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From Zeroing Dynamics to Zeroing-Gradient Dynamics for Solving Tracking Control Problem of Robot Manipulator Dynamic System with Linear Output or Nonlinear Output

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Abstract: With the vigorous development of mechanical intelligence in industrial manufacturing, tracking control dynamic systems have been widely applied in many aspects of industry. In this paper, we present one theorem to discuss the validity condition of a ZD model with order-n for solving the tracking control problem of a nonlinear problem by utilizing a Lie derivative. Moreover, we also give the unified formula of the ZD model with order-n and rigorously prove it mathematically. In addition, we present three other theorems to give the global exponential convergence property of the ZD controller u(t), and the steady-state tracking error bound of the ZGD controller u(t), and the radius bound where the steady-state tracking error converges exponentially. Finally, simulations are conducted to demonstrate the validity and parameter influences of the ZD model and ZGD model for solving the tracking control problem with a single linear or nonlinear output of the single-link manipulator with flexible joints.

Keywords: zeroing dynamics; zeroing-gradient dynamics; tracking control; flexible joint robot manipulator

MSC: 93C10; 93C15; 93C95

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1. Introduction

In the past decades, many control systems in the real world have been inherently nonlinear, which has attracted many scholars in the industry to analyze and study the characteristics of nonlinearity in different fields, such as robot manipulators [1], aerospace [2], unmanned aerial vehicles [3], and military [4]. In these studies, nonlinear trajectory tracking control is one of the most important research objectives in the control systems.

By far, many methods have been developed, such as input–output linearization (IOL) [5], backstepping [6], phase plane [7], sliding mode control (SMC) [8], and proportional-integral-derivative (PID) [9], to solve control system problems on a variety of industrial applications. For example, in [10], Freeman proposed a tracking control scheme for upper limb stroke rehabilitation by utilizing the IOL method. In [11], Nehrir et al. proposed a tracking controller for DC motors by utilizing the IOL method. In [12], Madani et al. designed a nonlinear dynamic model for a quadrotor helicopter by using the backstepping control method. In [13], Hua et al. designed a controller for chemical reactor systems by using the backstepping control method. In [14], Liu et al. developed a fast setpoint altitude tracking controller for Hypersonic Flight Vehicle by implementing the phase plane method. In [15], Hao el al. developed a control method for co-plane formation by implementing the phase plane method. In [16], Komurcugil et al. presented the application of the SMC method to power converters. In [17], Zhang et al. presented the application of the SMC method in a missile tracking control controller. In [18], Xu et al. proposed a novel precise control algorithm for a four-wheel mobile robot by using the PID control method. In [19], Loucif et al. proposed a whale optimizer algorithm for a robot manipulator by using the PID control method. Furthermore, event-triggered controls [20–22] have also made some progress in the last couple decades, solving problems such as image encryption [23].

Although the above methods have achieved success in many fields of nonlinear trajectory tracking control, they have their own limitations when faced with problems such as singularity [5], the chattering phenomenon [24], large time-delay processes [25], and a large number of model parameters [26]. To overcome the above difficulties, neural network-related [27] content is introduced into the nonlinear control method. In [28], Man et al. developed a new tracking controller using Radial Basis Function (RBF) neural works [29], which can make it adaptive to learn the uncertainty of the nonlinear tracking system, and its output can ensure that the tracking error converges gradually. In [30], Zheng et al. developed a singularity-free controller for nonlinear control systems by using a neural network. In [31], Kumar et al. developed a neural network-based controller for the trajectory tracking control of redundant robot manipulators. In [32], Muñoz et al. improved the tracking control capability of an autonomous underwater vehicle based on dynamic neural networks. However, these neural network-based nonlinear tracking control methods sometimes also have limitations of synchronization, instability, excessive time consumption, noise disturbance or parameter perturbation. Therefore, a series of neural dynamic-based methods and zeroing dynamic (ZD) method [33–36] have been proposed to deal with such problems. In [37], Li et al. developed a new controller to solve the synchronization, and parameter perturbation problem of the chaotic system by using the zeroing dynamic (ZD) approach. In [38], Hu et al. investigated two tracking controllers for varactor systems and analysis of their stability. In [39], Li et al. designed a sign-bi-power activation function to accelerate convergence time by using the ZD method. In [40], Li et al. developed the tracking controller to deal with multiple-integrator systems with noise disturbance by using the ZD method. By far, the ZD-based trajectory tracking control method has been proven to be effective, and its accuracy has also been verified. However, we found that few studies have considered how to choose the most appropriate model order for the ZD method, while the choice of model order is one of the important factors in practical trajectory tracking control applications.

Therefore, in this paper, we established and proved the applicable conditions of the ZD model of order-n through the Lie derivative. In addition, in the process of deducing the applicable conditions of the ZD model with order-n, we found that at some point, the controller u(t) may disappear or be inapplicable due to the problems with the divisor becoming zero, which we called divided by zero (DBZ) problems. In order to conquer DBZ problems, we rewrite the zeroing-gradient dynamic (ZGD) model by using the Lie derivative.

To demonstrate the effectiveness of our proposed method, we designed two different simulation scenarios to verify it. First, we transform the operating problem in the narrow space into the tracking problem of the dynamic system of the single-link manipulator with flexible joints. Second, we transform the calibration problem of the robot manipulator in the three dimensional space into the tracking control problem of the dynamic system of the single-link manipulator with flexible joints. The simulation results of two different simulation schemes show that in the ZD model, the manipulator can track the desired path very well after a short time, and the tracking error converges to zero after a very short time. In addition, the result also shows that the input controller u(t) is smooth enough during the whole control process. However, the simulation results show that the ZGD model with the DBZ problem is impossible to converge to 0 at the exponential level. Therefore, after careful research, we found that we can only give an acceptable steady-state tracking error bound for the ZGD model with the DBZ problem, and the radius bound where the steady-state tracking error converges exponentially.

The contributions of our study can be summarized as follows:

- In the first theorem, the unified formula of the ZD model with order-n with rigorous mathematical proof and its applicable conditions are given by utilizing the Lie derivative.
- The exponential convergence property of the ZD model is given by using the Lie derivative and proved mathematically in the second theorem.
- The steady-state tracking error of the ZGD model is given by the Lie derivative in the third theorem.
- The radius bound where the steady-state tracking error converges exponentially is given in the fourth theorem.

The remainder of this paper is organized into the following five sections, and the main content of this paper is shown in the Figure 1. Section 2 presents the related zeroing dynamics and zeroing-gradient dynamics models for the tracking control theory. Meanwhile, we establish and prove our first theorem about the applicable conditions of the ZD model with order-n through the Lie derivative, give a unified formula for ZD models with order-n, and formulate the dynamic system of the single-link manipulator with flexible joints. Section 3 presents the ZD model for the tracking control system with a single linear output, and the ZGD model for the tracking control system with a single nonlinear output of a single link manipulator with flexible joints. Section 4 presents the convergence property of the ZD and ZGD models for solving the tracking control problem with single linear or nonlinear outputs of the single-link manipulator with flexible joints. In addition, we also present our other three theorems about the global exponential convergence property of the ZD controller u(t) and the steady-state tracking error bound of the ZGD controller u(t) and the radius bound for exponential convergence of the steady-state tracking error. Section 5 conducts several relative extreme simulations to show the validity and parameter influences of the ZD model and ZGD model for solving the tracking control problem with single linear or nonlinear outputs of the single-link manipulator with flexible joints. We conclude this paper in Section 6, as follows:

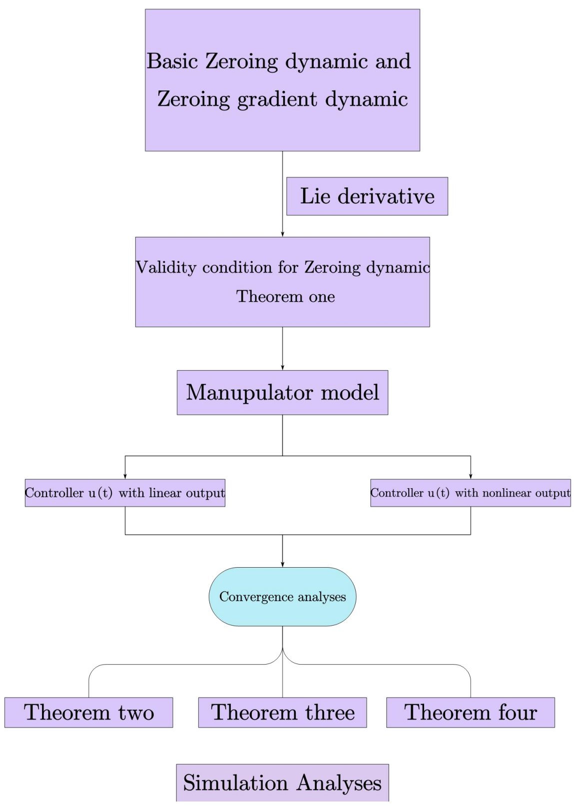


Figure 1. Structure of the main finding of this paper.

2. Zeroing Dynamics and Zeroing-Gradient Dynamics Models

In this section, we present the related zeroing dynamics and zeroing-gradient dynamics models for the tracking control theory, and formulate the dynamic system of the single-link manipulator with flexible joints.

2.1. Zeroing Dynamics and Zeroing-Gradient Dynamics Models for Tracking Control Problem of Nonlinear System

The tracking problem of a nonlinear system with a single output is defined as follows:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + G(\mathbf{x})\mathbf{u}(t), \\ y = h(\mathbf{x}), \end{cases} \tag{1}$$

where $\dot{\mathbf{x}}$ is the derivative vector of the state vector $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a nonlinear map, $G : \mathbb{R}^n \longrightarrow \mathbb{R}^{n \times r}$ is a linear map, and $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a linear or nonlinear map. Let y_d denote the desired output. Then, the target of the tracking control problem of the nonlinear system with a single output is to find a controller $\mathbf{u}(t)$, which makes the tracking error $y - y_d$ converge to zero as time t evolving.

For solving the tracking problem of the nonlinear system, we set the tracking error $e_1 = y - y_d$. By using the tracking control design formula $\dot{e}_1 = -\lambda_1 e_1$, we formulate the ZD model with order-1 as the following ordinary differential equation with constant coefficients:

$$\dot{e}_1 + \lambda_1 e_1 = 0, \tag{2}$$

where $\lambda_1 \in \mathbb{R}^+$ is the ZD model design parameter. From Equation (2), we can see that the ZD model with order-1 is equivalent to the proportional derivative controller design model with the error dynamic form. If the controller u is not contained in the ZD model with order-1 (2), we need to construct the ZD model with a higher order to solve the tracking control problem. By the iterative method, the tracking error with order-n is constructed by the following equation group:

$$\begin{cases} e_1 = y - y_d, \\ e_2 = \dot{e}_1 + \lambda_1 e_1, \\ \cdots \quad \cdots \quad \cdots, \\ e_i = \dot{e}_{i-1} + \lambda_{i-1} e_{i-1}, \\ \cdots \quad \cdots \quad \cdots, \\ e_n = \dot{e}_{n-1} + \lambda_{n-1} e_{n-1}, \end{cases} \tag{3}$$

where $\lambda_2, \cdots, \lambda_{n-1} \in \mathbb{R}^+$ are the ZD model parameters. Then, the ZD model with order-n, which is constructed by the tracking control design formula, is formulated as follows:

$$\dot{e}_n + \lambda_n e_n = 0, \tag{4}$$

where $\lambda_n \in \mathbb{R}^+$ is the ZD model design parameter. In addition, the ZD model with order-n (4) is linked to the original tracking error e_1 by the following ordinary differential equation with constant coefficients:

$$e_1^{(n)} + c_1 e_1^{(n-1)} + \cdots + c_{n-i} e_1^{(i)} + \cdots + c_n e_1 = 0, \tag{5}$$

where $e_1^{(n)}$ is the derivative with order-n, $c_{n-i} = \sum_{l_1, \cdots, l_{n-i} \in \mathrm{I}} \lambda_{l_1} \cdots \lambda_{l_{n-i}}$ $i = 1, 2, \cdots, n$, and the index set I is $\{1, 2, \cdots, n\}$. Furthermore, we say that the ZD model can solve the problem of the tracking control of the nonlinear system, if there exists a number n, such that its corresponding ZD model with order-n (4) contains the system controller $\mathbf{u}(t)$.

