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Nonlocality of Star-Shaped Correlation Tensors Based on the Architecture of a General Multi-Star-Network

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Abstract: In this work, we study the nonlocality of star-shaped correlation tensors (SSCTs) based on a general multi-star-network $MSN(m, n_1, \ldots, n_m)$. Such a network consists of $1 + m + n_1 + \cdots + n_m$ nodes and one center-node A that connects to m star-nodes $B^1, B^2, \ldots, B^m$ while each star-node B^j has $n_j + 1$ star-nodes $A, C_1^j, C_2^j, \ldots, C_{n_j}^j$. By introducing star-locality and star-nonlocality into the network, some related properties are obtained. Based on the architecture of such a network, SSCTs including star-shaped probability tensors (SSPTs) are proposed and two types of localities in SSCTs and SSPTs are mathematically formulated, called D-star-locality and C-star-locality. By establishing a series of characterizations, the equivalence of these two localities is verified. Some necessary conditions for a star-shaped CT to be D-star-local are also obtained. It is proven that the set of all star-local SSCTs is a compact and path-connected subset in the Hilbert space of tensors over the index set Δ_S and has least two types of star-convex subsets. Lastly, a star-Bell inequality is proved to be valid for all star-local SSCTs. Based on our inequality, two examples of star-nonlocal $MSN(m, n_1, \ldots, n_m)$ are presented.

Keywords: multi-star-network; star-shaped correlation tensor; star-locality; star-Bell inequality

MSC: 81P45; 81P40

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1. Introduction

As promising platforms for quantum information processing, quantum networks (QNs) [1] have recently attracted much interest [2–7]. It is important to understand the quantum correlations that arise in a QN. Recent developments have shown that the topological structure of a QN leads to novel notions of nonlocality [8,9] and new concepts of entanglement and separability [10–12]. These new concepts and definitions are different from the traditional ones [13,14] and thus need to be analysed using new theoretical tools, such as mutual information [10,11], fidelity with pure states [11,12], and covariance matrices built from measurement probabilities [15,16].

According to Bell's local causality assumption [17,18], the joint probability $P(o_1o_2 \ldots o_n|m_1m_2 \ldots m_n)$ of obtaining measurement outcomes $o_1, o_2, \ldots, o_n$ of systems $A_1, A_2, \ldots, A_n$ can be obtained in terms of a local hidden variable model (LHVM) with just one "hidden variable", or "hidden state", λ. Such a probability distribution is said to be Bell local. Focusing on QNs, completely different approaches to multipartite nonlocality were proposed [19–23]. That means that network nonlocalities are fundamentally different from standard multipartite nonlocalities. Carvacho et al. [24] investigated a quantum network consisting of three spatially separated nodes and experimentally witnessed quantum correlations in the network. Due to the complex topological structure of a network, it is possible to detect the quantum nonlocality in experiments by performing just one fixed measurement [8,25–28].

Quantum coherence originated from the superposition principle originally pointed out by Schrödinger [29] and is a fundamentally quantum property [30,31]. Quantum nonlocality is a correlation property of subsystems of a multipartite system, exhibited by a set of local measurements. It is also a powerful tool for analyzing correlations in a quantum network [32] and a direct link between the theory of multisubspace coherence [33] and the approach to quantum networks with covariance matrices [15,16].

Patricia et al. [34] found some sufficient conditions for nonlocality in QNs and showed that any network with shared pure entangled states is genuinelu multipartite nonlocal. Šupić et al. [35] proposed a concept of genuine network quantum nonlocality and proved several examples of genuine network nonlocal correlations.

Recently, Tavakoli et al. [36] discussed the main concepts, methods, results, and future challenges of network nonlocality with a list of open problems. More recently, Xiao et al. [37] discussed two types of trilocality in probability tensors (PTs), $P = [\![P(a_1a_2a_3)]\!]$ and that of correlation tensors (CTs) $P = [\![P(a_1a_2a_3|x_1x_2x_3)]\!]$, based on the triangle network [8] and described by continuous (integral) and discrete (sum) trilocal hidden variable models (C-triLHVMs and D-triLHVMs).

Haddadi et al. [38] studied the thermal evolution of the entropic uncertainty bound in the presence of quantum memory for an inhomogeneous, four-qubit, spin-star system and proved that the entropic uncertainty bound can be controlled and suppressed by adjusting the inhomogeneity parameter of the system. Related research on spin-star systems can be found in [39,40] and the references therein. As a generalization of star-networks [22,23], Yang et al. [41] considered the nonlocality of $(2^n - 1)$-partite tree-tensor networks (referring to Figure 1 for the case where $n = 2$) and derived the Bell-type inequalities.

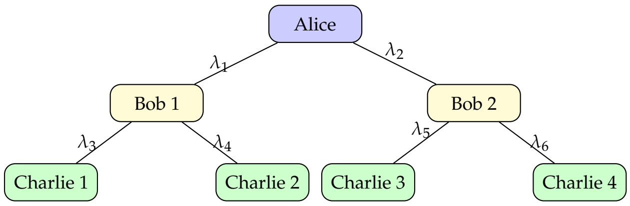

Figure 1. The six-local tree-tensor networks consisting of seven parties and six independent sources $S_1, S_2, \ldots, S_6$ characterized by hidden variables $\lambda_1, \lambda_2, \ldots, \lambda_6$, respectively [41].

Extending the scenario in [41], Yang et al. [42] discussed the nonlocality of a type of multi-star-shaped QNs (Figure 2), called 3-layer m-star QNs (3-m-SQNWs), and established related Bell-type inequalities.

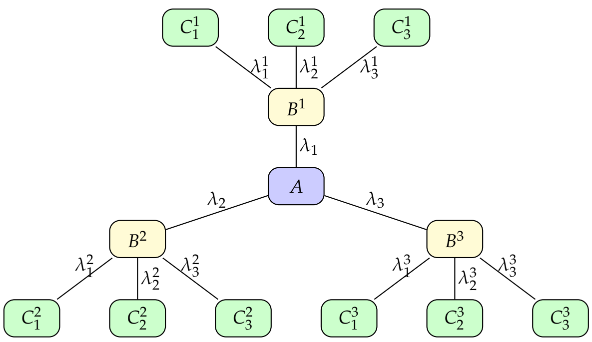

Figure 2. A 3-layer m-star quantum network (3-m-SQNW) for $m = 3$ consisting of a node A, m star-nodes $B^1, B^2, \ldots, B^m$, and m^2 star-nodes $C_1^j, C_2^j, \ldots, C_m^j (j = 1, 2, \ldots, m)$ [42].

In this work, we study the nonlocality of star-shaped CTs and star-shaped PTs based on a more general multi-star network $MSN(m, n_1, \ldots, n_m)$ depicted in Figure 3.

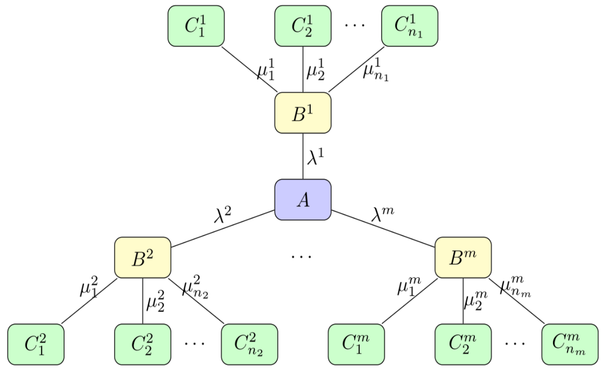

Figure 3. The multi-star-network scenario, denoted by $MSN(m, n_1, \ldots, n_m)$. When $m = 1, n_1 = n - 1$, it reduces to $MSN(1, n-1)$, which is just an n-local scenario [22,43]; when $m = n_1 = 1$, it becomes $MSN(1, 1)$, reducing to the bi-local scenario [20,43].

Such a network consists of $1 + m + n_1 + \cdots + n_m$ nodes and one center-node A that connects to m star-nodes $B^1, B^2, \ldots, B^m$ while each star-node B^j has $n_j + 1$ star-nodes $A, C_1^j, C_2^j, \ldots, C_{n_j}^j$.

In Section 2, we will introduce the star-locality and star-nonlocality of the multi-star-network $MSN(m, n_1, \ldots, n_m)$ and give some related properties. In Section 3, we will first introduce star-shaped CTs (SSCTs), including star-shaped PTs (SSPTs), and discuss two types of localities of SSCTs and SSPTs, called D-star-locality and C-star-locality. Then, we establish a series of characterizations of D-star-localities and C-star-localities, show the equivalence of these two types of localities, and give some necessary conditions for star-shaped CT to be D-star-local. At the end of this section, we will show that the set $\mathcal{CT}^{\text{star-local}}(\Delta_S)$ of all star-local SSCTs over the index set Δ_S is a compact and path-connected subset in the Hilbert space $\mathcal{T}^{\text{star}}(\Delta_S)$ of all tensors over Δ_S and contains at least two types of subsets that are star-convex. In Section 4, we shall establish an inequality that holds for all star-local SSCTs, called a star-Bell inequality. Based on our inequality, two examples are given. The first example is a star-nonlocal $MSN(m, n_1, \ldots, n_m)$, in which the shared states are all entangled pure states, and the second one gives a star-nonlocal $MSN(m, n_1, \ldots, n_m)$ in which the shared states are all entangled mixed states. In Section 5, we will give a summary and conclusions.

2. Multi-Star-Network Scenario

2.1. Notations and Concepts

In what follows, we consider the multi-star-network scenario as depicted in Figure 3, denoted by $MSN(m, n_1, \ldots, n_m)$. The network involves $1 + m + \sum_{j=1}^m n_j$ parties

$$A, B^1, \ldots, B^m, C_1^1, \ldots, C_{n_1}^1, \ldots, C_1^m, \ldots, C_{n_m}^m$$

and $m + \sum_{j=1}^m n_j$ sources

$$S^1, \ldots, S^m, S_1^1, \ldots, S_{n_1}^1, \ldots, S_1^m, \ldots, S_{n_m}^m,$$

which are characterized by hidden variables $\lambda^j \in D_j$ and $\mu_k^j \in F_j(k)(j \in [m], k \in [n_j])$, where $[n] := \{1, 2, \ldots, n\}$.

We use $\rho_{A_j B_0^j} \in \mathcal{D}(\mathcal{H}_{A^j} \otimes \mathcal{H}_{B_0^j})$ to denote the states shared by A and B^j for all $j \in [m]$, and $\rho_{B_k^j C_k^j} \in \mathcal{D}(\mathcal{H}_{B_k^j} \otimes \mathcal{H}_{C_k^j})$ to denote the states shared by B^j and C_k^j for all $j \in [m]$ and

$k \in [n_j]$. We get $\mathcal{H}_A = \bigotimes_{j=1}^m \mathcal{H}_{A^j}$, $\mathcal{H}_{B^j} = \mathcal{H}_{B_0^j} \otimes (\bigotimes_{k=1}^{n_j} \mathcal{H}_{B_k^j})(j = 1, 2, \ldots, m)$. Then we define the system state as

$$\Gamma = \left(\bigotimes_{j=1}^{m} \rho_{A^j B_0^j} \right) \otimes \left(\bigotimes_{j=1}^{m} (\rho_{B_1^j C_1^j} \otimes \rho_{B_2^j C_2^j} \otimes \ldots \otimes \rho_{B_{n_j}^j C_{n_j}^j}) \right). \tag{1}$$

Consider the measurement assemblages

$$\left. \begin{array}{l} \mathcal{M}(A) = \left\{ M(x) := \{M_{a|x}\}_{a=1}^{o(A)} : x = 1, 2, \ldots, m(A) \right\}, \\ \mathcal{N}(B^j) = \left\{ N^j(y_j) := \{N^j_{b_j|y_j}\}_{b_j=1}^{o(B^j)} : y_j = 1, 2, \ldots, m(B^j) \right\}, \\ \mathcal{L}(C_k^j) = \left\{ L_k^j(z_{j,k}) := \{L^{j,k}_{c_{j,k}|z_{j,k}}\}_{c_{j,k}=1}^{o(C_k^j)} : z_{j,k} = 1, 2, \ldots, m(C_k^j) \right\} \end{array} \right\} \tag{2}$$

consisting of positive-operator-valued measures (POVMs), on systems A, B^j and C_k^j, respectively, where $j \in [m]$ and $k \in [n_j]$, consisting of positive operators satisfying the normalization conditions:

$$\sum_{a=1}^{o(A)} M_{a|x} = I_A, \sum_{b_j=1}^{o(B^j)} N^j_{b_j|y_j} = I_{B^j}, \sum_{c_{j,k}=1}^{o(C_k^j)} L^{j,k}_{c_{j,k}|z_{j,k}} = I_{C_k^j}.$$

Then, we can obtain a measurement assemblage (MA)

$$\mathcal{M} := \mathcal{M}(A) \otimes \left(\bigotimes_{j=1}^{m} \mathcal{N}(B^j) \right) \otimes \left(\bigotimes_{j=1}^{m} (\mathcal{L}(C_1^j) \otimes \mathcal{L}(C_2^j) \otimes \ldots \otimes \mathcal{L}(C_{n_j}^j)) \right) \tag{3}$$

of the quantum network with measurement operators

$$M_{a\mathbf{bc}|x\mathbf{yz}} := M_{a|x} \otimes \left(\bigotimes_{j=1}^{m} N^j_{b_j|y_j} \right) \otimes \left(\bigotimes_{j=1}^{m} (L^{j,1}_{c_{j,1}|z_{j,1}} \otimes L^{j,2}_{c_{j,2}|z_{j,2}} \otimes \ldots \otimes L^{j,n_j}_{c_{j,n_j}|z_{j,n_j}}) \right), \tag{4}$$

where $x \in [m(A)]$, $y_j \in [m(B^j)]$ and $z_k^j \in [m(C_k^j)]$ denote the inputs of parties A, B^j and C_k^j with the corresponding outputs $a \in [o(A)]$, $b_j \in [o(B_j)]$ and $c_k^j \in [o(C_k^j)]$, respectively, and

$$\mathbf{y} = (y_1, y_2, \ldots, y_m) \equiv \{y_j\}_{j=1}^m, \ \mathbf{b} = (b_1, b_2, \ldots, b_m) \equiv \{b_j\}_{j=1}^m,$$

$$\mathbf{z} = (z_{1,1}, \ldots, z_{1,n_1}, z_{2,1}, , \ldots, z_{2,n_2}, \ldots, z_{m,1}, \ldots, z_{m,n_m}) \equiv \{z_{j,k}\}_{j \in [m], k \in [n_j]},$$

$$\mathbf{c} = (c_{1,1}, \ldots, c_{1,n_1}, c_{2,1}, \ldots, c_{2,n_2}, \ldots, c_{m,1}, \ldots, c_{m,n_m}) \equiv \{c_{j,k}\}_{j \in [m], k \in [n_j]}.$$

Clearly, the measurement operators $M_{abc|xyz}$ are positive operators acting on the Hilbert space

$$\mathcal{H}_{\mathrm{MHS}} := \mathcal{H}_A \otimes \left(\bigotimes_{j=1}^{m} \mathcal{H}_{B^j} \right) \otimes \left(\bigotimes_{j=1}^{m} (\mathcal{H}_{C_1^j} \otimes \mathcal{H}_{C_2^j} \otimes \ldots \otimes \mathcal{H}_{C_{n_j}^j}) \right),$$

while the system state Γ given by (1) is an operator acting on the Hilbert space

$$\mathcal{H}_{\mathrm{SHS}} := \left(\bigotimes_{j=1}^{m} (\mathcal{H}_{A_j} \otimes \mathcal{H}_{B_0^j}) \right) \otimes \left(\bigotimes_{j=1}^{m} (\mathcal{H}_{B_1^j} \otimes \mathcal{H}_{C_1^j} \otimes \ldots \otimes \mathcal{H}_{B_{n_j}^j} \otimes \mathcal{H}_{C_{n_j}^j}) \right).$$

Generally, $\mathcal{H}_{\mathrm{MHS}} \neq \mathcal{H}_{\mathrm{SHS}}$ due to the non-commutativity of tensor product, and in that case, the product $M_{a\mathbf{bc}|x\mathbf{yz}}\Gamma$ does not work well. Therefore, we have to change the system state Γ to a state $\tilde{\Gamma}$ acting on the space $\mathcal{H}_{\text{MHS}}$ in order to make the tensor product $M_{a\mathbf{bc}|x\mathbf{yz}}\tilde{\Gamma}$ reasonable. To do this, we define a swapping operation $U : \mathcal{H}_{\text{SHS}} \to \mathcal{H}_{\text{MHS}}$ by $|\Psi\rangle \mapsto U|\Psi\rangle$, where

$$\begin{aligned} U|\Psi\rangle &= \left(\bigotimes_{j=1}^{m}|\psi_{A_j}\rangle\right) \otimes \left(\bigotimes_{j=1}^{m}(|\psi_{B_0^j}\rangle \otimes |\psi_{B_1^j}\rangle \otimes \ldots \otimes |\psi_{B_m^j}\rangle)\right) \otimes \\ &\quad \left(\bigotimes_{j=1}^{m}(|\psi_{C_1^j}\rangle \otimes \ldots \otimes |\psi_{C_{n_j}^j}\rangle)\right) \\ &\quad \in \mathcal{H}_{\text{MHS}} \end{aligned}$$

for all

$$\begin{aligned} |\Psi\rangle &= \left(\bigotimes_{j=1}^{m}\Big(|\psi_{A_j}\rangle \otimes |\psi_{B_0^j}\rangle\Big)\right) \\ &\quad \otimes \left(\bigotimes_{j=1}^{m}\Big(|\psi_{B_1^j}\rangle \otimes |\psi_{C_1^j}\rangle \otimes \ldots \otimes |\psi_{B_{n_j}^j}\rangle \otimes |\psi_{C_{n_j}^j}\rangle\Big)\right) \\ &\quad \in \mathcal{H}_{\text{SHS}}. \end{aligned}$$

Then, we obtain a new state $\tilde{\Gamma} = U\Gamma U^{\dagger}$ acting the Hilbert space $\mathcal{H}_{\text{MHS}}$ so that the operator product $M_{a\mathbf{bc}|x\mathbf{yz}}\tilde{\Gamma}$ works well. Furthermore, it is easy to see that

$$\text{tr}[M_{a\mathbf{bc}|x\mathbf{yz}}\tilde{\Gamma}] = \text{tr}[\widetilde{M}_{abc|xyz}\Gamma], \tag{5}$$

where $\widetilde{M}_{a\mathbf{bc}|x\mathbf{yz}} = U^{\dagger}M_{a\mathbf{bc}|x\mathbf{yz}}U$, which is an operator acting on the Hilbert space $\mathcal{H}_{\text{SHS}}$ for every index $(a, \mathbf{b}, \mathbf{c}, x, \mathbf{y}, \mathbf{z})$. Thus, the joint probability distribution $P(abc|xyz)$ of obtaining a, b, c reads:

$$P^{\Gamma}_{\mathcal{M}}(a\mathbf{bc}|x\mathbf{yz}) := \text{tr}[M_{a\mathbf{bc}|x\mathbf{yz}}\tilde{\Gamma}] = \text{tr}[\widetilde{M}_{a\mathbf{bc}|x\mathbf{yz}}\Gamma]. \tag{6}$$

With these preparations, we can describe the locality and nonlocality of our quantum network $MSN(m, n_1, \ldots, n_m)$ as follows.

