

Article

A New Approach for Solving Nonlinear Fractional Ordinary Differential Equations

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Abstract: Recently, researchers have been interested in studying fractional differential equations and their solutions due to the wide range of their applications in many scientific fields. In this paper, a new approach called the Hussein–Jassim (HJ) method is presented for solving nonlinear fractional ordinary differential equations. The new method is based on a power series of fractional order. The proposed approach is employed to obtain an approximate solution for the fractional differential equations. The results of this study show that the solutions obtained from solving the fractional differential equations are highly consistent with those obtained by exact solutions.

Keywords: fractional differential equations; new iterative method; convergence; new approximate method; power series

MSC: 34A08; 81Q05



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1. Introduction

Engineering-related problems, applied mathematics, and physics are significantly impacted by nonlinear phenomena. Many of these physical phenomena are represented by nonlinear differential equations. In both physics and mathematics, differential equations remain an important issue requiring innovative approaches to find precise or approximate solutions. Since the majority of brand-new linear and nonlinear equations lack an exact analytic solution, numerical techniques have mostly been employed to solve them [1].

During the past decades, fractional differential equations (FDEs) have appeared more and more frequently in different research areas, and they can offer an improved description of many vital phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, cosmology, and materials science. Consequently, considerable attention has been given to the solution of the fractional differential equations [2].

The exploration and development of numerical techniques particularly designed to solve fractional differential equations have been inspired by the growing interest in applications of fractional calculus. It is more difficult to find analytical solutions for FDEs than it is to solve conventional ordinary differential equations (ODEs), and most of the time, the answer can only be approximated numerically [3].

Mathematical models called nonlinear differential equations (NDEs) are used to explain complicated events that appear in our environment. Numerous applications of science and engineering, including those involving fluid dynamics, plasma physics, hydrodynamics, solid state physics, optical fibers, acoustics, and other fields, use nonlinear equations. Recently, numerous researchers have focused their on NDEs solutions utilizing a variety of techniques, including the Adomian decomposition method [4], the variational iteration method [5], the homotopy perturbation method [6], the homotopy analysis method [7], the differential transform method [8], the F-expansion method [9], the

Exp-function method [10], the sine–cosine method [11], the reduced differential transform method [12], the Sumudu homotopy perturbation method [13], the Sumudu Adomian decomposition method [14], the Daftardar–Jafari method [15], and others [16–39].

In this paper, a new iterative method for solving fractional ordinary differential equations (FODEs) is presented and discussed. This method is mainly based on fractional power series. In order to introduce the method, we must mention several concepts and definitions.

Definition 1 ([40–42]). If $f(x) \in C([a, b])$, $\alpha > 0$, and $a < x < b$, then the Riemann–Liouville fractional integral of order α is given by as

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt,$$

where Γ is the well-known Gamma function.

The properties of the Riemann–Liouville fractional integral are as follows:

1. $I_x^\alpha I_x^\sigma f(x) = I_x^{\alpha+\sigma} f(x);$
2. $I_x^\alpha I_x^\sigma f(x) = I_x^\sigma I_x^\alpha f(x);$
3. $I_x^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta};$

where α and σ are greater than zero and β is a real number.

Definition 2 ([41,42]). The fractional derivative of $f(x)$ in the Caputo sense is defined as

$${}_a^C D_x^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau,$$

where $m-1 < \alpha \leq m$, $m \in \mathbb{N}$.

The following are the basic properties of the operator ${}_a^C D_x^\alpha$:

1. ${}_a^C D_x^\alpha \lambda = 0;$
2. ${}_a^C D_x^\alpha I^\alpha f(x) = f(x);$
3. ${}_a^C D_x^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha};$
4. ${}_a^C D_x^\alpha {}_a^C D_x^\sigma f(x) = {}_a^C D_x^{\alpha+\sigma} f(x) = {}_a^C D_x^\sigma {}_a^C D_x^\alpha f(x);$
5. ${}_a^C D_x^\alpha (\lambda f(x) + \delta g(x)) = \lambda {}_a^C D_x^\alpha f(x) + \delta {}_a^C D_x^\alpha g(x);$
6. $I_a^\alpha {}_a^C D_x^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{t^k}{k!};$

where λ and δ are constants.

Definition 3 ([42]). The Mittag–Leffler function $E_\alpha(x)$ is defined as

$$E_\alpha(x) = \sum_{m=0}^{\infty} \frac{x^m}{\Gamma(m\alpha + 1)}.$$

For special values α , the Mittag–Leffler function is given by the following:

1. $E_0(x) = \frac{1}{1-x};$
2. $E_1(x) = e^x;$
3. $E_2(x^2) = \cosh(x);$
4. $E_2(-x^2) = \cos(x).$

2. Analysis of the New Method

Consider the following initial value problem in the FODE sense

$${}_t^C D_t^\sigma \psi(t) + \mathcal{N}(\psi(t)) = g(t^\sigma), \quad 0 < \sigma \leq 1 \quad (1)$$

with initial condition $\psi(0) = \lambda$, where ψ is an analytical function, ${}^C D_t^\sigma$ is the Caputo operator, \mathcal{N} is a nonlinear operator, and $g(t^\sigma)$ is a known function.

By taking the Riemann–Liouville operator of fractional integration to both side of Equation (1), we obtain

$$\psi(t) = \lambda + I_t^\sigma(g(t^\sigma) - \mathcal{N}(\psi(t))). \quad (2)$$

Now, we rewrite the right side of Equation (2) as an infinite fractional power series:

$$\lambda + I_t^\sigma(g(t^\sigma) - \mathcal{N}(\psi(t))) = a_0 + a_1 t^\sigma + a_2 t^{2\sigma} + \dots = \sum_{i=0}^{\infty} a_i t^{i\sigma}, \quad (3)$$

where a_0, a_1, \dots are coefficients.