In order to develop the validity condition of the ZD model for solving the tracking control problem of the nonlinear problem, we present the iterative definition of the Lie derivative with order-l of $h(\mathbf{x})$ with respect to vector function $\mathbf{f}(\mathbf{x})$ as follows:

$$L_{\mathbf{f}}^{l}h(\mathbf{x}) = \left(\frac{\partial L_{\mathbf{f}}^{l-1}h(\mathbf{x})}{\partial \mathbf{x}}\right)^{\mathrm{T}} \mathbf{f}(\mathbf{x}), \tag{6}$$

where $L_{\mathbf{f}}^{0}h(\mathbf{x}) = h(\mathbf{x})$. In addition, the Lie derivative of $L_{\mathbf{f}}^{l}h(\mathbf{x})$ with respect to $G(\mathbf{x})$ is formulated as

$$L_G L_{\mathbf{f}}^{l}h(\mathbf{x}) = \left(\frac{\partial L_{\mathbf{f}}^{l}h(\mathbf{x})}{\partial \mathbf{x}}\right)^{\mathrm{T}} G(\mathbf{x}). \tag{7}$$

The nonlinear control system (1) is said to have a relative degree m in the domain $\mathbf{U}$, if for any $\mathbf{x} \in \mathbf{U}$, $L_G L_{\mathbf{f}}^{l}h(\mathbf{x}) = 0, 0 \le l < m-1$, and $L_G L_{\mathbf{f}}^{m-1}h(\mathbf{x})$ is not a zero constant function.

Theorem 1. *Suppose that the $\mathbf{f}(\mathbf{x})$, $G(\mathbf{x})$ and $h(\mathbf{x})$ are smooth maps. If the nonlinear control system (1) has a relative degree n in the domain $\mathbf{U}$, then there exists a ZD model with order-n, which can be used to design a controller $\mathbf{u}(t)$ to tracking the desired path y_d.*

Proof of Theorem 1. By the states dynamic equation in the nonlinear control system (1), we calculate the derivative of output function y as follows:

$$\dot{y} = \frac{\partial h(\mathbf{x})}{\partial \mathbf{x}}\dot{\mathbf{x}} = \frac{\partial h(\mathbf{x})}{\partial \mathbf{x}}(\mathbf{f}(\mathbf{x}) + G(\mathbf{x})\mathbf{u}(t)) = L_{\mathbf{f}}^{1}h(\mathbf{x}) + L_G L_{\mathbf{f}}^{0}h(\mathbf{x})\mathbf{u}(t). \tag{8}$$

Since the nonlinear control system (1) has a relative degree n in the domain $\mathbf{U}$, the second term $L_G L_{\mathbf{f}}^{0}h(\mathbf{x})\mathbf{u}(t)$ is equal to zero. Then, we obtain $\dot{y} = L_{\mathbf{f}}^{1}h(\mathbf{x})$. We continue to calculate the second order derivative of output function y as follows:

$$\ddot{y} = \frac{\partial L_{\mathbf{f}}^{1}h(\mathbf{x})}{\partial \mathbf{x}}\dot{\mathbf{x}} = \frac{\partial L_{\mathbf{f}}^{1}h(\mathbf{x})}{\partial \mathbf{x}}(\mathbf{f}(\mathbf{x}) + G(\mathbf{x})\mathbf{u}(t)) = L_{\mathbf{f}}^{2}h(\mathbf{x}) + L_G L_{\mathbf{f}}^{1}h(\mathbf{x})\mathbf{u}(t). \tag{9}$$

Since the nonlinear control system (1) has a relative degree n in the domain $\mathbf{U}$, the second term $L_G L_{\mathbf{f}}^{1}h(\mathbf{x})\mathbf{u}(t)$ is equal to zero. Then, we obtain $\dot{y} = L_{\mathbf{f}}^{2}h(\mathbf{x})$. For $l = 3, \cdots, n$, we similarly calculate the order-lth derivative $y^{(l)}$ of output function y as

$$y^{(l)} = L_{\mathbf{f}}^{l}h(\mathbf{x}) + L_G L_{\mathbf{f}}^{l-1}h(\mathbf{x})\mathbf{u}(t). \tag{10}$$

Since the nonlinear control system (1) has a relative degree n in the domain $\mathbf{U}$, we obtain the following equations:

$$y^{(l)} = L_{\mathbf{f}}^{l}h(\mathbf{x}), l = 3, \cdots, n-1, \tag{11}$$

and

$$y^{(n)} = L_{\mathbf{f}}^{n}h(\mathbf{x}) + L_G L_{\mathbf{f}}^{n-1}h(\mathbf{x})\mathbf{u}(t). \tag{12}$$

By the tracking error equation $e_1 = y - y_d$, we find that $e_1^{(l)} = y^{(l)} - y_d^{(l)} = L_{\mathbf{f}}^{l}h(\mathbf{x}) - y_d^{(l)}$ for $l = 1, \cdots, n-1$, and $e_1^{(n)} = y^{(n)} - y_d^{(n)} = L_{\mathbf{f}}^{n}h(\mathbf{x}) + L_G L_{\mathbf{f}}^{n-1}h(\mathbf{x})\mathbf{u}(t) - y_d^{(n)}$. We obtain the following equation by substituting $e_1^{(l)}$ with $l = 1, \cdots, n$ into the ordinary differential equation form of the ZD model with order-n (5):

$$\begin{aligned} &L_G L_{\mathbf{f}}^{n-1}h(\mathbf{x})\mathbf{u}(t) + L_{\mathbf{f}}^{n}h(\mathbf{x}) + c_1 L_{\mathbf{f}}^{n-1}h(\mathbf{x}) + \cdots + c_{n-i}L_{\mathbf{f}}^{i}h(\mathbf{x}) + \cdots + c_n L_{\mathbf{f}}^{0}h(\mathbf{x}) \\ &- (y_d^{(n)} + c_1 y_d^{(n-1)} + \cdots + c_{n-i}y_d^{(i)} + \cdots + c_n y_d) = 0. \end{aligned} \tag{13}$$

Since Equation (13), which is developed from the ordinary differential equation form of the ZD model, contains the controller $\mathbf{u}(t)$, we conclude that this ZD model with order-n can be adopted to solve the tracking control problem of nonlinear system (1). □

From Theorem 1, we can see that the ZD model with order-n can be used to design a controller $\mathbf{u}(t)$ to track the desired path y_d, if the nonlinear control system (1) has a relative degree n in the domain $\mathbf{U}$. Moreover, when the dimension of $\mathbf{u}(t)$ is equal to 1, the controller $\mathbf{u}(t)$ is represented as the following formula by Equation (13):

$$\begin{aligned}\mathbf{u}(t) =& \frac{-1}{L_G L_{\mathbf{f}}^{n-1} h(\mathbf{x})}[(L_{\mathbf{f}}^{n} h(\mathbf{x}) + c_1 L_{\mathbf{f}}^{n-1} h(\mathbf{x}) + \cdots + c_{n-i} L_{\mathbf{f}}^{i} h(\mathbf{x}) + \cdots + c_n L_{\mathbf{f}}^{0} h(\mathbf{x})) \\ &- (y_d^{(n)} + c_1 y_d^{(n-1)} + \cdots + c_{n-i} y_d^{(i)} + \cdots + c_n y_d)].\end{aligned} \tag{14}$$

In this paper, we only consider the situation of $\dim(\mathbf{u}(t)) = 1$. Since $L_G L_{\mathbf{f}}^{n-1} h(\mathbf{x})$ is the denominator of the controller $\mathbf{u}(t)$, we may encounter the divided by zero (DBZ) problem. Therefore, we consider the following two cases for solving the tracking control problem of nonlinear system (1):

Case I: If $L_G L_{\mathbf{f}}^{n-1} h(\mathbf{x})$ is a non-zero constant or greater than zero for any $\mathbf{x} \in \mathbf{U}$, then we adopt Equation (14), which is developed from the ZD model with order-n, to solve the the tracking control problem of nonlinear system (1).

Case II: If $L_G L_{\mathbf{f}}^{n-1} h(\mathbf{x})$ is a function of n-dimension vector $\mathbf{x}$, and there exists $\mathbf{x}_s \in \mathbf{U}$ such that $L_G L_{\mathbf{f}}^{n-1} h(\mathbf{x}_s) = 0$ (DBZ problem), then we need the following zeroing-gradient dynamic (ZGD) model to solve the tracking control problem of nonlinear system (1).

In order to solve the DBZ problem in Case II, we develop the ZGD model to solve the the tracking control problem of nonlinear system (1) as follows. For convenience, we denote $[(L_{\mathbf{f}}^{n} h(\mathbf{x}) + c_1 L_{\mathbf{f}}^{n-1} h(\mathbf{x}) + \cdots + c_{n-i} L_{\mathbf{f}}^{i} h(\mathbf{x}) + \cdots + c_n L_{\mathbf{f}}^{0} h(\mathbf{x})) - (y_d^{(n)} + c_1 y_d^{(n-1)} + \cdots + c_{n-i} y_d^{(i)} + \cdots + c_n y_d)]$ as $J(\mathbf{x})$. Then, Equation (13) is reformulated as follows:

$$L_G L_{\mathbf{f}}^{n-1} h(\mathbf{x}) \mathbf{u}(t) + J(\mathbf{x}) = 0. \tag{15}$$

Let us denote $L_G L_{\mathbf{f}}^{n-1} h(\mathbf{x}) \mathbf{u}(t) + J(\mathbf{x})$ as $\Phi(\mathbf{x}, \mathbf{u})$. In order to construct the ZGD model, we define the energy function as $\varepsilon(t) = |\Phi(\mathbf{x}, \mathbf{u})|^2/2$. By using the gradient dynamics (GD) design formula, we obtain the ZGD model for solving the tracking control problem of nonlinear system (1) as follows:

$$\dot{\mathbf{u}} = -\eta \frac{\partial \varepsilon(t)}{\partial \mathbf{u}} = -\eta L_G L_{\mathbf{f}}^{n-1} h(\mathbf{x})(L_G L_{\mathbf{f}}^{n-1} h(\mathbf{x}) \mathbf{u} + J(\mathbf{x})), \tag{16}$$

where $\eta \in \mathbb{R}^+$ is the GD design parameter. By combining the ZGD model Equation (16) and the states equations in nonlinear system (1), we can obtain the controller $\mathbf{u}(t)$ for solving the tracking control problem of nonlinear system (1).

2.2. The Single-Link Manipulator with Flexible Joint Dynamic System

To tackle the problem of tracking control, the dynamic model of the single-link manipulator with flexible joints [41] (as shown in Figure 2) is used as a simulation model to analyze the method we proposed. From Figure 2, we know that the manipulator has two degrees of freedom, where x_1 represents the rotation between the manipulator and the base, and x_3 represents the movement relationship between the end-effector and the manipulator. Meanwhile, its model can be described as follows:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \frac{K_s}{J_h} x_3 - \frac{K_m^2 K_g^2}{R_m J_h} x_2 + \frac{K_m K_g}{R_m J_h} u, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = -\frac{K_s}{J_h} x_3 + \frac{K_m^2 K_g^2}{R_m J_h} x_2 - \frac{K_m K_g}{R_m J_h} u - \frac{K_s}{J_l} x_3 + \frac{mgh}{J_l} \sin(x_1 + x_3), \end{cases} \tag{17}$$

where x_1,x_2,x_3, and x_4 are elements of the state vector, and x_2 and x_4 are the time derivative for x_1 and x_3. Moreover, $u \in \mathbb{R}$ is the control input. The values of parameters $K_s > 0 \in \mathbb{R}$, $J_h > 0 \in \mathbb{R}$, $J_l > 0 \in \mathbb{R}$, $K_m > 0 \in \mathbb{R}$, $K_g > 0 \in \mathbb{R}$, $R_m > 0 \in \mathbb{R}$, $m > 0 \in \mathbb{R}$, $g < 0 \in \mathbb{R}$, and $h > 0 \in \mathbb{R}$ of the single-link manipulator with a flexible joint can be found in Table 1 [41]. We also denoted y as the system output, and the desired output is denoted as y_d. Therefore, the task of the tracking control problem is to design a controller u to track the desired trajectory y_d, and the tracking error e_1 can be written as y-y_d, and it approaches zero as time t evolves. In addition, we correspond the dynamic model (17) of a single link manipulator with a flexible joints with the generalized nonlinear system (1) by the following rewriting vectors:

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} x_2 \\ \frac{K_s}{J_h} x_3 - \frac{K_m^2 K_g^2}{R_m J_h} x_2 \\ x_4 \\ -\frac{K_s}{J_h} x_3 + \frac{K_m^2 K_g^2}{R_m J_h} x_2 - \frac{K_s}{J_l} x_3 + \frac{mgh}{J_l} \sin(x_1 + x_3) \end{bmatrix}, G(\mathbf{x}) = \begin{bmatrix} 0 \\ \frac{K_m K_g}{R_m J_h} \\ 0 \\ -\frac{K_m K_g}{R_m J_h} \end{bmatrix}. \tag{18}$$

In summary, by the serviceability conditions of different models, we can see that the ZD model can be adopted to deal with the linear output situation, while the ZGD model can be adopted to tackle the nonlinear output situation for solving the tracking control problem of nonlinear system (1), especially the dynamic system (17) of the single-link manipulator with flexible joints.

