mathematics

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Recognition and Implementation of Contact Simple Map Germs from $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$

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Abstract: The classification of contact simple map germs from $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ was given by Dimca and Gibson. In this article, we give a useful criteria to recognize this classification of contact simple map germs of holomorphic mappings with finite codimension. The recognition is based on the computation of explicit numerical invariants. By using this characterization, we implement an algorithm to compute the type of the contact simple map germs without computing the normal form and also give its implementation in the computer algebra system SINGULAR.

Keywords: simple map germ; $\mathcal{K}$-equivalence; codimension

MSC: 58Q05; 14H20

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Citation: Xu, P.; Binyamin, M.A.; Aslam, A.; Shahbaz, M.; Aslam, S.; Kanwal, S. Recognition and Implementation of Contact Simple Map Germs from $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. *Mathematics* **2023**, *11*, 1575. https://doi.org/10.3390/math11071575

Academic Editors: Elena Guardo, Emeritus Mario Gionfriddo, Salvatore Milici and Carsten Schneider

Received: 17 January 2023
Revised: 19 February 2023
Accepted: 21 March 2023
Published: 23 March 2023

1. Introduction and Preliminaries

Currently, SINGULAR is one of the most widely used computer algebra systems for the computation of commutative algebra, algebraic geometry, and singularity theory. The motivation behind the development of SINGULAR was mainly driven by mathematical problems in singularity theory. For more details about the history of SINGULAR, see [1]. The main aim of this paper is to implement a classifier in the computer algebra system SINGULAR which compute the type of simple contact map germs from plane to plane.

Let K be a field. Let Aut$K[x_1, x_2, \ldots, x_n]$ and Aut$K < x_1, x_2, \ldots, x_n >$ be K-algebra automorphism groups of the commutative polynomial algebra and free associative algebra with n generators, respectively. This is equivalent to formed by one-to-one mappings of all polynomials of affine space $\mathbb{A}_k^n$. Both groups admits a representation as a colimit of algebraic sets of automorphisms filtered by total degree which turns them in to topological spaces with Zariski topology compatible with group structure. These groups also possess a power series topology. To study more about these concepts, the readers can consult [2–4].

Classification and recognition of singularities of map germs up to some equivalence relation are well understood terms and have been a subject of large number of investigation in previous literature (cf. [5–16]). It is very interesting to discuss the recognition problem for the classification of singularities due to some equivalence relation. Classification for map germs under some equivalence relation, means finding a list of map germs and showing that all map germs satisfying certain conditions are equivalent to a map germ in the list. Recognition means finding some criteria which describe a given map germ and which map germ it is equivalent to in the list.

Dimca and Gibson [5,17] gave the classification of map germs of Boardman Symbol $(2,0)$ and $(2,1)$ from $(\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$. They also classified the contact simple map germs from $(\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$. Our aim is to recognize the contact simple map germs by using suitable numerical invariants, such as contact codimension, an integer σ_f which gives the bound for k-determinancy of map germ f, double fold number and multiplicity of f. Moreover, an algorithm is presented to compute the type of contact simple map germs without computing the normal form. Finally, this algorithm is implemented in the computer algebra system SINGULAR.

Let $S(2,2) =< x,y > \mathbb{C}[[x,y]]^2$ and $\mathcal{K} = Aut_{\mathbb{C}}(\mathbb{C}[[x,y]]) \times Gl_2(\mathbb{C}[[x,y]])$, where $\mathbb{C}$ denote the set of complex numbers. The contact group $\mathcal{K}$ acts $S(2,2)$ by

$$\mathcal{K} \times S(2,2) \to S(2,2),$$

such that

$$((\alpha, A), h) \mapsto \alpha^{-1} \circ Ah.$$

Any two elements $f_1, f_2 \in S(2,2)$ are $\mathcal{K}$-equivalent $(f_1 \sim_{\mathcal{K}} f_2)$ if they lie in the same orbit under the group action $\mathcal{K}$.