By taking the fractional derivative of Caputo ${}^C D_t^{i\sigma}$ where $i = 0, 1, 2, \dots$, for both sides of Equation (3) at $t = 0$, we obtain

$$\begin{cases} a_0 = \psi(0) = \lambda, \\ a_1 = \frac{(g(t^\sigma) - \mathcal{N}(\psi(t)))_{t=0}}{\Gamma(\sigma+1)}, \\ a_2 = \frac{{}^C D_t^\sigma(g(t^\sigma) - \mathcal{N}(\psi(t)))_{t=0}}{\Gamma(2\sigma+1)}, \\ a_3 = \frac{{}^C D_t^{2\sigma}(g(t^\sigma) - \mathcal{N}(\psi(t)))_{t=0}}{\Gamma(3\sigma+1)}, \\ \vdots \end{cases} \quad (4)$$

Substituting Equation (4) into Equation (3) results in

$$\psi(t) = \sum_{i=0}^{\infty} \frac{{}^C D_t^{(i-1)\sigma}(g(t^\sigma) - \mathcal{N}(\psi(t)))_{t=0}}{\Gamma(i\sigma + 1)} t^{i\sigma}. \quad (5)$$

Now, suppose that $\psi(t)$ is a solution of Equation (5), which can be expressed as

$$\psi(t) = \sum_{i=0}^{\infty} \psi_i \quad (6)$$

Substituting Equation (6) into Equation (5) yields

$$\sum_{i=0}^{\infty} \psi_i = \sum_{i=0}^{\infty} \frac{{}^C D_t^{(i-1)\sigma}(g(t^\sigma) - \mathcal{N}(\sum_{i=0}^{\infty} \psi_i))_{t=0}}{\Gamma(i\sigma + 1)} t^{i\sigma}. \quad (7)$$

By setting $i = i + 1$ in the left side of Equation (7), we obtain

$$\psi_0 + \sum_{i=0}^{\infty} \psi_{i+1} = \lambda + \sum_{i=1}^{\infty} \frac{{}^C D_t^{(i-1)\sigma}(g(t^\sigma) - \mathcal{N}(\sum_{i=0}^{\infty} \psi_i))_{t=0}}{\Gamma(i\sigma + 1)} t^{i\sigma}. \quad (8)$$

We compare the two sides of Equation (8):

$$\begin{cases} \psi_0 = \psi(0) = \lambda, \\ \psi_1 = \frac{(g(t^\sigma) - \mathcal{N}(\psi_0))_{t=0}}{\Gamma(\sigma+1)} t^\sigma, \\ \psi_2 = \frac{{}^C D_t^\sigma(g(t^\sigma) - \mathcal{N}(\psi_0 + \psi_1))_{t=0}}{\Gamma(2\sigma+1)} t^{2\sigma}, \\ \psi_3 = \frac{{}^C D_t^{2\sigma}(g(t^\sigma) - \mathcal{N}(\psi_0 + \psi_1 + \psi_2))_{t=0}}{\Gamma(3\sigma+1)} t^{3\sigma}, \\ \vdots \\ \psi_{i+1} = \frac{{}^C D_t^{i\sigma}(g(t^\sigma) - \mathcal{N}(\sum_{j=0}^i \psi_j))_{t=0}}{\Gamma((i+1)\sigma+1)} t^{(i+1)\sigma}. \end{cases} \quad (9)$$

Therefore, the approximate solution can be formulated as follows:

$$\mathcal{Y}(t) = \mathcal{Y}_0 + \mathcal{Y}_1 + \mathcal{Y}_3 + \dots = \sum_{i=0}^{\infty} \mathcal{Y}_i.$$

3. Convergence of the New Method

Theorem 1. The proposed method used to solve Equation (1) is equivalent to determine the following sequence:

$$\begin{aligned} \mathcal{S}_{\eta} &= \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3 + \dots + \mathcal{Y}_{\eta}, \\ \mathcal{S}_0 &= 0 \end{aligned} \quad (10)$$

By using the iterative scheme:

$$\mathcal{S}_{\eta+1} = \sum_{i=1}^{\eta+1} \frac{{}^C D_t^{(i-1)\sigma} \left(g(t^\sigma) - \mathcal{N} \left(\sum_{j=0}^{i-1} \mathcal{Y}_j \right) \right)_{t=0} t^{i\sigma}}{\Gamma(i\sigma + 1)}. \quad (11)$$

Proof. For $\eta = 0$, Equation (11) can be written as

$$\mathcal{S}_1 = \frac{(g(t^\sigma) - \mathcal{N}(\mathcal{Y}_0))_{t=0} t^\sigma}{\Gamma(\sigma + 1)},$$

then,

$$\mathcal{Y}_1 = \frac{(g(t^\sigma) - \mathcal{N}(\mathcal{Y}_0))_{t=0} t^\sigma}{\Gamma(\sigma + 1)}.$$

For $\eta = 1$, Equation (11) can be expressed as

$$\begin{aligned} \mathcal{S}_2 &= \sum_{i=1}^2 \frac{{}^C D_t^{(i-1)\sigma} \left(g(t^\sigma) - \mathcal{N} \left(\sum_{j=0}^{i-1} \mathcal{Y}_j \right) \right)_{t=0} t^{i\sigma}}{\Gamma(i\sigma + 1)}, \\ &= \frac{(g(t^\sigma) - \mathcal{N}(\mathcal{Y}_0))_{t=0} t^\sigma}{\Gamma(\sigma + 1)} + \frac{{}^C D_t^\sigma \left(g(t^\sigma) - \mathcal{N}(\mathcal{Y}_0 + \mathcal{Y}_1) \right)_{t=0} t^{2\sigma}}{\Gamma(2\sigma + 1)}, \\ &= \mathcal{Y}_1 + \frac{{}^C D_t^\sigma \left(g(t^\sigma) - \mathcal{N}(\mathcal{Y}_0 + \mathcal{Y}_1) \right)_{t=0} t^{2\sigma}}{\Gamma(2\sigma + 1)} \end{aligned}$$