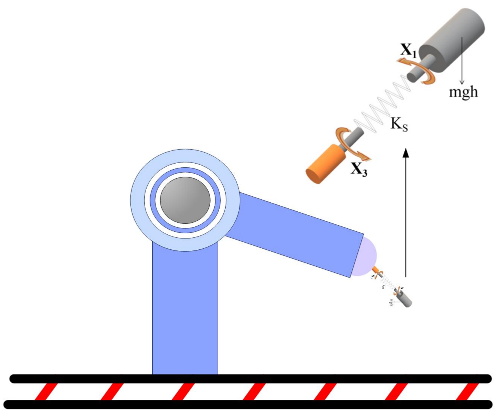


Figure 2. The model of the single-link manipulator with flexible joints, with key parameters.

Table 1. Parameters value of the single-link with a flexible joint manipulator.

Variable Name	Parameter	Value
Spring Stiffness	K_s	1.61 [N/m]
Inertia of hub	J_h	0.0021 [Kgm^2]
Load Inertia	J_l	0.0059 [Kgm^2]
Motor Const.	K_m	0.00767 [N/rad/s]
Gear Ratio	K_g	70
Motor Resist.	R_m	2.6 [Ω]
Link Mass	m	0.403 [Kg]
Grav. Const.	g	−9.81 [N/m]
Height of C.M.	h	0.06 [m]

3. Tracking Control of Single-Link Manipulator with Flexible Joints System

In this section, we construct the ZD model for the tracking control system with a single linear output, and the ZGD model for the tracking control system with a single nonlinear output of the single-link manipulator with flexible joints.

3.1. Tracking Control with Single Linear Output of Single-Link Manipulator with Flexible Joints

Based on the dynamic system of the single-link manipulator with flexible joints, we set the output $y_1 = h_1(\mathbf{x}) = x_1 + x_3$, and the desired path as y_{1d}, where y_{1d} is differentiable with order-6. In order to track control this system, we define the tracking error as $e_{lo1} = y_1 - y_{1d}$.

Next, we need to check if there exists a ZD model with a certain order, which can be adopted to solve the above tracking control problem. According to Theorem 1, we should calculate the relative degree of the tracking control system with output $y_1 = h_1(\mathbf{x}) = x_1 + x_3$ of the single link manipulator with flexible joints. By the definition of the Lie derivative with order-l of $h(\mathbf{x})$ with respect to $\mathbf{f}(\mathbf{x})$ and the Lie derivative of $L^l_{\mathbf{f}} h(\mathbf{x})$ with respect to $G(\mathbf{x})$, we find that $L^0_{\mathbf{f}} h_1(\mathbf{x}) = x_1 + x_3$, $L_G L^0_{\mathbf{f}} h_1(\mathbf{x}) = 0$, $L^1_{\mathbf{f}} h_1(\mathbf{x}) = x_2 + x_4$, $L_G L^1_{\mathbf{f}} h_1(\mathbf{x}) = 0$, $L^2_{\mathbf{f}} h_1(\mathbf{x}) = -(K_s x_3)/J_l + (mgh\sin(x_1 + x_3))/J_l$, $L_G L^2_{\mathbf{f}} h_1(\mathbf{x}) = 0$, and $L^3_{\mathbf{f}} h_1(\mathbf{x}) = -(K_s x_4)/J_l + (mgh(x_2 + x_4)\cos(x_1 + x_3))/J_l$. Then, the Lie derivative of $L^3_{\mathbf{f}} h_1(\mathbf{x})$ with respect to $G(\mathbf{x})$ is explicitly calculated as follows:

$$L_G L^3_{\mathbf{f}} h_1(\mathbf{x}) = \begin{bmatrix} -\frac{mgh}{J_l}(x_2 + x_4)\sin(x_1 + x_3) \\ \frac{mgh}{J_l}\cos(x_1 + x_3) \\ -\frac{mgh}{J_l}(x_2 + x_4)\sin(x_1 + x_3) \\ -\frac{K_s}{J_l} + \frac{mgh}{J_l}\cos(x_1 + x_3) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 \\ \frac{K_m K_g}{R_m J_h} \\ 0 \\ -\frac{K_m K_g}{R_m J_h} \end{bmatrix} = \frac{K_s K_m K_g}{R_m J_h J_l} \neq 0. \tag{19}$$

On the basis of the relative degree definition, we discover that the tracking control system with a single linear output of the single-link manipulator with flexible joints has a relative degree of 4 in the domain $\mathbb{R}^4$. Therefore, there exists a ZD model with order-4, which can be used to design a controller $u(t)$ to track the desired path y_{1d}, based on the result of Theorem 1. In order to adopt the Equation (13) for computing the controller $u(t)$, we calculate the Lie derivative $L^4_{\mathbf{f}} h_1(\mathbf{x})$ as follows:

$$\begin{aligned} L^4_{\mathbf{f}} h_1(\mathbf{x}) &= \begin{bmatrix} -\frac{mgh}{J_l}(x_2 + x_4)\sin(x_1 + x_3) \\ \frac{mgh}{J_l}\cos(x_1 + x_3) \\ -\frac{mgh}{J_l}(x_2 + x_4)\sin(x_1 + x_3) \\ -\frac{K_s}{J_l} + \frac{mgh}{J_l}\cos(x_1 + x_3) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} x_2 \\ \frac{K_s}{J_h}x_3 - \frac{K_m^2 K_g^2}{R_m J_h}x_2 \\ x_4 \\ -\frac{K_s}{J_h}x_3 + \frac{K_m^2 K_g^2}{R_m J_h}x_2 - \frac{K_s}{J_l}x_3 + \frac{mgh}{J_l}\sin(x_1 + x_3) \end{bmatrix} \\ &= -\frac{mgh}{J_l}(x_2 + x_4)^2\sin(x_1 + x_3) + (\frac{K_s^2}{J_h J_l} + \frac{K_s^2}{J_l^2})x_3 - \frac{K_s mgh}{J_l^2}\sin(x_1 + x_3) \\ &\quad - \frac{K_m^2 K_g^2 K_s}{R_m J_h J_l}x_2 - \frac{K_s mgh}{J_l^2}x_3\cos(x_1 + x_3) + \frac{(mgh)^2}{J_l^2}\sin(x_1 + x_3)\cos(x_1 + x_3). \end{aligned} \tag{20}$$

By using the Equation (13), we obtain the controller of the tracking control system with a single linear output of the single-link manipulator with flexible joints as follows:

$$\begin{aligned} u(t) =& \frac{-1}{L_G L_{\mathbf{f}}^3 h_1(\mathbf{x})}[(L_{\mathbf{f}}^4 h_1(\mathbf{x}) + c_{lo1} L_{\mathbf{f}}^3 h_1(\mathbf{x}) + c_{lo2} L_{\mathbf{f}}^2 h_1(\mathbf{x}) + c_{lo3} L_{\mathbf{f}}^1 h_1(\mathbf{x}) + c_{lo4} L_{\mathbf{f}}^0 h_1(\mathbf{x})) \\ & - (y_{1d}^{(4)} + c_{lo1} y_{1d}^{(3)} + c_{lo2} y_{1d}^{(2)} + c_{lo3} y_{1d}^{(1)} + c_{lo4} y_{1d})] \\ =& - \frac{R_m J_h J_l}{K_s K_m K_g}[-\frac{mgh}{J_l}(x_2+x_4)^2 \sin(x_1+x_3) + (\frac{K_s^2}{J_h J_l} + \frac{K_s^2}{J_l^2})x_3 - \frac{K_s mgh}{J_l^2}\sin(x_1+x_3) \\ & - \frac{K_m^2 K_g^2 K_s}{R_m J_h J_l} x_2 - \frac{K_s mgh}{J_l^2} x_3 \cos(x_1+x_3) + \frac{(mgh)^2}{J_l^2}\sin(x_1+x_3)\cos(x_1+x_3) \\ & + c_{lo1}(\frac{mgh}{J_l}(x_2+x_4)\cos(x_1+x_3) - \frac{K_s}{J_l}x_4) + c_{lo2}(\frac{mgh}{J_l}\sin(x_1+x_3) - \frac{K_s}{J_l}x_3) \\ & + c_{lo3}(x_2+x_4) + c_{lo4}(x_1+x_3) - (y_{1d}^{(4)} + c_{lo1} y_{1d}^{(3)} + c_{lo2} y_{1d}^{(2)} + c_{lo3} y_{1d}^{(1)} + c_{lo4} y_{1d})] \\ =& - \frac{R_m J_h J_l}{K_s K_m K_g}[-\frac{mgh}{J_l}(x_2+x_4)^2 \sin(x_1+x_3) + (\frac{K_s^2}{J_h J_l} + \frac{K_s^2}{J_l^2} - \frac{c_{lo2} K_s}{J_l})x_3 + c_{lo4} x_1 \\ & + (\frac{c_{lo2} mgh}{J_l} - \frac{K_s mgh}{J_l^2})\sin(x_1+x_3) + (c_{lo3} - \frac{K_m^2 K_g^2 K_s}{R_m J_h J_l})x_2 + (c_{lo3} - \frac{c_{lo1} K_s}{J_l})x_4 \\ & + \frac{mgh}{J_l}(c_{lo1} x_2 + c_{lo1} x_4 - \frac{K_s}{J_l} x_3)\cos(x_1+x_3) + \frac{(mgh)^2}{J_l^2}\sin(x_1+x_3)\cos(x_1+x_3) \\ & - (y_{1d}^{(4)} + c_{lo1} y_{1d}^{(3)} + c_{lo2} y_{1d}^{(2)} + c_{lo3} y_{1d}^{(1)} + c_{lo4} y_{1d})], \end{aligned} \tag{21}$$

where $c_{lo1} = \lambda_{lo1} + \lambda_{lo2} + \lambda_{lo3} + \lambda_{lo4}$, $c_{lo2} = \lambda_{lo1}\lambda_{lo2} + \lambda_{lo1}\lambda_{lo3} + \lambda_{lo1}\lambda_{lo4} + \lambda_{lo2}\lambda_{lo3} + \lambda_{lo2}\lambda_{lo4} + \lambda_{lo3}\lambda_{lo4}$, $c_{lo3} = \lambda_{lo1}\lambda_{lo2}\lambda_{lo3} + \lambda_{lo1}\lambda_{lo2}\lambda_{lo4} + \lambda_{lo1}\lambda_{lo3}\lambda_{lo4} + \lambda_{lo2}\lambda_{lo3}\lambda_{lo4}$, $c_{lo4} = \lambda_{lo1}\lambda_{lo2}\lambda_{lo3}\lambda_{lo4}$, and $\lambda_{lo1}, \lambda_{lo2}, \lambda_{lo3}, \lambda_{lo4} \in \mathbb{R}^+$ are ZD model parameters. Since $L_G L_{\mathbf{f}}^3 h_1(\mathbf{x}) = K_s K_m K_g / R_m J_h J_l$ is a nonlinear constant for any $t \in [0, +\infty)$, we can directly adopt the ZD controller (21) to solve the tracking control problem with a single linear output of the single-link manipulator with flexible joints. It is evident that the output of this tracking control problem is nonlinear.

3.2. Tracking Control with Single Nonlinear Output of Single-Link Manipulator with Flexible Joints

Based on the dynamic system of the single link manipulator with flexible joints, we set the output $y_2 = h_2(\mathbf{x}) = x_1^2 + x_3^2$, and the desired path as y_{2d}, where y_{2d} is differentiable with order-4. In order to track control this system, we define the tracking error as $e_{no1} = y_2 - y_{2d}$.