Definition 1. *Let $f \in S(2,2)$, we define the orbit map $\theta_f : \mathcal{K} \to S(2,2)$ as $\theta_f(\phi, \mathrm{M}) = \phi^{-1} \circ \mathrm{M}f$, $\phi : \mathbb{C}[[x,y]] \to \mathbb{C}[[x,y]]$ and $\mathrm{M} = (\mathrm{M}_{ij})$. Exceptionally, we have $\theta_f(id) = f$. The orbit of f under the action of $\mathcal{K}$ is the image of θ_f, assume $\mathcal{K}_f := Im(\theta_f)$. The tangent space to the orbit at f, $\mathcal{T}_{\mathcal{K}_f,f}$ is defined as:*

$$\mathcal{T}_{\mathcal{K}_f,f} =< x,y >_{\mathbb{C}[[x,y]]} < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >_{\mathbb{C}[[x,y]]} + < f_1, f_2 >_{\mathbb{C}[[x,y]]} \mathbb{C}[[x,y]]^2.$$

The $\mathcal{K}$-codimension of f is defined as:

$$c_f := dim_{\mathbb{C}} \frac{S(2,2)}{T_{\mathcal{K}_f,f}}.$$

Note that c_f exists with the restriction that f is $\mathcal{K}$-finite.

Definition 2. *Let $f \in S(2,2)$, then the tangent space at $f = (f_1, f_2)$ is an integer σ_f which gives the bound for k-determinancy of map germ f and is defined as:*

$$\sigma_f = min\{l :< x,y >^l \mathbb{C}[[x,y]]^2 \subset \mathrm{T}_{\mathcal{K}_f,id}\}.$$

Definition 3. *The $\mathcal{K}$-modality of $f \in S(2,2)$ is the smallest integer m such that a sufficiently small neighborhood of f can be covered by a finite number of $m-$parameter families of orbits under the action of group $\mathcal{K}$ in the space of map germs. In particular, if $\mathcal{K}$-modality of f is 0, then it is called $\mathcal{K}$-simple map germ.*

Definition 4. *Let I_f be the ideal associated to finitely determined map germ $f \in \mathrm{A}(2,2)$. Then, its double fold number is denoted by d_f and is defined as the number of 2's, occurs in the decreasing sequence $2 = h_{I_f}(1) \geq h_{I_f}(2) \geq \ldots \geq h_{I_f}(k) \geq \ldots$, where h_{I_f} denote the Hilbert function of the graded algebra associated to $\frac{\mathbb{C}[[x,y]]}{I_f}$.*

The Thom–Boardman symbols are non increasing sequences of non-negative integers which were introduced by Thom and Boardman to classify singularities of differentiable maps [18]. In general, the computation of these numbers is an extremely difficult task.

Definition 5. *Let $I =< f_1, \ldots, f_p >$ be an ideal in $\mathbb{C}[[x_1, \ldots, x_n]]$. Then, the s-th Jacobian extension of I to be the ideal $\Delta^s(I) = I + I_1$, where I_1 is the ideal in $\mathbb{C}[[x_1, \ldots, x_n]]$ generated by* *all $(n-s+1)\times(n-s+1)$ minors of the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})_{j=1\ldots n}^{i=1\ldots p}$. Then, one has a sequence of inclusions*

$$I=\Delta^0(I)\subset\Delta^1(I)\subset\cdots\subset\Delta^n(I).$$

If I is proper then the critical Jacobian extension of I is the last ideal $\Delta^{i_1}(I)$ in the sequence which is proper. This ideal $\Delta^{i_1}(I)$ has in turn its critical Jacobian extension $\Delta^{i_2}(\Delta^{i_1}(I))$ and so on. The sequence of integers $(i_1,i_2,\ldots)$ obtained in this way is called Boardman symbol of the ideal I.

Definition 6. *The Boardman symbol of a map germ $f:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ is the Boardman symbol of the ideal I_f generated by the components $f_1,f_2,\ldots,f_p$ of f.*

Motivation: There are many classification results for singularities of map germs, however, studies for recognition problems are not so much. To compute the normal form of a map germ is a space and time-consuming process, therefore the new investigation for this problem is to give criteria to identify a map germ independent to compute the normal form.

2. Computation of Numerical Invariants under Contact Equivalence

In the following, we give a sequence of algorithms to compute the numerical invariants used for the recognition of contact simple map germs. We implement these algorithms in the computer algebra system SINGULAR [19]. Additionally, we explain how someone can compute the invariants in SINGULAR by using the implemented codes.

By using Algorithm 1, we can compute an important invariant, which measures the complexity of map germs called the codimension.

Algorithm 1 $\mathcal{K}$-codimension of a map.

Input: A map germ $f=(f_1,f_2)\in\mathbb{C}[[x,y]]^2$.

Output: An integer number c_f, the $\mathcal{K}$-codimension of f.