According to $\mathcal{S}_2 = \mathcal{Y}_1 + \mathcal{Y}_2$, the latter equation can be written as

$$\mathcal{Y}_2 = \frac{{}^C D_t^\sigma (g(t^\sigma) - \mathcal{N}(\mathcal{Y}_0 + \mathcal{Y}_1))_{t=0} t^{2\sigma}}{\Gamma(2\sigma + 1)}.$$

This theorem will be proved by strong induction. Let us assume that

$$\mathcal{Y}_{m+1} = \frac{{}^C D_t^{m\sigma} \left(g(t^\sigma) - \mathcal{N} \left(\sum_{j=0}^m \mathcal{Y}_j \right) \right)_{t=0} t^{(m+1)\sigma}}{\Gamma((m+1)\sigma + 1)},$$

where $m = 1, 2, 3, \dots, \eta - 1$, so

$$\begin{aligned} \mathcal{S}_{\eta+1} &= \sum_{i=1}^{\eta+1} \frac{{}^C D_t^{(i-1)\sigma} \left(g(t^\sigma) - \mathcal{N} \left(\sum_{j=0}^{i-1} \mathcal{Y}_j \right) \right)_{t=0} t^{i\sigma}}{\Gamma(i\sigma + 1)}, \\ &= \frac{(g(t^\sigma) - \mathcal{N}(\mathcal{Y}_0))_{t=0} t^\sigma}{\Gamma(\sigma + 1)} + \dots + \frac{{}^C D_t^{\eta\sigma} \left(g(t^\sigma) - \mathcal{N} \left(\sum_{j=0}^{\eta} \mathcal{Y}_j \right) \right)_{t=0} t^{(\eta+1)\sigma}}{\Gamma((\eta+1)\sigma + 1)}, \\ &= \mathcal{Y}_1 + \mathcal{Y}_2 + \dots + \mathcal{Y}_{\eta} + \frac{{}^C D_t^{\eta\sigma} \left(g(t^\sigma) - \mathcal{N} \left(\sum_{j=0}^{\eta} \mathcal{Y}_j \right) \right)_{t=0} t^{(\eta+1)\sigma}}{\Gamma((\eta+1)\sigma + 1)}. \end{aligned}$$

Using Equation (10), the following equation can be obtained:

$$\psi_{\eta+1} = \frac{{}^C D_t^{\eta\sigma} \left(g(t^\sigma) - \mathcal{N} \left(\sum_{j=0}^{\eta} \psi_j \right) \right)_{t=0} t^{(\eta+1)\sigma}}{\Gamma((\eta+1)\sigma+1)}.$$

Hence, the theorem is proved, as the latter equation is similar to Equation (9). \square

Theorem 2. Let \mathcal{B} be a Banach space.

I. $\sum_{i=0}^{\infty} \psi_i$ obtained by Equation (9) convergence to $\mathcal{S} \in \mathcal{B}$, if

$$\exists (0 \leq \mathcal{E} < 1), \text{ such that } (\forall \eta \in \mathbb{N} \Rightarrow \|\psi_\eta\| \leq \mathcal{E} \|\psi_{\eta-1}\|). \quad (12)$$

II. $\mathcal{S} = \sum_{\eta=1}^{\infty} \psi_\eta$, satisfies in

$$\mathcal{S} = \sum_{i=1}^{\infty} \frac{{}^C D_t^{(i-1)\sigma} (g(t^\sigma) - \mathcal{N}(\sum_{i=0}^{\infty} \psi_i))_{t=0}}{\Gamma(i\sigma+1)} t^{i\sigma}. \quad (13)$$

Proof.

I. We must prove that $\{\mathcal{S}\}_{\eta=0}^{\infty}$ is a Cauchy sequence in the Banach space \mathcal{B} .

From Equation (12), we have

$$\begin{aligned} \|\mathcal{S}_{\eta+1} - \mathcal{S}_\eta\| &= \left\| \sum_{i=0}^{\eta+1} \psi_i - \sum_{i=0}^{\eta} \psi_i \right\| = \|\psi_{\eta+1}\|, \\ &\leq \mathcal{E} \|\psi_\eta\| \leq \mathcal{E}^2 \|\psi_{\eta-1}\| \leq \mathcal{E}^3 \|\psi_{\eta-2}\| \leq \dots \leq \mathcal{E}^{\eta+1} \|\psi_0\|. \end{aligned} \quad (14)$$

For all $\eta, m \in \mathbb{N}, \eta \geq m$,

$$\begin{aligned} \|\mathcal{S}_\eta - \mathcal{S}_m\| &= \left\| (\mathcal{S}_\eta - \mathcal{S}_{\eta-1}) + (\mathcal{S}_{\eta-1} - \mathcal{S}_{\eta-2}) + \dots + (\mathcal{S}_{m+1} - \mathcal{S}_m) \right\|, \\ &\leq \|\mathcal{S}_\eta - \mathcal{S}_{\eta-1}\| + \|\mathcal{S}_{\eta-1} - \mathcal{S}_{\eta-2}\| + \dots + \|\mathcal{S}_{m+1} - \mathcal{S}_m\|, \\ &\leq \mathcal{E}^\eta \|\psi_0\| \leq \mathcal{E}^{\eta-1} \|\psi_0\| \leq \mathcal{E}^{\eta-2} \|\psi_0\| \leq \dots \leq \mathcal{E}^{m+1} \|\psi_0\|, \\ &\leq \mathcal{E}^{m+1} \|\psi_0\| (\mathcal{E}^{\eta-m-1} + \mathcal{E}^{\eta-m-2} + \dots + 1), \\ &= \frac{1-\mathcal{E}^{\eta-m}}{1-\mathcal{E}} \mathcal{E}^{m+1} \|\psi_0\|. \end{aligned} \quad (15)$$

Since $(\mathcal{E}^{\eta-m-1} + \dots + 1)$ is a geometric series and $0 \leq \mathcal{E} < 1$, then $\lim_{\eta, m \rightarrow \infty} \|\mathcal{S}_\eta - \mathcal{S}_m\| = 0$.