Next, we need to check if there exists a ZD model with a certain order, which can be adopted to solve the tracking control problem with single nonlinear output in this subsection. According to Theorem 1, we should calculate the relative degree of the tracking control system with output $y_2 = h_2(\mathbf{x}) = x_1^2 + x_3^2$ of the single-link manipulator with flexible joints. By the definition of the Lie derivative with order-l of $h(\mathbf{x})$ with respect to $\mathbf{f}(\mathbf{x})$ and the Lie derivative of $L_{\mathbf{f}}^l h(\mathbf{x})$ with respect to $G(\mathbf{x})$, we find that $L_{\mathbf{f}}^0 h_2(\mathbf{x}) = x_1^2 + x_3^2$, $L_G L_{\mathbf{f}}^0 h_2(\mathbf{x}) = 0$, $L_{\mathbf{f}}^1 h_2(\mathbf{x}) = x_1 x_2 + x_3 x_4$. Then, the Lie derivative of $L_{\mathbf{f}}^1 h_2(\mathbf{x})$ with respect to $G(\mathbf{x})$ is explicitly calculated as follows:

$$L_G L_{\mathbf{f}}^1 h_2(\mathbf{x}) = \begin{bmatrix} x_2 \\ x_1 \\ x_4 \\ x_3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 \\ \frac{K_m K_g}{R_m J_h} \\ 0 \\ -\frac{K_m K_g}{R_m J_h} \end{bmatrix} = \frac{K_m K_g}{R_m J_h}(x_1 - x_3), \tag{22}$$

which is not a zero constant function. On the basis of the relative degree definition, we find that the tracking control system with a single nonlinear output of the single-link manipulator with flexible joints has a relative degree of 2 in the domain $\mathbb{R}^4$. Therefore, there exists a ZD model with order-2, which can be used to design a controller $u(t)$ to track the desired path y_{2d}, based on the result of Theorem 1. In order to adopt Equation (13) for computing the controller $u(t)$, we calculate the Lie derivative $L_{\mathbf{f}}^2 h_2(\mathbf{x})$ as follows:

$$L_{\mathbf{f}}^2 h_2(\mathbf{x}) = \begin{bmatrix} x_2 \\ x_1 \\ x_4 \\ x_3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} x_2 \\ \frac{K_s}{J_h} x_3 - \frac{K_m^2 K_g^2}{R_m J_h} x_2 \\ x_4 \\ -\frac{K_s}{J_h} x_3 + \frac{K_m^2 K_g^2}{R_m J_h} x_2 - \frac{K_s}{J_l} x_3 + \frac{mgh}{J_l} \sin(x_1 + x_3) \end{bmatrix}$$
$$= x_2^2 + x_4^2 + \frac{K_s}{J_h} x_1 x_3 - \frac{K_m^2 K_g^2}{R_m J_h} x_1 x_2 - (\frac{K_s}{J_h} + \frac{K_s}{J_l}) x_3^2 + \frac{K_m^2 K_g^2}{R_m J_h} x_2 x_3$$
$$+ \frac{mgh}{J_l} x_3 \sin(x_1 + x_3). \tag{23}$$

Substituting the above Lie derivatives into Equation (14), we obtain the following equation for the controller $u(t)$:

$$L_G L_{\mathbf{f}}^1 h(\mathbf{x}) u(t) + L_{\mathbf{f}}^2 h_2(\mathbf{x}) + c_{no1} L_{\mathbf{f}}^1 h_2(\mathbf{x}) + c_{no2} L_{\mathbf{f}}^0 h(\mathbf{x}) - (y_{2d}^{(2)} + c_{no1} y_{2d}^{(1)} + c_{no2} y_{2d})$$
$$= \frac{K_m K_g}{R_m J_h}(x_1 - x_3) u + x_2^2 + x_4^2 + \frac{K_s}{J_h} x_1 x_3 + (c_{no1} - \frac{K_m^2 K_g^2}{R_m J_h}) x_1 x_2 - (\frac{K_s}{J_h} + \frac{K_s}{J_l}) x_3^2 + \frac{K_m^2 K_g^2}{R_m J_h} x_2 x_3 \tag{24}$$
$$+ \frac{mgh}{J_l} x_3 \sin(x_1 + x_3) + c_{no1} x_3 x_4 + c_{no2}(x_1^2 + x_3^2) - \ddot{y}_{2d} + c_{no1} \dot{y}_{2d} + c_{no2} y_{2d}) = 0,$$

where $c_{no1} = \lambda_{no1} + \lambda_{no2}$, and $c_{no1} = \lambda_{no1}\lambda_{no2}$. Since there exist $\mathbf{x}_s \in \mathbb{R}^4$, such that $L_G L_{\mathbf{f}}^1 h(\mathbf{x}_s) = K_m K_g (x_1 - x_3)/R_m J_h = 0$, we need to construct the ZGD model to tackle the DBZ problem. Let us denote

$$J_{no}(\mathbf{x}) = x_2^2 + x_4^2 + \frac{K_s}{J_h} x_1 x_3 + (c_{no1} - \frac{K_m^2 K_g^2}{R_m J_h}) x_1 x_2 - (\frac{K_s}{J_h} + \frac{K_s}{J_l}) x_3^2 + \frac{K_m^2 K_g^2}{R_m J_h} x_2 x_3$$
$$+ \frac{mgh}{J_l} x_3 \sin(x_1 + x_3) + c_{no1} x_3 x_4 + c_{no2}(x_1^2 + x_3^2) - \ddot{y}_{2d} + c_{no1} \dot{y}_{2d} + c_{no2} y_{2d}).$$

Then, the controller equation of $u(t)$ (24) is rewritten as follows:

$$\frac{K_m K_g}{R_m J_h}(x_1 - x_3) u + J_{no}(\mathbf{x}) = 0, \tag{25}$$

and we denote $K_m K_g (x_1 - x_3)/R_m J_h u + J_{no}(\mathbf{x})$ as $\Phi_{no}(\mathbf{x}, \mathbf{u})$. To develop the ZGD model for solving the tracking control problem with single nonlinear output of the single-link manipulator with a flexible joints, we define the energy function as $\varepsilon_{no}(t) = |\Phi_{no}(\mathbf{x}, \mathbf{u})|^2/2$. By using the gradient design formula, we obtain the ZGD model for solving the tracking control problem with a single nonlinear output of the single-link manipulator with flexible joints as follows:

$$\dot{u} = -\eta_{no} \frac{\partial \varepsilon_{no}(t)}{\partial u} = -\eta_{no} \frac{K_m K_g}{R_m J_h}(x_1 - x_3)(\frac{K_m K_g}{R_m J_h}(x_1 - x_3) u + J_{no}(\mathbf{x})), \tag{26}$$

where $\eta_{no} > 0 \in \mathbb{R}$ is the GD design parameter. By combining the ZGD model Equation (26) and the dynamic system (17) of the single link manipulator with flexible joints, we can obtain the controller $u(t)$ for solving the tracking control problem of the single-link manipulator with flexible joints.

4. Convergence Analyses

In this section, we analyze the convergence property of the ZD and ZGD models for solving the tracking control problem with single linear or nonlinear outputs of the single-link manipulator with flexible joints.

Firstly, we give the convergence property of the ZD model for solving the tracking control problem with a single linear output of the single link manipulator with flexible joints by the following theorem.

Theorem 2. *Suppose that the desired path y_{1d} is differentiable with order-5. The ZD controller $u(t)$ (21) for solving the tracking control problem of the single-link manipulator with flexible joints (17) makes the linear output $y_1 = x_1 + x_3$ globally exponentially converge to the desired path y_{1d}.*

Proof of Theorem 2. When we construct the controller $u(t)$ (21) for solving the tracking control problem with linear output $y_1 = x_1 + x_3$ of the single-link manipulator with flexible joints, we have shown that the tracking control system has a relative degree of 4 in the domain $\mathbb{R}^4$. Then, since the controller $u(t)$ makes the equation the following holds based on Equation (21):

$$\begin{aligned}&L_G L_{\mathbf{f}}^3 h_1(\mathbf{x})u(t) + (L_{\mathbf{f}}^4 h_1(\mathbf{x}) + c_{lo1}L_{\mathbf{f}}^3 h_1(\mathbf{x}) + c_{lo2}L_{\mathbf{f}}^2 h_1(\mathbf{x}) + c_{lo3}L_{\mathbf{f}}^1 h_1(\mathbf{x}) + c_{lo4}L_{\mathbf{f}}^0 h_1(\mathbf{x}))\\ &-(y_{1d}^{(4)} + c_{lo1}y_{1d}^{(3)} + c_{lo2}y_{1d}^{(2)} + c_{lo3}y_{1d}^{(1)} + c_{lo4}y_{1d}) = 0\end{aligned} \tag{27}$$

Then, the tracking error $e_{lo1} = y_1 - y_{1d}$ satisfies the following constant coefficient ordinary differential equation with order-4:

$$e_{lo1}^{(4)} + c_{lo1}\dddot{e}_{lo1} + c_{lo2}\ddot{e}_{lo1} + c_{lo3}\dot{e}_{lo1} + c_{lo4}e_{lo1} = 0. \tag{28}$$

The characteristic polynomial of the constant coefficient ordinary differential Equation (28) is formulated as follows:

$$z^4 + c_{lo1}z^3 + c_{lo2}z^2 + c_{lo3}z + c_{lo4} = 0. \tag{29}$$

Then, the characteristic roots of the constant coefficient ordinary differential Equation (28) are $z_1 = -\lambda_{lo1}$, $z_2 = -\lambda_{lo2}$, $z_3 = -\lambda_{lo3}$, and $z_4 = -\lambda_{lo4}$, on the basis of the definition of $c_{lo1}, c_{lo2}, c_{lo3}$, and c_{lo4} [42,43]. Then, the general solution of the constant coefficient ordinary differential Equation (28) is calculated as

$$e_{lo1}(t) = \sum_{i=1}^{m}\sum_{\lambda_{loj}\in I_i}^{n_i=|I_i|}(C_{i0} + C_{i1}t + \cdots + C_{i(n_i-1)}t^{n_i-1})\exp(-\lambda_{loi}t), \tag{30}$$

where $I = \{z_1, z_2, z_3, z_4\} = \bigcup_{i=1}^{m} I_i$ is a partition of characteristic roots set I based on the equality relation. As $t \longrightarrow \infty$, $e_{lo1}(t)$ converges to 0, for any initial error $e_{lo1}(0) = \sum_{i=1}^{m} C_{i0}$. Therefore, the ZD controller $u(t)$ (21) for solving the tracking control problem of the single-link manipulator with a flexible joints (17) makes the linear output $y_1 = x_1 + x_3$ globally exponentially converge to the desired path y_{1d}. □

Secondly, we give the tracking error bound of the ZGD model for solving the tracking control problem with a single nonlinear output of the single-link manipulator with flexible joints by the following theorem.

Theorem 3. *Suppose that the desired path y_{2d} is differentiable with order-2. Starting with the initial states vector $\mathbf{x}(0) = [x_1(0), x_2(0), x_3(0), x_4(0)]^{\mathrm{T}} \in \mathbb{R}^4$ and initial input $u(0) \in \mathbb{R}$, we then obtain the following results about the tracking error $e_{no1} = y_2 - y_{2d} = x_1^2 + x_3^2$ of the ZGD controller $u(t)$ (26) for solving the tracking control problem of the single-link manipulator with flexible joints (17).*

- *For the situation $x_1 - x_3 \neq 0$, if $0 < \sqrt{\alpha} \leq |x_1 - x_3| \leq \beta \leq \infty$, and $|\dot{u}| \leq \gamma \leq \infty$, the steady-state tracking error satisfies the following bound condition:*

$$\limsup_{t\longrightarrow\infty} |e_{no1}(t)| \leq \frac{R_m J_h \gamma\beta}{\alpha\eta_{no}k_m K_g \lambda_{no1}\lambda_{no2}}. \tag{31}$$

- *For the situation $x_1 - x_3 = 0$, the steady-state tracking error is bounded.*

Proof of Theorem 3. We consider the following situations.

(i) For the situation $x_1 - x_3 \neq 0$.