1: Compute the module M:= $\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\rangle$.
2: Compute the module $\mathcal{T}_{\mathcal{K}_f,f}:=<x,y>_{\mathbb{C}[[x,y]]}M+\langle f_1,f_2\rangle\mathbb{C}[[x,y]]^2$.
3: Compute the module $\mathcal{ST}_{\mathcal{K}_f,f}:=std(\mathcal{T}_{\mathcal{K}_f,f})$.
4: **return** $c_f:=vdim(\mathcal{ST}_{\mathcal{K}_f,f})-2$.

2.1. Singular Example

In SINGULAR declaration of a ring can be made as follows:

```
ring R=0,(x,y),ds;
```

In the following example, we have as an input the map $f(x,y)=(f_1,f_2)$, where

$$f_1=x^2+2xy+y^2+y^3-3xy^3+3x^2y^3-x^3y^3,$$

$$f_2=xy^2+y^3-2x^2y^2-2xy^3+x^3y^2+x^2y^3.$$

In SINGULAR, this can be written as:

```
> poly f1=x2+2xy+y2+y3-3xy3+3x2y3-x3y3;
> poly f2=xy2+y3-2x2y2-2xy3+x3y2+x2y3;
> ideal J=f1,f2;
```

To compute the codimension of given map germ, we use the procedure:

```
> KcoDim(J);
  8
```

By using Algorithm 2, we can compute an important numerical invariant of map germs $f \in S(2,2)$, which is closely connected to the determinacy of map germs.

Algorithm 2 Sigma of a map germ.

Input: A map germ $f = (f_1, f_2) \in \mathbb{C}[[x,y]]^2$.

Output: An integer number σ_f, which is closely connected to the determinacy of map germ f.

1: Compute the module M:= $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$.
2: Compute the module $\mathcal{T}_{\mathcal{K}_f,f} :=< x,y >_{\mathbb{C}[[x,y]]} M + \langle f_1, f_2 \rangle \mathbb{C}[[x,y]]^2$.
3: Compute the module $\mathcal{ST}_{\mathcal{K}_f,f} := std(\mathcal{T}_{\mathcal{K}_f,f})$.
4: Compute σ_f:= size(reduce($< x,y >_{\mathbb{C}[[x,y]]} + \langle f_1, f_2 \rangle \mathbb{C}[[x,y]]^2, \mathcal{ST}_{\mathcal{K}_f,f}$)).
5: **return** σ_f.

2.2. Singular Example

```
ring R=0,(x,y),ds;
```

In the following example, we have as an input the map $f(x,y) = (f_1, f_2)$, where

$$f_1 = x^2 - 2x^2y + x^2y^2 + y^4 - 4xy^4 - 4y^5 + 6x^2y^4 + 12xy^5 + 6y^6 - 4x^3y^4 - 12x^2y^5 - 12xy^6 - 4y^7 + x^4y^4 + 4x^3y^5 + 6x^2y^6 + 4xy^7 + y^8,$$
$$f_2 = xy^2 - 2x^2y^2 - 3xy^3 + x^3y^2 + 4x^2y^3 + 3xy^4 - x^3y^3 - 2x^2y^4 - xy^5.$$

In SINGULAR, this can be written as:

```
poly f1=x2-2x2y+x2y2+y4-4xy4-4y5+6x2y4+12xy5+6y6-4x3y4-12x2y5-12xy6
         -4y7+x4y4+4x3y5+6x2y6+4xy7+y8;
> poly f2=xy2-2x2y2-3xy3+x3y2+4x2y3+3xy4-x3y3-2x2y4-xy5;
> ideal J=f1,f2;
> siGmaa(J);
4
```

Implementing the Algorithm 3 in SINGULAR, we generate a SINGULAR command `dFoldn` to compute the double fold number for a given map germ, as follows:

```
ring R=0,(x,y),ds;

> poly f1=x2-2x2y+x2y2+y4-4xy4-4y5+6x2y4+12xy5+6y6-4x3y4-12x2y5-12xy6
         -4y7+x4y4+4x3y5+6x2y6+4xy7+y8;
> poly f2=xy2-2x2y2-3xy3+x3y2+4x2y3+3xy4-x3y3-2x2y4-xy5;
> ideal J=f1,f2;
> dFoldn(J);
2
```

Algorithm 3 Double fold number of a map germ.

Input: A map germ $f = (f_1, f_2) \in \mathbb{C}[[x,y]]^2$.