Definition 1. *A quantum network $MSN(m, n_1, \ldots, n_m)$ with the state (1) is said to be star-local for an MA $\mathcal{M}$ given by (3) if there exists a probability distribution (PD)*

$$p(\lambda, \mu_1, \ldots, \mu_m) = \prod_{j=1}^{m} p(\lambda^j) \times \prod_{j=1}^{m}\prod_{k=1}^{n_j} p(\mu_k^j), \tag{7}$$

where $\{p_j(\lambda^j)\}_{\lambda^j}$ and $\{p_{j,k}(\mu_k^j)\}_{\mu_k^j}$ are respectively probability distributions (PDs) of λ^j and μ_k^j such that for all $a, \mathbf{b}, \mathbf{c}, x, \mathbf{y}, \mathbf{z}$, it holds that

$$\begin{aligned} P^{\Gamma}_{\mathcal{M}}(a\mathbf{bc}|x\mathbf{yz}) &= \sum_{\lambda\in D, \mu_1\in F_1, \ldots, \mu_m\in F_m} p(\lambda, \mu_1, \ldots, \mu_m)P_A(a|x, \lambda) \\ &\quad \times \prod_{j=1}^{m} P_{B^j}(b_j|y_j, \lambda^j, \mu_j) \times \prod_{j=1}^{m}\prod_{k=1}^{n_j} P_{C_k^j}(c_{j,k}|z_{j,k}, \mu_k^j), \end{aligned} \tag{8}$$

where

$$\lambda = (\lambda^1, \ldots, \lambda^m) \in D, \mu_j = (\mu_1^j, \ldots, \mu_{n_j}^j) \in F_j (j \in [m]) \text{ (local hidden variables(LHVs))};$$

$$D = D_1 \times \ldots \times D_m,\ F_j = F_1^j \times \ldots \times F_{n_j}^j (j \in [m]) \text{ (finite sets of LHVs)},$$

$\{P_A(a|x,\lambda)\}, \{P_{B^j}(b_j|y_j,\lambda^j,\mu_j)\}$ and $\{P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j)\}$ are PDs of a, b_j and $c_{j,k}$, respectively. Otherwise, $MSN(m,n_1,\dots,n_m)$ is said to be star-nonlocal for $\mathcal{M}$.

$MSN(m,n_1,\dots,n_m)$ is said to be star-local if it is star-local for any $\mathcal{M}$, and it is said to be star-nonlocal if it is not star-local, i.e., it is star-nonlocal for some $\mathcal{M}$.

2.2. Properties

Similar to the reference [42], we can obtain the following results:

Proposition 1. *If a network $MSN(m,n_1,\dots,n_m)$ with the state (1) is star-local for $\mathcal{M}$ given by Equation (3), then the $\tilde{\Gamma}$ as a state of system $AB_1\cdots B_mC_1^1\cdots C_{n_1}^1\cdots C_1^m\cdots C_{n_m}^m$ is Bell-local for $\mathcal{M}$.*

Proposition 2. *The reduced states of $\tilde{\Gamma}$ on subsystems $A_jB_0^j$ and $B_k^jC_k^j$ are $\tilde{\Gamma}_{A_jB_0^j} = \rho_{A_jB_0^j}$ and $\tilde{\Gamma}_{B_k^jC_k^j} = \rho_{B_k^jC_k^j}$, respectively, for all $j \in [m]$ and $k \in [n_j]$.*

Proposition 3. *If the network $MSN(m,n_1,\dots,n_m)$ with the state (1) is star-local, then the bipartite states $\rho_{B_t^jC_t^j}$ and $\rho_{A_jB_0^j}$ are Bell-local for all $s \in [m]$ and $t \in [n_j]$. Furthermore, the m-partite reduced state $(\tilde{\Gamma})_{B^1B^2\dots B^m}$ is Bell-local.*

Consequently, if one of bipartite states $\rho_{B_t^jC_t^j}$ and $\rho_{A_jB_0^j}$ is Bell-nonlocal, then the network $MSN(m,n_1,\dots,n_m)$ must be star-nonlocal. Especially, if one of the shared states is a pure entangled state, then the network $MSN(m,n_1,\dots,n_m)$ is star-nonlocal. See Examples 1 and 2 in Section 4.

Proposition 4. *Every separable (i.e., all of the shared states are separable) $MSN(m,n_1,\dots,n_m)$ is star-local.*

Proof. Since the shared states $\rho_{A_jB_0^j}$ and $\rho_{B_k^jC_k^j}$ are separable, they can be written as

$$\rho_{A_jB_0^j} = \sum_{\lambda^j=1}^{d_j} p_j(\lambda^j)|s'_{\lambda^j}\rangle\langle s'_{\lambda^j}| \otimes |s''_{\lambda^j}\rangle\langle s''_{\lambda^j}|,$$

$$\rho_{B_k^jC_k^j} = \sum_{\mu_k^j=1}^{d_k^j} p_{j,k}(\mu_k^j)|t'_{\mu_k^j}\rangle\langle t'_{\mu_k^j}| \otimes |t''_{\mu_k^j}\rangle\langle t''_{\mu_k^j}|,$$

where $p_j(\lambda^j)$ and $p_{j,k}(\mu_k^j)$ are PDs of λ^j and μ_k^j. Put

$$\lambda = (\lambda^1,\lambda^2,\dots,\lambda^m), \mu_j = (\mu_1^j,\mu_2^j,\dots,\mu_{n_j}^j),$$

$$D = [d_1]\times\dots\times[d_m],\ F_j = [d_1^j]\times\dots\times[d_{n_j}^j](j\in[m]),$$

then

$$\begin{aligned}
\Gamma &= \left(\bigotimes_{j=1}^m \rho_{A^jB_0^j}\right)\otimes\left(\bigotimes_{j=1}^m(\rho_{B_1^jC_1^j}\otimes\rho_{B_2^jC_2^j}\otimes\dots\otimes\rho_{B_{n_j}^jC_{n_j}^j})\right)\\
&= \sum_{\lambda\in D}\sum_{\mu_1\in F_1,\dots,\mu_m\in F_m}\prod_{j=1}^m p_j(\lambda^j)\times\prod_{j=1}^m\prod_{k=1}^{n_j}p_{j,k}(\mu_k^j)\\
&\quad\bigotimes_{j=1}^m\left(|s'_{\lambda^j}\rangle\langle s'_{\lambda^j}|\otimes|s''_{\lambda^j}\rangle\langle s''_{\lambda^j}|\right)\otimes\left(\bigotimes_{j=1}^m\bigotimes_{k=1}^{n_j}|t'_{\mu_k^j}\rangle\langle t'_{\mu_k^j}|\otimes|t''_{\mu_k^j}\rangle\langle t''_{\mu_k^j}|\right),
\end{aligned}$$

which induces the measurement state

$$\widetilde{\Gamma} = U\Gamma U^{\dagger} = \sum_{\lambda\in D}\sum_{\mu_1\in F_1,\ldots,\mu_m\in F_m}\prod_{j=1}^{m}p_j(\lambda^j)\times\prod_{j=1}^{m}\prod_{k=1}^{n_j}p_{j,k}(\mu_k^j)\times\Gamma'(\lambda,\mu_1,\ldots,\mu_m),$$

where

$$\begin{aligned}\Gamma'(\lambda,\mu_1,\ldots,\mu_m) &= \left(\bigotimes_{j=1}^{m}|s'_{\lambda^j}\rangle\langle s'_{\lambda^j}|\right)\otimes\left(\bigotimes_{j=1}^{m}\left(|s''_{\lambda^j}\rangle\langle s''_{\lambda^j}|\otimes\bigotimes_{k=1}^{n_j}|t'_{\mu_k^j}\rangle\langle t'_{\mu_k^j}|\right)\right)\\ &\quad\otimes\left(\bigotimes_{j=1}^{m}\bigotimes_{k=1}^{n_j}|t''_{\mu_k^j}\rangle\langle t''_{\mu_k^j}|\right).\end{aligned}$$

Thus, for any MA $\mathcal{M}$ given by (3), we compute that

$$\begin{aligned}P_{\mathcal{M}}^{\Gamma}(a\mathbf{bc}|x\mathbf{yz}) &= \mathrm{tr}[(M_{a|x}\otimes N_{\mathbf{b}|\mathbf{y}}\otimes L_{\mathbf{c}|\mathbf{z}})\widetilde{\Gamma}]\\ &= \sum_{\lambda\in D}\sum_{\mu_1\in F_1,\ldots,\mu_m\in F_m}\prod_{j=1}^{m}p_j(\lambda^j)\times\prod_{j=1}^{m}\prod_{k=1}^{n_j}p_{j,k}(\mu_k^j)\\ &\quad\times\mathrm{tr}[(M_{a|x}\otimes N_{\mathbf{b}|\mathbf{y}}\otimes L_{\mathbf{c}|\mathbf{z}})\Gamma'(\lambda,\mu_1,\ldots,\mu_m)]\\ &= \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m}\prod_{j=1}^{m}p_j(\lambda^j)\times\prod_{j=1}^{m}\prod_{k=1}^{n_j}p_{j,k}(\mu_k^j)\times P_A(a|x,\lambda)\\ &\quad\times\prod_{j=1}^{m}P_{B^j}(b_j|y_j,\lambda^j,\mu_j)\times\prod_{j=1}^{m}\prod_{k=1}^{n_j}P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j),\end{aligned}\tag{9}$$

where

$$P_A(a|x,\lambda) = \mathrm{tr}[M_{a|x}\left(\bigotimes_{j=1}^{m}|s'_{\lambda^j}\rangle\langle s'_{\lambda^j}|\right)];$$

$$P_{B^j}(b_j|y_j,\lambda^j,\mu_j) = \mathrm{tr}[N_{b_j|y_j}(|s''_{\lambda^j}\rangle\langle s''_{\lambda^j}|\otimes\bigotimes_{k=1}^{n_j}|t'_{\mu_k^j}\rangle\langle t'_{\mu_k^j}|)];$$

$$P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j) = \mathrm{tr}[L_{c_k^j|z_k^j}|t''_{\mu_k^j}\rangle\langle t''_{\mu_k^j}|)].$$

This shows that Equation (8) holds and then the network is star-local. The proof is completed. □

3. Star-Locality of Star-Shaped Cts

When a multi-star network given by Figure 3 for the case that $m = 3$ is measured by parties

$$A, B^1,\ldots,B^m, C_1^1,\ldots,C_{n_1}^1,\ldots,C_1^m,\ldots,C_{n_m}^m,$$

the conditional probabilities $P(a\mathbf{bc}|x\mathbf{yz})$ of obtaining result $(a,\mathbf{b},\mathbf{c})$ conditioned on the measurement choice $(x,\mathbf{y},\mathbf{z})$ form a correlation tensor (CT) [44] $\mathbf{P} = [\![P(a\mathbf{bc}|x\mathbf{yz})]\!]$ over the index set

$$\Delta_S = [o(A)]\times\prod_{j=1}^{m}[o(B^j)]\times\prod_{j=1}^{m}\prod_{k=1}^{n_j}[o(C_k^j)]\times[m_A]\times\prod_{j=1}^{m}[m(B^j)]\times\prod_{j=1}^{m}\prod_{k=1}^{n_j}[m(C_k^j)],\tag{10}$$

which is a non-negative function defined on Δ_S satisfying the following completeness condition:

$$\sum_{a,\mathbf{b},\mathbf{c}}P(a\mathbf{bc}|x\mathbf{yz}) = 1,\ \forall x,\mathbf{y},\mathbf{z}.\tag{11}$$

We call such a $\mathbf{P}$ a *star-shaped CT* over Δ_S. Let $\mathcal{CT}^{\text{star}}(\Delta_S)$ be the set of all star-shaped CTs over Δ_S.

To discuss the algebraic and topological properties of the $\mathcal{CT}^{\text{star}}(\Delta_S)$, we have to make it live in a Hilbert space. To accomplish this, we let $\mathcal{T}^{\text{star}}(\Delta_S)$ be the set of all real tensors $\mathbf{P} = [\![P(a\mathbf{bc}|x\mathbf{yz})]\!]$ over Δ_S. That is, $\mathbf{P} \in \mathcal{T}^{\text{star}}(\Delta_S)$ if and only if it is a real-valued function defined on Δ_S with the value $P(a\mathbf{bc}|x\mathbf{yz})$ and a point $(a, \mathbf{b}, \mathbf{c}, x, \mathbf{y}, \mathbf{z})$ in Δ_S. Clearly, $\mathcal{T}^{\text{star}}(\Delta_S)$ becomes a finite-dimensional Hilbert space over $\mathbb{R}$ with respect to the following operation and inner product:

$$s\mathbf{P}_1 + t\mathbf{P}_2 = [\![sP_1(a\mathbf{bc}|x\mathbf{yz}) + tP_2(a\mathbf{bc}|x\mathbf{yz})]\!],$$

$$\langle \mathbf{P}_1, \mathbf{P}_2 \rangle = \sum_{a,\mathbf{b},\mathbf{c},x,\mathbf{y},\mathbf{z}} P_1(a\mathbf{bc}|x\mathbf{yz})P_2(a\mathbf{bc}|x\mathbf{yz}).$$

The norm induced by the inner product reads

$$\|\mathbf{P}\| := \sqrt{\langle \mathbf{P}, \mathbf{P} \rangle} = \left\{ \sum_{a,\mathbf{b},\mathbf{c},x,\mathbf{y},\mathbf{z}} (P(a\mathbf{bc}|x\mathbf{yz}))^2 \right\}^{\frac{1}{2}}.$$

Especially, when $m(A) = m(B^j) = m(C_k^j) = 1$ for all k, j, we denote $\mathbf{P} = [\![P(a\mathbf{bc}|x\mathbf{yz})]\!]$ by $\mathbf{P} = [\![P(a\mathbf{bc})]\!]$ and call it a *star-shaped probability tensor (PT)* over

$$\Omega_S = [o(A)] \times \prod_{j=1}^{m} [o(B^j)] \times \prod_{j=1}^{m} \prod_{k=1}^{n_j} [o(C_k^j)].$$

Let $\mathcal{PT}^{\text{star}}(\Omega_S)$ be the set of all star-shaped PTs over Ω_S and let $\mathcal{T}^{\text{star}}(\Omega_S)$ be the set of all real tensors $\mathbf{P} = [\![P(a\mathbf{bc})]\!]$ over Ω_S, which is a finite-dimensional Hilbert space over $\mathbb{R}$ with respect to the following operation and inner product:

$$s\mathbf{P}_1 + t\mathbf{P}_2 = [\![sP_1(a\mathbf{bc}) + tP_2(a\mathbf{bc})]\!],$$

$$\langle \mathbf{P}_1, \mathbf{P}_2 \rangle = \sum_{a,\mathbf{b},\mathbf{c}} P_1(a\mathbf{bc})P_2(a\mathbf{bc}).$$

The norm induced by the inner product reads

$$\|\mathbf{P}\| := \sqrt{\langle \mathbf{P}, \mathbf{P} \rangle} = \left\{ \sum_{a,\mathbf{b},\mathbf{c}} (P(a\mathbf{bc}))^2 \right\}^{\frac{1}{2}}.$$

3.1. Concepts

Definition 2. *A star-shaped CT $\mathbf{P} = [\![P(a\mathbf{bc}|x\mathbf{yz})]\!]$ over Δ_S is said to be C-star-local if it admits a "C-star-shaped LHVM":*

$$\begin{aligned} P(a\mathbf{bc}|x\mathbf{yz}) \quad = \quad & \int_{D\times F_1\times\ldots\times F_m} p(\lambda, \mu_1, \ldots, \mu_m) P_A(a|x,\lambda) \prod_{j=1}^{m} P_{B^j}(b_j|y_j, \lambda^j, \mu_j) \\ & \times \prod_{j=1}^{m} \prod_{k=1}^{n_j} P_{C_k^j}(c_{j,k}|z_{j,k}, \mu_k^j)\, \mathrm{d}\gamma(\lambda) \mathrm{d}\tau_1(\mu_1) \ldots \mathrm{d}\tau_m(\mu_m) \end{aligned} \tag{12}$$

for all $a, \mathbf{b}, \mathbf{c}, x, \mathbf{y}, \mathbf{z}$, where

(i) $(\Lambda, \Omega, \mu) \equiv \left(D \times \prod_{j=1}^{m} F_j, \sigma \times \prod_{j=1}^{m} \delta_j, \gamma \times \prod_{j=1}^{m} \tau_j\right)$ is a product measure space with

$$\lambda = (\lambda^1, \ldots, \lambda^m) \in D, \mu_j = (\mu_1^j, \ldots, \mu_{n_j}^j) \in F_j (j \in [m]) \text{ (LHVs)};$$

$$D = D_1 \times \ldots \times D_m,\ F_j = F_1^j \times \ldots \times F_{n_j}^j (j \in [m]) \text{ (spaces of LHVs)} ;$$

$$\sigma = \prod_{j=1}^{m} \sigma_j, \delta_j = \prod_{k=1}^{n_j} \delta_k^j (j \in [m]) \text{ (product } \sigma\textit{-algebras)} ;$$

$$\gamma = \prod_{j=1}^{m} \gamma_j, \tau_j = \prod_{k=1}^{n_j} \tau_k^j (j \in [m]) \text{ (product measures)} ;$$

(ii) All of the local hidden variables (LHVs) $\lambda^1, \ldots, \lambda^m, \mu_1^j, \ldots, \mu_{n_j}^j (\forall j \in [m])$ are independent, i.e.,

$$p(\lambda, \mu_1, \ldots, \mu_m) = \prod_{j=1}^{m} p_j(\lambda^j) \times \prod_{j=1}^{m} \prod_{k=1}^{n_j} p_{j,k}(\mu_k^j), \tag{13}$$

where $p_j(\lambda^j)$ and $p_{j,k}(\mu_k^j)$ are density functions (DFs) of λ_j and μ_k^j, respectively, i.e., they are non-negative and satisfy

$$\int_{D_j} p_j(\lambda^j) \mathrm{d}\gamma_j(\lambda^j) = 1,\ \int_{F_k^j} p_{j,k}(\mu_k^j) \mathrm{d}\tau_k^j(\mu_k^j) = 1;$$

(iii) $P_A(a|x, \lambda), P_{B^j}(b_j|y_j, \lambda^j, \mu_j)$ and $P_{C_k^j}(c_{j,k}|z_{j,k}, \mu_k^j)$ are PDs of a, b_j and $c_{j,k}$, respectively, and are measurable with respect to λ, (λ^j, μ_j) and μ_k^j, respectively.

A star-shaped CT $\mathbf{P} = [\![P(a\mathbf{bc}|x\mathbf{yz})]\!]$ over Δ_S is said to be *C-star-nonlocal* if it is not C-star-local.

We use $\mathcal{CT}^{\text{C-star-local}}(\Delta_S)$ and $\mathcal{CT}^{\text{C-star-nonlocal}}(\Delta_S)$ to denote the sets of all C-star-local CTs and all C-star-nonlocal CTs over Δ_S, respectively.