Assume that $u^*(t)$ is the theoretical solution of the ZGD model for solving the tracking control problem with a single nonlinear output of the single-link manipulator with flexible joints, and this theoretical solution satisfies $\frac{K_m K_g}{R_m J_h}(x_1 - x_3)u^* + J_{no}(\mathbf{x}) = 0$. Let us define the controller error as $\mathcal{E}(t) = u(t) - u^*(t)$. Then, the derivative of the controller error is calculated as follows:

$$\begin{aligned}\dot{\mathcal{E}} =& \dot{u} - \dot{u}^* \\ =& -\eta_{no}\frac{K_m K_g}{R_m J_h}(x_1 - x_3)(\frac{K_m K_g}{R_m J_h}(x_1 - x_3)u + J_{no}(\mathbf{x})) - \dot{u}^* \\ =& -\eta_{no}\frac{K_m K_g}{R_m J_h}(x_1 - x_3)(\frac{K_m K_g}{R_m J_h}(x_1 - x_3)u - \frac{K_m K_g}{R_m J_h}(x_1 - x_3)u^*) - \dot{u}^* \\ =& -\frac{\eta_{no}K_m^2 K_g^2}{R_m^2 J_h^2}(x_1 - x_3)^2\mathcal{E} - \dot{u}^*.\end{aligned} \tag{32}$$

By computing the derivative of the controller error, we transform the states dynamic system into the error dynamic system. In order to develop the stability and convergence property of the ZGD model, we define the Lyapunov function candidate as $\mathcal{L}(t) = \mathcal{E}^2(t)/2$, which is evidently a positive-definite function. Then, the derivative of the Lyapunov function candidate is formulated as

$$\dot{\mathcal{L}}(t) = \mathcal{E}\dot{\mathcal{E}} = -\frac{\eta_{no}K_m^2 K_g^2}{R_m^2 J_h^2}(x_1 - x_3)^2\mathcal{E}^2 - \mathcal{E}\dot{u}^*. \tag{33}$$

We analyze $\dot{\mathcal{L}}$ term by term. Since $\sqrt{\alpha} \leq |x_1 - x_3|$, we obtain that the first term of $\dot{\mathcal{L}}$ is controlled by following inequality:

$$-\frac{\eta_{no}K_m^2 K_g^2}{R_m^2 J_h^2}(x_1 - x_3)^2\mathcal{E}^2 \leq -\frac{\alpha\eta_{no}K_m^2 K_g^2}{R_m^2 J_h^2}\mathcal{E}^2. \tag{34}$$

Due to the condition $|\dot{u}| \leq \gamma$, we obtain that the second term of $\dot{\mathcal{L}}$ is controlled by following inequality:

$$-\mathcal{E}\dot{u}^* \leq |\mathcal{E}||\dot{u}^*| \leq \gamma|\mathcal{E}|. \tag{35}$$

Substituting (34) and (35) into (33), we obtain the following inequality:

$$\dot{\mathcal{L}}(t) \leq -\frac{\alpha\eta_{no}K_m^2 K_g^2}{R_m^2 J_h^2}\mathcal{E}^2 + \gamma|\mathcal{E}| = -|\mathcal{E}|(\frac{\alpha\eta_{no}K_m^2 K_g^2}{R_m^2 J_h^2}|\mathcal{E}| - \gamma). \tag{36}$$

As time t evolves, the absolute value $|\mathcal{E}|$ falls into the following three cases:

Case I: If $\alpha\eta_{no}K_m^2 K_g^2|\mathcal{E}|/R_m^2 J_h^2 - \gamma > 0$, then $\dot{\mathcal{L}}(t) < 0$. By the Lyapunov stability theorem, we find that $\mathcal{E}$ converges to zero as $t \longrightarrow \infty$.

Case II: If $\alpha\eta_{no}K_m^2 K_g^2|\mathcal{E}|/R_m^2 J_h^2 - \gamma = 0$, then $\dot{\mathcal{L}}(t) \leq 0$. When $\dot{\mathcal{L}}(t) < 0$, $\mathcal{E}$ converges to zero as $t \longrightarrow \infty$. When $\dot{\mathcal{L}}(t) = 0$, $|\mathcal{E}|$ stays on the ball with radius $\gamma R_m^2 J_h^2/\alpha\eta_{no}K_m^2 K_g^2$.

Case III: If $\alpha\eta_{no}K_m^2 K_g^2|\mathcal{E}|/R_m^2 J_h^2 - \gamma > 0$, then $\dot{\mathcal{L}}(t) \leq L_0$, where L_0 is a positive constant. When $\dot{\mathcal{L}}(t) \leq 0$, $\mathcal{E}$ will never go outside the ball with radius $\gamma R_m^2 J_h^2/\alpha\eta_{no}K_m^2 K_g^2$. When $\dot{\mathcal{L}}(t) > 0$, the function $\mathcal{L}(t)$ is an increase function, which also makes $|\mathcal{E}|$ increase. Then, there exists a time instant t_z such that $\alpha\eta_{no}K_m^2 K_g^2|\mathcal{E}|/R_m^2 J_h^2 - \gamma = 0$, which goes back to Case II.

By the above analyses, we obtain the steady-state of the controller error of the ZGD controller $u(t)$ (26) for solving the tracking control problem of the single-link manipulator with flexible joints (17) as follows:

$$\limsup_{t \longrightarrow \infty} |\mathcal{E}| \leq \frac{R_m^2 J_h^2 \gamma}{\alpha\eta_{no}k_m^2 K_g^2}. \tag{37}$$

Based on the construction process of the ZGD controller $u(t)$ (26) for solving the tracking control problem of the single-link manipulator with flexible joints (17), we have

$$\begin{aligned}\Phi_{no}(\mathbf{x},\mathbf{u}) =& \ddot{e}_{no1} + (\lambda_{no1} + \lambda_{no2})\dot{e}_{no1} + \lambda_{no1}\lambda_{no2}e_{no1} \\ =& \frac{K_m K_g}{R_m J_h}(x_1 - x_3)u + J_{no}(\mathbf{x}) \\ =& \frac{K_m K_g}{R_m J_h}(x_1 - x_3)u - \frac{K_m K_g}{R_m J_h}(x_1 - x_3)u^* = \frac{K_m K_g}{R_m J_h}(x_1 - x_3)\mathcal{E}.\end{aligned} \tag{38}$$

Then, the tracking error of the ZGD model for solving the tracking control problem of the single-link manipulator with flexible joints (17) satisfies the following ordinary differential inequality by Equation (38) and $|x_1 - x_3| \leq \beta$:

$$\begin{aligned}-\frac{R_m J_h \gamma\beta}{\alpha\eta_{no}k_m K_g} \leq -\frac{K_m K_g}{R_m J_h}|x_1 - x_3||\mathcal{E}| \leq& \ddot{e}_{no1} + (\lambda_{no1} + \lambda_{no2})\dot{e}_{no1} + \lambda_{no1}\lambda_{no2}e_{no1} \\ \leq& \frac{K_m K_g}{R_m J_h}|x_1 - x_3||\mathcal{E}| \leq \frac{R_m J_h \gamma\beta}{\alpha\eta_{no}k_m K_g}.\end{aligned} \tag{39}$$

By the Gronwall inequality, when $t \geq t_z$, the tracking error of the ZGD model for solving the tracking control problem of the single-link manipulator with flexible joints (17) is controlled by the following inequalities:

- If $\lambda_{no1} \neq \lambda_{no2}$, then

$$\begin{aligned}&-\hat{c}_1\exp(-\lambda_{no1}t) - \hat{c}_2\exp(-\lambda_{no2}t) - \frac{R_m J_h \gamma\beta}{\alpha\eta_{no}k_m K_g\lambda_{no1}\lambda_{no2}} \\ \leq& e_{no1} \leq \hat{c}_1\exp(-\lambda_{no1}t) + \hat{c}_2\exp(-\lambda_{no2}t) + \frac{R_m J_h \gamma\beta}{\alpha\eta_{no}k_m K_g\lambda_{no1}\lambda_{no2}};\end{aligned} \tag{40}$$

- if $\lambda_{no1} = \lambda_{no2}$, then

$$\begin{aligned}&-(\hat{c}_1 + \hat{c}_2 t)\exp(-\lambda_{no1}t) - \frac{R_m J_h \gamma\beta}{\alpha\eta_{no}k_m K_g\lambda_{no1}^2} \\ \leq& e_{no1} \leq (\hat{c}_1 + \hat{c}_2 t)\exp(-\lambda_{no1}t) + \frac{R_m J_h \gamma\beta}{\alpha\eta_{no}k_m K_g\lambda_{no1}^2},\end{aligned} \tag{41}$$

where $\hat{c}_1, \hat{c}_2$ are constants. Based on the property of exponential function, we have the following inequality:

$$|e_{no1}| \leq \begin{cases} |\bar{c}_1|\exp(-\lambda_{no1}t) + |\bar{c}_2|\exp(-\lambda_{no2}t) + \dfrac{R_m J_h \gamma\beta}{\alpha\eta_{no}k_m K_g\lambda_{no1}lambda_{no2}}, \text{if } \lambda_{no1} \neq \lambda_{no2}, \\ (|\bar{c}_1| + |\bar{c}_1|t)\exp(-\lambda_{no1}t) + \dfrac{R_m J_h \gamma\beta}{\alpha\eta_{no}k_m K_g\lambda_{no1}^2}, \text{if } \lambda_{no1} = \lambda_{no2}, \end{cases} \tag{42}$$

Then, we obtain the steady-state tracking error of the ZGD model for solving the tracking control problem of the single-link manipulator with flexible joints (17) as follows:

$$\limsup_{t\longrightarrow\infty}|e_{no1}(t)| \leq \frac{R_m J_h \gamma\beta}{\alpha\eta_{no}k_m K_g\lambda_{no1}\lambda_{no2}}. \tag{43}$$

(ii) For the situation $x_1 - x_3 = 0$.

Since the ZGD controller (26) is used for solving the tracking control problem of the single-link manipulator with flexible joints (17), it is easy to draw a conclusion that $\lim_{t\to t_s}\dot{u}$ = $\lim_{x_1-x_3\to 0}\dot{u}$ = 0 given $\dot{u} = -\eta_{no}K_m K_g(x_1 - x_3)(K_m K_g(x_1 - x_3)u/R_m J_h + J_{no}(x))/R_m J_h$. Additionally, it implies that $u(t_{s-})$ = $u(t_s)$ = $u(t_{s+})$ regardless of the value of the time instant $u(t_{s-})$,$u(t_s)$,$u(t_{s+})$ as they are bounded. Due to the bounded control input, the output of the tracking control problem of the single-link manipulator with flexible joints (17) is also bounded. Because the desired trajectory y_d is bounded, the tracking error will be limited at time instant t_{s-}, t_s, t_{s+}. Even after passing the DBZ point, the tracking error will again converge to the error bound, which means that the tracking control problem of the single link manipulator with flexible joints (17) equipped with the ZGD controller (26) finally overcomes the DBZ problem. □

Theorem 4. *Suppose that the desired path y_{2d} is differentiable with order-2. Starting with initial states vector $\mathbf{x}(0) = [x_1(0), x_2(0), x_3(0), x_4(0)]^{\mathrm{T}} \in \mathbb{R}^4$ and initial input $u(0) \in \mathbb{R}$, we then obtain the following results about the tracking error $e_{no1} = y_2 - y_{2d} = x_1^2 + x_3^2$ of the ZGD controller $u(t)$ (26) for solving the tracking control problem of the single-link manipulator with flexible joints (17).*

- *For the situation $x_1 - x_3 \neq 0$, if $0 < \sqrt{\alpha} \leq |x_1 - x_3| \leq \beta \leq \infty$, and $|\dot{u}| \leq \gamma \leq \infty$, then, for any loosening parameter $\varepsilon \in (0, 1)$, the steady-state tracking error exponentially converges to the closed ball with radius $R_m J_h \gamma \beta / (\varepsilon \alpha \eta_{no} k_m K_g \lambda_{no1} \lambda_{no2})$.*
- *For the situation $x_1 - x_3 = 0$, the steady-state tracking error is bounded.*

Proof of Theorem 4. We consider the following situations.

(i) For the situation $x_1 - x_3 \neq 0$.