Output: An integer number d_f, the double fold number of f.

1: Compute the ideals $J := f+ < x,y >^k$ and $K := f+ < x,y >^{k+1}$.
2: Construct the decreasing sequence:
   h(k):=$dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{J} - dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{K} \leq 2$.
3: Count the number d_f of 2's occuring in h(k).
4: **return** d_f.

An important invariant, which describe the geometry of map germs is the Boardman symbol. This can be computed by using Algorithm 4.

Algorithm 4 Boardman symbol of a map germ.

Input: A map germ $f = (f_1, f_2) \in \mathbb{C}[[x, y]]^2$.
Output: A sequence of integer numbers $(i_1, i_2, \ldots)$, the Boardman symbol of f.

1: Compute jacobian matrix $M := \left(\frac{\partial f_i}{\partial x_j}\right)_{j=1,2}^{i=1,2}$.
2: Compute the sequence of inclusion of sth jacobian extension of I: $\Delta^s(I) = I + I_t$,
3: Compute the critical jacobian extension: $\Delta^{i_2}(\Delta^{i_1}(I))\ldots$
4: **return** $(i_1, i_2, \ldots)$.

We generate SINGULAR command `Brsymbol` and apply it to compute the contact invariant, second order Boardman symbol for a given map germ, as follows:

2.3. Singular Example

```
ring R=0,(x,y),ds;
> poly f1=x2-2x3+y3+x4-3x2y2-3xy3+3y4+3x4y+9x3y2-6x2y3-6xy4+3y5-x6-
          9x5y+17x3y3-6x2y4-3xy5+y6+3x7+6x6y-15x5y2+9x3y4-3x2y5-3x8+
          3x7y+6x6y2-9x5y3+3x4y4+x9-3x8y+3x7y2-x6y3;
> poly f2=xy2-2x3y-3x2y2+2xy3+x5+6x4y-x3y2-4x2y3+xy4-3x6-4x5y+7x4y2-
          x2y4+3x7-2x6y-3x5y2+2x4y3-x8+2x7y-x6y2;
> ideal J=f1,f2;
> Brsymbol(J);
2,1

> poly f1=xy-3y2-3x3+18x2y-25xy2+16y3-11x4+34x3y-42x2y2+16xy3-
          5y4-7x5+19x4y-10x3y2+4x2y3-3x6+2x5y-x4y2;
> poly f2=x2-6xy+10y2+6x3-28x2y+26xy2-16y3+18x4-52x3y+84x2y2-64xy3+
        26y4+12x5-50x4y+68x3y2-40x2y3+10x6-28x5y+26x4y2+4x7-8x6y+x8;
> ideal J=f1,f2;
> Brsymbol(J);
2,0
```

3. Recognition of Contact Simple Map Germs from $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$

In this section, we give a characterization of simple map germs with respect to contact equivalence in terms of the codimension, the Milnor number, a number closely connected to the determinacy of f and the Boardman symbol of f. The k-th jet of f is denoted by $j^k(f)$, this is the Taylor expansion of components of f up to degree k.

Proposition 1. *Let $f \in S(2, 2)$, then $\mathcal{K}$-simple map germs are $\mathcal{K}$-equivalent to one of the following in the Table 1.*

Table 1. Normal form of simple map germs.

Type	Normal Form	Conditions
A_k	(x, y^{k+1})	$k \geq 1$
$B_{p,q}$	$(xy, x^p + y^q)$	$q \geq p \geq 2$
C_k	(x^2, y^k)	$k = 3, 4$
$E_{3,q}$	$(x^2 + y^3, y^q)$	$q > 3$
$F_{3,q}$	$(x^2 + y^3, xy^q)$	$q \geq 2$
$F_{p,2}$	$(x^2 + y^p, xy^2)$	$p \geq 4$

Proof. For proof see the article [17]. □

Table 2 contains all the invariants used to characterize the contact simple map germs from $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ classified by Dimca and Gibson [17].

Table 2. Invariants of simple map germs.

Type	c_f	μ_f	σ_f
A_k	k	k	$k+1$
$B_{p,q}$	$p+q$	$p+q-1$	q
C_k	$3k-2$	$2k-1$	k
$E_{3,q}$	$2q+1$	$2q-1$	q
$F_{3,q}$	$2q+4$	$2q+2$	$q+2$
$F_{p,2}$	$p+5$	$p+3$	p

Let f be a map germs from $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. According to Dimca and Gibson's classification if $j^1 f = 0$, then possible $j^2 f$ and the corresponding second order Boardman symbols are given in the Table 3.