Specifically, when $D_1, \ldots, D_m, F_1^j, \ldots, F_{n_j}^j (j \in [m])$ are finite sets with the counting measures, a C-star-shaped-LHVM (12) becomes a "D-star-shaped-LHVM":

$$\begin{aligned} P(a\mathbf{bc}|x\mathbf{yz}) &= \sum_{\lambda \in D, \mu_1 \in F_1, \ldots, \mu_m \in F_m} p(\lambda, \mu_1, \ldots, \mu_m) P_A(a|x, \lambda) \\ &\quad \times \prod_{j=1}^{m} P_{B^j}(b_j|y_j, \lambda^j, \mu_j) \times \prod_{j=1}^{m} \prod_{k=1}^{n_j} P_{C_k^j}(c_{j,k}|z_{j,k}, \mu_k^j), \end{aligned} \tag{14}$$

where $\{P_A(a|x, \lambda)\}, \{P_{B^j}(b_j|y_j, \lambda^j, \mu_j)\}$, and $\{P_{C_k^j}(c_{j,k}|z_{j,k}, \mu_k^j)\}$ are PDs of a, b_j and $c_{j,k}$, respectively, and the joint PD $p(\lambda, \mu_1, \ldots, \mu_m)$ is given by (13). In this case, we say that $\mathbf{P}$ is *D-star-local*. If $\mathbf{P}$ has no D-star-shaped LHVMs of the form (14), then we say that it is *D-star-nonlocal*.

We use $\mathcal{CT}^{\text{D-star-local}}(\Delta_S)$ and $\mathcal{CT}^{\text{D-star-nonlocal}}(\Delta_S)$ to denote the sets of all D-star-local CTs and all D-star-nonlocal CTs over Δ_S, respectively. Clearly,

$$\mathcal{CT}^{\text{D-star-local}}(\Delta_S) \subset \mathcal{CT}^{\text{C-star-local}}(\Delta_S).$$

Definition 3. *A star-shaped PT $\mathbf{P} = [\![P(a\mathbf{bc})]\!]$ over Ω_S is said to be C-star-local if it admits a "C-star-shaped LHVM":*

$$\begin{aligned} P(a\mathbf{bc}) &= \int_{D \times F_1 \times \ldots \times F_m} p(\lambda, \mu_1, \ldots, \mu_m) P_A(a|\lambda) \prod_{j=1}^{m} P_{B^j}(b_j|\lambda^j, \mu_j) \\ &\quad \times \prod_{j=1}^{m} \prod_{k=1}^{n_j} P_{C_k^j}(c_{j,k}|\mu_k^j) \, \mathrm{d}\gamma(\lambda) \mathrm{d}\tau_1(\mu_1) \ldots \mathrm{d}\tau_m(\mu_m) \end{aligned} \tag{15}$$

for all $a, \mathbf{b}, \mathbf{c}$, where $p(\lambda, \mu_1, \ldots, \mu_m)$ is a DF of the form (13). It is said to be C-star-nonlocal if it is not C-star-local.

Definition 4. *A star-shaped PT $\mathbf{P} = [\![P(a\mathbf{bc})]\!]$ over Ω_S is said to be D-star-local if it admits a "D-star-shaped LHVM":*

$$\begin{aligned} P(a\mathbf{bc}) &= \sum_{\lambda \in D, \mu_1 \in F_1, \ldots, \mu_m \in F_m} p(\lambda, \mu_1, \ldots, \mu_m) P_A(a|\lambda) \\ &\quad \times \prod_{j=1}^{m} P_{B^j}(b_j|\lambda^j, \mu_j) \times \prod_{j=1}^{m} \prod_{k=1}^{n_j} P_{C_k^j}(c_{j,k}|\mu_k^j) \end{aligned} \tag{16}$$

for all $a, \mathbf{b}, \mathbf{c}$, where $p(\lambda, \mu_1, \ldots, \mu_m)$ is a PD of the form (13). It is said to be D-star-nonlocal if it is not D-star-local.

Definition 5. *A star-shaped PT $\mathbf{P} = [\![P(a\mathbf{bc})]\!]$ over Ω_S is said to be star-local if it is either C-star-local or D-star-local. It is said to be star-nonlocal if is neither C-star-local nor D-star-local.*

We use $\mathcal{PT}^{\text{C-star-local}}(\Omega_S)$ (resp., $\mathcal{PT}^{\text{D-star-local}}(\Omega_S)$) to denote the set of all C-star-local (resp., D-star-local) star-shaped PTs over Ω_S.

Clearly,

$$\mathcal{PT}^{\text{D-star-local}}(\Omega_S) \subset \mathcal{PT}^{\text{C-star-local}}(\Omega_S).$$

3.2. Characterizations

To show every C-star-local CT (especially every PT) is D-star-local, we need the following lemma [37,43]. Recall that an $m \times n$ function matrix $B(\lambda) = [b_{ij}(\lambda)]$ on Λ is said to be *row-statistic* (RS) if, for each $\lambda \in \Lambda$, $b_{ij}(\lambda) \geq 0$ for all i, j and $\sum_{j=1}^{n} b_{ij}(\lambda) = 1$.

Lemma 1. *Let (Λ, Ω) be a measurable space and let $B(\lambda) = [b_{ij}(\lambda)]$ be an $m \times n$ RS function matrix whose entries b_{ij} are Ω-measurable on Λ. Then, $B(\lambda)$ can be written as:*

$$B(\lambda) = \sum_{k=1}^{n^m} \alpha_k(\lambda)[\delta_{j,J_k(i)}], \ \forall \lambda \in \Lambda, \tag{17}$$

where $\alpha_k (k = 1, 2, \ldots, n^m)$ are all non-negative and Ω-measurable functions on Λ with $\sum_{k=1}^{n^m} \alpha_k(\lambda) = 1$ for all $\lambda \in \Lambda$, and $\{J_k\}_{k=1}^{n^m}$ denotes the set of all maps from $[m]$ into $[n]$.

Put

$$N(A) = o(A)^{m(A)}, N(B^j) = o(B^j)^{m(B^j)}, N(C_k^j) = o(C_k^j)^{m(C_k^j)}$$

and let $\{J_i\}_{i=1}^{N(A)}$ be the set of all maps from $[m(A)]$ into $[o(A)]$, $\{K_{s_j}^j\}_{s_j=1}^{N(B^j)}$ the set of all maps from $[m(B^j)]$ into $[o(B^j)]$, and let $\{L_{t_{jk}}^{j,k}\}_{t_{jk}=1}^{N(C_k^j)}$ be the set of all maps from $[m(C_k^j)]$ into $[o(C_k^j)]$.

Let $\mathbf{P} = [\![P(a\mathbf{bc}|x\mathbf{yz})]\!]$ be a C-star-local CT over Δ_S. Then, it has a C-star-shaped LHVM (12). Since function matrices

$$M(\lambda) := [P_A(a|x, \lambda)]_{x,a}, M(\lambda^j, \mu_j) := [P_{B^j}(b_j|y_j, \lambda^j, \mu_j)]_{y_j,b_j}, M(\mu_k^j) := [P_{C_k^j}(c_{j,k}|z_{j,k}, \mu_k^j)]_{z_{j,k},c_{j,k}}$$

are RS for each parameters $\lambda, (\lambda^j, \mu_j), \mu_k^j$ and their entries are measurable with respect to the related parameters, respectively, it follows from Lemma 1 that they have the following decompositions:

$$M(\lambda) = \sum_{i=1}^{N(A)} \alpha(i|\lambda)[\delta_{a,J_i(x)}],$$

$$M(\lambda^j,\mu_j) = \sum_{s_j=1}^{N(B^j)} \beta^j(s_j|\lambda^j,\mu_j)[\delta_{b_j,K^j_{s_j}(y_j)}],$$

$$M(\mu^j_k) = \sum_{t_{jk}=1}^{N(C^j_k)} f^{j,k}(t_{jk}|\mu^j_k)[\delta_{c_{j,k},L^{j,k}_{t_{jk}}(z_{j,k})}];$$

equivalently,

$$P_A(a|x,\lambda) = \sum_{i=1}^{N(A)} \alpha(i|\lambda)\delta_{a,J_i(x)}, \tag{18}$$

$$P_{B^j}(b_j|y_j,\lambda^j,\mu_j) = \sum_{s_j=1}^{N(B^j)} \beta^j(s_j|\lambda^j,\mu_j)\delta_{b_j,K^j_{s_j}(y_j)}, \tag{19}$$

$$P_{C^j_k}(c_{j,k}|z_{j,k},\mu^j_k) = \sum_{t_{jk}=1}^{N(C^j_k)} f^{j,k}(t_{jk}|\mu^j_k)\delta_{c_{j,k},L^{j,k}_{t_{jk}}(z_{j,k})}, \tag{20}$$

where $\alpha_i(\lambda)$, $\beta^j_{s_j}(\lambda^j,\mu_j)$ and $f^{j,k}_{t_{jk}}(\mu^j_k)$ are PDs of i, s_j and t_{jk}, respectively, and are measurable with respect to λ, (λ^j,μ_j) and μ^j_k, respectively. It follows from Equations (12) and (18)–(20) that

$$P(a\mathbf{bc}|x\mathbf{yz}) = \sum_{i,s_j,t_{jk}} \pi(i,\mathbf{s},\mathbf{t})\delta_{a,J_i(x)} \prod_{j=1}^{m} \delta_{b_j,K^j_{s_j}(y_j)} \times \prod_{j=1}^{m}\prod_{k=1}^{n_j} \delta_{c_{j,k},L^{j,k}_{t_{jk}}(z_{j,k})} \tag{21}$$

for all $a, \mathbf{b}, \mathbf{c}, x, \mathbf{y}, \mathbf{z}$, where $\mathbf{s} = (s_1, s_2, \ldots, s_m) \equiv \{s_j\}_{j=1}^m$,

$$\mathbf{t} = (t_{11}, t_{12}, \ldots, t_{1n_1}, t_{21}, t_{22}, \ldots, t_{2n_2}, \ldots, t_{m1}, t_{m2}, \ldots, t_{mn_m}) \equiv \{t_{jk}\}_{j\in[m],k\in[n_j]},$$

and

$$\begin{aligned}\pi(i,\mathbf{s},\mathbf{t}) &= \int_{D\times F_1\times\ldots\times F_m} p(\lambda,\mu_1,\ldots,\mu_m)\alpha(i|\lambda)\prod_{j=1}^{m}\beta^j(s_j|\lambda^j,\mu_j) \\ &\quad \times \prod_{j=1}^{m}\prod_{k=1}^{n_j} f^{j,k}(t_{jk}|\mu^j_k)\,\mathrm{d}\gamma(\lambda)\mathrm{d}\tau_1(\mu_1)\ldots\mathrm{d}\tau_m(\mu_m),\end{aligned} \tag{22}$$

with $p(\lambda,\mu_1,\ldots,\mu_m)$ given by (13). Clearly, $\mathbf{p} = [\![\pi(i,\mathbf{s},\mathbf{t})]\!]$ is a C-star-local PT over

$$\Gamma_S = [N(A)] \times \prod_{j=1}^{m}[N(B^j)] \times \prod_{j=1}^{m}\prod_{k=1}^{n_i}[N(C^j_k)],$$

which generates $\mathbf{P}$ in terms of Equation (21).

Conversely, if (21) holds for some completely independent PD (13) and a C-star-local PT $\mathbf{p} = [\![\pi(i,\mathbf{s},\mathbf{t})]\!]$ with a C-star-shaped LHVM (22), then (12) holds for P_A, P_{B^j} and $P_{C^j_k}$ given by Equations (18)–(20). Thus, $\mathbf{P}$ is C-star-local.

This shows that (12) $\Leftrightarrow$ (21) and leads to the following.

Theorem 1. *A star-shaped CT* $\mathbf{P}$ *over* Δ_S *is C-star-local if and only if it has the following decomposition:*

$$\mathbf{P} = \sum_{i,\mathbf{s},\mathbf{t}} \pi(i,\mathbf{s},\mathbf{t})\mathbf{D}_{i,\mathbf{s},\mathbf{t}}, \tag{23}$$

where $\mathbf{p} = [\![\pi(i,\mathbf{s},\mathbf{t})]\!]$ *is a C-star-local PT over* Γ_S *given by* (22) *and* $\mathbf{D}_{i,\mathbf{s},\mathbf{t}} = [\![D_{i,\mathbf{s},\mathbf{t}}(a\mathbf{bc}|x\mathbf{yz})]\!]$ *is given by*

$$D_{i,\mathbf{s},\mathbf{t}}(a\mathbf{bc}|x\mathbf{yz}) = \delta_{a,J_i(x)}\prod_{j=1}^{m}\delta_{b_j,K^j_{s_j}(y_j)} \times \prod_{j=1}^{m}\prod_{k=1}^{n_j}\delta_{c_{j,k},L^{j,k}_{t_{jk}}(z_{j,k})}.$$

As an application of Theorem 1, we obtain the following relationship between C-star-local CTs and C-star-local PTs:

$$\mathcal{CT}^{\text{C-star-local}}(\Delta_S) = \left\{\sum_{i,\mathbf{s},\mathbf{t}}\pi(i,\mathbf{s},\mathbf{t})\mathbf{D}_{i,\mathbf{s},\mathbf{t}} : \mathbf{p} = [\![\pi(i,\mathbf{s},\mathbf{t})]\!] \in \mathcal{PT}^{\text{C-star-local}}(\Gamma_S)\right\} \tag{24}$$

Again, we let $\mathbf{P}$ be a C-star-local CT over Δ_S. We aim to prove that $\mathbf{P}$ is D-star-local. First, it has a C-star-shaped LHVM (12). Since

$$p(\lambda,\mu_1,\ldots,\mu_m) = \prod_{j=1}^{m}p_j(\lambda^j) \times \prod_{j=1}^{m}\prod_{k=1}^{n_j}p_{j,k}(\mu^j_k),$$

we obtain from (12) and (20) that

$$\begin{aligned} P(a\mathbf{bc}|x\mathbf{yz}) \;=\; & \sum_{t_{jk}\in[N(C^j_{n_j})](j\in[m])}\int_D\prod_{j=1}^{m}p_j(\lambda^j) \times P_A(a|x,\lambda)\mathrm{d}\gamma(\lambda) \\ & \times\int_{F_1\times\ldots\times F_m}\prod_{j=1}^{m}P_{B^j}(b_j|y_j,\lambda^j,\mu_j) \times \prod_{j=1}^{m}\prod_{k=1}^{n_j}p_{j,k}(\mu^j_k)f^{j,k}_{t_{jk}}(\mu^j_k)\,\mathrm{d}\tau_1(\mu_1)\ldots\mathrm{d}\tau_m(\mu_m) \\ & \times\prod_{j=1}^{m}\prod_{k=1}^{n_j}\delta_{c_{j,k},L^{j,k}_{t_{jk}}(z_{j,k})}. \end{aligned} \tag{25}$$

Put

$$q_{j,k}(t_{jk}) = \int_{F^j_k}f^{j,k}_{t_{jk}}(\mu^j_k)p_{j,k}(\mu^j_k)\mathrm{d}\tau^j_k(\mu^j_k),$$

which are PDs of t_{jk} and satisfy

$$\prod_{k=1}^{n_j}q_{j,k}(t_{jk}) = \int_{F_j}\prod_{k=1}^{n_j}(f^{j,k}_{t_{jk}}(\mu^j_k)p_{j,k}(\mu^j_k))\,\mathrm{d}\tau_j(\mu_j),$$

and define

$$P_{B^j}(b_j|y_j,\lambda^j,t_{j1},\ldots,t_{jn_j}) \;=\; \frac{1}{\prod_{k=1}^{n_j}q_{j,k}(t_{jk})}\int_{F_j}P_{B^j}(b_j|y_j,\lambda^j,\mu_j) \times \left(\prod_{k=1}^{n_j}f^{j,k}_{t_{jk}}(\mu^j_k)p_{j,k}(\mu^j_k)\right)\mathrm{d}\tau_j(\mu^j)$$

if $\prod_{k=1}^{n_j}q_{j,k}(t_{jk}) > 0$; and

$$P_{B^j}(b_j|y_j,\lambda^j,t_{j1},\ldots,t_{jn_j}) = \frac{1}{o(B^j)},$$

otherwise. Clearly, $P_{B^j}(b_j|y_j,\lambda^j,t_{j1},\ldots,t_{jn_j})$ is a PD of b_j for each $(y_j,\lambda^j,t_{j1},\ldots,t_{jn_j})$, and when $\prod_{k=1}^{n_j}q_{j,k}(t_{jk}) > 0$, we have

$$\prod_{k=1}^{n_j}q_{j,k}(t_{jk}) \times P_{B^j}(b_j|y_j,\lambda^j,t_{j1},\ldots,t_{jn_j}) = \int_{F_j}P_{B^j}(b_j|y_j,\lambda^j,\mu_j)\left(\prod_{k=1}^{n_j}f^{j,k}_{t_{jk}}(\mu^j_k)p_{j,k}(\mu^j_k)\right)\mathrm{d}\tau_j(\mu^j). \tag{26}$$

Note that the right-hand side of above equation is less than equal to $\prod_{k=1}^{n_j} q_{j,k}(t_{jk})$ and is equal to zero when $\prod_{k=1}^{n_j} q_{j,k}(t_{jk}) = 0$. Thus, Equation (26) is valid in any case. Using Equation (26) yields that

$$\begin{aligned}
&\prod_{j=1}^{m}\prod_{k=1}^{n_j} q_{j,k}(t_{jk}) \times \prod_{j=1}^{m} P_{B^j}(b_j|y_j,\lambda^j,t_{j1},\ldots,t_{jn_j}) \\
=\ &\prod_{j=1}^{m}\int_{F_j} P_{B^j}(b_j|y_j,\lambda^j,\mu_j) \times \left(\prod_{k=1}^{n_j} f_{t_{jk}}^{j,k}(\mu_k^j)p_{j,k}(\mu_k^j)\right)\mathrm{d}\tau_j(\mu^j) \\
=\ &\int_{F_1\times\ldots\times F_m}\prod_{j=1}^{m} P_{B^j}(b_j|y_j,\lambda^j,\mu_j)\left(\prod_{j=1}^{m}\prod_{k=1}^{n_j} f_{t_{jk}}^{j,k}(\mu_k^j)p_{j,k}(\mu_k^j)\right)\mathrm{d}\tau_1(\mu_1)\ldots\mathrm{d}\tau_m(\mu_m).
\end{aligned}$$

Combining Equation (25) yields that

$$\begin{aligned}
P(a\mathbf{bc}|x\mathbf{yz}) =\ &\sum_{t_{jk}\in[N(C_{n_j}^j)](j\in[m],j\in[m])}\prod_{j=1}^{m}\prod_{k=1}^{n_j} q_{j,k}(t_{jk}) \\
&\times\int_D\prod_{j=1}^{m} p_j(\lambda^j)\times\prod_{j=1}^{m} P_{B^j}(b_j|y_j,\lambda^j,t_{j1},\ldots,t_{jn_j})\times P_A(a|x,\lambda)\mathrm{d}\gamma(\lambda) \\
&\times\prod_{j=1}^{m}\prod_{k=1}^{n_j}\delta_{c_{j,k},L_{t_{jk}}^{j,k}(z_{j,k})}.
\end{aligned}\tag{27}$$