Thus, $\{\mathcal{S}_\eta\}$ is a Cauchy sequence in the Banach space, and it is convergent. In other words $\exists \mathcal{S} \in \mathcal{B}$, such that $\lim_{\eta \rightarrow \infty} \mathcal{S}_\eta = \sum_{\eta=1}^{\infty} \psi_\eta = \mathcal{S}$.

II. Equation (11) can be written as

$$\begin{aligned} \lim_{\eta \rightarrow \infty} \mathcal{S}_{\eta+1} &= \lim_{\eta \rightarrow \infty} \sum_{i=1}^{\eta+1} \frac{{}^C D_t^{(i-1)\sigma} (g(t^\sigma) - \mathcal{N}(\sum_{j=0}^{i-1} \psi_j))_{t=0}}{\Gamma(i\sigma+1)} t^{i\sigma}, \\ &= \sum_{i=1}^{\infty} \frac{{}^C D_t^{(i-1)\sigma} (g(t^\sigma) - \mathcal{N}(\sum_{j=0}^{i-1} \psi_j))_{t=0}}{\Gamma(i\sigma+1)} t^{i\sigma}. \end{aligned}$$

Since the upper bound of the sum approaches infinity, we obtain the following equation:

$$\mathcal{S} = \sum_{i=1}^{\infty} \frac{{}^C D_t^{(i-1)\sigma} (g(t^\sigma) - \mathcal{N}(\sum_{i=0}^{\infty} \psi_i))_{t=0}}{\Gamma(i\sigma+1)} t^{i\sigma} = \sum_{i=1}^{\infty} \psi_i.$$

\square

Theorem 3. The following equation (Equation (13))

$$\mathcal{S} = \sum_{i=1}^{\infty} \frac{{}^C D_t^{(i-1)\sigma} (g(t^\sigma) - \mathcal{N}(\sum_{i=0}^{\infty} \psi_i))_{t=0}}{\Gamma(i\sigma + 1)} t^{i\sigma}$$

is equivalent to Equation (1):

$${}^C D_t^\sigma \psi(t) + \mathcal{N}(\psi(t)) = g(t^\sigma), \quad 0 < \sigma \leq 1$$

Proof . Equation (13) can be rewritten as follows:

$$\begin{aligned} \lambda + \mathcal{S} &= \lambda + \sum_{i=1}^{\infty} \frac{{}^C D_t^{(i-1)\sigma} (g(t^\sigma) - \mathcal{N}(\sum_{i=0}^{\infty} \psi_i))_{t=0}}{\Gamma(i\sigma + 1)} t^{i\sigma}, \\ &= \sum_{i=0}^{\infty} \frac{{}^C D_t^{(i-1)\sigma} (g(t^\sigma) - \mathcal{N}(\sum_{i=0}^{\infty} \psi_i))_{t=0}}{\Gamma(i\sigma + 1)} t^{i\sigma} \end{aligned}$$

By considering $\psi = \mathcal{S} + \lambda = \sum_{i=1}^{\infty} \psi_i + \psi_0 = \sum_{i=0}^{\infty} \psi_i$, we obtain

$$\sum_{i=0}^{\infty} \psi_i = \sum_{i=0}^{\infty} \frac{{}^C D_t^{(i-1)\sigma} (g(t^\sigma) - \mathcal{N}(\sum_{i=0}^{\infty} \psi_i))_{t=0}}{\Gamma(i\sigma + 1)} t^{i\sigma}. \quad (16)$$

It seems clear that the right-hand side of Equation (16) is equivalent to the right-hand side of Equation (5), so Equation (16) can be written in the following form:

$$\psi(t) = \sum_{i=0}^{\infty} \frac{{}^C D_t^{(i-1)\sigma} (g(t^\sigma) - \mathcal{N}(\psi(t)))_{t=0}}{\Gamma(i\sigma + 1)} t^{i\sigma}. \quad (17)$$

Using Equations (3) and (4), Equation (17) can be written as

$$\psi(t) = \lambda + I_t^\sigma (g(t^\sigma) - \mathcal{N}(\psi(t))). \quad (18)$$

Taking ${}^C D_t^\sigma$ for both sides of Equation (18) results in

$${}^C D_t^\sigma \psi(t) + \mathcal{N}(\psi(t)) = g(t^\sigma).$$

Hence, the solution of Equation (13) is similar to the solution of Equation (1). \square

4. Illustrative Examples

Example 1. Consider the following fractional ordinary differential equation:

$${}^C D_t^\sigma \psi(t) - \psi(t) = 0, \quad (19)$$

with initial condition $\psi(0) = 1$.

