrm{d}s \right) \left(1 - \frac{k}{\tan kt} \right) + \int_{s_l}^{t} \varphi_p(s)\, \mathrm{d}s \right] \right\}. \end{aligned} \tag{A7}$$

Now, let $F(\boldsymbol{\varphi}) = TI_{h,\text{phase}}(\boldsymbol{\varphi} + \mathrm{i}\boldsymbol{\xi}_t)$. Notice that $F(\boldsymbol{\varphi})$ is of second-order polynomials in $\boldsymbol{\varphi}$ and is thus holomorphic. Hence, $F(\boldsymbol{\varphi})$ fulfills the first condition of Definition 1. Furthermore, it has already been shown in [17] that $TI_{h,\text{phase}}(\boldsymbol{\varphi})$ is holomorphic. Therefore, we only need to see the absolute bounds of $\Psi_{n,\text{phase}}(\boldsymbol{\varphi}, \boldsymbol{\xi}_t) \exp\left(x \sum_{l=1}^{n} \alpha_l\right)$:

$$\begin{aligned} \left| \Psi_{n,\text{phase}}(\boldsymbol{\varphi}, \boldsymbol{\xi}_t) \exp\left(x \sum_{l=1}^{n} \alpha_l \right) \right| &\leq \exp\left(|x| \sum_{l=1}^{n} |\alpha_l| \right) \exp\left(\sum_{l=1}^{n} |\alpha_l| \left| \frac{k}{\tan kt} x(t - s_l) \right| \right) \\ &\quad \times \exp\left\{ \sum_{l=1}^{n} |\alpha_l| \left| \left(\int_{s_l}^{t} \varphi_x(s)\, \mathrm{d}s \right) \left(-\frac{k}{\tan kt} \right) \right| \right\} \\ &\quad \times \exp\left\{ \sum_{l=1}^{n} |\alpha_l| \left| \left(\int_{s_l}^{t} (\varphi_x(s) + \varphi_p(s))\, \mathrm{d}s \right) \right| \right\} \\ &\leq \exp\left(|x| \sum_{l=1}^{n} |\alpha_l| \right) \exp\left\{ \left[\frac{k}{\tan kt} \left(|x| t + \sqrt{t} |\varphi_{x,\Delta}|_0 \right) \right.\right. \\ &\quad \left.\left. + \sqrt{t} \left(|\varphi_{x,\Delta}|_0 + |\varphi_{p,\Delta}|_0 \right) \right] \sum_{l=1}^{n} |\alpha_l| \right\} \\ &= \prod_{l=1}^{n} \exp(C\, |\alpha_l|) \qquad \text{with} \qquad C = C(t, |\boldsymbol{\varphi}_\Delta|_0, |x|). \end{aligned} \tag{A8}$$

The remaining part of the proof follows a similar fashion to the procedures already shown in the main text above, differing only in the value of C as in Equation (A8) compared with that in Equation (36). The details have been omitted. □

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