Table 3. Normal form of second jet of simple map germs.

$j^2 f$	**Br**(f)
(x^2, y^2)	$(2,0)$
(xy, x^2+y^2)	$(2,0)$
(x^2, xy)	$(2,0)$
(xy, y^2)	$(2,0)$
$(xy, 0)$	$(2,0)$
$(x^2, 0)$	$(2,1)$
$(0,0)$	$(2,2)$

Remark 1. *This is easy to see that* $(x^2, y^2) \sim_{\mathcal{K}} (xy, x^2+y^2)$ *and* $(x^2, xy) \sim_{\mathcal{K}} (xy, y^2)$.

Proposition 2. *Let* $f(x,y)$ *be a map germ from* $(\mathbb{C}^2, 0)$ *to* $(\mathbb{C}^2, 0)$. *Then, if* $j^1(f) \sim_{\mathcal{K}} (x, 0)$ *and* $c_f = \mu_f = k \geq 1$, *then* f *is contact simple of type* A_k,

Proof. Since $f(x,y)$ be a map germ with $j^1(f) \sim_{\mathcal{K}} (x, 0)$ therefore

$$f(x,y) \sim_{\mathcal{K}} (x + a_{0,2}y^2 + a_{1,1}xy + h.o.t., b_{0,2}y^2 + b_{1,1}xy + h.o.t.).$$

Note that, by using x, all terms having factor x can be cancelled in the ideal I_f, so we obtain

$$f(x,y) \sim_{\mathcal{K}} (x + a_{0,2}y^2 + a_{0,3}y^3 + \ldots, b_{0,2}y^2 + b_{0,3}y^3 + \ldots).$$

If $c_f = \mu_f = 1$, then $b_{0,2} \neq 0$, so we can take $b_{0,2} = 1$. This gives

$$f(x,y) \sim_{\mathcal{K}} (x + a_{0,2}y^2 + a_{0,3}y^3 + \ldots, y^2 + b_{0,3}y^3 + \ldots).$$

By using y^2, all terms having factor y^2 can be cancelled in the ideal I_f and we obtain

$$f(x,y) \sim_{\mathcal{K}} (x, y^2).$$

In a similar way, we can show that if $c_f = \mu_f = k \geq 2$ then $f(x,y) \sim_{\mathcal{K}} (x, y^{k+1})$. □

The following results are due to Dimca and Gibson:

Proposition 3. *Let a finitely $\mathcal{K}$-determined map germ f from $(\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ such that $Br(f) = (2,0)$. Then f is contact simple and contact equivalent to $B_{p,q} : (xy, x^p + y^q)$, with $q \geq p \geq 2$.*

Proposition 4. *Let f be a map germ from $(\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ such that $Br(f) = (2,0)$. Then f is $\mathcal{K}$-simple and $\mathcal{K}$-equivalent to the map germ of the form $f = (x^2 + y^k, xy^l + y^m R(y))$, where $R \in E_1$ with $R(0) \neq 0$, $k \geq 3$, $m \geq 3$, $l \geq 2$. Moreover the following holds:*

1. *If $m \leq l$, then f is $\mathcal{K}$-equivalent to the normal form*

$$C_m : (x^2, y^m),\ m \geq 3\ if\ k \geq m$$

and $\mathcal{K}$-equivalent to the normal form

$$E_{k,m} : (x^2 + y^k, y^m),\ m > k \geq 3\ if\ k < m.$$

2. *If $m \geq 2l$, then f is $\mathcal{K}$-equivalent to the normal form*

$$F_{p,l} : (x^2 + y^p, xy^l),\ p \geq 3\ if\ l \geq 2.$$

with $p = ord(y^k + R^2 y^{2(m-l)})$.

Proposition 5. *Let f be a map germ from $(\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ such that $Br(f) = (2,1)$. Then f is $\mathcal{K}$-simple if, and only if, f is $\mathcal{K}$-equivalent to the map germ of type $C_k, k = 3,4, E_{3,q}$, $q > 3, F_{3,q}, q > 1$ and $F_{q,2}, q > 3$.*

Proposition 6. *Let f be a map germ from $(\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ such that $Br(f) = (2,2)$. Then f is not a $\mathcal{K}$-simple map germ.*

The map germ of type $B_{p,q}$ can be characterized in terms of invariants by using the following proposition.