Using the algorithm developed in the present work, Equation (19) can be expressed as

$$\psi = 1 + I_t^\sigma (\psi). \quad (20)$$

By using Equation (5), we obtain

$$\psi = \sum_{i=0}^{\infty} {}^C D_t^{i\sigma} (1 + I_t^\sigma (\psi))_{t=0} \frac{t^{i\sigma}}{\Gamma(i\sigma + 1)}. \quad (21)$$

Suppose that $\psi(t)$ is the solution of Equation (19), which can be expressed as

$$\psi = \sum_{i=0}^{\infty} \psi_i. \quad (22)$$

Substituting Equation (22) into Equation (21) results in

$$\sum_{i=0}^{\infty} \psi_i = 1 + \sum_{i=1}^{\infty} {}^C D_t^{(i-1)\sigma} \left(\sum_{j=0}^{i-1} \psi_j \right)_{t=0} \frac{t^{i\sigma}}{\Gamma(i\sigma + 1)}. \quad (23)$$

By comparing both side of Equation (23)

$$\begin{aligned} \psi_0 &= 1, \\ \psi_1 &= (\psi_0)_{t=0} \frac{t^\sigma}{\Gamma(\sigma+1)} = \frac{t^\sigma}{\Gamma(\sigma+1)}, \\ \psi_2 &= {}^C D_t^\sigma (\psi_0 + \psi_1)_{t=0} \frac{t^{2\sigma}}{\Gamma(2\sigma+1)} = \frac{t^{2\sigma}}{\Gamma(2\sigma+1)}, \\ \psi_3 &= {}^C D_t^{2\sigma} (\psi_0 + \psi_1 + \psi_2)_{t=0} \frac{t^{3\sigma}}{\Gamma(3\sigma+1)} = \frac{t^{3\sigma}}{\Gamma(3\sigma+1)}, \\ &\vdots \\ \psi_{i+1} &= {}^C D_t^{i\sigma} \left(\sum_{j=0}^i \psi_j \right)_{t=0} \frac{t^{(i+1)\sigma}}{\Gamma((i+1)\sigma+1)} = \frac{t^{(i+1)\sigma}}{\Gamma((i+1)\sigma+1)}. \end{aligned}$$

Thus, the approximate solution of Equation (19) is given by

$$\psi_a(t) = 1 + \frac{t^\sigma}{\Gamma(\sigma+1)} + \frac{t^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{t^{3\sigma}}{\Gamma(3\sigma+1)} + \cdots = \sum_{i=0}^{\infty} \frac{t^{i\sigma}}{\Gamma(i\sigma+1)},$$

Hence, the exact solution of Equation (19) at $\sigma = 1$ is $\psi_e = \exp(t)$.

Example 2. Assume the non-linear differential equation:

$${}^C D_t^\sigma \psi(t) + \psi^2(t) = 1, \quad (24)$$

with initial condition $\psi(0) = 0$.

By applying the new algorithm presented in this work, Equation (24) can be written as

$$\sum_{i=0}^{\infty} \psi_i = \sum_{i=1}^{\infty} \frac{{}^C D_t^{(i-1)\sigma} \left(1 - \left(\sum_{j=0}^{i-1} \psi_j \right)^2 \right)_{t=0}}{\Gamma(i\sigma+1)} t^{i\sigma}. \quad (25)$$

By comparing both side of Equation (25), we obtain

$$\begin{aligned} \psi_0 &= 0, \\ \psi_1 &= \left(1 - (\psi_0)^2 \right)_{t=0} \frac{t^\sigma}{\Gamma(\sigma+1)} = \frac{t^\sigma}{\Gamma(\sigma+1)}, \\ \psi_2 &= {}^C D_t^\sigma \left(1 - (\psi_0 + \psi_1)^2 \right)_{t=0} \frac{t^{2\sigma}}{\Gamma(2\sigma+1)} = 0, \\ \psi_3 &= \frac{{}^C D_t^{2\sigma} \left(1 - (\psi_0 + \psi_1 + \psi_2)^2 \right)_{t=0}}{\Gamma(3\sigma+1)} t^{3\sigma} = -\frac{\Gamma(2\sigma+1)}{\Gamma^2(\sigma+1)} \frac{t^{3\sigma}}{\Gamma(3\sigma+1)}, \\ &\vdots \\ \psi_{i+1} &= {}^C D_t^{i\sigma} \left(1 - \left(\sum_{j=0}^i \psi_j \right)^2 \right)_{t=0} \frac{t^{(i+1)\sigma}}{\Gamma((i+1)\sigma+1)}. \end{aligned}$$

Thus, the approximate solution of Equation (24) is given by

$$\psi_a = \frac{t^\sigma}{\Gamma(\sigma+1)} - \frac{\Gamma(2\sigma+1)}{\Gamma^2(\sigma+1)} \frac{t^{3\sigma}}{\Gamma(3\sigma+1)} + \cdots,$$

Therefore, the exact solution of Equation (24) at $\sigma = 1$ is given by the following formula:

$$\psi_e(t) = \frac{\exp(2t)-1}{\exp(2t)+1}$$

Example 3. Suppose that nonlinear differential equation

$${}^C D_t^\sigma \psi(t) = t^\sigma \psi^2(t), \quad 0 < \sigma \leq 1 \quad (26)$$

with initial condition $\psi(0) = 1$.

Applying the new approach presented in this work, Equation (26) can be formulated as

$$\sum_{i=0}^{\infty} \psi_i = 1 + \sum_{i=1}^{\infty} {}^C D_t^{(i-1)\sigma} \left(t^\sigma \left(\sum_{j=0}^{i-1} \psi_j \right)^2 \right)_{t=0} \frac{t^{i\sigma}}{\Gamma(i\sigma + 1)}. \quad (27)$$