Based on the inequality (36), we obtain the following inequality for any loosening parameter $\varepsilon \in (0, 1)$:

$$\dot{\mathcal{L}}(t) \leq -\frac{(1-\varepsilon)\alpha\eta_{no}K_m^2K_g^2}{R_m^2J_h^2}\mathcal{E}^2 + (-\frac{\varepsilon\alpha\eta_{no}K_m^2K_g^2}{R_m^2J_h^2}\mathcal{E}^2 + \gamma|\mathcal{E}|). \tag{44}$$

Since the first term of the right side of inequality (44) $(-(1-\varepsilon)\alpha\eta_{no}K_m^2K_g^2\mathcal{E}^2)/(R_m^2J_h^2)$ is always less than 0, the residual error $\mathcal{E}$ falls into the following two situations as time evolving:

- If $-(\varepsilon\alpha\eta_{no}K_m^2K_g^2\mathcal{E}^2)/(R_m^2J_h^2) + \gamma|\mathcal{E}| \leq 0$ (i.e., $|\mathcal{E}| \geq (R_m^2J_h^2\gamma)/(\varepsilon\alpha\eta_{no}K_m^2K_g^2)$), then we obtain the following inequality:

$$\begin{aligned}\dot{\mathcal{L}}(t) &\leq -\frac{(1-\varepsilon)\alpha\eta_{no}K_m^2K_g^2}{R_m^2J_h^2}\mathcal{E}^2 + (-\frac{\varepsilon\alpha\eta_{no}K_m^2K_g^2}{R_m^2J_h^2}\mathcal{E}^2 + \gamma|\mathcal{E}|) \\ &\leq -\frac{(1-\varepsilon)\alpha\eta_{no}K_m^2K_g^2}{R_m^2J_h^2}\mathcal{E}^2 \\ &= -\frac{2(1-\varepsilon)\alpha\eta_{no}K_m^2K_g^2}{R_m^2J_h^2}\mathcal{L}(t).\end{aligned} \tag{45}$$

By using the Gronwall inequality, we obtain

$$\mathcal{L}(t) \leq \mathcal{L}(0)\exp(-\frac{2(1-\varepsilon)\alpha\eta_{no}K_m^2K_g^2}{R_m^2J_h^2}t),$$

and

$$|\mathcal{E}(t)| \leq |\mathcal{E}(0)|\exp(-\frac{(1-\varepsilon)\alpha\eta_{no}K_m^2K_g^2}{R_m^2J_h^2}t),\ t \in [0, t_c], \tag{46}$$

where the exponential convergence rate is at least $((1-\varepsilon)\alpha\eta_{no}K_m^2K_g^2)/(R_m^2J_h^2)$, and the convergence time is at least $t_c = (R_m^2J_h^2 \ln(\varepsilon\alpha\eta_{no}K_m^2K_g^2|\mathcal{E}(0)|)/(R_m^2J_h^2\gamma))/\ ((1-\varepsilon)\alpha\eta_{no}K_m^2K_g^2)$. Therefore, if $|\mathcal{E}(0)| \geq (R_m^2J_h^2\gamma)/(\varepsilon\alpha\eta_{no}K_m^2K_g^2)$, then

$$|\mathcal{E}(t)| \leq \begin{cases} |\mathcal{E}(0)|\exp(-\frac{(1-\varepsilon)\alpha\eta_{no}K_m^2K_g^2}{R_m^2J_h^2}t),\ t \in [0, t_c], \\ \frac{R_m^2J_h^2\gamma}{\varepsilon\alpha\eta_{no}K_m^2K_g^2},\ t \in [t_c, +\infty). \end{cases} \tag{47}$$

- If $-(\varepsilon\alpha\eta_{no}K_m^2K_g^2\mathcal{E}^2)/(R_m^2J_h^2)+\gamma|\mathcal{E}|\geq 0$, then $|\mathcal{E}|\leq(R_m^2J_h^2\gamma)/(\varepsilon\alpha\eta_{no}K_m^2K_g^2)$. Therefore, if $|\mathcal{E}(0)|\leq(R_m^2J_h^2\gamma)/(\varepsilon\alpha\eta_{no}K_m^2K_g^2)$, then $|\mathcal{E}(t)|\leq(R_m^2J_h^2\gamma)/(\varepsilon\alpha\eta_{no}K_m^2K_g^2)$, $t\in[0,+\infty)$.

  In summary, we obtain the exponential bound of the residual error as follows:

$$|\mathcal{E}(t)|\leq\frac{R_m^2J_h^2\gamma}{\varepsilon\alpha\eta_{no}K_m^2K_g^2},\ t\in[t_c,+\infty). \tag{48}$$

Similar to the derivation of tight error bound differential inequality (39), we obtain the following exponential error bound differential inequality:

$$-\frac{R_mJ_h\gamma\beta}{\varepsilon\alpha\eta_{no}k_mK_g}\leq\ddot{e}_{no1}+(\lambda_{no1}+\lambda_{no2})\dot{e}_{no1}+\lambda_{no1}\lambda_{no2}e_{no1}\leq\frac{R_mJ_h\gamma\beta}{\varepsilon\alpha\eta_{no}k_mK_g}. \tag{49}$$

Similar to the derivation of tight error bound (42), we obtain the following exponential error bound:

$$|e_{no1}|\leq\begin{cases}|\tilde{c}_1|\exp(-\lambda_{no1}t)+|\tilde{c}_2|\exp(-\lambda_{no2}t)+\dfrac{R_mJ_h\gamma\beta}{\varepsilon\alpha\eta_{no}k_mK_g},\text{if }\lambda_{no1}\neq\lambda_{no2},\\(|\tilde{c}_1|+|\tilde{c}_1|t)\exp(-\lambda_{no1}t)+\dfrac{R_mJ_h\gamma\beta}{\varepsilon\alpha\eta_{no}k_mK_g},\text{if }\lambda_{no1}=\lambda_{no2},\end{cases} \tag{50}$$

Then, we obtain the steady-state tracking exponential error of the ZGD model for solving the tracking control problem of the single-link manipulator with flexible joints (17) as follows:

$$\limsup_{t\longrightarrow\infty}|e_{no1}(t)|\leq\frac{R_mJ_h\gamma\beta}{\varepsilon\alpha\eta_{no}k_mK_g\lambda_{no1}\lambda_{no2}},\ t\in[t_c,+\infty). \tag{51}$$

Thus, for any loosening parameter $\varepsilon\in(0,1)$, the steady-state tracking error exponentially converges to the closed ball with radius $R_mJ_h\gamma\beta/(\varepsilon\alpha\eta_{no}k_mK_g\lambda_{no1}\lambda_{no2})$.

(ii) For the situation $x_1-x_3=0$. Based on the proof of Theorem 3, we similarly obtain the bounded property of the tracking control system. □

Remark 1. *The error bound expression for the ZGD model, $R_mJ_h\gamma\beta/(\varepsilon\alpha\eta_{no}k_mK_g\lambda_{no1}\lambda_{no2})$, reveals the ZGD model versus parameters. The parameter λ_{no1},λ_{no2} is on the denominator of the expression, which indicates that the larger the parameter, the tighter the error bound, i.e., the smaller the radius. In addition, through experiments, we further found the relationship between the ZGD model parameters λ and the convergence rate. That is, the larger the parameters, the faster the convergence rate of the ZGD model is also achieved.*

5. Simulations and Discussion

In this section, we conduct several simulations to show the validity and parameter influences of the ZD model and ZGD model for solving the tracking control problem with single linear or nonlinear outputs of the single-link manipulator with flexible joints (17). For comparison, we also conduct simulations using Gradient Dynamic (GD) to solve the tracking control problem. The simulations are carried out on the MATLAB R2021a simulation platform. The hardware environment for the manipulator systems was a desktop with an Intel® i7-12700H CPU (2.30 GHz) and 16.00 GB RAM. The simulation data were obtained by using the ode15s function in the Matlab toolbox. For the simulation purpose, we set two different desired paths as $y_{1d,s1}=\pi\sin(t)/4+\pi/4$, and $y_{1d,s2}=\sin(t)\exp(-t/20)$.

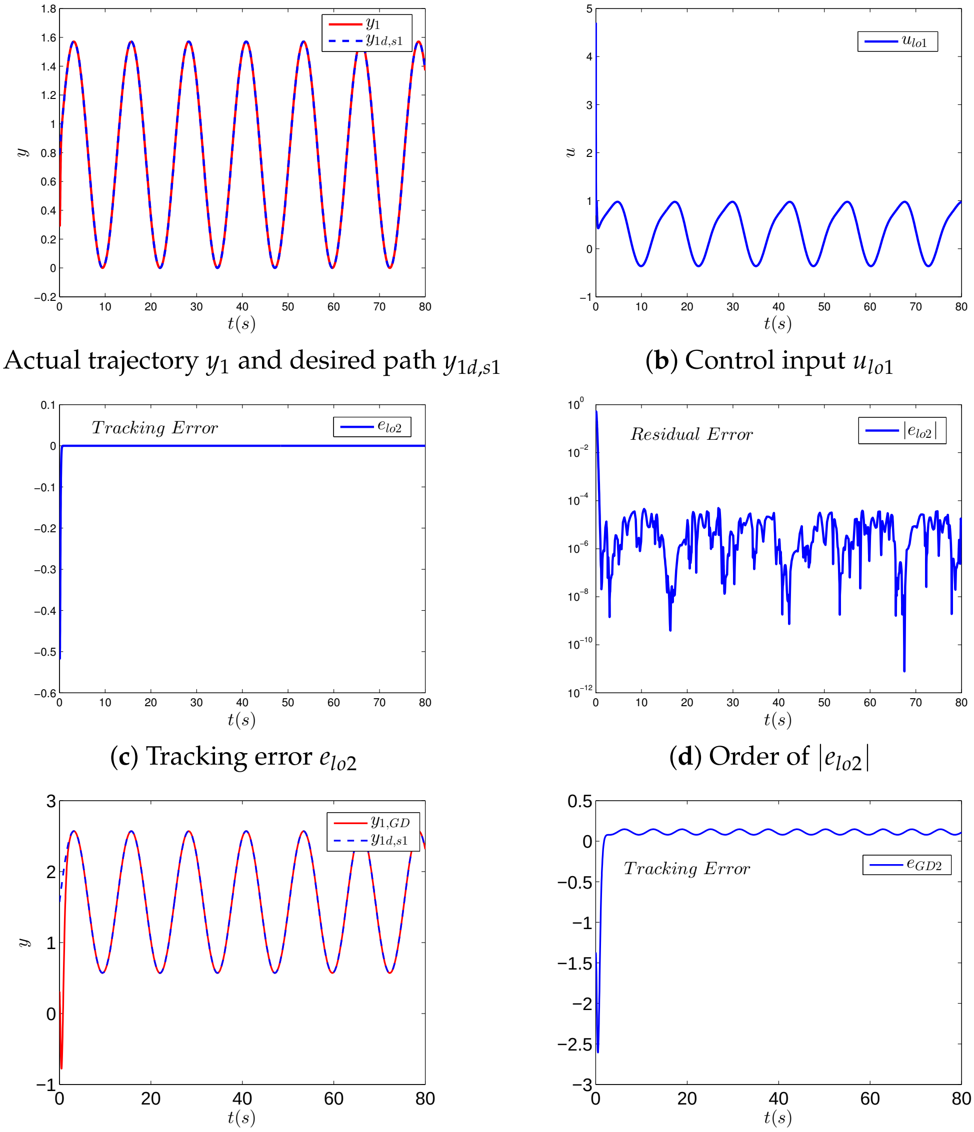
5.1. Test for Tracking Control with Single Linear Output of Single-Link Manipulator with Flexible Joints

In this subsection, we conduct two simulations to show the validity and parameter influences of the ZD model for solving the tracking control problem with a single linear output of the single link manipulator with flexible joints (17).

5.1.1. Test for Tracking Control with Single Linear Output of Single-Link Manipulator with Flexible Joints in Narrow Space

In real world applications, sometimes we need to operate the robot manipulator in certain extreme environments, such as narrow spaces, high noise spaces and so on. In this part, we conduct a simulation to operate the robot manipulator in a narrow space. Firstly, we transform the operating problem of the robot manipulator in the narrow space into the tracking control problem of the dynamic system of a single link manipulator with flexible joints (17). When we suppose that the narrow space only endures the range of the sum of the joint angle in the interval $[0, \pi/2]$, the operating problem of the robot manipulator in the narrow space is formulated as the tracking control problem with a single linear output of the single-link manipulator with flexible joints (17) in Section 3.1, and the desired path is set as $y_{1d,s1} = \pi\sin(t)/4 + \pi/4$.

Secondly, we adopt the ZD controller $u(t)$ (21) to solve the tracking control problem of the single-link manipulator with flexible joints (17) for the desired path $y_{1d,s1} = \pi\sin(t)/4 + \pi/4$. We illustrate the simulation results of the ZD controller $u(t)$ (21) (denoted by u_{lo1} in this case) to solve the tracking control problem of the single-link manipulator with flexible joints (17) for the desired path $y_{1d,s1} = \pi\sin(t)/4 + \pi/4$ in Figure 3. Specifically, we plot the actual trajectory and desired path $y_{1d,s1}$ in Figure 3a where the red solid line represents the actual trajectory $y_1 = x_1 + x_3$, and the blue dashed line represents the desired path $y_{1d,s1} = \pi\sin(t)/4 + \pi/4$. From Figure 3a, we can see that the output y_1 tracks the desired path $y_{1d,s1} = \pi\sin(t)/4 + \pi/4$ very well after a short time. The controller u_{lo1} is visualized in Figure 3b. From Figure 3b, we can see that the whole control process is smooth enough. The tracking error is illustrated in Figure 3c, and we can see that the tracking error $e_{lo2} = y_1 - y_{1d,s1} = x_1 + x_3 - (\pi\sin(t)/4 + \pi/4)$ converges to zero after a short time. In order to show the precision of the proposed ZD model, we define the residual error as $|e_{lo2}| = |y_1 - y_{1d,s1}| = |x_1 + x_3 - (\pi\sin(t)/4 + \pi/4)|$. In addition, we plot the logarithm diagram of the residual error $|e_{lo2}|$ in Figure 3d, and we can see that the maximal steady-state residual error is about 10^{-5}, which means that the ZD controller u_{lo1} has a good performance on solving the tracking control problem of the single-link manipulator with flexible joints (17) for the desired path $y_{1d,s1} = \pi\sin(t)/4 + \pi/4$.