Using Lemma 1 for the RS function matrix $[P_{B^j}(b_j|y_j,\lambda^j,t_{j1},\ldots,t_{jn_j})]$ with $(y_jt_{j1}\cdots t_{jn_j},b_j)$-entry $P_{B^j}(b_j|y_j,\lambda^j,t_{j1},\ldots,t_{jn_j})$, we get that

$$P_{B^j}(b_j|y_j,\lambda^j,t_{j1},\ldots,t_{jn_j}) = \sum_{r^j=1}^{N^*(B^j)} g_{r^j}^j(\lambda^j)\delta_{b_j,E_{r^j}^j(y_j,t_{j1},\ldots,t_{jn_j})},\tag{28}$$

where

$$N^*(B^j) = o(B_j)^{m(B_j)N(C_1^j)\cdots N(C_{n_j}^j)},$$

$g_{r^j}^j(\lambda^j)$ is a PD of r^j and is measurable with respect to λ^j, and $\{E_{r^j}^j\}_{r^j\in[N^*(B^j)]}$ denotes the set of all maps from $[m(B_j)N(C_1^j)\cdots N(C_{n_j}^j)]$ into $[o(B_j)]$. Thus, we see from Equation (28) that

$$\begin{aligned}
\prod_{j=1}^{m} P_{B^j}(b_j|y_j,\lambda^j,t_{j1},\ldots,t_{jn_j}) &= \prod_{j=1}^{m}\sum_{r^j=1}^{N^*(B^j)} g_{r^j}^j(\lambda^j)\delta_{b_j,E_{r^j}^j(y_j,t_{j1},\ldots,t_{jn_j})} \\
&= \sum_{r_1=1}^{N^*(B^1)}\cdots\sum_{r_m=1}^{N^*(B^m)}\prod_{j=1}^{m} g_{r^j}^j(\lambda^j)\times\prod_{j=1}^{m}\delta_{b_j,E_{r^j}^j(y_j,t_{j1},\ldots,t_{jn_j})}.
\end{aligned}\tag{29}$$

It follows from Equations (27) and (29) that

$$\begin{aligned}
P(a\mathbf{bc}|x\mathbf{yz}) =\ &\sum_{r^1=1}^{N^*(B^1)}\cdots\sum_{r^m=1}^{N^*(B^m)}\sum_{t_{jk}\in[N(C_{n_j}^j)](j\in[m],j\in[m])}\prod_{j=1}^{m}\prod_{k=1}^{n_j} q_{j,k}(t_{jk}) \\
&\times\int_D\prod_{j=1}^{m} p_j(\lambda^j)\times\prod_{j=1}^{m} g_{r^j}^j(\lambda^j)\times P_A(a|x,\lambda)\mathrm{d}\gamma(\lambda) \\
&\times\prod_{j=1}^{m}\delta_{b_j,E_{r^j}^j(y_j,t_{j1},\ldots,t_{jn_j})}\times\prod_{j=1}^{m}\prod_{k=1}^{n_j}\delta_{c_{j,k},L_{t_{jk}}^{j,k}(z_{j,k})}.
\end{aligned}\tag{30}$$

Put

$$h_j(r^j) = \int_{D_j} p_j(\lambda^j) g^j_{r^j}(\lambda^j) \mathrm{d}\tau_j(\lambda^j), \tag{31}$$

then we obtain a PD $h_j(r^j)$ of r^j for every j. Define $r = (r^1, r^2, \ldots, r^m)$ and put

$$P_A(a|x,r) = \frac{1}{\prod_{j=1}^m h_j(r^j)} \int_D \prod_{j=1}^m p_j(\lambda^j) \times \prod_{j=1}^m g^j_{r^j}(\lambda^j) \times P_A(a|x,\lambda) \mathrm{d}\gamma(\lambda)$$

if $\prod_{j=1}^m h_j(r^j) > 0$; otherwise, define $P_A(a|x,r) = \frac{1}{o_A}$ for all a, x, then $P_A(a|x,r)$ is a PD of a and

$$\int_D \prod_{j=1}^m p_j(\lambda^j) \times \prod_{j=1}^m g^j_{r^j}(\lambda^j) \times P_A(a|x,\lambda) \mathrm{d}\gamma(\lambda) = \prod_{j=1}^m h_j(r^j) \times P_A(a|x,r). \tag{32}$$

Thus, from Equations (30) and (32), we get that

$$\begin{aligned} P(a\mathbf{bc}|x\mathbf{yz}) &= \sum_{r \in R, t_1 \in T_1, \ldots, t_m \in T_m} \prod_{j=1}^m h_j(r^j) \times \prod_{j=1}^m \prod_{k=1}^{n_j} q_{j,k}(t_{jk}) \times P_A(a|x,r) \\ &\quad \times \prod_{j=1}^m \delta_{b_j, K^j_{r^j}(y_j, t_{j1}, \ldots, t_{jn_j})} \times \prod_{j=1}^m \prod_{k=1}^{n_j} \delta_{c_{j,k}, L^{j,k}_{t_{jk}}(z_{j,k})}, \end{aligned} \tag{33}$$

where $t_j = (t_{j1}, \ldots, t_{jn_j})$, and

$$R = \prod_{j=1}^m [N^*(B^j)],\ T_j = [N(C^j_1)] \times \cdots \times [N(C^j_{n_j})] (j = 1, 2, \ldots, m).$$

Put

$$P_{B^j}(b_j|y_j, r^j, t_j) = \delta_{b_j, K^j_{r^j}(y_j, t_{j1}, \ldots, t_{jn_j})},\ P_{C^i_k}(c_{j,k}|z_{j,k}, t_{jk}) = \delta_{c_{j,k}, L^{j,k}_{t_{jk}}(z_{j,k})},$$

which are of PDs of b_j and $c_{j,k}$, respectively. Then Equation (33) becomes

$$\begin{aligned} P(a\mathbf{bc}|x\mathbf{yz}) &= \sum_{r \in R, t_1 \in T_1, \ldots, t_m \in T_m} \prod_{j=1}^m h_j(r^j) \times \prod_{j=1}^m \prod_{k=1}^{n_j} q_{j,k}(t_{jk}) \times P_A(a|x,r) \\ &\quad \times \prod_{j=1}^m P_{B^j}(b_j|y_j, r^j, t_j) \times \prod_{j=1}^m \prod_{k=1}^{n_j} P_{C^i_k}(c_{j,k}|z_{j,k}, t_{jk}). \end{aligned} \tag{34}$$

This shows that $\mathbf{P}$ is D-star-local.

From this discussion, we have the following conclusion.

Theorem 2. *A star-shaped CT* $\mathbf{P}$ *over* Δ_S *is C-star-local if and only if it is D-star-local, that is,*

$$\mathcal{CT}^{\text{C-star-local}}(\Delta_S) = \mathcal{CT}^{\text{D-star-local}}(\Delta_S) \equiv \mathcal{CT}^{\text{star-local}}(\Delta_S).$$

Due to this conclusion, we say that a star-shaped CT $\mathbf{P}$ over Δ_S is *star-local* if it is C-star-local, equivalently, if it is D-star-local.

As a special case of $m = n_1 = n_2 = 2$, Theorem 2 implies the following result, which is an equivalent characterization of the six-locality discussed in [41].

Corollary 1. *The correlations* $P(a, b_1, b_2, c_1, c_2, c_3, c_4|x, y_1, y_2, z_1, z_2, z_3, z_4)$ *discussed in [41] are six-local if and only if the following decomposition is valid:*

$$\begin{aligned}
&P(a, b_1, b_2, c_1, c_2, c_3, c_4|x, y_1, y_2, z_1, z_2, z_3, z_4) \\
&= \sum_{\lambda_k \in [n_k](\forall k)} \prod_{k=1}^{6} p_k(\lambda_k) \times P_1(a|x, \lambda_1\lambda_2) P_2(b_1|y_1, \lambda_1\lambda_3\lambda_4) P_3(b_2|y_2, \lambda_2\lambda_5\lambda_6) \\
&\times P_4(c_1|z_1, \lambda_3) P_5(c_2|z_2, \lambda_4) P_6(c_3|z_3, \lambda_5) P_7(c_4|z_4, \lambda_6),
\end{aligned} \tag{35}$$

for all possible $a, b_1, b_2, c_1, c_2, c_3, c_4, x, y_1, y_2, z_1, z_2, z_3, z_4$, *where* $p_k(\lambda_k)$*'s are PDs of* λ_k, *and* $P_1, P_2, \ldots, P_7$ *are PDs of* $a, b_1, b_2, c_1, c_2, c_3, c_4$, *respectively.*

Theorem 3. *A star-shaped CT* $\mathbf{P} = [\![P(a\mathbf{bc}|x\mathbf{yz})]\!]$ *over* Δ_S *is star-local if and only if it is "separable star-quantum", i.e., it can be generated by an MA (3) together with some separable states* $\rho_{A_j B_0^j} \in \mathcal{D}(\mathcal{H}_{A^j} \otimes \mathcal{H}_{B_0^j})$ *and* $\rho_{B_k^j C_k^j} \in \mathcal{D}(\mathcal{H}_{B_k^j} \otimes \mathcal{H}_{C_k^j})$, *in such a way that*

$$P(a\mathbf{bc}|x\mathbf{yz}) = \mathrm{tr}[(M_{a|x} \otimes N_{\mathbf{b}|\mathbf{y}} \otimes L_{\mathbf{c}|\mathbf{z}})\tilde{\Gamma}], \ \forall x, a, \mathbf{y}, \mathbf{b}, \mathbf{z}, \mathbf{c}, \tag{36}$$

where the network state Γ *is given by Equation (1).*

Proof. To show the necessity, we let $\mathbf{P} = [\![P(a\mathbf{bc}|x\mathbf{yz})]\!]$ be star-local. Then, it can be written as (14), that is,

$$\begin{aligned}
P(a\mathbf{bc}|x\mathbf{yz}) \;=\; & \sum_{\lambda \in D, \mu_1 \in F_1, \ldots, \mu_m \in F_m} p(\lambda, \mu_1, \ldots, \mu_m) P_A(a|x, \lambda) \\
& \times \prod_{j=1}^{m} P_{B^j}(b_j|y_j, \lambda^j, \mu_j) \times \prod_{j=1}^{m} \prod_{k=1}^{n_j} P_{C_k^j}(c_{j,k}|z_{j,k}, \mu_k^j),
\end{aligned} \tag{37}$$

where $\{P_A(a|x, \lambda)\}, \{P_{B^j}(b_j|y_j, \lambda^j, \mu_j)\}$ and $\{P_{C_k^j}(c_{j,k}|z_{j,k}, \mu_k^j)\}$ are PDs of a, b_j and $c_{j,k}$, respectively, and

$$p(\lambda, \mu_1, \ldots, \mu_m) = \prod_{j=1}^{m} p_j(\lambda^j) \times \prod_{j=1}^{m} \prod_{k=1}^{n_j} p_{j,k}(\mu_k^j), \tag{38}$$

in which $p_j(\lambda^j)$ and $p_{j,k}(\mu_k^j)$ are PDs of λ_j and μ_k^j, respectively. Choose Hilbert spaces

$$\mathcal{H}_{A^j} = \mathcal{H}_{B_0^j} = \mathbb{C}^{|D_j|}, \quad \mathcal{H}_{B_k^j} = \mathcal{H}_{C_k^j} = \mathbb{C}^{|F_k^j|}, \forall j, k,$$

where $|S|$ denotes the cardinality of a finite set S; take their orthonormal bases $\{|s_{\lambda^j}\rangle\}_{\lambda^j=1}^{|D_j|}$ and $\{|t_{\mu_k^j}\rangle\}_{\mu_k^j=1}^{|F_k^j|} (\forall j, k)$, respectively; and put

$$\mathcal{H}_A = \bigotimes_{j=1}^{m} \mathcal{H}_{A^j}, \quad \mathcal{H}_{B^j} = \mathcal{H}_{B_0^j} \otimes \left(\bigotimes_{k=1}^{n_j} \mathcal{H}_{B_k^j} \right).$$

Choose separable states

$$\rho_{A_j B_0^j} = \sum_{\lambda^j=1}^{|D_j|} p_j(\lambda^j) |s_{\lambda^j}\rangle\langle s_{\lambda^j}| \otimes |s_{\lambda^j}\rangle\langle s_{\lambda^j}|, \; \rho_{B_k^j C_k^j} = \sum_{\mu_k^j=1}^{|F_k^j|} p_{j,k}(\mu_k^j) |t_{\mu_k^j}\rangle\langle t_{\mu_k^j}| \otimes |t_{\mu_k^j}\rangle\langle t_{\mu_k^j}|.$$

Then, we can obtain a network state

$$\Gamma = \left(\bigotimes_{j=1}^{m} \rho_{A^j B_0^j}\right) \otimes \left(\bigotimes_{j=1}^{m} (\rho_{B_1^j C_1^j} \otimes \rho_{B_2^j C_2^j} \otimes \ldots \otimes \rho_{B_{n_j}^j C_{n_j}^j})\right),$$

which induces the measurement state

$$\widetilde{\Gamma} \;=\; \sum_{\lambda\in D} \sum_{\mu_1\in F_1,\ldots,\mu_m\in F_m} \prod_{j=1}^{m} p_j(\lambda^j) \times \prod_{j=1}^{m}\prod_{k=1}^{n_j} p_{j,k}(\mu_k^j) \times \Gamma'(\lambda,\mu_1,,\ldots,\mu_m),$$

where

$$\Gamma'(\lambda,\mu_1,,\ldots,\mu_m) = \left(\bigotimes_{j=1}^{m} |s_{\lambda^j}\rangle\langle s_{\lambda^j}|\right) \otimes \left(\bigotimes_{j=1}^{m} [|s_{\lambda^j}\rangle\langle s_{\lambda^j}| \otimes \bigotimes_{k=1}^{n_j} |t_{\mu_k^j}\rangle\langle t_{\mu_k^j}|]\right) \otimes \left(\bigotimes_{j=1}^{m}\bigotimes_{k=1}^{n_j} |t_{\mu_k^j}\rangle\langle t_{\mu_k^j}|\right).$$

To define an MA (3), we put

$$M_{a|x} = \sum_{\lambda\in D} P_A(a|x,\lambda) \bigotimes_{j=1}^{m} |s_{\lambda^j}\rangle\langle s_{\lambda^j}|,$$

$$N_{b_j|y_j} = \sum_{\mu^j\in F_j} P_{B^j}(b_j|y_j,\lambda^j,\mu_j)|s_{\lambda^j}\rangle\langle s_{\lambda^j}| \otimes \left(\bigotimes_{k=1}^{n_j} |t_{\mu_k^j}\rangle\langle t_{\mu_k^j}|\right),$$

$$L_{c_{j,k}|c_{j,k}} = \sum_{\mu_k^j\in F_k^j} P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j)|t_{\mu_k^j}\rangle\langle t_{\mu_k^j}|.$$

It can be checked that

$$P(a\mathbf{bc}|x\mathbf{yz}) = \mathrm{tr}[(M_{a|x}\otimes N_{\mathbf{b}|\mathbf{y}}\otimes L_{\mathbf{c}|\mathbf{z}})\widetilde{\Gamma}]$$

for all possible variables $a, \mathbf{b}, \mathbf{c}, x, \mathbf{y}$, and $\mathbf{z}$. This proves that $\mathbf{P}$ is separable star-quantum.

Conversely, we suppose that $\mathbf{P}$ can be written as the form of (36). Then, from the proof of Proposition 4, we see that $\mathbf{P}$ has a D-star-shaped LHVM (9) and then is star-local. The proof is completed. □

Theorem 4. *Let a star-shaped CT* $\mathbf{P} = [\![P(a\mathbf{bc}|x\mathbf{yz})]\!]$ *over* Δ_S *be star-local. Then, for each* $1 \le j_0 \le m$ *and* $(j_0,k_0) \in [m]\times[n_{j_0}]$*, the following conclusions are valid.*

(a) The marginal $\mathbf{P}_{AB^{j_0}C_{k_0}^{j_0}} = [\![P_{AB^{j_0}C_{k_0}^{j_0}}(ab_{j_0}c_{j_0,k_0}|xy_{j_0}z_{j_0,k_0})]\!]$ *of* $\mathbf{P}$ *on subsystem* $AB^{j_0}C_{k_0}^{j_0}$ *is bilocal.*

(b) The marginal $\mathbf{P}_{AC_{k_0}^{j_0}} = [\![P_{AC_{k_0}^{j_0}}(ac_{j_0,k_0}|xz_{j_0,k_0})]\!]$ *of* $\mathbf{P}$ *on subsystem* $AC_{k_0}^{j_0}$ *is product:* $\mathbf{P}_{AC_{k_0}^{j_0}} = \mathbf{P}_A \otimes \mathbf{P}_{C_{k_0}^{j_0}}$*, i.e.,*

$$P_{AC_{k_0}^{j_0}}(ac_{j_0,k_0}|xz_{j_0,k_0}) = P_A(a|x)P_{C_{k_0}^{j_0}}(c_{j_0,k_0}|z_{j_0,k_0}). \tag{39}$$

(c) The (n_0+1)*-partite CT*

$$\begin{aligned}\mathbf{P}_{C_1^{j_0}\cdots C_{n_{j_0}}^{j_0}B^{j_0}} &= [\![P_{C_1^{j_0}\cdots C_{n_{j_0}}^{j_0}B^{j_0}}(c_{j_0,1}\cdots c_{j_0,n_0}b_{j_0}|z_{j_0,1}\cdots z_{j_0,n_0}y_{j_0})]\!] \\ &:= [\![P_{AB^{j_0}C_{k_0}^{j_0}}(b_{j_0}c_{j_0,1}\cdots c_{j_0,n_0}|y_{j_0}z_{j_0,1}\cdots z_{j_0,n_0})]\!]\end{aligned}$$

is n_0*-local.*

Proof. Since **P** is star-local, it has a D-star-shaped LHVM (14):

$$P(a\mathbf{bc}|x\mathbf{yz}) = \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} p(\lambda,\mu_1,\ldots,\mu_m)P_A(a|x,\lambda) \times \prod_{j=1}^{m} P_{B^j}(b_j|y_j,\lambda^j,\mu_j) \times \prod_{j=1}^{m}\prod_{k=1}^{n_j} P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j), \tag{40}$$

where

$$p(\lambda,\mu_1,\ldots,\mu_m) = \prod_{j=1}^{m} p_j(\lambda^j) \times \prod_{j=1}^{m}\prod_{k=1}^{n_j} p_{j,k}(\mu_k^j), \tag{41}$$

in which $p_j(\lambda^j)$ and $p_{j,k}(\mu_k^j)$ are PDs of λ_j and μ_k^j, respectively.