By comparing both sides of Equation (27), we obtain the following:

$$\begin{aligned} \psi_0 &= 1, \\ \psi_1 &= \left(t^\sigma (\psi_0)^2 \right)_{t=0} \frac{t^\sigma}{\Gamma(\sigma+1)} = 0, \\ \psi_2 &= {}^C D_t^\sigma \left(t^\sigma (\psi_0 + \psi_1)^2 \right)_{t=0} \frac{t^{2\sigma}}{\Gamma(2\sigma+1)} = \frac{\Gamma(\sigma+1)}{\Gamma(2\sigma+1)} t^{2\sigma}, \\ \psi_3 &= {}^C D_t^{2\sigma} \left(t^\sigma (\psi_0 + \psi_1 + \psi_2)^2 \right)_{t=0} \frac{t^{3\sigma}}{\Gamma(3\sigma+1)} = 0, \\ \psi_4 &= {}^C D_t^{2\sigma} \left(t^\sigma (\psi_0 + \psi_1 + \psi_2 + \psi_3)^2 \right)_{t=0} \frac{t^{4\sigma}}{\Gamma(4\sigma+1)} = \frac{2\Gamma(\sigma+1)}{\Gamma(2\sigma+1)} \frac{\Gamma(3\sigma+1)}{\Gamma(4\sigma+1)} t^{4\sigma}, \\ &\vdots \\ \psi_{i+1} &= {}^C D_t^{i\sigma} \left(t^\sigma \left(\sum_{j=0}^i \psi_j \right)^2 \right)_{t=0} \frac{t^{(i+1)\sigma}}{\Gamma((i+1)\sigma+1)}. \end{aligned}$$

Thus, the approximate solution of Equation (26) is given by

$$\psi_a(t) = 1 + \frac{\Gamma(\sigma+1)}{\Gamma(2\sigma+1)} t^{2\sigma} + \frac{2\Gamma(\sigma+1)}{\Gamma(2\sigma+1)} \frac{\Gamma(3\sigma+1)}{\Gamma(4\sigma+1)} t^{4\sigma} + \dots$$

Therefore, the exact solution of Equation (26) at $\sigma = 1$ is given by the following formula $\psi_e(t) = \frac{2}{2-t^2}$.

Example 4. Consider the nonlinear ordinary differential equation:

$${}^C D_t^\sigma \psi(t) = \frac{2-t^\sigma}{1+\psi}, \quad 0 < \sigma \leq 1 \quad (28)$$

with initial condition $\psi(0) = 1$.

By applying the new approach, Equation (28) can be expressed as

$$\sum_{i=0}^{\infty} \psi_i = 1 + \sum_{i=1}^{\infty} {}^C D_t^{(i-1)\sigma} \left(\frac{2-t^\sigma}{1+\sum_{j=0}^{i-1} \psi_j} \right)_{t=0} \frac{t^{i\sigma}}{\Gamma(i\sigma + 1)}. \quad (29)$$

By comparing both sides of Equation (29), we obtain

$$\begin{aligned} \psi_0 &= 1, \\ \psi_1 &= \left(\frac{2-t^\sigma}{1+\psi_0} \right)_{t=0} \frac{t^\sigma}{\Gamma(\sigma+1)} = \frac{t^\sigma}{\Gamma(\sigma+1)}, \\ \psi_2 &= {}^C D_t^\sigma \left(\frac{2-t^\sigma}{1+\psi_0+\psi_1} \right)_{t=0} \frac{t^{2\sigma}}{\Gamma(2\sigma+1)} = -\frac{1+\Gamma(\sigma+1)}{2} \frac{t^{2\sigma}}{\Gamma(2\sigma+1)}, \\ &\vdots \\ \psi_{i+1} &= {}^C D_t^{i\sigma} \left(\frac{2-t^\sigma}{1+\sum_{j=0}^i \psi_j} \right)_{t=0} \frac{t^{(i+1)\sigma}}{\Gamma((i+1)\sigma+1)}. \end{aligned}$$

Thus, the approximate solution of Equation (28) is given by

$$\mathcal{Y}_a(t) = 1 + \frac{t^\sigma}{\Gamma(\sigma+1)} - \frac{1 + \Gamma(\sigma+1)}{2} \frac{t^{2\sigma}}{\Gamma(2\sigma+1)}.$$

Therefore, the exact solution of Equation (28) is given by the following formula:
 $\mathcal{Y}_e(t) = -1 + \sqrt{-t^2 + 4t + 4}.$

Remark 1. From Tables 1–4 and Figures 1–4, it is clear that the approximate solution of the new method converges with the analytical solution.

Table 1. Values of the approximate and exact solutions of Equation (19) at different values of \sqcup .

t	$\mathcal{Y}_{\sigma=0.8}$	$\mathcal{Y}_{\sigma=0.9}$	$\mathcal{Y}_{\sigma=1}$	\mathcal{Y}_e	$ \mathcal{Y}_e - \mathcal{Y}_{0.8} $	$ \mathcal{Y}_e - \mathcal{Y}_{0.9} $	$ \mathcal{Y}_e - \mathcal{Y}_1 $
0.1039	1.1818	1.1415	1.1095	1.1095	0.0723	0.0320	0.0000
0.2034	1.3258	1.2721	1.2255	1.2256	0.1002	0.0465	0.0001
0.3030	1.4719	1.4103	1.3536	1.3539	0.1180	0.0564	0.0004
0.4026	1.6267	1.5595	1.4945	1.4957	0.1310	0.0639	0.0012
0.5021	1.7939	1.7220	1.6493	1.6523	0.1416	0.0698	0.0029
0.6017	1.9762	1.8996	1.8191	1.8253	0.1509	0.0744	0.0062
0.7013	2.1761	2.0940	2.0047	2.0164	0.1597	0.0776	0.0117
0.8009	2.3957	2.3066	2.2072	2.2275	0.1682	0.0791	0.0203
0.9004	2.6369	2.5389	2.4275	2.4607	0.1762	0.0782	0.0332
1.0000	2.9016	2.7924	2.6667	2.7183	0.1833	0.0741	0.0516

Table 2. Values of the approximate and exact solutions of Equation (24) at different values of \sqcup .