(**a**) Actual trajectory y_1 and desired path $y_{1d,s1}$ (**b**) Control input u_{lo1}

(**c**) Tracking error e_{lo2} (**d**) Order of $|e_{lo2}|$

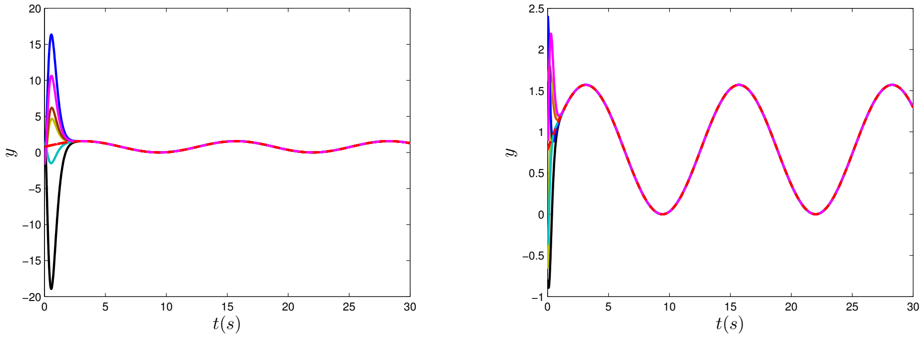
(**e**) Actual trajectory y_1 and desired path $y_{1d,s1}$ (**f**) Tracking error e_{lo2}

Figure 3. Tracking performance of ZD and GD controller (21) to solve the tracking control problem with single linear output of the single-link manipulator with flexible joints for the desired path $y_{1d,s1} = \pi \sin(t)/4 + \pi/4$.

In addition, we presented the performance of the Gradient Dynamic (GD) controller to solve the tracking control problem with the desired output $y_{1d,s1} = \pi \sin(t)/4 + \pi/4$ for comparison. From Figure 3e, it illustrates how long it takes for output $y_{1,GD}$ to track the desired path $y_{1d,s1} = \pi \sin(t)/4 + \pi/4$ by using the GD controller. In addition, Figure 3f illustrates that how long it takes for the tracking error to converge to near zero by using the GD controller. If we compare the convergence time for Figure 3a with Figure 3e, it is evident that the ZD method takes less time to converge than the GD method. Moreover, if we compare the tracking error for Figure 3c with Figure 3f, it can be seen that the error of the ZD method is basically stable at 0 when it converges, but the error of the GD method is constantly changing around the zero point, even when it converges. Therefore, the ZD method outperformed the GD method when the desired output is $y_{1d,s1} = \pi \sin(t)/4 + \pi/4$.

Moreover, we also illustrate the tracking performance of the ZD controller (21) with different parameter values to solve the tracking control problem with a single linear output of the single-link manipulator with flexible joints for the desired path $y_{1d,s1} = \pi \sin(t)/4 +$

$\pi/4$ in Figure 4, where Figure 4a shows the tracking performance of the ZD controller (21) with different parameter values $\lambda_{lo1} = \lambda_{lo2} = \lambda_{lo3} = \lambda_{lo4} = 5$, and Figure 4b shows the tracking performance of the ZD controller (21) with different parameter values $\lambda_{lo1} = \lambda_{lo2} = \lambda_{lo3} = \lambda_{lo4} = 10$. From Figure 4, we can see that the convergence time decreases, as the model parameter values increase.



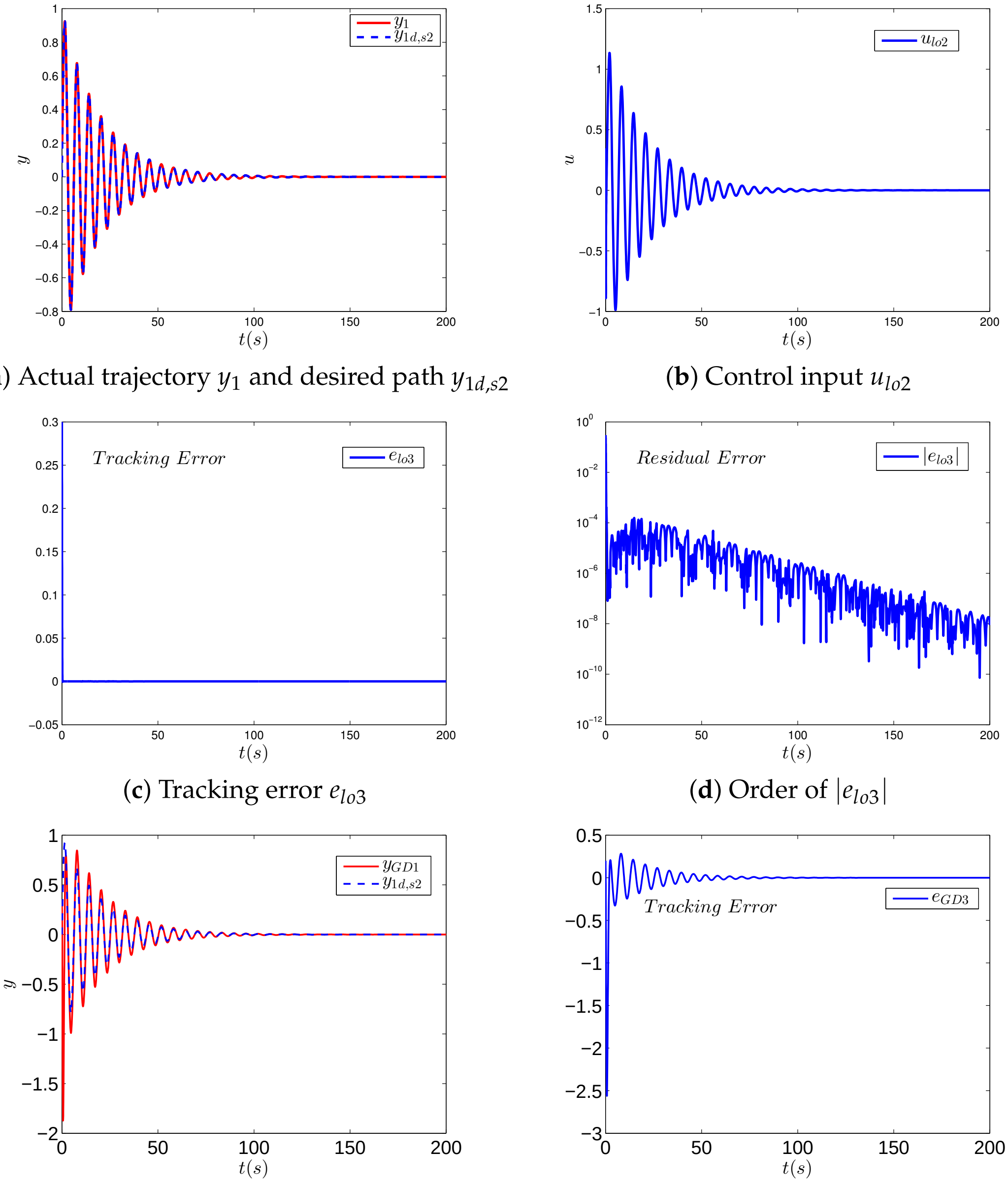
(**a**) $\lambda_{lo1} = \lambda_{lo2} = \lambda_{lo3} = \lambda_{lo4} = 5$ (**b**) $\lambda_{lo1} = \lambda_{lo2} = \lambda_{lo3} = \lambda_{lo4} = 10$

Figure 4. Tracking performance of ZD controller (21) with different model parameters values to solve the tracking control problem with the single-link manipulator with a flexible joints for the desired path $y_{1d,s1} = \pi \sin(t)/4 + \pi/4$. The various trajectories generated by distinct initial values are depicted by the diverse colored lines.

5.1.2. Test for Tracking Control with Single Linear Output of Single-Link Manipulator with Flexible Joints with Application in Calibration

In robot manipulator applications, sometimes we firstly need to calibration the robot manipulator in three-dimensional space. In this part, we conduct a simulation to calibration the robot manipulator in three-dimensional space. Firstly, we transform the calibration problem of the robot manipulator in the three-dimensional space into the tracking control problem of the dynamic system of the single-link manipulator with flexible joints (17). When we suppose that the end-effector periodically converges to the target point, the calibration problem of the robot manipulator in the three-dimensional space is formulated as the tracking control problem with a single linear output of the single-link manipulator with flexible joints (17) in Section 3.1, and the desired path can be set as $y_{1d,s2} = \sin(t)\exp(-t/20)$. In real world applications, the period and phase of sine function and the decay rate of the exponential function can be estimated (or self-adapted) by historical data collected by sensors.

Secondly, we adopt the ZD controller $u(t)$ (21) to solve the tracking control problem of the single-link manipulator with flexible joints (17) for the desired path $y_{1d,s2} = \sin(t)\exp(-t/20)$. We illustrate the simulation results of the ZD controller $u(t)$ (21) (denoted by u_{lo2} in this case) to solve the tracking control problem of the single-link manipulator with flexible joints (17) for the desired path $y_{1d,s2} = \sin(t)\exp(-t/20)$ in Figure 5. Specifically, we plot the actual trajectory and desired path $y_{1d,s2}$ in Figure 5a, where the red solid line represents the actual trajectory $y_1 = x_1 + x_3$, and the blue dashed line represents the desired path $y_{1d,s2} = \sin(t)\exp(-t/20)$. From Figure 5a, we can see that the output y_1 tracks the desired path $y_{1d,s2} = \sin(t)\exp(-t/20)$ very well after a short time. The controller u_{lo2} is visualized in Figure 5b. From Figure 5b, we can see that the whole control process is smooth enough. The tracking error is illustrated in Figure 5c, and we can see that the tracking error $e_{lo3} = y_1 - y_{1d,s1} = x_1 + x_3 - \sin(t)\exp(-t/20)$ converges to zero after a short time. In order to show the precision of the proposed ZD model, we define the residual error as $|e_{lo3}| = |y_1 - y_{1d,s2}| = |x_1 + x_3 - \sin(t)\exp(-t/20)|$. In addition, we plot the logarithm diagram of the residual error $|e_{lo3}|$ in Figure 5d, and we can see that the maximal steady-state residual error is about 10^{-5}, and the steady-state residual error decreases as the end-effector encloses the target point, which means that the ZD controller u_{lo2} has a good performance on solving the tracking control problem of the single-link manipulator with flexible joints (17) for the desired path $y_{1d,s2} = \sin(t)\exp(-t/20)$.



(**a**) Actual trajectory y_1 and desired path $y_{1d,s2}$ (**b**) Control input u_{lo2}

(**c**) Tracking error e_{lo3} (**d**) Order of $|e_{lo3}|$

(**e**) Actual trajectory y_1 and desired path $y_{1d,s2}$ (**f**) Tracking error e_{lo3}

Figure 5. Tracking performance of ZD and GD controllers to solve the tracking control problem with single linear output of the single-link manipulator with flexible joints for the desired path $y_{1d,s2} = \sin(t)\exp(-t/20)$.

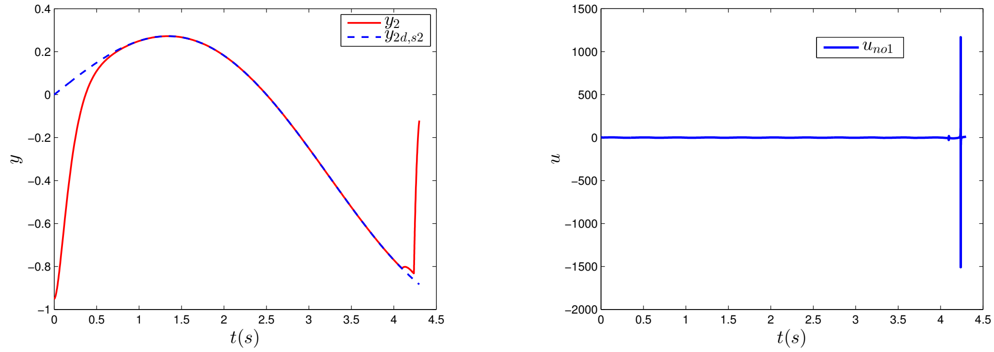
Regarding the comparison, as shown in Figure 5, it demonstrates that the ZD method outperforms the GD method when the desired output is $y_{1d,s2} = \sin(t)\exp(-t/20)$ if we compared the Figure 5a with Figure 5e, and Figure 5c with Figure 5d.

5.2. Test for Tracking Control with Single Nonlinear Output of Single-Link Manipulator with Flexible Joints

In this subsection, we conduct two simulations to show the validity and parameter influences of the ZGD model for solving the tracking control problem with a single nonlinear output of the single-link manipulator with flexible joints (17).