(a) Using (40) implies that

$$\begin{aligned}
& P_{AB^{j_0}C_{k_0}^{j_0}}(ab_{j_0}c_{j_0,k_0}|xy_{j_0}z_{j_0,k_0}) \\
= & \sum_{b_j,c_{j,k}(j\neq j_0,k\neq k_0)} P(a\mathbf{bc}|x\mathbf{yz}) \\
= & \sum_{\lambda^{j_0}} \sum_{\mu_1^{j_0}\cdots\mu_{n_{j_0}}^{j_0}} p_{j_0}(\lambda^{j_0})p_{j_0,1}(\mu_1^{j_0})\cdots p_{j_0,n_{j_0}}(\mu_{n_{j_0}}^{j_0})P_A(a|x,\lambda^{j_0}) \\
& \times P_{B^{j_0}}(b_{j_0}|y_{j_0},\lambda^{j_0},\mu_1^{j_0}\cdots\mu_{n_{j_0}}^{j_0})P_{C_{k_0}^{j_0}}(c_{j_0,k_0}|z_{j_0,k_0},\mu_{k_0}^{j_0}) \\
= & \sum_{\lambda^{j_0}}\sum_{\mu_{k_0}^{j_0}} p_{j_0}(\lambda^{j_0})p_{j_0,k_0}(\mu_{k_0}^{j_0})P_A(a|x,\lambda^{j_0})P_{B^{j_0}}(b_{j_0}|y_{j_0},\lambda^{j_0},\mu_{k_0}^{j_0})P_{C_{k_0}^{j_0}}(c_{j_0,k_0}|z_{j_0,k_0},\mu_{k_0}^{j_0}),
\end{aligned}$$

where

$$P_A(a|x,\lambda^{j_0}) = \sum_{\lambda_j\in F_j(j\neq j_0)} p_j(\lambda_j)P_A(a|x,\lambda),$$

$$P_{B^{j_0}}(b_{j_0}|y_{j_0},\lambda^{j_0},\mu_{k_0}^{j_0}) = \sum_{\mu_k^{j_0}(k\neq k_0)} \prod_{\mu_k^{j_0}(k\neq k_0)} p_{j_0,k}(\mu_k^{j_0}) \times P_{B^{j_0}}(b_{j_0}|y_{j_0},\lambda^{j_0},\mu_1^{j_0}\mu_2^{j_0}\cdots\mu_{n_{j_0}}^{j_0}).$$

This shows that $\mathbf{P}_{AB^{j_0}C_{k_0}^{j_0}}$ is bilocal [43]

(b) Using Equation (42) implies that

$$\begin{aligned}
P_{AC_{k_0}^{j_0}}(ac_{j_0,k_0}|xz_{j_0,k_0}) &= \sum_{b_{j_0}} P_{AB^{j_0}C_{k_0}^{j_0}}(ab_{j_0}c_{j_0,k_0}|xy_{j_0}z_{j_0,k_0}) \\
&= \sum_{\lambda^{j_0},\mu_{k_0}^{j_0}} p_{j_0}(\lambda^{j_0})p_{j_0,k_0}(\mu_{k_0}^{j_0})P_A(a|x,\lambda^{j_0})P_{C_{k_0}^{j_0}}(c_{j_0,k_0}|z_{j_0,k_0},\mu_{k_0}^{j_0}) \\
&= P_A(a|x)P_{C_{k_0}^{j_0}}(c_{j_0,k_0}|z_{j_0,k_0}),
\end{aligned}$$

implying Equation (39).

(c) Using the definition of $\mathbf{P}_{C_1^{j_0}\cdots C_{n_{j_0}}^{j_0} B^{j_0}}$ and (14), we have

$$\begin{aligned}
& P_{C_1^{j_0}\cdots C_{n_{j_0}}^{j_0} B^{j_0}}(c_{j_0,1}\cdots c_{j_0,n_0}b_{j_0}|z_{j_0,1}\cdots z_{j_0,n_0}y_{j_0}) \\
= \ & P_{AB^{j_0}C_{k_0}^{j_0}}(b_{j_0}c_{j_0,1}\cdots c_{j_0,n_0}|y_{j_0}z_{j_0,1}\cdots z_{j_0,n_0}) \\
= \ & \sum_a \sum_{b_j(j\neq j_0)} \sum_{c_{j,k}(k\in[n_j],j\neq j_0)} P(a\mathbf{bc}|x\mathbf{yz}) \\
= \ & \sum_{\lambda^{j_0}} \sum_{\mu_1^{j_0}\mu_2^{j_0}\cdots\mu_{n_{j_0}}^{j_0}} p_{j_0}(\lambda^{j_0})p_{j_0,1}(\mu_1^{j_0})\cdots p_{j_0,n_{j_0}}(\mu_{n_{j_0}}^{j_0}) \\
& \times \prod_{k=1}^{n_{j_0}} P_{C_k^{j_0}}(c_{j_0,k}|z_{j_0,k},\mu_k^{j_0}) \times P_{B^{j_0}}(b_{j_0}|y_{j_0},\lambda^{j_0},\mu_1^{j_0}\cdots\mu_{n_{j_0}}^{j_0})
\end{aligned}$$

for all possible $c_{j_0,1},\ldots,c_{j_0,n_0},b_{j_0},z_{j_0,1},\ldots,z_{j_0,n_0},y_{j_0}$. This shows that the (n_0+1)-partite CP $\mathbf{P}_{C_1^{j_0}\cdots C_{n_{j_0}}^{j_0} B^{j_0}}$ is n_0-local [43]. The proof is completed. □

For a star-shaped CT $\mathbf{P}$ over Δ_S, the conclusion (a) of Theorem 4 ensures that if there exists an index $(j_0,k_0)\in[m]\times[n_0]$ such that the marginal $\mathbf{P}_{AB^{j_0}C_{k_0}^{j_0}}$ is not bilocal, and conclusion (b) implies that if some of the marginal $\mathbf{P}_{AC_{k_0}^{j_0}}$ is not a product, then $\mathbf{P}$ must be star-nonlocal. Using conclusion (c) shows that when some marginal $\mathbf{P}_{C_1^{j_0}C_2^{j_0}\cdots C_{n_{j_0}}^{j_0} B^{j_0}}$ is not n_0-local [43], $\mathbf{P}$ must be star-nonlocal.

3.3. Global Properties

As the end of this section, let us give some properties of the set $\mathcal{CT}^{\text{star-local}}(\Delta_S)$. First, since all elements of $\mathcal{CT}^{\text{star-local}}(\Delta_S)$ admit their D-star-shaped LHVMs (34) with the *unified form* $\sum_{r\in R,t_1\in T_1,\ldots,t_m\in T_m}$ of summation, in which the index sets $R,T_1,\ldots,T_m$ are independent of $\mathbf{P}$, the following conclusion can be checked easily.

Theorem 5. *$\mathcal{CT}^{\text{star-local}}(\Delta_S)$ is a compact subset of the Hilbert space $\mathcal{T}^{\text{star}}(\Delta_S)$.*

This conclusion ensures that the set $\mathcal{CT}^{\text{star-nonlocal}}(\Delta_S)$ forms a relative open set in the Hilbert space $\mathcal{T}^{\text{star}}(\Delta_S)$. That means that any star-shaped CTs near a star-nonlocal CT are all star-nonlocal.

Theorem 6. *$\mathcal{CT}^{\text{star-local}}(\Delta_S)$ is a path-connected set in the Hilbert space $\mathcal{T}^{\text{star}}(\Delta_S)$.*

Proof. Put

$$I(a\mathbf{bc}|x\mathbf{yz}) \equiv \left\{ o(A)\prod_{j=1}^{m}\left(o(B^j)\prod_{k=1}^{n_j} o(C_k^j)\right)\right\}^{-1},$$

then $\mathbf{I} := [\![I(a\mathbf{bc}|x\mathbf{yz})]\!]$ is an element of $\mathcal{CT}^{\text{star-local}}(\Delta_S)$. Let $\mathbf{P} = [\![P(a\mathbf{bc}|x\mathbf{yz})]\!]$ and $\mathbf{Q} = [\![Q(a\mathbf{bc}|x\mathbf{yz})]\!]$ be any two elements of $\mathcal{CT}^{\text{star-local}}(\Delta_S)$. Then, $\mathbf{P}$ and $\mathbf{Q}$ admit D-star-shaped-LHVMs:

$$\begin{aligned}
P(a\mathbf{bc}|x\mathbf{yz}) \ = \ & \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} p(\lambda,\mu_1,\ldots,\mu_m)P_A(a|x,\lambda) \\
& \times \prod_{j=1}^{m} P_{B^j}(b_j|y_j,\lambda^j,\mu_j) \times \prod_{j=1}^{m}\prod_{k=1}^{n_j} P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j),
\end{aligned}$$

where

$$p(\lambda,\mu_1,\ldots,\mu_m) = \prod_{j=1}^{m} p_j(\lambda^j) \times \prod_{j=1}^{m}\prod_{k=1}^{n_j} p_{j,k}(\mu_k^j), \tag{42}$$

in which $p_j(\lambda^j)$ and $p_{j,k}(\mu_k^j)$ are PDs of λ_j and μ_k^j, respectively, and

$$\begin{aligned} Q(a\mathbf{bc}|x\mathbf{yz}) &= \sum_{\eta\in D',\xi_1\in F_1',\ldots,\xi_m\in F_m'} q(\eta,\xi_1,\ldots,\xi_m)Q_A(a|x,\eta) \\ &\times\prod_{j=1}^m Q_{B^j}(b_j|y_j,\eta^j,\xi_j)\times\prod_{j=1}^m\prod_{k=1}^{n_j} Q_{C_k^j}(c_{j,k}|z_{j,k},\xi_k^j), \end{aligned}$$

where $\eta=(\eta^1,\ldots,\eta^m)$, $\xi_j=(\xi_1^j,\ldots\xi_{n_j}^j)$, and

$$q(\eta,\xi_1,\ldots,\xi_m)=\prod_{j=1}^m q_j(\eta^j)\times\prod_{j=1}^m\prod_{k=1}^{n_j} q_{j,k}(\xi_k^j), \tag{43}$$

in which $q_j(\eta^j)$ and $q_{j,k}(\xi_k^j)$ are PDs of η^j and ξ_k^j, respectively.

For every $t\in[0,1/2]$, set

$$P_A^t(a|x,\lambda)=(1-2t)P_A(a|x,\lambda)+2t\frac{1}{o(A)},$$

$$P_{B^j}^t(b_j|y_j,\lambda^j)=(1-2t)P_{B^j}(b_j|y_j,\lambda^j)+2t\frac{1}{o(B^j)}(j\in[m]),$$

$$P_{C_k^j}^t(c_{j,k}|z_{j,k},\mu_k^j)=(1-2t)P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j)+2t\frac{1}{o(C_k^j)}(j\in[m],k\in[n_j]),$$

which are clearly PDs of a, b_j, and $c_{j,k}$, respectively. Put

$$\begin{aligned} P^t(a\mathbf{bc}|x\mathbf{yz}) &= \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} p(\lambda,\mu_1,\ldots,\mu_m)P_A^t(a|x,\lambda) \\ &\times\prod_{j=1}^m P_{B^j}^t(b_j|y_j,\lambda^j,\mu_j)\times\prod_{j=1}^m\prod_{k=1}^{n_j} P_{C_k^j}^t(c_{j,k}|z_{j,k},\mu_k^j), \end{aligned}$$

then $f(t):=[\![P^t(a\mathbf{bc}|x\mathbf{yz})]\!]$ is a star-local CT over Δ_S for all $t\in[0,1/2]$ with $f(0)=\mathbf{P}$ and $f(1/2)=\mathbf{I}$. Obviously, the map $t\mapsto f(t)$ from $[0,1/2]$ into $\mathcal{CT}^{\text{star-local}}(\Delta_S)$ is continuous. Similarly, for every $t\in[1/2,1]$, set

$$Q_A^t(a|x,\eta)=(2t-1)Q_A(a|x,\eta)+2(1-t)\frac{1}{o(A)},$$

$$Q_{B^j}^t(b_j|y_j,\eta^j)=(2t-1)Q_{B^j}(b_j|y_j,\eta^j)+2(1-t)\frac{1}{o(B^j)}(j\in[m]),$$

$$Q_{C_k^j}^t(c_{j,k}|z_{j,k},\xi_k^j)=(2t-1)Q_{C_k^j}(c_{j,k}|z_{j,k},\xi_k^j)+2(1-t)\frac{1}{o(C_k^j)}(j\in[m],k\in[n_j]),$$

which are clearly PDs of a, b^j, and c_k^j, respectively. Put

$$\begin{aligned} Q^t(a\mathbf{bc}|x\mathbf{yz}) &= \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} q(\lambda,\mu_1,\ldots,\mu_m)Q_A^t(a|x,\lambda) \\ &\times\prod_{j=1}^m Q_{B^j}^t(b_j|y_j,\lambda^j,\mu_j)\times\prod_{j=1}^m\prod_{k=1}^{n_j} Q_{C_k^j}^t(c_{j,k}|z_{j,k},\mu_k^j), \end{aligned}$$

then $g(t) := [\![Q^t(a\mathbf{bc}|x\mathbf{yz})]\!]$ is a star-local CT over Δ_S for all $t \in [1/2, 1]$ with $g(1/2) = \mathbf{I}$ and $g(1) = \mathbf{Q}$. Obviously, the map $t \mapsto g(t)$ from $[1/2, 1]$ into $\mathcal{CT}^{\text{star-local}}(\Delta_S)$ is continuous. Thus, the function $p : [0, 1] \to \mathcal{CT}^{\text{star-local}}(\Delta_S)$ defined by

$$p(t) = \begin{cases} f(t), & t \in [0, 1/2]; \\ g(t), & t \in (1/2, 1], \end{cases}$$

is continuous everywhere and then induces a path p in $\mathcal{CT}^{\text{star-local}}(\Delta_S)$ with $p(0) = \mathbf{P}$ and $p(1) = \mathbf{Q}$. This shows that $\mathcal{CT}^{\text{star-local}}(\Delta_S)$ is path-connected. The proof is completed. □

Next, we discuss the "quasi-convexity" of the set $\mathcal{CT}^{\text{star-local}}(\Delta_S)$ by finding two classes of subsets of $\mathcal{CT}^{\text{star-local}}(\Delta_S)$ that are star-convex.

For any fixed $1 \le u \le m$ and $1 \le v \le n_u$, by taking a star-shaped CT $\mathbf{E} = [\![E(a\mathbf{bc}|x\mathbf{yz})]\!]$ such that the marginal $\mathbf{E}_{\widehat{C_v^u B^u}}$ is completely product:

$$E_{\widehat{C_v^u B^u}}(a\mathbf{b}^u\widehat{\mathbf{c}}_v^u|x\mathbf{y}^u\widehat{\mathbf{z}}_v^u) = E_A(a|x) \times \prod_{j \neq u} E_{B^j}(b_j|y_j) \times \prod_{(j,k)\neq(u,v)} E_{C_k^j}(c_{j,k}|z_{j,k}),$$

where

$$\mathbf{b}^u = \{b_j\}_{j\neq u}, \widehat{\mathbf{c}}_v^u = \{c_{j,k}\}_{(j,k)\neq(u,v)}, \mathbf{y}^u = \{y_i\}_{i\neq u}, \widehat{\mathbf{z}}_v^u = \{z_{j,k}\}_{(j,k)\neq(u,v)},$$

we define a star-shaped CT $\mathbf{S}_{u,v} = [\![S_{u,v}(a\mathbf{bc}|x\mathbf{yz})]\!]$ by

$$S_{u,v}(a\mathbf{bc}|x\mathbf{yz}) = E_{\widehat{C_v^u B^u}}(a\mathbf{b}^u\widehat{\mathbf{c}}_v^u|x\mathbf{y}^u\widehat{\mathbf{z}}_v^u) \times \frac{1}{o(C_v^u)} \times \frac{1}{o(B^u)}. \tag{44}$$

Put

$$\mathcal{CT}^{\text{star-local}}_{\mathbf{E}_{\widehat{C_v^u B^u}}}(\Delta_S) = \left\{\mathbf{P} \in \mathcal{CT}^{\text{star-local}}(\Delta_S) :\ \mathbf{P}_{\widehat{C_v^u B^u}} = \mathbf{E}_{\widehat{C_v^u B^u}}\right\}, \tag{45}$$

which is just the set of all star-local CTs over Δ_S with a fixed marginal distribution $\mathbf{E}_{\widehat{C_v^u B^u}}$ on the subsystem $\widehat{C_v^u B^u} = A\prod_{j\neq u} B^j \prod_{(j,k)\neq(u,v)} C_v^u$. Clearly, $(\mathbf{S}_{u,v})_{\widehat{C_v^u B^u}} = \mathbf{E}_{\widehat{C_v^u B^u}}$ and $\mathbf{S}_{u,v} \in \mathcal{CT}^{\text{star-local}}_{\mathbf{E}_{\widehat{C_v^u B^u}}}(\Delta_S)$.

Using these notations, we obtain the following.

Theorem 7. *The set $\mathcal{CT}^{\text{star-local}}_{\mathbf{E}_{\widehat{C_v^u B^u}}}(\Delta_S)$ is star-convex with a sun $\mathbf{S}_{u,v}$, i.e., for all $t \in [0, 1]$, it holds that*

$$(1-t)\mathbf{S}_{u,v} + t\mathcal{CT}^{\text{star-local}}_{\mathbf{E}_{\widehat{C_v^u B^u}}}(\Delta_S) \subset \mathcal{CT}^{\text{star-local}}_{\mathbf{E}_{\widehat{C_v^u B^u}}}(\Delta_S). \tag{46}$$

Proof. Let $t \in [0, 1]$ and $\mathbf{P} \in \mathcal{CT}^{\text{star-local}}_{\mathbf{E}_{\widehat{C_v^u B^u}}}(\Delta_S)$. Then, $\mathbf{P} \in \mathcal{CT}^{\text{star-local}}(\Delta_S)$ and $\mathbf{P}_{\widehat{C_v^u B^u}} = \mathbf{E}_{\widehat{C_v^u B^u}}$. Since $\mathbf{P}$ has a D-star-shaped-LHVM:

$$\begin{aligned} P(a\mathbf{bc}|x\mathbf{yz}) \quad = \quad & \sum_{\lambda\in D, \mu_1\in F_1,\ldots,\mu_m\in F_m} \prod_{j=1}^m p_j(\lambda^j) \times \prod_{j=1}^m \prod_{k=1}^{n_j} p_{j,k}(\mu_k^j) \\ & \times P_A(a|x,\lambda) \prod_{j=1}^m P_{B^j}(b_j|y_j, \lambda^j, \mu_j) \times \prod_{j=1}^m \prod_{k=1}^{n_j} P_{C_k^j}(c_{j,k}|z_{j,k}, \mu_k^j), \end{aligned}$$

we get that

$$\begin{aligned}
P_{\widehat{C_v^u B^u}}(a\mathbf{b}^u\widehat{\mathbf{c}}_v^u|x\mathbf{y}^u\widehat{\mathbf{z}}_v^u) &= \sum_{c_{u,v},b_u} P(a\mathbf{bc}|x\mathbf{yz}) \\
&= \sum_{\lambda,\mu_k^j((j,k)\neq(u,v))} \prod_{j=1}^m p_j(\lambda^j) \times \prod_{(j,k)\neq(u,v)} p_{j,k}(\mu_k^j) \\
&\quad \times P_A(a|x,\lambda) \prod_{j\neq u} P_{B^j}(b_j|y_j,\lambda^j,\mu_j) \\
&\quad \times \prod_{(j,k)\neq(u,v)} P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j).
\end{aligned}$$

For every $t \in [0,1]$, put

$$\mu_u(s) = (\mu_1^u, \ldots, \mu_{v-1}^u, (\mu_v^u, s), \mu_{v+1}^u, \ldots, \mu_{n_u}^u),$$

and define

$$f_{u,v}^t(\mu_v^u, s) = \begin{cases} p_{u,v}(\mu_v^u)(1-t), & s=0; \\ p_{u,v}(\mu_v^u)t, & s=1, \end{cases} \tag{47}$$

$$P_{B^u}(b_u|y_u,\lambda^u,\mu_u(s)) = \begin{cases} \frac{1}{o(B^u)}, & s=0; \\ P_{B^u}(b_u|y_u,\lambda^u,\mu_u), & s=1, \end{cases} \tag{48}$$

$$P_{C_v^u}(c_{u,v}|z_{u,v},(\mu_v^u,s)) = \begin{cases} \frac{1}{o(C_v^u)}, & s=0; \\ P_{C_v^u}(c_{u,v}|z_{u,v},\mu_v^u), & s=1, \end{cases} \tag{49}$$

which are PDs of $(\mu_v^u, s), b_u$ and $c_{u,v}$, respectively. Put

$$\begin{aligned}
Q^t(a\mathbf{bc}|x\mathbf{yz}) &= \sum_{s=0,1} \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} \prod_{j=1}^m p_j(\lambda^j) \times \prod_{(j,k)\neq(u,v)} p_{j,k}(\mu_k^j) \times f_{u,v}^t(\mu_v^u,s) \\
&\quad \times P_A(a|x,\lambda) \prod_{j\neq u} P_{B^j}(b_j|y_j,\lambda^j,\mu_j) \times P_{B^u}(b_u|y_u,\lambda^u,\mu_u(s)) \\
&\quad \times \prod_{(j,k)\neq(u,v)} P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j) \times P_{C_v^u}(c_{u,v}|z_{u,v},(\mu_v^u,s)),
\end{aligned}$$

then $\mathbf{Q}^t = [\![Q^t(a\mathbf{bc}|x\mathbf{yz})]\!] \in \mathcal{CT}^{\text{star-local}}(\Delta_S)$.