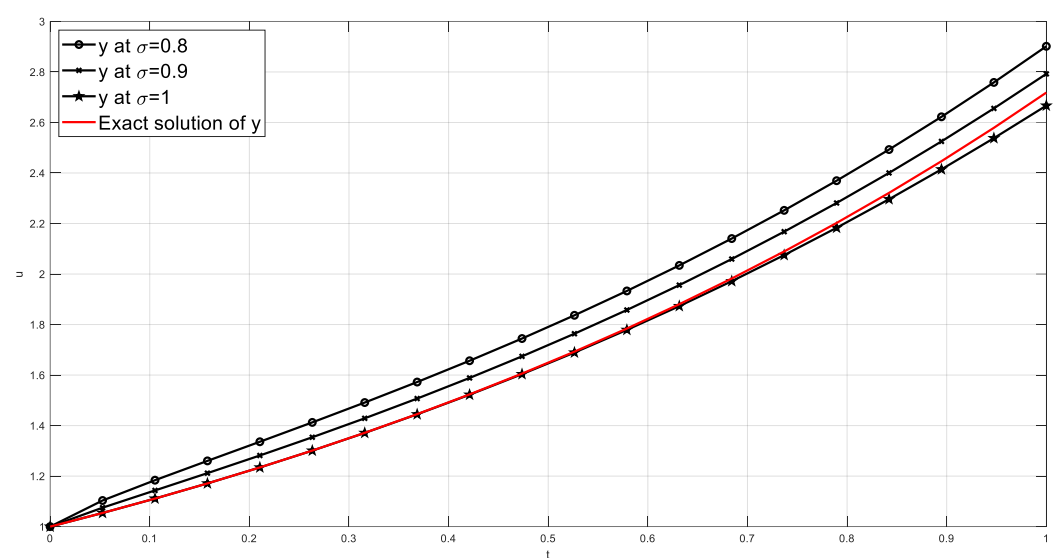
t	$\mathcal{Y}_{\sigma=0.8}$	$\mathcal{Y}_{\sigma=0.9}$	$\mathcal{Y}_{\sigma=1}$	\mathcal{Y}_e	$ \mathcal{Y}_e - \mathcal{Y}_{0.8} $	$ \mathcal{Y}_e - \mathcal{Y}_{0.9} $	$ \mathcal{Y}_e - \mathcal{Y}_1 $
0.1000	0.1697	0.1305	0.0997	0.0997	0.0701	0.0308	0.0000
0.2000	0.2927	0.2411	0.1973	0.1974	0.0954	0.0438	0.0000
0.3000	0.3979	0.3413	0.2910	0.2913	0.1065	0.0500	0.0003
0.4000	0.4875	0.4308	0.3787	0.3799	0.1076	0.0508	0.0013
0.5000	0.5614	0.5083	0.4583	0.4621	0.0993	0.0462	0.0038
0.6000	0.6180	0.5721	0.5280	0.5370	0.0809	0.0350	0.0090
0.7000	0.6555	0.6201	0.5857	0.6044	0.0511	0.0157	0.0187
0.8000	0.6717	0.6503	0.6293	0.6640	0.0077	0.0137	0.0347
0.9000	0.6645	0.6606	0.6570	0.7163	0.0518	0.0557	0.0593
1.0000	0.6314	0.6486	0.6667	0.7616	0.1302	0.1130	0.0949

Table 3. Values of the approximate and exact solutions of Equation (26) at different values of \sqcup .

t	$\Psi_{\sigma=0.8}$	$\Psi_{\sigma=0.9}$	$\Psi_{\sigma=1}$	Ψ_e	$ \Psi_e - \Psi_{0.8} $	$ \Psi_e - \Psi_{0.9} $	$ \Psi_e - \Psi_1 $
0.1180	1.0073	1.0073	1.0070	1.0070	0.0003	0.0002	0.0000
0.2160	1.0252	1.0248	1.0239	1.0239	0.0013	0.0009	0.0000
0.3140	1.0553	1.0540	1.0517	1.0519	0.0034	0.0022	0.0001
0.4120	1.1000	1.0969	1.0921	1.0927	0.0073	0.0042	0.0007
0.5100	1.1627	1.1561	1.1470	1.1495	0.0132	0.0066	0.0025
0.6080	1.2474	1.2348	1.2190	1.2267	0.0207	0.0081	0.0077
0.7060	1.3593	1.3373	1.3113	1.3319	0.0274	0.0054	0.0206
0.8040	1.5043	1.4682	1.4277	1.4776	0.0268	0.0093	0.0499
0.9020	1.6892	1.6331	1.5723	1.6858	0.0035	0.0527	0.1135
1.0000	1.9218	1.8382	1.7500	2.0000	0.0782	0.1618	0.2500

Table 4. Values of the approximate and exact solutions of Equation (28) at different values of \sqcup .

t	$\Psi_{\sigma=0.8}$	$\Psi_{\sigma=0.9}$	$\Psi_{\sigma=1}$	Ψ_e	$ \Psi_e - \Psi_{0.8} $	$ \Psi_e - \Psi_{0.9} $	$ \Psi_e - \Psi_1 $
0.1180	1.1867	1.1446	1.1110	1.1114	0.0753	0.0332	0.0004
0.2160	1.2899	1.2372	1.1927	1.1948	0.0950	0.0424	0.0022
0.3140	1.3717	1.3147	1.2647	1.2710	0.1008	0.0437	0.0063
0.4120	1.4365	1.3787	1.3271	1.3406	0.0959	0.0381	0.0134
0.5100	1.4860	1.4302	1.3800	1.4041	0.0818	0.0261	0.0242
0.6080	1.5213	1.4698	1.4232	1.4622	0.0592	0.0076	0.0390
0.7060	1.5433	1.4976	1.4568	1.5151	0.0283	0.0175	0.0583
0.8040	1.5524	1.5140	1.4808	1.5631	0.0107	0.0491	0.0823
0.9020	1.5490	1.5192	1.4952	1.6066	0.0576	0.0874	0.1114
1.0000	1.5333	1.5132	1.5000	1.6458	0.1125	0.1326	0.1458