Proposition 7. *Let $f(x,y)$ be a map germ from $(\mathbb{C}^2,0)$ to $(\mathbb{C}^2,0)$. Then, if $Br(f) = (2,0)$, $c_f = \mu_f + 1$ and $\sigma_f \geq 2$, then f is contact simple of type $B_{c_f - \sigma_f, \sigma_f}$.*

Proof. Let $f(x,y)$ be a map germ with $\mathrm{Br}(f) = (2,0)$, then $j^2(f)$ is $\mathcal{K}$-equivalent to $(xy, x^2 + y^2), (xy, x^2)$ or $(xy, 0)$. If $\sigma_f = 2$, then $j^2(f)$ is $\mathcal{K}$-equivalent to $(xy, x^2 + y^2)$. We can write

$$f(x,y) \sim_{\mathcal{K}} (xy + a_{3,0}x^3 + a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3 + h.o.t, x^2 + y^2 + b_{3,0}x^3 + b_{2,1}x^2y + b_{1,2}xy^2 + b_{0,3}y^3 + h.o.t),$$

All terms having factor xy or x^2 can be cancelled in ideal I_f, so we obtain

$$f(x,y) \sim_{\mathcal{K}} (xy + a_{0,3}y^3 + a_{0,4}y^4 + \ldots, x^2 + y^2 + b_{0,3}y^3 + b_{0,4}y^4 + \ldots).$$

Now the transformation $x \to y$ and $y \to x$, gives

$$f(x,y) \sim_{\mathcal{K}} (xy + a_{0,3}x^3 + a_{0,4}x^4 + \ldots, x^2 + y^2 + b_{0,3}x^3 + b_{0,4}x^4 + \ldots).$$

Again all terms with factor x^2 can be cancelled in ideal I_f, so we obtain

$$f(x,y) \sim_{\mathcal{K}} (xy, x^2 + y^2).$$

If $\sigma_f > 2$, then $j^2(f)$ is $\mathcal{K}$-equivalent to (xy, x^2) or $(xy, 0)$. Now if $c_f = \mu_f + 1$ and $j^2(f)$ is $\mathcal{K}$-equivalent to (xy, x^2) (resp. $(xy, 0)$), then by (Proposition 9.25 [20]), f is $\mathcal{K}$-equivalent to $(xy, x^2 + y^{\sigma_f})$ (resp. $(xy, x^{c_f - \sigma_f} + y^{\sigma_f})$) . □

The map germ of type C, E and F can be characterized in terms of invariants by using the following propositions.

Proposition 8. *Let $f(x,y)$ be a map germ from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$ such that $Br(f) = (2,1)$. If $c_f = 3\sigma_f - 2$ and $\mu_f = 2\sigma_f - 1$, then f is $\mathcal{K}$-simple of type C_k, $k = 3,4$.*

Proof. Let $f(x,y)$ be a map germ with $\text{Br}(f) = (2,1)$, then $j^2(f)$ is $\mathcal{K}$-equivalent to $(x^2, 0)$. Then by Lemma 2.2 [13], we can write

$$f(x,y) \sim_{\mathcal{K}} (x^2 + a_{3,0}x^3 + a_{2,1}x^2y + a_{0,3}y^3 + h.o.t, b_{3,0}x^3 + b_{1,2}xy^2 + b_{0,3}y^3 + h.o.t).$$

All the terms with factor x^2 can be cancelled in ideal I_f, so we obtain

$$f(x,y) \sim_{\mathcal{K}} (x^2 + a_{0,3}y^3 + a_{0,4}y^4 + \ldots, b_{0,3}y^3 + b_{0,4}y^4 + \ldots).$$

Now if $\mu_f = 2\sigma_f - 1$, then $b_{0,\sigma_f - 1} = 0$ and if $c_f = 3\sigma_f - 2$, then $a_{0,\sigma_f - 1} = 0$, so we obtain

$$f(x,y) \sim_{\mathcal{K}} (x^2 + a_{0,\sigma_f}y^{\sigma_f} + \ldots, b_{0,\sigma_f}y^{\sigma_f} + \ldots).$$

Now all the terms with factor y^{σ_f} can be cancelled in ideal I_f, so we obtain $f(x,y) \sim_{\mathcal{K}} (x^2, y^{\sigma_f})$. $\square$

Proposition 9. *Let $f(x,y)$ be a map germ from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$ such that $Br(f) = (2,1)$ and $c_f = \mu_f + 2$. Then*