On the other hand, for all $a, \mathbf{b}, \mathbf{c}, x, \mathbf{y}, \mathbf{z}$, we compute that

$$\begin{aligned} Q^t(a\mathbf{bc}|x\mathbf{yz}) &= \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} \prod_{j=1}^m p_j(\lambda^j) \times \prod_{(j,k)\neq(u,v)} p_{j,k}(\mu_k^j) \times f_{u,v}^t(\mu_v^u, 0) \\ &\quad \times P_A(a|x,\lambda) \prod_{j\neq u} P_{B^j}(b_j|y_j,\lambda^j,\mu_j) \times P_{B^u}(b_u|y_u,\lambda^u,\mu_u(0)) \\ &\quad \times \prod_{(j,k)\neq(u,v)} P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j) \times P_{C_v^u}(c_{u,v}|z_{u,v},(\mu_v^u,0)) \\ &\quad + \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} \prod_{j=1}^m p_j(\lambda^j) \times \prod_{(j,k)\neq(u,v)} p_{j,k}(\mu_k^j) \times f_{u,v}^t(\mu_v^u, 1) \\ &\quad \times P_A(a|x,\lambda) \prod_{j\neq u} P_{B^j}(b_j|y_j,\lambda^j,\mu_j) \times P_{B^u}(b_u|y_u,\lambda^u,\mu_u(1)) \\ &\quad \times \prod_{(j,k)\neq(u,v)} P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j) \times P_{C_v^u}(c_{u,v}|z_{u,v},(\mu_v^u,1)). \end{aligned}$$

Using Equations (47)–(49), we obtain that

$$\begin{aligned} Q^t(a\mathbf{bc}|x\mathbf{yz}) &= (1-t) \sum_{\lambda,\mu_k^j((j,k)\neq(u,v))} \prod_{j=1}^m p_j(\lambda^j) \times \prod_{(j,k)\neq(u,v)} p_{j,k}(\mu_k^j) \\ &\quad \times P_A(a|x,\lambda) \prod_{j\neq u} P_{B^j}(b_j|y_j,\lambda^j,\mu_j) \times \frac{1}{o(B^u)} \\ &\quad \times \prod_{(j,k)\neq(u,v)} P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j) \times \frac{1}{o(C_v^u)} \\ &\quad +t \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} \prod_{j=1}^m p_j(\lambda^j) \times \prod_{(j,k)} p_{j,k}(\mu_k^j) \\ &\quad \times P_A(a|x,\lambda) \prod_{j=1}^m P_{B^j}(b_j|y_j,\lambda^j,\mu_j) \times \prod_{(j,k)} P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j) \\ &= (1-t)S_{u,v}(a\mathbf{bc}|x\mathbf{yz}) + tP(a\mathbf{bc}|x\mathbf{yz}). \end{aligned}$$

This shows that

$$(1-t)\mathbf{S}_{u,v} + t\mathbf{P} = \mathbf{Q}^t \in \mathcal{CT}^{\text{star-local}}(\Delta_S), \quad \forall t \in [0,1].$$

Since $(\mathbf{S}_{u,v})_{\widehat{C_v^u B^u}} = \mathbf{P}_{\widehat{C_v^u B^u}} = \mathbf{E}_{\widehat{C_v^u B^u}}$, we have $\mathbf{Q}^t_{\widehat{C_v^u B^u}} = (1-t)(\mathbf{S}_{u,v})_{\widehat{C_v^u B^u}} + t\mathbf{P}_{\widehat{C_v^u B^u}} = \mathbf{E}_{\widehat{C_v^u B^u}}$. This shows that $\mathbf{Q}^t \in \mathcal{CT}^{\text{star-local}}_{\mathbf{E}_{\widehat{C_v^u B^u}}}(\Delta_S)$. The proof is completed. $\square$

Next, let us find another star-convex subset of $\mathcal{CT}^{\text{star-local}}(\Delta_S)$. Fixed $1 \le u \le m$ and taken a star-shaped CT $\mathbf{F} = [\![F(a\mathbf{bc}|x\mathbf{yz})]\!]$ such that

$$F_{\widehat{AB^u}}(\mathbf{b}^u\mathbf{c}|\mathbf{y}^u\mathbf{z}) := \sum_{a,b_u} F(a\mathbf{bc}|x\mathbf{yz}) = \prod_{j\neq u} F_{B^j}(b_j|y_j) \times \prod_{j,k} F_{C_k^j}(c_{j,k}|z_{j,k}),$$

where $\mathbf{b}^u = \{b_j\}_{j\neq u}, \mathbf{y}^u = \{y_j\}_{j\neq u}$, we define a star-shaped CT $\mathbf{S}_u = [\![S_u(a\mathbf{bc}|x\mathbf{yz})]\!]$ by

$$S_u(a\mathbf{bc}|x\mathbf{yz}) = \frac{1}{o(A)} \times F_{\widehat{AB^u}}(\mathbf{b}^u\mathbf{c}|\mathbf{y}^u\mathbf{z}) \times \frac{1}{o(B^u)} \times \prod_{j,k} F_{C_k^j}(c_{j,k}|z_{j,k}). \tag{50}$$

Put

$$\mathcal{CT}^{\text{star-local}}_{\mathbf{F}_{\widehat{AB^u}}}(\Delta_S) = \left\{\mathbf{P} \in \mathcal{CT}^{\text{star-local}}(\Delta_S) : \ \mathbf{P}_{\widehat{AB^u}} = \mathbf{F}_{\widehat{AB^u}}\right\}, \tag{51}$$

which is just the set of all star-local CTs over Δ_S with fixed marginal distribution $\mathbf{F}_{\widehat{AB^u}}$ on the subsystem $\widehat{AB^u} = (\prod_{j\neq u} B^j)C$. Clearly, $(\mathbf{S}_u)_{\widehat{AB^u}} = \mathbf{F}_{\widehat{AB^u}} = [\![F_{\widehat{AB^u}}(\mathbf{b}^u\mathbf{c}|\mathbf{y}^u\mathbf{z})]\!]$ and then $\mathbf{S}_u \in \mathcal{CT}^{\text{star-local}}_{\mathbf{F}_{\widehat{AB^u}}}(\Delta_S)$.

With these notations, we have the following.

Theorem 8. *The set $\mathcal{CT}^{\text{star-local}}_{\mathbf{F}_{\widehat{AB^n}}}(\Delta_S)$ is star-convex with a sun $\mathbf{S}_u$, i.e., for all $t \in [0,1]$, it holds that*

$$(1-t)\mathbf{S}_{u,v} + t\mathcal{CT}^{\text{star-local}}_{\mathbf{F}_{\widehat{AB^u}}}(\Delta_S) \subset \mathcal{CT}^{\text{star-local}}_{\mathbf{F}_{\widehat{AB^u}}}(\Delta_S). \tag{52}$$

Proof. Let $\mathbf{P} \in \mathcal{CT}^{\text{star-local}}_{\mathbf{F}_{\widehat{AB^u}}}(\Delta_S)$. Then, $\mathbf{P} \in \mathcal{CT}^{\text{star-local}}(\Delta_S)$ and $\mathbf{P}_{\widehat{AB^u}} = \mathbf{F}_{\widehat{AB^u}}$. Since $\mathbf{P}$ has a D-star-shaped LHVM

$$\begin{aligned} P(a\mathbf{bc}|x\mathbf{yz}) \;\; = \;\; & \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} \prod_{j=1}^m p_j(\lambda^j) \times \prod_{j=1}^m \prod_{k=1}^{n_j} p_{j,k}(\mu_k^j) \\ & \times P_A(a|x,\lambda) \prod_{j=1}^m P_{B^j}(b_j|y_j,\lambda^j,\mu_j) \times \prod_{j=1}^m \prod_{k=1}^{n_j} P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j), \end{aligned}$$

we get that

$$\begin{aligned} P_{\widehat{AB^u}}(\mathbf{b}^u\mathbf{c}|\mathbf{y}^u\mathbf{z}) \;\; = \;\; & \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} \prod_{j=1}^m p_j(\lambda^j) \times \prod_{j=1}^m \prod_{k=1}^{n_j} p_{j,k}(\mu_k^j) \times \\ & \times \prod_{j\neq u} P_{B^j}(b_j|y_j,\lambda^j,\mu_j) \times \prod_{j=1}^m \prod_{k=1}^{n_j} P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j). \end{aligned}$$

For every $t \in [0,1]$, put

$$g_u^t(\lambda^u,s) = \begin{cases} p_u(\lambda^u)(1-t), & s=0; \\ p_n(\lambda^u)t, & s=1, \end{cases}$$

$$\lambda' = (\lambda^1,\lambda^2,\lambda^{u-1},(\lambda^u,s),\lambda^{u+1},\ldots,\lambda^m),$$

$$P'(a|x,\lambda') = \begin{cases} \frac{1}{o(A)}, & s=0; \\ P(a|x,\lambda), & s=1, \end{cases}$$

$$P'_{B^u}(b_u|y_u,(\lambda^u,s),\mu_u) = \begin{cases} \frac{1}{o(B^n)}, & s=0; \\ P_{B^n}(b_u|y_u,\lambda^u,\mu_u), & s=1, \end{cases}$$

and define

$$\begin{aligned} Q^t(a\mathbf{bc}|x\mathbf{yz}) \;\; = \;\; & \sum_{s=0,1} \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} \prod_{j\neq u} p_j(\lambda^j) \times g_u^t(\lambda^u,s) \times \prod_{j=1}^m \prod_{k=1}^{n_j} p_{j,k}(\mu_k^j) \times \\ & \times P'_A(a|x,\lambda') \times \prod_{j\neq u} P_{B^j}(b_j|y_j,\lambda^j,\mu_j) \times P'_{B^u}(b_u|y_u,(\lambda^u,s),\mu_u) \\ & \times \prod_{j=1}^m \prod_{k=1}^{n_j} P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j). \end{aligned}$$

Clearly, $\mathbf{Q}^t := [\![Q^t(a\mathbf{bc}|x\mathbf{yz})]\!] \in \mathcal{CT}^{\text{star-local}}(\Delta_S)$.

On the other hand, for all $a, \mathbf{b}, \mathbf{c}, x, \mathbf{y}, \mathbf{z}$, we compute that

$$\begin{aligned} Q^t(a\mathbf{b}\mathbf{c}|x\mathbf{y}\mathbf{z}) &= (1-t)\sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m}\prod_{j=1}^{m} p_j(\lambda^j)\times\prod_{j=1}^{m}\prod_{k=1}^{n_j} p_{j,k}(\mu_k^j)\times \\ &\quad \times\frac{1}{o(A)}\times\prod_{j\neq u} P_{B^j}(b_j|y_j,\lambda^j,\mu_j)\times\frac{1}{o(B^u)} \\ &\quad \times\prod_{j,k} P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j) \\ &\quad +t\sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m}\prod_{j=1}^{m} p_j(\lambda^j)\times\prod_{j=1}^{m}\prod_{k=1}^{n_j} p_{j,k}(\mu_k^j)\times \\ &\quad \times P_A(a|x,\lambda)\times\prod_{j=1}^{m} P_{B^j}(b_j|y_j,\lambda^j,\mu_j)\times\prod_{j=1}^{m}\prod_{k=1}^{n_j} P_{C_k^j}(c_{j,k}|z_{j,k},\mu_k^j) \\ &= (1-t)S_u(a\mathbf{b}\mathbf{c}|x\mathbf{y}\mathbf{z})+tP(a\mathbf{b}\mathbf{c}|x\mathbf{y}\mathbf{z}). \end{aligned}$$

This shows that

$$(1-t)\mathbf{S}_u + t\mathbf{P} = \mathbf{Q}^t \in \mathcal{CT}^{\text{star-local}}(\Delta_S), \quad \forall t\in[0,1].$$

Clearly, $\mathbf{Q}^t_{\widehat{AB^u}} = \mathbf{F}_{\widehat{AB^u}}$. Hence, $(1-t)\mathbf{S}_u + t\mathbf{P} = \mathbf{Q}^t \in \mathcal{CT}^{\text{star-local}}_{\mathbf{F}_{\widehat{AB^u}}}(\Delta_S)$. The proof is completed. $\square$

4. A Star-Bell Inequality

In this section, we derive an inequality (56) that holds for all star-local star-shaped CTs, called a star-Bell inequality. Consider a star-shaped CT

$$\mathbf{P} = [\![P(a\mathbf{b}\mathbf{c}|x\mathbf{y}\mathbf{z})]\!] = [\![P(a,b_1\cdots b_m,\mathbf{c}|x,y_1\cdots y_m,\mathbf{z})]\!] \tag{53}$$

with inputs $x, y_j, z_{j,k} \in \{0,1\}$ and outcomes $a, b_j, c_{j,k}, \in \{0,1\}$, where $j\in[m], k\in[n_j]$. Put $N = \sum_{j=1}^{m} n_j$. For all $\alpha_0, \alpha_j, z_{j,k} \in \{0,1\}$, we define the following two quantities

$$\begin{aligned} I_{\alpha_0\alpha_1\ldots\alpha_m}(\mathbf{P}) &= \frac{1}{2^N}\sum_{z_{j,k}=0,1}\sum_{a,b_j,c_{j,k}=0,1}(-1)^{a+\sum_j b_j+\sum_{j,k}c_{j,k}} \\ &\quad \times P(a,b_1\cdots b_m,\mathbf{c}|\alpha_0,\alpha_1\cdots\alpha_m,\mathbf{z}), \end{aligned} \tag{54}$$

$$\begin{aligned} J_{\beta_0\beta_1\ldots\beta_m}(\mathbf{P}) &= \frac{1}{2^N}\sum_{z_{j,k}=0,1}(-1)^{\sum\limits_{j,k} z_{j,k}}\sum_{a,b_j,c_{j,k}=0,1}(-1)^{a+\sum_j b_j+\sum_{j,k}c_{j,k}} \\ &\quad \times P(a,b_1\cdots b_m,\mathbf{c}|\beta_0,\beta_1\cdots\beta_m,\mathbf{z}). \end{aligned} \tag{55}$$

Theorem 9. *If a star-shaped CT* $\mathbf{P}$ *given by Equation (53) is star-local, then*

$$|I_{\alpha_0\alpha_1\ldots\alpha_m}(\mathbf{P})|^{\frac{1}{N}} + |J_{\beta_0\beta_1\ldots\beta_m}(\mathbf{P})|^{\frac{1}{N}} \leq 1,\ \ \forall \alpha_j,\beta_j\in\{0,1\}. \tag{56}$$

Proof. Since **P** is star-local, it has a D-star-shaped LHVM (14). Thus,

$$\begin{aligned}
& \sum_{a,b_j,c_{j,k}=0,1} (-1)^{a+\sum_j b_j+\sum_{j,k} c_{j,k}} P(a, b_1 \cdots b_m, \mathbf{c}|\alpha_0, \alpha_1 \cdots \alpha_m, \mathbf{z}) \\
= & \sum_{a,b_j,c_{j,k}=0,1} (-1)^{a+\sum_j b_j+\sum_{j,k} c_{j,k}} \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} p(\lambda, \mu_1, \ldots, \mu_m) \\
& \times P_A(a|\alpha_0, \lambda) \times \prod_{j=1}^{m} P_{B^j}(b_j|\alpha_j, \lambda^j, \mu_j) \prod_{j=1}^{m}\prod_{k=1}^{n_j} P_{C^j_k}(c_{j,k}|z_{j,k}, \mu^j_k) \\
= & \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} p(\lambda, \mu_1, \ldots, \mu_m) \sum_{a=0,1} (-1)^a P_A(a|\alpha_0, \lambda) \\
& \times \prod_{j=1}^{m} \sum_{b_j=0,1} (-1)^{b_j} P_{B^j}(b_j|\alpha_j, \lambda^j, \mu_j) \\
& \times \prod_{j=1}^{m}\prod_{k=1}^{n_j} \sum_{c_{j,k}=0,1} (-1)^{c_{j,k}} P_{C^j_k}(c_{j,k}|z_{j,k}, \mu^j_k) \\
= & \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} p(\lambda, \mu_1, \ldots, \mu_m)\langle A_{\alpha_0}\rangle_\lambda \prod_{j=1}^{m} \langle B^j_{\alpha_j}\rangle_{\lambda^j,\mu_j} \\
& \times \prod_{j=1}^{m}\prod_{k=1}^{n_j} \langle C^j_{z_{j,k}}\rangle_{\mu^j_k},
\end{aligned}$$

where

$$\begin{cases}
\langle A_{\alpha_0}\rangle_\lambda = \sum_{a=0,1}(-1)^a P_A(a|\alpha_0, \lambda), \\
\langle B^j_{\alpha_j}\rangle_{\lambda^j,\mu_j} = \sum_{b_j=0,1}(-1)^{b_j} P_{B^j}(b_j|\alpha_j, \lambda^j, \mu_j), \\
\langle C^j_{z_{j,k}}\rangle_{\mu^j_k} = \sum_{c_{j,k}=0,1}(-1)^{c_{j,k}} P_{C^j_k}(c_{j,k}|z_{j,k}, \mu^j_k).
\end{cases}$$