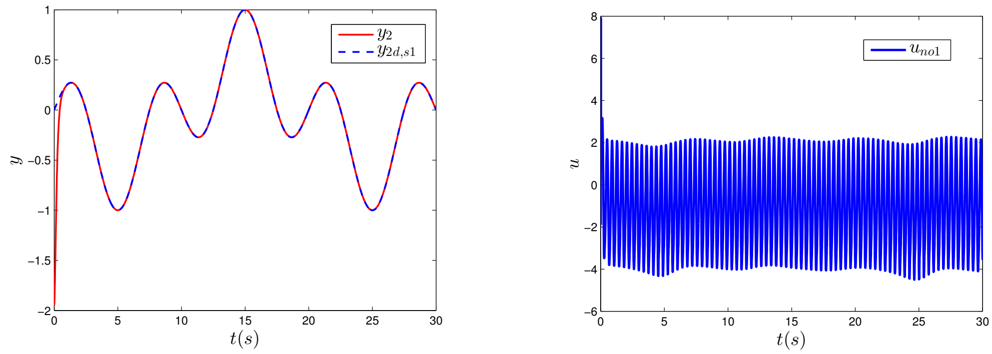
Firstly, we adopt the ZD controller $u(t)$ (21) to solve the tracking control problem with single nonlinear output $y_2 = x_1^2 + x_3^2$ of the single-link manipulator with flexible joints (17) for the desired path $y_{2d,s1} = \sin(0.1\pi t)\cos(0.2\pi t)$. We illustrate the simulation results of the ZD controller $u(t)$ (21) (denoted by u_{no1} in this case) to solve the tracking control problem with a single nonlinear output of the single-link manipulator with flexible joints (17) for the desired path $y_{2d,s1} = \sin(0.1\pi t)\cos(0.2\pi t)$ in Figure 6. Specifically, we plot the actual trajectory and desired path $y_{1d,s2}$ in Figure 6a, where the red solid line represents the actual trajectory $y_2 = x_1^2 + x_3^2$, and the blue dashed line represents the desired path $y_{2d,s1} = \sin(0.1\pi t)\cos(0.2\pi t)$. The controller u_{no1} is visualized in Figure 6b. From Figure 6, we can see that the ZD controller $u(t)$ (21) cannot solve the tracking control problem with single nonlinear output $y_2 = x_1^2 + x_3^2$ of the single-link manipulator with flexible joints (17) for the desired path $y_{2d,s1} = \sin(0.1\pi t)\cos(0.2\pi t)$, due to the DBZ problem. Therefore, we adopt the ZGD controller u_{no1} to solve the tracking control problem with single nonlinear output $y_2 = x_1^2 + x_3^2$ of the single link manipulator with flexible joints (17) for the desired path $y_{2d,s1} = \sin(0.1\pi t)\cos(0.2\pi t)$. We illustrate the simulation results of the ZGD controller u_{no1} to solve the tracking control problem with single nonlinear output $y_2 = x_1^2 + x_3^2$ of the single-link manipulator with flexible joints (17) for the desired path $y_{2d,s1} = \sin(0.1\pi t)\cos(0.2\pi t)$ in Figure 7. Specifically, we plot the actual trajectory and desired path $y_{2d,s1}$ in Figure 7a, where the red solid line represents the actual trajectory $y_2 = x_1^2 + x_3^2$, and the blue dashed line represents the desired path $y_{2d,s1} = \sin(0.1\pi t)\cos(0.2\pi t)$. From Figure 7a, we can see that the output y_2 tracks the desired path $y_{2d,s1} = \sin(0.1\pi t)\cos(0.2\pi t)$ very well after a short time. The controller u_{no1} is visualized in Figure 7b. From Figure 7b, we can see that the whole control process is smooth enough. The tracking error is illustrated in Figure 7c, and we can see that the tracking error $e_{no2} = y_2 - y_{2d,s1} = x_1^2 + x_3^2 - \sin(0.1\pi t)\cos(0.2\pi t)$ converges to zero after a short time. In order to show the precision of the proposed ZD model, we define the residual error as $|e_{no2}| = |y_1 - y_{2d,s1}| = |x_1^2 + x_3^2 - \sin(0.1\pi t)\cos(0.2\pi t)|$. In addition, we plot the logarithm diagram of the residual error $|e_{no2}|$ in Figure 7d, and we can see that the maximal steady-state residual error is about 10^{-4}, which means that the ZD controller u_{no1} has a good performance on solving the tracking control problem of the single-link manipulator with flexible joints (17) for the desired path $y_{2d,s1} = \sin(0.1\pi t)\cos(0.2\pi t)$.

Secondly, we adopt the ZGD controller $u(t)$ (26) to solve the tracking control problem with single nonlinear output $y_2 = x_1^2 + x_3^2$ of the single-link manipulator with flexible joints (17) for the desired path $y_{2d,s2} = \sin(0.5t)\exp(-t/5)$. We illustrate the simulation results of the ZGD controller $u(t)$ (26) ((denoted by u_{no2} in this case)) to solve the tracking control problem with single nonlinear output $y_2 = x_1^2 + x_3^2$ of the single-link manipulator with flexible joints (17) for the desired path $y_{2d,s2} = \sin(0.5t)\exp(-t/5)$ in Figure 8. Specifically, we plot the actual trajectory and desired path $y_{2d,s2}$ in Figure 8a, where the red solid line represents the actual trajectory $y_2 = x_1^2 + x_3^2$, and the blue dashed line represents the desired path $y_{2d,s2} = \sin(0.5t)\exp(-t/5)$. From Figure 8a, we can see that the output y_2 tracks the desired path $y_{2d,s2} = \sin(0.5t)\exp(-t/5)$ very well after a short time. The controller u_{no2} is visualized in Figure 8b. From Figure 8b, we can see that the whole control process is smooth enough. The tracking error is illustrated in Figure 8c, and we can see that the tracking error $e_{no3} = y_2 - y_{2d,s2} = x_1^2 + x_3^2 - \sin(0.5t)\exp(-t/5)$ converges to zero after a short time. In order to show the precision of the proposed ZD model, we define the residual error as $|e_{no3}| = |y_1 - y_{2d,s2}| = |x_1^2 + x_3^2 - \sin(0.5t)\exp(-t/5)|$. In addition, we plot the logarithm diagram of the residual error $|e_{no3}|$ in Figure 8d, and we can see that maximal steady-state residual error is about 10^{-4}, which means that the ZD controller u_{no2} has a good performance on solving the tracking control problem of the single-link manipulator with flexible joints (17) for the desired path $y_{2d,s2} = \sin(0.5t)\exp(-t/5)$.

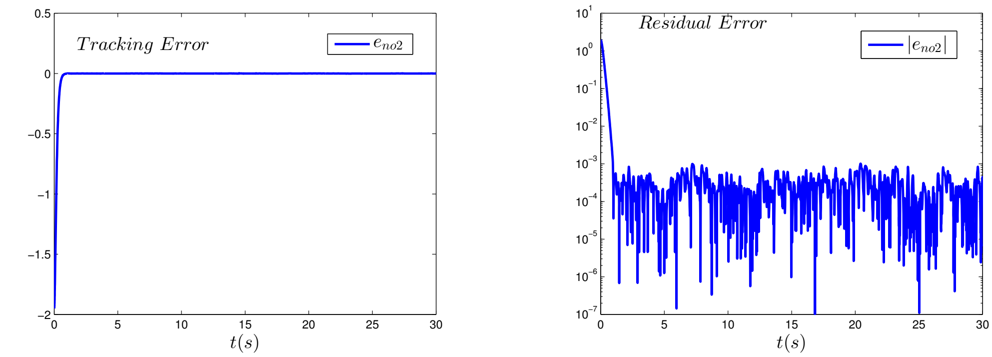


(**a**) Actual trajectory y_2 and desired path $y_{2d,s1}$ (**b**) Control input u_{no1}

Figure 6. Tracking performance of ZD controller (25) to solve the tracking control problem with single nonlinear output of the single-link manipulator with flexible joints for the desired path $y_{2d,s1} = \sin(0.1\pi t)\cos(0.2\pi t)$.

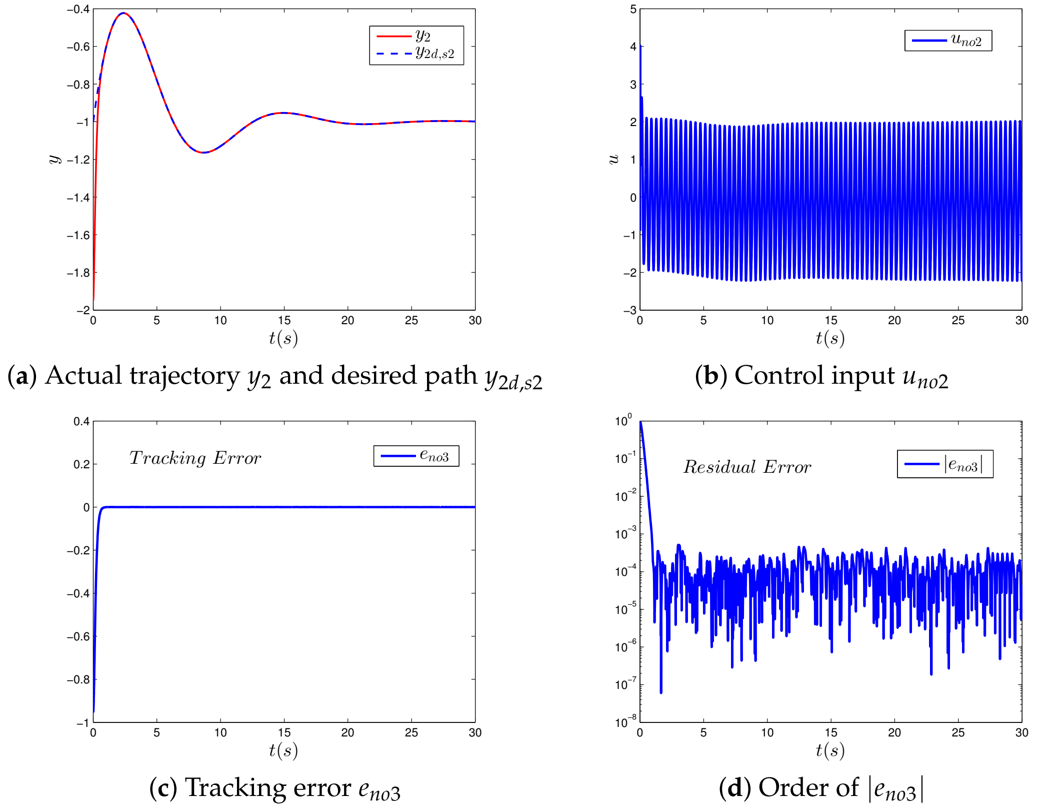


(**a**) Actual trajectory y_2 and desired path $y_{2d,s1}$ (**b**) Control input u_{no1}



(**c**) Tracking error e_{no2} (**d**) Order of $|e_{no2}|$

Figure 7. Tracking performance of ZGD controller (26) to solve the tracking control problem with single nonlinear output of the single-link manipulator with flexible joints for the desired path $y_{2d,s1} = \sin(0.1\pi t)\cos(0.2\pi t)$.



(**a**) Actual trajectory y_2 and desired path $y_{2d,s2}$

(**b**) Control input u_{no2}

(**c**) Tracking error e_{no3}

(**d**) Order of $|e_{no3}|$

Figure 8. Tracking performance of ZGD controller (26) to solve the tracking control problem with single nonlinear output of the single-link manipulator with flexible joints for the desired path $y_{2d,s2} = \sin(0.5t)\exp(-t/5)$.

6. Conclusions

In this paper, we have established and rigorously proven the applicable conditions of ZD models with order-n through Lie derivatives, and we have also provided a unified formula for ZD models with order-n for any dynamic systems in Theorem 1. That is, if there is a relative degree n in the domain U, then the corresponding u(t) can be calculated using the unified formula. In addition, we have presented the global exponential convergence property of the ZD controller u(t), and the steady-state tracking error bound of the ZGD controller u(t), and the radius bound of the steady-state tracking error exponentially converges in the other three theorems. With these four theorems, the zeroing dynamics makes the systems reach equilibrium automatically based on the principle of error reduction. Moreover, in both the ZD method and ZGD method, we have utilized the error function to replace the gradient function of the gradient neural network, making the iteration independent of the specific cost function and reducing the demand for the number of parameters. Therefore, our method is helpful for the future study of high-dimensional dynamic systems with too many parameters. As for the future research direction, we will focus on developing more robustness algorithms to overcome the disturbance and noises problems in high-order dynamic models.

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