1. *If $c_f = 2\sigma_f + 1$ and $d_f > 2$, then f is $\mathcal{K}$-simple of type E_{3,σ_f},*
2. *If $c_f = 2\sigma_f$, then f is $\mathcal{K}$-simple of type $F_{3,\sigma_f - 2}$,*
3. *If $c_f = \sigma_f + 5$ and $d_f = 2$, then f is $\mathcal{K}$-simple of type $F_{\sigma_f,2}$.*

Proof. Let $f(x,y)$ be a map germ with $\text{Br}(f) = (2,1)$. Then, $j^2(f)$ is $\mathcal{K}$-equivalent to $(x^2, 0)$ and we can write
$f(x,y) \sim_{\mathcal{K}} (x^2 + a_{3,0}x^3 + a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3 + h.o.t, b_{3,0}x^3 + b_{2,1}x^2y + b_{1,2}xy^2 + b_{0,3}y^3 + h.o.t)$.
All the terms with factor x^2 can be cancelled in ideal I_f, so we obtain

$$f(x,y) \sim_{\mathcal{K}} (x^2 + a_{1,2}xy^2 + a_{0,3}y^3 + a_{0,4}y^4 + h.o.t, b_{1,2}xy^2 + b_{0,3}y^3 + b_{0,4}y4 + h.o.t).$$

(1) If $\mu_f \geq 7$, $c_f = 2\sigma_f + 1 \geq 9$ and $\sigma_f \geq 4$, then $a_{1,2} = b_{1,2} = 0$, so we obtain

$$f(x,y) \sim_{\mathcal{K}} (x^2 + a_{0,3}y^3 + a_{0,4}y^4 + \ldots, b_{0,3}y^3 + b_{0,4}y^4 + \ldots).$$

Now if $\mu_f = 7$ and $c_f = 9$, then $b_{0,3} = 0$, $a_{0,3} \neq 0$ and $b_{0,4} \neq 0$, so we obtain

$$f(x,y) \sim_{\mathcal{K}} (x^2 + a_{0,3}y^3 + a_{0,4}y^4 + \ldots, y^4 + b_{0,5}y^5 + \ldots).$$

All the terms with factor y^4 can be cancelled in ideal I_f, so we obtain $f(x,y) \sim_{\mathcal{K}} (x^2 + y^3, y^4)$.
If $c_f = 2\sigma_f + 1$ and $\sigma_f > 4$, then $f(x,y) \sim_{\mathcal{K}} (x^2 + y^3, y^{\sigma_f})$.
(2) If $\mu_f = 2\sigma_f + 2$ and $c_f = 2\sigma_f \geq 8$ and $\sigma_f \geq 4$, then $a_{1,2} = b_{0,\sigma_f - 1} = 0$, moreover $a_{0,3} \neq$ and $b_{1,\sigma_f - 2} \neq 0$, so we obtain

$$f(x,y) \sim_{\mathcal{K}} (x^2 + y^3, xy^{\sigma_f - 2}).$$

(3) If $\mu_f = \sigma_f + 3$ and $c_f = \sigma_f + 5$ and $\sigma_f = 4$, then $a_{1,2} = b_{0,3} = a_{0,\sigma_f - 1} = 0$, moreover $a_{1,2} \neq 0$ and $a_{0,\sigma_f} \neq 0$, so we obtain

$$f(x,y) \sim_{\mathcal{K}} (x^2 + y^{\sigma_f}, xy^2).$$

□

In Algorithm 5, we give a classifier for contact simple map germs from plane to the plane in terms of co-dimension, Milnor number and fold number.

Algorithm 5 Contact simple plane to plane maps (KSimPlePlaneGerms).

Input: A contact simple map germ $f(x,y)$ from the plane to the plane.

Output: Normal form $g(x,y)$ of simple map germ, or 0 if f is not contact simple.