Hence,

$$\begin{aligned}
|I_{\alpha_0\alpha_1\ldots\alpha_m}(\mathbf{P})| \leq & \frac{1}{2^N} \sum_{\substack{z_{j,k}=0,1 \\ j=1,\ldots m,k=1,\ldots,n_j}} \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} p(\lambda, \mu_1, \ldots, \mu_m) \\
& \times \left| \langle A_{\alpha_0}\rangle_\lambda \prod_{j=1}^{m} \langle B^j_{\alpha_j}\rangle_{\lambda^j,\mu_j} \times \prod_{j=1}^{m}\prod_{k=1}^{n_j} \langle C^j_{z_{j,k}}\rangle_{\mu^j_k} \right| \\
= & \frac{1}{2^N} \sum_{\substack{z_{j,k}=0,1 \\ j=1,\ldots m,k=1,\ldots,n_j}} \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} p(\lambda, \mu_1, \ldots, \mu_m) \\
& \times |\langle A_{\alpha_0}\rangle_\lambda| \times \prod_{j=1}^{m} \left|\langle B^j_{\alpha_j}\rangle_{\lambda^j,\mu_j}\right| \times \prod_{j=1}^{m}\prod_{k=1}^{n_j} \left|\langle C^j_{z_{j,k}}\rangle_{\mu^j_k}\right|.
\end{aligned}$$

Note that $|\langle A_{\alpha_0}\rangle_\lambda| \leq 1, |\langle B^j_{\alpha_j}\rangle_{\lambda^j,\mu_j}| \leq 1$, we have

$$|I_{\alpha_0\alpha_1\ldots\alpha_m}(\mathbf{P})| \leq \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} p(\lambda, \mu_1, \ldots, \mu_m) f(\mu_1, \ldots, \mu_m), \tag{57}$$

where

$$f(\mu_1, \ldots, \mu_m) = \prod_{j=1}^{m}\prod_{k=1}^{n_j} \left| \frac{1}{2} \sum_{z_{j,k}=0,1} \langle C^j_{z_{j,k}}\rangle_{\mu^j_k} \right|. \tag{58}$$

Analogously, we can get

$$|J_{\beta_0\beta_1\ldots\beta_m}(\mathbf{P})| \leq \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} p(\lambda,\mu_1,\ldots,\mu_m)g(\mu_1,\ldots,\mu_m), \tag{59}$$

where

$$g(\mu_1,\ldots,\mu_m) = \prod_{j=1}^{m}\prod_{k=1}^{n_j}\left|\frac{1}{2}\sum_{z_{j,k}=0,1}(-1)^{z_{j,k}}\langle C^j_{z_{j,k}}\rangle_{\mu^j_k}\right|. \tag{60}$$

Since

$$p(\lambda,\mu_1,\ldots,\mu_m) = \prod_{j=1}^{m} p_j(\lambda^j) \times \prod_{j=1}^{m}\prod_{k=1}^{n_j} p_{j,k}(\mu^j_k),$$

where $\{p_j(\lambda^j)\}_{\lambda_j}$ and $\{p_{j,k}(\mu^j_k)\}_{\mu^j_k}$ are probability distributions, we have from Equation (57) that

$$\begin{aligned}
|I_{\alpha_0\alpha_1\ldots\alpha_m}(\mathbf{P})| &\leq \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m} p(\lambda,\mu_1,\ldots,\mu_m)f(\mu_1,\ldots,\mu_m) \\
&= \sum_{\lambda\in D,\mu_1\in F_1,\ldots,\mu_m\in F_m}\prod_{j=1}^{m} p_j(\lambda^j) \times \prod_{j=1}^{m}\prod_{k=1}^{n_j} p_{j,k}(\mu^j_k) \\
&\quad \times \prod_{j=1}^{m}\prod_{k=1}^{n_j}\left|\frac{1}{2}\sum_{z_{j,k}=0,1}\langle C^j_{z_{j,k}}\rangle_{\mu^j_k}\right| \\
&= \sum_{\mu_1\in F_1,\ldots,\mu_m\in F_m}\prod_{j=1}^{m}\left(\sum_{\lambda^j} p_j(\lambda^j)\right) \\
&\quad \times \prod_{j=1}^{m}\prod_{k=1}^{n_j}\left(p_{j,k}(\mu^j_k)\left|\frac{1}{2}\sum_{z_{j,k}=0,1}\langle C^j_{z_{j,k}}\rangle_{\mu^j_k}\right|\right).
\end{aligned}$$

Note that $\sum_{\lambda^j} p_j(\lambda^j) = 1$ for all $j = 1,2,\ldots,m$, we obtain that

$$|I_{\alpha_0\alpha_1\ldots\alpha_m}(\mathbf{P})| \leq \prod_{j=1}^{m}\prod_{k=1}^{n_j}\left(\sum_{\mu^j_k} p_{j,k}(\mu^j_k)\left|\frac{1}{2}\sum_{z_{j,k}=0,1}\langle C^j_{z_{j,k}}\rangle_{\mu^j_k}\right|\right).$$

Similarly, using inequality (59) implies that

$$|J_{\beta_0\beta_1\ldots\beta_m}(\mathbf{P})| \leq \prod_{j=1}^{m}\prod_{k=1}^{n_j}\left(\sum_{\mu^j_k} p_{j,k}(\mu^j_k)\left|\frac{1}{2}\sum_{z_{j,k}=0,1}(-1)^{z_{j,k}}\langle C^j_{z_{j,k}}\rangle_{\mu^j_k}\right|\right).$$

Using the following inequality [22] Lemma 1 :

$$\sum_{k=1}^{m}\left(\prod_{i=1}^{n} x_i^k\right)^{\frac{1}{n}} \leq \prod_{i=1}^{n}(x_i^1 + x_i^2 + \ldots + x_i^m)^{\frac{1}{n}},\ \forall x_i^k \geq 0,$$

we have

$$\begin{aligned}
& (|I_{\alpha_0\alpha_1\ldots\alpha_m}(\mathbf{P})|)^{\frac{1}{N}} + (|J_{\beta_0\beta_1\ldots\beta_m}(\mathbf{P})|)^{\frac{1}{N}} \\
\leq & \left(\prod_{j=1}^{m}\prod_{k=1}^{n_j}\sum_{\mu_k^j} p_{j,k}(\mu_k^j)\left|\frac{1}{2}\sum_{z_{j,k}=0,1}\langle C^j_{z_{j,k}}\rangle_{\mu_k^j}\right|\right)^{\frac{1}{N}} \\
& + \left(\prod_{j=1}^{m}\prod_{k=1}^{n_j}\sum_{\mu_k^j} p_{j,k}(\mu_k^j)\left|\frac{1}{2}\sum_{z_{j,k}=0,1}(-1)^{z_{j,k}}\langle C^j_{z_{j,k}}\rangle_{\mu_k^j}\right|\right)^{\frac{1}{N}} \\
\leq & \prod_{j=1}^{m}\prod_{k=1}^{n_j}\left(\sum_{\mu_k^j} p_{j,k}(\mu_k^j)\left(\left|\frac{1}{2}\sum_{z_{j,k}=0,1}\langle C^j_{z_{j,k}}\rangle_{\mu_k^j}\right| + \left|\frac{1}{2}\sum_{z_{j,k}=0,1}(-1)^{z_{j,k}}\langle C^j_{z_{j,k}}\rangle_{\mu_k^j}\right|\right)\right)^{\frac{1}{N}} \\
= & \prod_{j=1}^{m}\prod_{k=1}^{n_j}\left(\sum_{\mu_k^j} p_{j,k}(\mu_k^j)\left(\left|\frac{\langle C^j_0\rangle_{\mu_k^j} + \langle C^j_1\rangle_{\mu_k^j}}{2}\right| + \left|\frac{\langle C^j_0\rangle_{\mu_k^j} - \langle C^j_1\rangle_{\mu_k^j}}{2}\right|\right)\right)^{\frac{1}{N}} \\
\leq & 1.
\end{aligned}$$

This shows that inequality (56) is valid and completes the proof. □

The validity of the inequality (56) is a necessary condition for a star-shaped CT $\mathbf{P}$ to be star-local. So, we call it a star-Bell inequality (SBI). Thus, a violation of SBI for some parameters $\alpha_0, \alpha_1, \ldots, \alpha_m$ and $\beta_0, \beta_1, \ldots, \beta_m$ shows that $\mathbf{P}$ is star-nonlocal.

Let us return to the network situation. Let A_x, $B^j_{y_j}$ and $C^{j,k}_{z_{j,k}}$ be $\{+1,-1\}$-valued observables of $\mathcal{H}_A$, $\mathcal{H}_{B^j}$, and $\mathcal{H}_{C^j_k}$. Then, we have the following spectrum decompositions:

$$\begin{cases} A_x = M_{0|x} - M_{1|x} = \sum_{a=0,1}(-1)^a M_{a|x}, \\ B^j_{y_j} = N^j_{0|y_j} - N^j_{1|y_j} = \sum_{b_j=0,1}(-1)^{b_j} N^j_{b_j|y_j}, \\ C^{j,k}_{z_{j,k}} = L^{j,k}_{0|z_{j,k}} - L^{j,k}_{1|z_{j,k}} = \sum_{z_{j,k}=0,1}(-1)^{c_{j,k}} L^{j,k}_{c_{j,k}|z_{j,k}}. \end{cases} \tag{61}$$

Put

$$M(x) = \{M_{0|x}, M_{1|x}\}, N^j(y_j) = \left\{N^j_{0|y_j}, N^j_{1|y_j}\right\}, L^{j,k}(z_{j,k}) = \left\{L^{j,k}_{0|z_{j,k}}, L^{j,k}_{1|z_{j,k}}\right\},$$

which are clearly POVMs of $\mathcal{H}_A$, $\mathcal{H}_{B^j}$, and $\mathcal{H}_{C^j_k}$, respectively. Then, we can get a measurement assemblage

$$\mathcal{M} = \left\{M(x)\otimes\left(\bigotimes_{j=1}^{m} N^j(y_j)\right)\otimes\left(\bigotimes_{j=1}^{m}\bigotimes_{k=1}^{n_j} L^{j,k}(z_{j,k})\right) : x, y_j, z_{j,k} = 0,1\right\} \tag{62}$$

of the quantum network with measurement operators

$$M_{a\mathbf{bc}|x\mathbf{yz}} := M_{a|x}\otimes\left(\bigotimes_{j=1}^{m} N^j_{b_j|y_j}\right)\otimes\left(\bigotimes_{j=1}^{m}(L^{j,1}_{c_{j,1}|z_{j,1}}\otimes L^{j,2}_{c_{j,2}|z_{j,2}}\otimes\ldots\otimes L^{j,n_j}_{c_{j,n_j}|z_{j,n_j}})\right),$$

where

$$a\in\{0,1\}, \mathbf{b} = (b_1,\ldots,b_m)\in\{0,1\}^m, \mathbf{c} = \{c_{j,k}\}_{k\in[n_j],j\in[m]}(c_{j,k} = 0,1),$$

$$x \in \{0,1\}, \mathbf{y} = (y_1, \ldots, y_m) \in \{0,1\}^m, \mathbf{z} = \{z_{j,k}\}_{k \in [n_j], j \in [m]} (z_{j,k} = 0,1).$$

For all $\alpha_j \in \{0,1\}$, it is computed that

$$\begin{aligned} I_{\alpha_0\alpha_1\ldots\alpha_m}(\mathbf{P}^{\Gamma}_{\mathcal{M}}) &= \frac{1}{2^N}\sum_{z_{j,k}}\sum_{a,b_j,c_{j,k}}(-1)^{a+\sum_j b_j+\sum_{j,k} c_{j,k}} P(a, b_1\cdots b_m, \mathbf{c}|\alpha_0, \alpha_1\cdots\alpha_m, \mathbf{z}) \\ &= \frac{1}{2^N}\sum_{z_{j,k}}\sum_{a,b_j,c_{j,k}}(-1)^{a+\sum_j b_j+\sum_{j,k} c_{j,k}} \\ &\quad \times \mathrm{tr}\left[\left(M_{a|\alpha_0} \otimes \left(\bigotimes_{j=1}^m N^j_{b_j|\alpha_j}\right) \otimes \left(\bigotimes_{j=1}^m \bigotimes_{k=1}^{n_j} L^{j,k}_{c_{j,k}|z_{j,k}}\right)\right)\tilde{\Gamma}\right] \\ &= \frac{1}{2^N}\sum_{z_{j,k}}\left\langle A_{\alpha_0} \otimes \left(\bigotimes_{j=1}^m B^j_{\alpha_j}\right) \otimes \left(\bigotimes_{j=1}^m \bigotimes_{k=1}^{n_j} C^{j,k}_{z_{j,k}}\right)\right\rangle_{\tilde{\Gamma}}. \end{aligned} \tag{63}$$

Similarly, for all $\beta_j \in \{0,1\}$, we have

$$\begin{aligned} J_{\beta_0\beta_1\ldots\beta_m}(\mathbf{P}^{\Gamma}_{\mathcal{M}}) &= \frac{1}{2^N}\sum_{z_{j,k}}(-1)^{\sum_{j,k} z_{j,k}}\sum_{a,b_j,c_{j,k}}(-1)^{a+\sum_j b_j+\sum_{j,k} c_{j,k}} \\ &\quad \times P(a, b_1\cdots b_m, \mathbf{c}|\beta_0, \beta_1\cdots\beta_m, \mathbf{z}) \\ &= \frac{1}{2^N}\sum_{z_{j,k}}(-1)^{\sum_{j,k} z_{j,k}}\left\langle A_{\beta_0} \otimes \left(\bigotimes_{j=1}^m B^j_{\beta_j}\right) \otimes \left(\bigotimes_{j=1}^m \bigotimes_{k=1}^{n_j} C^{j,k}_{z_{j,k}}\right)\right\rangle_{\tilde{\Gamma}}. \end{aligned} \tag{64}$$

This shows that the SBI (56) becomes

$$|I_{\alpha_0\alpha_1\ldots\alpha_m}((\mathbf{P}^{\Gamma}_{\mathcal{M}}))|^{\frac{1}{N}} + |J_{\beta_0\beta_1\ldots\beta_m}((\mathbf{P}^{\Gamma}_{\mathcal{M}}))|^{\frac{1}{N}} \le 1, \ \forall \alpha_j, \beta_j \in \{0,1\}. \tag{65}$$

It is valid whenever the network with state Γ is star-local for the given MA $\mathcal{M}$. Hence, to explore the star-nonlocality of the $MSN(m, n_1, \ldots, n_m)$, it suffices to choose some specific states distributed in the network and to choose specific measurements for each party such that the corresponding SBI (56) is violated for some $\alpha_0, \alpha_1, \ldots, \alpha_m$ and $\beta_0, \beta_1, \ldots, \beta_m$.

Example 1. *Let us consider the situation that the states distributed in the network are pure entangled states. Denote*

$$\begin{cases} |\psi\rangle_{A_j B^j_0} = p^j_1|00\rangle + p^j_2|11\rangle (j \in [m]), \\ |\psi\rangle_{B^j_k C^j_k} = q^{j,k}_1|00\rangle + q^{j,k}_2|11\rangle (j \in [m], k \in [n_j]), \end{cases} \tag{66}$$

the normalized pure states shared by A *and* B^j *and by* B^j *and* C^j_k*, respectively, with real and positive coefficients* p^j_1, p^j_2 *and* $q^{j,k}_1, q^{j,k}_1$ *with* $(p^j_1)^2 + (p^j_2)^2 = 1$ *and* $(q^{j,k}_1)^2 + (q^{j,k}_2)^2 = 1$*. Thus,*

$$\Lambda := \prod_{j=1}^m (2p^j_1 p^j_2) \times \prod_{j=1}^m \prod_{k=1}^{n_j} (2q^{j,k}_1 q^{j,k}_2) > 0.$$

Then, we can get

$$\rho_{A_j B^j_0} = |\psi\rangle_{A_j B^j_0}\langle\psi|, \ \rho_{B^j_k C^j_k} = |\psi\rangle_{B^j_k C^j_k}\langle\psi|, \tag{67}$$

Consider the $\{+1,-1\}$-valued observables of $\mathcal{H}_A = (\mathbb{C}^2)^{\otimes m}$, $\mathcal{H}_{B^j} = (\mathbb{C}^2)^{\otimes(1+n_j)}$, and $\mathcal{H}_{C_k^j} = \mathbb{C}^2$:

$$\begin{cases} X_0 = \sigma_1^{\otimes m}; \\ X_1 = \sigma_3^{\otimes m}, \end{cases} \quad \begin{cases} Y_0^j = \sigma_1^{\otimes(1+n_j)}; \\ Y_1^j = \sigma_3^{\otimes(1+n_j)}, \end{cases} \quad \begin{cases} Z_0^{j,k} = (\cos\eta^{j,k}, 0, \sin\eta^{j,k}) \cdot \vec{\sigma}; \\ Z_1^{j,k} = (\cos\theta^{j,k}, 0, \sin\theta^{j,k}) \cdot \vec{\sigma}, \end{cases} \tag{68}$$

where $j \in [m], k \in [n_j]$, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector composed of Pauli operators and $\eta^{j,k}, \theta^{j,k} \in [-\pi, \pi]$. The spectral projections form an MA $\mathcal{M}$ given by (62) for the network.