**Figure 1.** Approximate and exact solutions of Equation (19).

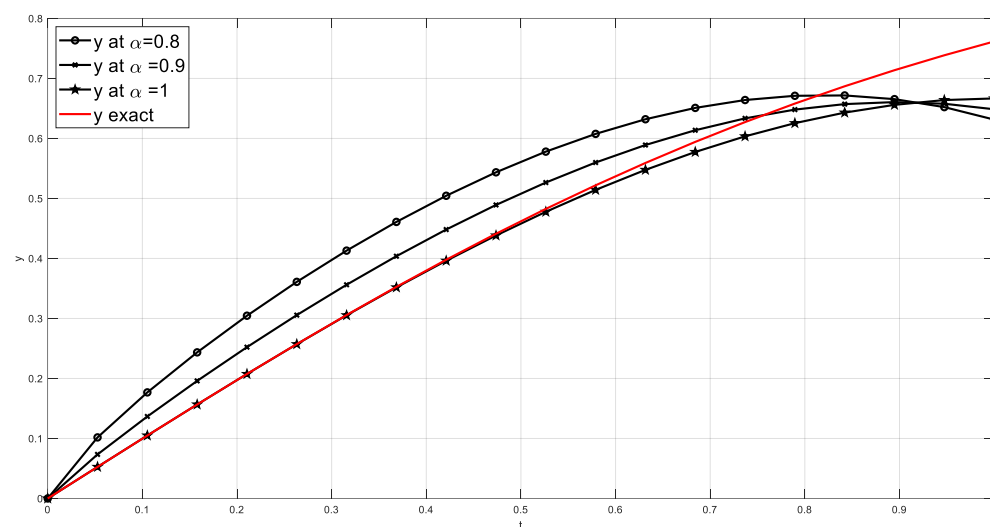


Figure 2. Approximate and exact solutions of Equation (24).

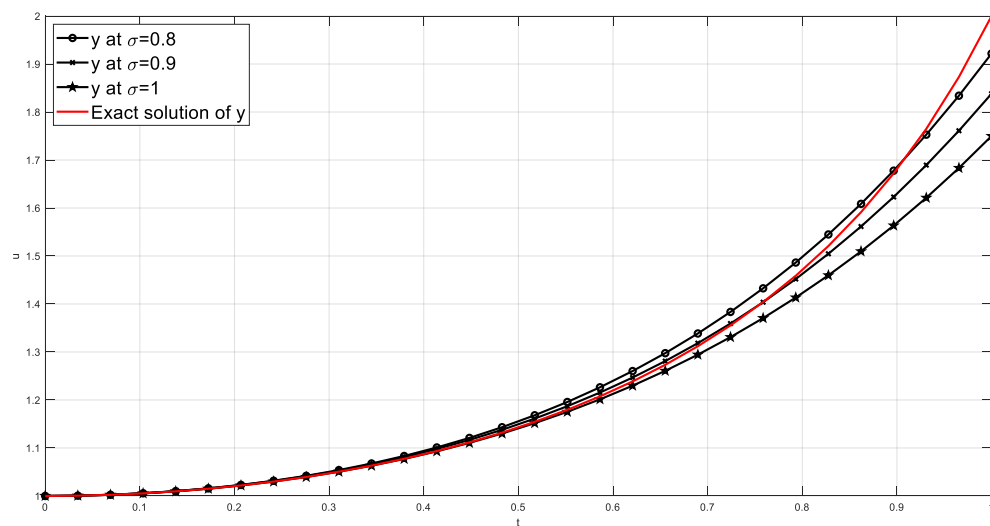


Figure 3. Approximate and exact solutions of Equation (26).

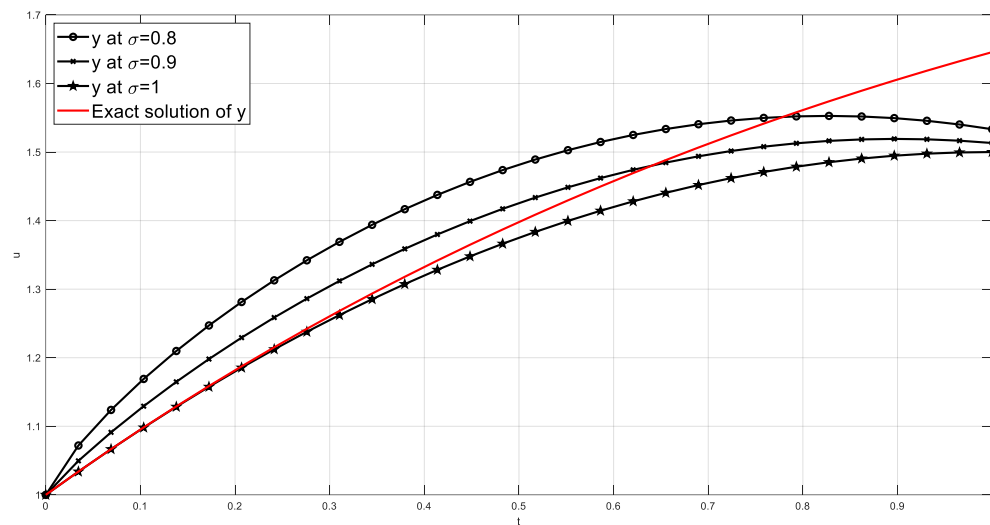


Figure 4. Approximate and exact solutions of Equation (28).

5. Conclusions

A new approach is proposed for solving fractional ordinary differential equations. The strength of this new method lies in its ability to solve different types of fractional ordinary differential equations, such as linear, nonlinear, homogeneous, and nonhomogeneous, of the order $0 < \sigma \leq 1$. The results of this study show that the approximate solutions obtained from the proposed algorithm are highly consistent with the exact solutions. Moreover, this method can be developed to solve ordinary differential equations, partial differential equations, integral equations, and fractional differential equations. It is also recommended for solving boundary value problems of differential equations.

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