Compute $\mathrm{Br}(f)$, the Boardman symbol of f, c_f, the codimension of f, μ_f, the Milnor number of f, d_f, the double fold number of f and σ_f.
if $j^1(f) \sim_{\mathcal{K}} (x,0)$ **then**
  return f is of type A_{σ_f};
if $\mathrm{Br}(f) = (2,0)$ and $c_f = \mu_f + 1$ and $\sigma_f \geq 2$ **then**
  return f is of type $B_{c_f - \sigma_f, \sigma_f}$;
if $\mathrm{Br}(f) = (2,1)$ **then**
  if $c_f = 3\mu_f - 2$ **then**
    if $c_f = 3\sigma_f - 2$ **then**
      return f is of type C_{σ_f};
  if $c_f = \mu_f + 2$ **then**
    if $c_f = 2\sigma_f + 1$ and $d_f > 2$ **then**
      return f is of type E_{3,σ_f};
    if $c_f = 2\sigma_f$ **then**
      return f is of type $F_{3,\sigma_f - 2}$;
    if $c_f = \sigma_f + 5$ and $d_f = 2$ **then**
      return f is of type $F_{\sigma_f,2}$;
else
return 0.

Singular Examples

In the first example, we have as an input the map $f(x,y) = (f_1, f_2)$, where

$f_1 = x^2 - 2x^2y + x^2y^2 + y^4 - 4xy^4 - 4y^5 + 6x^2y^4 + 12xy^5 + 6y^6 - 4x^3y^4 - 12x^2y^5 - 12xy^6 - 4y^7 + x^4y^4 + 4x^3y^5 + 6x^2y^6 + 4xy^7 + y^8$,
$f_2 = xy^2 - 2x^2y^2 - 3xy^3 + x^3y^2 + 4x^2y^3 + 3xy^4 - x^3y^3 - 2x^2y^4 - xy^5$.

In SINGULAR the type of $\mathcal{K}$-simple map germ from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$ can be obtained as:

```
ideal I=x2-2x2y+x2y2+y4-4xy4-4y5+6x2y4+12xy5+6y6-4x3y4-12x2y5-12xy6
       -4y7+x4y4+4x3y5+6x2y6+4xy7+y8, xy2-2x2y2-3xy3+x3y2+4x2y3+3xy4
       -x3y3-2x2y4-xy5;
> KSimPlePlaneGerms(I);
  f is of type F_4,2
```

In the second example we have as an input the map $f(x,y) = (f_1, f_2)$, where $f_1 = x^2 - 4x^2y + y^3 + 8x^2y^2 - 6y^4 - 10x^2y^3 + 18y^5 + 8x^2y^4 - 35y^6 - 4x^2y^5 + 48y^7 + x^2y^6 - 48y^8 + 35y^9 - 18y^{10} + 6y^{11} - y^{12}$,

$f_2 = y^4 - 8y^5 + 32y^6 - 84y^7 + 160y^8 - 232y^9 + 262y^{10} - 232y^{11} + 160y^{12} - 84y^{13} + 32y^{14} - 8y^{15} + y16.$

In SINGULAR the type of $\mathcal{K}$-simple map germ from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^3, 0)$ can be obtained as:

```
ideal I= x2-4x2y+y3+8x2y2-6y4-10x2y3+18y5+8x2y4-35y6-4x2y5+48y7+x2y6
         -48y8+35y9-18y10+6y11-y12,y4-8y5+32y6-84y7+160y8-232y9+262y10
         -232y11+160y12-84y13+32y14-8y15+y16;
> KSimPlePlaneGerms(I);
  f is of type E_3,4
```

4. Conclusions

In this work, a useful criteria is given to classify contact simple map germs from $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ based on the computation of explicit numerical invariants. We give an algorithm to implement this classification in computer algebra system SINGULAR.

Future work:

1. Implement a classifier for the classification of contact map germs in higher dimensions.
2. Implement a classifier for the classification of map germs with respect to left–right equivalence.

Author Contributions: Conceptualization, A.A. and S.A.; Methodology, P.X., M.A.B., A.A., M.S., S.A. and S.K.; Software, S.K.; Validation, P.X., M.A.B., A.A., M.S., S.A. and S.K.; Investigation, P.X. and M.A.B.; Writing—original draft, M.A.B., M.S. and S.A.; Writing—review & editing, P.X., A.A. and S.K. All authors have read and agreed to the published version of the manuscript.

Funding: This work is funded in part by the National Natural Science Foundation of China (62002079) and Natural Science Foundation of Guangdong Province of China (2023A1515011401). The research of the second and fourth authors is supported by Higher Education Commission of Pakistan by the project no. 7495 /Punjab/NRPU/R&D/HEC/2017.

Data Availability Statement: The code used in this paper can be downloaded from https://www.mathcity.org/files/ahsan/ProcedureKsimPle.txt (accessed on 10 March 2023).

Conflicts of Interest: The authors declare no conflict of interest.

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