Using Equations (67), (68) and (63) and taking $\alpha_j = 0 (j = 0, 1, \ldots, m)$, we can get

$$\begin{aligned} I_{00\ldots0}(\mathbf{P}_{\mathcal{M}}^{\Gamma}) &= \frac{1}{2^N} \sum_{z_{j,k}=0,1} \left\langle X_0 \otimes \left(\bigotimes_{j=1}^{m} Y_0^j \right) \otimes \left(\bigotimes_{j=1}^{m} \bigotimes_{k=1}^{n_j} Z_{z_{j,k}}^{j,k} \right) \right\rangle_{\tilde{\Gamma}} \\ &= \frac{1}{2^N} \sum_{z_{j,k}=0,1} \left\langle \sigma_1^{\otimes m} \otimes \left(\bigotimes_{j=1}^{m} \sigma_1^{\otimes(1+n_j)} \right) \otimes \left(\bigotimes_{j=1}^{m} \bigotimes_{k=1}^{n_j} C_{z_{j,k}}^{j,k} \right) \right\rangle_{\tilde{\Gamma}} \\ &= \frac{1}{2^N} \sum_{z_{j,k}=0,1} \left\langle \left(\bigotimes_{j=1}^{m} (\sigma_1 \otimes \sigma_1) \right) \otimes \left(\bigotimes_{j=1}^{m} \bigotimes_{k=1}^{n_j} (\sigma_1 \otimes C_{z_{j,k}}^{j,k}) \right) \right\rangle_{\Gamma} \\ &= \frac{1}{2^N} \prod_{j=1}^{m} \langle \sigma_1 \otimes \sigma_1 \rangle_{\rho_{A_j B_0^j}} \times \prod_{j=1}^{m} \prod_{k=1}^{n_j} \left\langle \sigma_1 \otimes \sum_{z_{j,k}=0,1} C_{z_{j,k}}^{j,k} \right\rangle_{\rho_{B_k^j C_k^j}} \\ &= \frac{1}{2^N} \prod_{j=1}^{m} (2 p_1^j p_2^j) \times \prod_{j=1}^{m} \prod_{k=1}^{n_j} 2(\cos\eta^{j,k} + \cos\theta^{j,k}) q_1^{j,k} q_2^{j,k} \\ &= \frac{\Lambda}{2^N} \prod_{j=1}^{m} \prod_{k=1}^{n_j} (\cos\eta^{j,k} + \cos\theta^{j,k}). \end{aligned}$$

Analogously, taking $\beta_j = 1 (j = 0, 1, \ldots, m)$, we have

$$\begin{aligned} J_{11\ldots1}(\mathbf{P}_{\mathcal{M}}^{\Gamma}) &= \frac{1}{2^N} \prod_{j=1}^{m} \langle \sigma_3 \otimes \sigma_3 \rangle_{\rho_{A_j B_0^j}} \prod_{j=1}^{m} \prod_{k=1}^{n_j} \left\langle \sigma_3 \otimes \sum_{z_{j,k}=0,1} (-1)^{z_{j,k}} C_{z_{j,k}}^{j,k} \right\rangle_{\rho_{B_k^j C_k^j}} \\ &= \frac{1}{2^N} \prod_{j=1}^{m} \prod_{k=1}^{n_j} (\sin\eta^{j,k} - \sin\theta^{j,k}). \end{aligned}$$

Putting

$$\eta = (\eta^{1,1}, \ldots, \eta^{1,n_1}, \ldots, \eta^{m,1}, \ldots, \eta^{m,n_m}), \theta = (\theta^{1,1}, \ldots, \theta^{1,n_1}, \ldots, \theta^{m,1}, \ldots, \theta^{m,n_m})$$

implies that

$$\begin{aligned} f(\eta, \theta) &:= |I_{00\ldots0}(\mathbf{P}_{\mathcal{M}}^{\Gamma})|^{\frac{1}{N}} + |J_{11\ldots1}(\mathbf{P}_{\mathcal{M}}^{\Gamma})|^{\frac{1}{N}} \\ &= \left| \frac{1}{2^N} \Lambda \prod_{j=1}^{m} \prod_{k=1}^{n_j} (\cos\eta^{j,k} + \cos\theta^{j,k}) \right|^{\frac{1}{N}} + \left| \frac{1}{2^N} \prod_{j=1}^{m} \prod_{k=1}^{n_j} (\sin\eta^{j,k} - \sin\theta^{j,k}) \right|^{\frac{1}{N}} \\ &= \frac{1}{2} \sqrt[N]{\Lambda} \left| \prod_{j=1}^{m} \prod_{k=1}^{n_j} (\cos\eta^{j,k} + \cos\theta^{j,k}) \right|^{\frac{1}{N}} + \frac{1}{2} \left| \prod_{j=1}^{m} \prod_{k=1}^{n_j} (\sin\eta^{j,k} - \sin\theta^{j,k}) \right|^{\frac{1}{N}}. \end{aligned}$$

Taking $\theta = -\eta$, i.e., $\theta^{j,k} = -\eta^{j,k}$ for all j, k yields that

$$f(\eta, -\eta) = \sqrt[N]{\Lambda}\left|\prod_{j=1}^{m}\prod_{k=1}^{n_j}\cos\eta^{j,k}\right|^{\frac{1}{N}} + \left|\prod_{j=1}^{m}\prod_{k=1}^{n_j}\sin\eta^{j,k}\right|^{\frac{1}{N}}.$$

By taking $\eta^{j,k} \in [0, \pi/2]$ such that

$$\sin\eta^{j,k} = \frac{1}{\sqrt{1+\Lambda^{\frac{2}{N}}}}, \cos\eta^{j,k} = \frac{\Lambda^{\frac{1}{N}}}{\sqrt{1+\Lambda^{\frac{2}{N}}}} \tag{69}$$

for each j, k, we get that

$$|I_{00\ldots0}(\mathbf{P}^{\Gamma}_{\mathcal{M}})|^{\frac{1}{N}} + |J_{11\ldots1}(\mathbf{P}^{\Gamma}_{\mathcal{M}})|^{\frac{1}{N}} = f(\eta, -\eta) = \sqrt{1+\Lambda^{\frac{2}{N}}} > 1$$

since $\Lambda > 0$. This shows that SBI (65) is violated for $(\alpha_j, \beta_j) = (0,1)(j = 0, 1, \ldots, m)$ and then the network with the shared states given by (66) is star-nonlocal.

The following example is about a situation in which the states distributed in the network are Werner states with noise parameters v_j and v_k^j.

Example 2. *Let us consider the Werner states distributed in the network:*

$$\rho_{A_jB_0^j} = v_j|\phi^+\rangle\langle\phi^+| + (1-v_j)\frac{I}{4}, \rho_{B_k^jC_k^j} = v_k^j|\phi^+\rangle\langle\phi^+| + (1-v_k^j)\frac{I}{4}, \tag{70}$$

where $v_j \in (0,1], v_k^j \in (0,1], j \in [m], k \in [n_j]$ and $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Consider the $\{+1, -1\}$-valued observables of $\mathcal{H}_A = (\mathbb{C}^2)^{\otimes m}$, $\mathcal{H}_{B^j} = (\mathbb{C}^2)^{\otimes(1+n_j)}$ and $\mathcal{H}_{C_k^j} = \mathbb{C}^2$:

$$\begin{cases} X_0 = \sigma_1^{\otimes m}; \\ X_1 = \sigma_3^{\otimes m}, \end{cases} \quad \begin{cases} Y_0^j = \sigma_1^{\otimes(1+n_j)}; \\ Y_1^j = \sigma_3^{\otimes(1+n_j)}, \end{cases} \quad \begin{cases} Z_0^{j,k} = \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3); \\ Z_1^{j,k} = \frac{1}{\sqrt{2}}(\sigma_1 - \sigma_3), \end{cases} \tag{71}$$

where $j \in [m], k \in [n_j]$ and σ_1, σ_3 are Pauli operators. The spectral projections form an MA $\mathcal{M}$ given by (62) for the network. Using Equation (70), Equation (71), and Equation (63) and taking $\alpha_j = 0(j = 0, 1, \ldots, m)$, we compute that

$$\begin{aligned} I_{00\ldots0}(\mathbf{P}^{\Gamma}_{\mathcal{M}}) &= \frac{1}{2^N}\sum_{z_{j,k}=0,1}\left\langle X_0 \otimes (\bigotimes_{j=1}^{m} Y_0^j) \otimes (\bigotimes_{j=1}^{m}\bigotimes_{k=1}^{n_j} Z_{z_{j,k}}^{j,k})\right\rangle_{\tilde{\Gamma}} \\ &= \frac{1}{2^N}\sum_{z_{j,k}=0,1}\left\langle \sigma_1^{\otimes m} \otimes (\bigotimes_{j=1}^{m} \sigma_1^{\otimes(1+n_j)}) \otimes (\bigotimes_{j=1}^{m}\bigotimes_{k=1}^{n_j} C_{z_{j,k}}^{j,k})\right\rangle_{\tilde{\Gamma}} \\ &= \frac{1}{2^N}\sum_{z_{j,k}=0,1}\left\langle \left(\bigotimes_{j=1}^{m}(\sigma_1\otimes\sigma_1)\right) \otimes \left(\bigotimes_{j=1}^{m}\bigotimes_{k=1}^{n_j}(\sigma_1 \otimes C_{z_{j,k}}^{j,k})\right)\right\rangle_{\Gamma} \\ &= \frac{1}{2^N}\prod_{j=1}^{m}\langle\sigma_1\otimes\sigma_1\rangle_{\rho_{A_jB_0^j}}\prod_{j=1}^{m}\prod_{k=1}^{n_j}\left\langle\sigma_1 \otimes \sum_{z_{j,k}=0,1} C_{z_{j,k}}^{j,k}\right\rangle_{\rho_{B_k^jC_k^j}} \\ &= \frac{V}{\sqrt{2^N}}, \end{aligned}$$

where $V = \prod_{j=1}^{m} v_j \prod_{j=1}^{m} \prod_{k=1}^{n_j} v_k^j$.

Analogously, taking $\beta_j = 1 (j = 0, 1, \ldots, m)$, *we have* $J_{11\ldots1}(\mathbf{P}_{\mathcal{M}}^{\Gamma}) = \frac{V}{\sqrt{2^N}}$. *Hence,*

$$|I_{00\ldots0}(\mathbf{P}_{\mathcal{M}}^{\Gamma})|^{\frac{1}{N}} + |J_{11\ldots1}(\mathbf{P}_{\mathcal{M}}^{\Gamma})|^{\frac{1}{N}} = \sqrt{2}V^{\frac{1}{N}}.$$

Thus, $|I_{00\ldots0}(\mathbf{P}_{\mathcal{M}}^{\Gamma})|^{\frac{1}{N}} + |J_{11\ldots1}(\mathbf{P}_{\mathcal{M}}^{\Gamma})|^{\frac{1}{N}} > 1$ *if and only if* $V > \frac{1}{\sqrt{2^N}}$. *Therefore, when the coefficients of the shared state (70) satisfy the condition* $1 > V > \frac{1}{\sqrt{2^N}}$, *Equation (65) is violated, and then the network* $MSN(m, n_1, \ldots, n_m)$ *is star-nonlocal.*

5. Summary and Conclusions

In this work, a more general multi-star-network $MSN(m, n_1, \ldots, n_m)$ was introduced. Such a network consists of $1 + m + n_1 + \cdots + n_m$ nodes and one center-node A that connects to m star-nodes $B^1, B^2, \ldots, B^m$ while each star-node B^j has $n_j + 1$ star-nodes $A, C_1^j, C_2^j, \ldots, C_{n_j}^j$. When $m = 1, n_1 = n - 1$, it reduces to $MSN(1, n-1)$, which is just an n-local scenario [22,43], and when $m = n_1 = 1$, it becomes $MSN(1,1)$, reducing to the bi-local scenario [20,43].

First, we have introduced the nonlocality of the star-locality and star-nonlocality of such a network and deduced some related properties. Based on the architecture of such a network, we have proposed the concepts of star-shaped correlation tensors (SSCTs) and star-shaped probability tensors (SSPTs) and mathematically formulated two types of localities of SSCTs and SSPTs, named "D-star-locality" and "C-star-locality". By definition, an SSCT/SSPT is said to be C-star-local (resp., D-star-local) if it admits an integral star-shaped LHVM (resp., a finite-sum star-shaped LHVM). By establishing a series of characterizations, we have proven the equivalence of these localities is verified and then called them "star-locality". We have also found some necessary conditions for a star-shaped CT to be star-local. For the global properties of star-local SSCTs, we have proved that the set of all star-local SSCTs forms a path-connected compact set in the Hilbert space of tensors over the index set Δ_S and has least two types of star-convex subsets. Lastly, we have established a star-Bell inequality, which is proven to be valid for all star-local SSCTs. Based on this inequality, we have given two examples of star-nonlocal multi-star-network $MSN(m, n_1, \ldots, n_m)$ with the shared pure and mixed entangled states, respectively.

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References

1. Cirac, J.I.; Zoller, P.; Kimble, H.J.; Mabuchi, H. Quantum state transfer and entanglement distribution among distant nodes in a quantum network. *Phys. Rev. Lett.* **1997**, *78*, 3221. [CrossRef]
2. Kimble, H.J. The quantum internet. *Nature* **2008**, *453*, 1023. [CrossRef] [PubMed]

3. Sangouard, N.; Simon, C.H.; de Riedmatten, H.; Gisin, N. Quantum repeaters based on atomic ensembles and linear optics. *Rev. Mod. Phys.* **2011** *83*, 33. [CrossRef]
4. Simon, C. Towards a global quantum network. *Nat. Phot.* **2017**, *11*, 678. [CrossRef]
5. Wehner, S.; Elkouss, D.; Hanson, R. Quantum internet: A vision for the road ahead. *Science* **2018**, *362*, 303. [CrossRef] [PubMed]
6. Cirac, J.I.; van Enk, S.J.; Zoller, P.; Kimble, H.J.; Mabuchi, H. Quantum communication in a quantum network. *Phys. Scr.* **1998**, *T76*, 223. [CrossRef]
7. Gisin, N.; Ribordy, G.; Tittel, W.; Zbinden, H. Quantum cryptography. *Rev. Mod. Phys.* **2002**, *74*, 145. [CrossRef]
8. Renou, M.O.; Bäumer, E.; Boreiri, S.; Brunner, N.; Gisin, N.; Beigi, S. Genuine quantum nonlocality in the triangle network. *Phys. Rev. Lett.* **2019**, *123*, 140401. [CrossRef]
9. Gisin, N.; Bancal, J.D.; Cai, Y.; Remy, P.;Tavakoli, A.; Cruzeiro, E.Z.; Popescu, S.; Brunner, N. Constraints on nonlocality in networks from no-signaling and independence. *Nat. Commun.* **2020**, *11*, 2378. [CrossRef]
10. Navascués, M.; Wolfe, E.; Rosset, D.; Pozas-Kerstjens, A. Genuine network multipartite entanglement. *Phys. Rev. Lett.* **2020**, *125*, 240505. [CrossRef]
11. Kraft, T.; Designolle, S.; Ritz, C.;Brunner, N.; Gühne, O.; Huber, M. Quantum entanglement in the triangle network. *Phys. Rev. A* **2021**, *103*, L060401. [CrossRef]
12. Luo, M.X. New genuinely multipartite entanglement. *Adv. Quantum Technol.* **2021**, *4*, 2000123. [CrossRef]
13. Acín, D.; Bruß, A.; Lewenstein, M.; Sanpera, A. Classification of mixed three-qubit states. *Phys. Rev. Lett.* **2001**, *87*, 040401. [CrossRef]
14. Gühne, O.; Tóth, G. Entanglement detection. *Phys. Rep.* **2009**, *474*, 1. [CrossRef]
15. Kela, A.; Von Prillwitz, K.; Åberg, J.; Chaves, R.; Gross, D. Semidefinite tests for latent causal structures. *IEEE Trans. Inf. Theory* **2020**, *66*, 339. [CrossRef]
16. Åberg, J.; Nery, R.; Duarte, C.; Chaves, R. Semidefinite tests for quantum network topologies. *Phys. Rev. Lett.* **2020**, *125*, 110505. [CrossRef]
17. Bell, J.S. On the Einstein Podolsky Rosen paradox. *Physics* **1964**, *1*, 195–200. [CrossRef]
18. Bell, J.S. *Speakable and Unspeakable in Quantum Mechanics*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2004.
19. Branciard, C.; Gisin, N.; Pironio, S. Characterizing the nonlocal correlations created via entanglement swapping. *Phys. Rev. Lett.* **2010**, *104*, 170401. [CrossRef]
20. Branciard, C.; Rosset, D.; Gisin, N.; Pironio, S. Bilocal versus nonbilocal correlations in entanglement-swapping experiments. *Phys. Rev. A* **2012**, *85*, 032119. [CrossRef]
21. Fritz, T. Beyond Bell's theorem: Correlation scenarios. *New J. Phys.* **2012**, *14*, 103001. [CrossRef]
22. Tavakoli, A.; Skrzypczyk, P.; Cavalcanti, D.; Acín, A. Nonlocal correlations in the star-network configuration. *Phys. Rev. A* **2014**, *90*, 062109. [CrossRef]
23. Mukherjee, K.;Paul, B.; Roy, A. Characterizing quantum correlations in a fixed-input n-local network scenario. *Phys. Rev. A* **2020**, *101*, 032328. [CrossRef]
24. Carvacho, G.; Andreoli, F.; Santodonato, L.; Bentivegna, M.; Chaves, R.; Sciarrino, F. Experimental violation of local causality in a quantum network. *Nat. Commun.* **2017**, *8*, 14775. [CrossRef]
25. Fraser, T.C.; Wolfe, E. Causal compatibility inequalities admitting quantum violations in the triangle structure. *Phys. Rev. A* **2018**, *98*, 022113. [CrossRef]
26. Wolfe, E.; Spekkens, R.W.; Fritz, T. The inflation technique for causal inference with latent variables. *J. Causal Infer.* **2019**, *7*, 20170020. [CrossRef]
27. Gisin, N. Entanglement 25 years after quantum teleportation: Testing joint measurements in quantum networks. *Entropy* **2019**, *21*, 325. [CrossRef]
28. Renou, M.O.; Wang, Y.; Boreiri, S.; Beigi, S.; Gisin, N.; Brunner, N. Limits on correlations in networks for quantum and no-signaling resources. *Phys. Rev. Lett.* **2019**, *123*, 070403. [CrossRef]
29. Schröedinger, E. Die gegenwärtige Situation in der Quantenmechanik. *Naturwissenschaften* **1935**, *23*, 807. [CrossRef]
30. Gour, G.; Spekkens, R.W. The resource theory of quantum reference frames: manipulations and monotones. *New J. Phys.* **2008**, *10*, 033023. [CrossRef]
31. Baumgratz, T.; Cramer, M.; Plenio, M.B. Quantifying coherence. *Phys. Rev. Lett.* **2014**, *113*, 140401. [CrossRef]
32. Kraft, T.; Spee, C.; Yu, X.D.; Gühne, O. Characterizing quantum networks: Insights from coherence theory. *Phys. Rev. A* **2021**, *103*, 052405. [CrossRef]
33. Kraft T.; Piani, M. Monogamy relations of quantum coherence between multiple subspaces. *arXiv* **2019**, arXiv:1911.10026.
34. Contreras-Tejada, P.; Palazuelos, C.; de Vicente, J.I. Genuine multipartite nonlocality is intrinsic to quantum networks. *Phys. Rev. Lett.* **2021**, *126*, 040501. [CrossRef]
35. Šupić, I.; Bancal, J.D.; Cai, Y.; Brunner, N. Genuine network quantum nonlocality and self-testing. *Phys. Rev. A* **2022**, *105*, 022206. [CrossRef]
36. Tavakoli, A.; Pozas-Kerstjens, A.; Luo, M.X.; Renou, M.O. Bell nonlocality in networks. *Rep. Prog. Phys.* **2022**, *85*, 056001. [CrossRef] [PubMed]
37. Xiao, S.; Cao, H.; Guo, Z.; Han, K. Two types of trilocality of probability and correlation tensors. *Entropy* **2023**, *25*, 273. [CrossRef]

38. Haddadi, S.; Ghominejad, M.; Akhound, A.; Pourkarimi, M. R. Suppressing measurement uncertainty in an inhomogeneous spin star system. *Sci. Rep.* **2021**, *11*, 22691. [CrossRef]
39. Militello, B.; Messina, A. Genuine tripartite entanglement in a spin-star network at thermal equilibrium. *Phys. Rev. A* **2011**, *83*, 042305. [CrossRef]
40. Haddadi, S.; Pourkarimi, M. R.; Akhound, A.; Ghominejad, M. Thermal quantum correlations in a two-dimensional spin star model. *Mod. Phys. Lett. A* **2019**, *34*, 1950175. [CrossRef]
41. Yang, L.H.; Qi, X.F.; Hou, J.C. Nonlocal correlations in the tree-tensor-network configuration. *Phys. Rev. A* **2021**, *104*, 042405. [CrossRef]
42. Yang, Y.; Xiao, S.; Cao, H.X. Nonlocality of a type of multi-star-shaped quantum networks. *J. Phys. A: Math. Theor.* **2022**, *55*, 025303. [CrossRef]
43. Xiao, S.; Cao, H.X.; Guo, Z.H.; Han, K.Y. Characterizations of Bilocality and n-Locality of Correlation Tensors. *arXiv* **2022**, arXiv:2210.04207. https://doi.org/10.48550/arXiv.2210.04207.
44. Bai, L.H.; Xiao, S.; Guo, Z.H.; Cao, H.X. Decompositions of n-partite nonsignaling correlation-type tensors with applications. *Front. Phys.* **2022**, *10*, 864452. [CrossRef]