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Estimating the Rate of Convergence of the PH/M/1 Model by Reducing to Quasi-Birth-Death Processes

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Abstract: We are studying the quasi-birth-death process and the property of weak ergodicity. Using the C-matrix method, we derive estimates for the rate of convergence to the limiting regime for the general case of the $PH/M/1$ model, as well as the particular case when $m = 3$. We provide a numerical example for the case $m = 3$, and construct graphs showing the probability of an empty queue and the probability of $p_1(t)$.

Keywords: bounds on the rate of convergence; C-matrix method; limiting regime; logarithmic norm method; queuing system

MSC: 60J27; 60J28; 60K25

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1. Introduction

The process of quasi-birth-death is a generalization of the usual process of birth and death. The study of such processes plays an important role in complex telecommunication systems, biology, and radio engineering. In modern software development, the microservice architecture of applications in the cloud infrastructure is increasingly used, where the receipt and service of requests can have their own characteristics. It is precisely such systems that can be studied with the help of quasi-birth-death processes.

Quasi-birth-and-death processes are used to model a wide variety of applications. Among those covered in the literature are assemble to order systems [1], production lines [2], wireless communications [3] and a variety of queuing systems [4].

In the first, the term quasi-birth-death processes (QBD processes) was introduced by V. Wallace and a computer program was written to analyze them. In the dissertation [5], the first algorithms were developed and the importance of matrix-geometric methods of solution was described. In [6], the $PH/M/c$ model was studied in the case of a finite and infinite state space, and estimates of the stationary distribution of the virtual waiting time were obtained using the matrix-geometric method. QBD processes also have applications in a multi-server queuing system, for example, see [7–11]

In this paper, we considered a single server queue, for which receipts occur in accordance with to the update process, and the service time is distributed exponentially. The intervals between receipts have the distribution $F_{m,\nu}(x)$. This is an example of a $PH/M/1$ queue for which interarrival times have an Erlang distribution. A more detailed description of the model has been described in [4]. In [12], for this model, the Toeplitz matrix was obtained, which specifies the associated random walk transition probabilities, but a numerical example with given intensity functions was not considered.

The contributions of this paper can be summarized as follows:

- Using the C-matrix method, we pass from the direct Kolmogorov system to a system of the form $\frac{d\mathbf{z}}{dt} = W(t)\mathbf{z}(t)$, for $t \geq 0$, (see [13]). Previously, this method was used to study one class of supercomputer systems in [14]. The general approach was discussed in [15]. We managed to obtain exhaustive estimates of the rate of convergence to the limiting regime for the general case of the model $PH/M/1$.
- In Section 5, we considered a particular case of the $PH/M/1$ model for $m = 3$ and obtained estimates based on the results obtained for the general case.
- Numerical examples are considered and the corresponding graphs are constructed.

In [14] considers a transitional analysis of a Markov two-server model of a supercomputer in which clients served by a random number of servers simultaneously.

One of the most important problems in the study of Markov models of queuing systems is the construction of their probabilistic characteristics. Their precise calculation is a rather difficult task even for stationary models, and in the non-stationary case, the situation becomes much more complicated, due to with which the use of approximation approaches with certain guarantees of their accuracy is inevitable. Using standard research approaches (diagonal and triangular transformations, see [16,17] for an example), it is not possible to obtain "good" estimates for the $PH/M/1$ model. There are other methods for studying quasi-birth-death processes, for example [18–20], but the comparison of methods is not the purpose of this article.

2. Basic Notions

Denote by $\|\cdot\|$ the l_1-norm and $\|\mathbf{x}\| = \sum |x_i|$, $\|B\| = \max_j \sum_i |b_{ij}|$ for matrix $B = (b_{ij})$.

Recall that a Markov chain $X(t)$ is called *weakly ergodic*, if $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $\mathbf{p}^*(0)$ and $\mathbf{p}^{**}(0)$, where $\mathbf{p}^*(t)$ and $\mathbf{p}^{**}(t)$ are the corresponding solutions of forward Kolmogorov system

$$\frac{d\mathbf{p}(t)}{dt} = A(t)\mathbf{p}(t).$$

3. Model Description

We will consider the queuing model $PH/M/1$ (Figure 1), which has the following specific features:

- The system has one server that serves customers;
- Service times are exponentially distributed;
- Intervals between arrivals have distribution $F_{m,\nu}(x)$ (Erlang distribution),

  where

$$F_{m,\nu}(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{(m-1)!}\nu^m x^{m-1} e^{-\nu x}, & x \geq 0, m \geq 1. \end{cases} \tag{1}$$

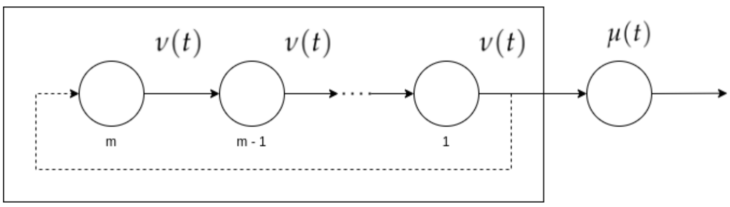

Figure 1. Transition graph for the $PH/M/1$ model.

This queue can be explored using a two-dimensional Markov process (we will reduce it to a one-dimensional process) $\{(N(t), \phi(t)), t \geq 0\}$ on the state space $\cup_{n\geq 0} l(n)$, with $l(n) = \{(n,1), ..., (n,m)\}$ for all $n \geq 0$. Let $N(t)$ represent the number of customers in the system and $\phi(t)$ be the position of the token at time t.

We assume that the following transitions are possible for a two-dimensional Markov process:

- $(n,j) \to (n-1,j)$ with intensity function $\mu(t)$ for $n \geq 1$;
- $(n,j) \to (n,j-1)$ with intensity function $\nu(t)$ for $n \geq 0, j \geq 2$;
- $(n,1) \to (n+1,m)$ with intensity function $\nu(t)$ for $n \geq 0$.

The intensity matrix has the following form:

$$Q(t) = \begin{pmatrix} \Omega & \omega\cdot\Psi & 0 & 0 & \cdots \\ \mu(t)I & \Omega-\mu(t)I & \omega\cdot\Psi & 0 & \cdots \\ 0 & \mu(t)I & \Omega-\mu(t)I & \omega\cdot\Psi & \cdots \\ 0 & 0 & \mu(t)I & \Omega-\mu(t)I & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2}$$

where the vectors ω and Ψ and the matrix Ω of order m are defined as follows:

$$\Psi = (0,0,\cdots,1),$$

$$\omega = \begin{pmatrix} \nu \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$\Omega = \begin{pmatrix} -\nu(t) & 0 & 0 & \cdots & 0 \\ \nu(t) & -\nu(t) & 0 & \cdots & 0 \\ 0 & \nu(t) & -\nu(t) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \nu(t) & -\nu(t) \end{pmatrix},$$

We write the matrix $A = Q^T$

$$A(t) = \left(\begin{array}{cccc|cccc|cccc|c} -\nu(t) & \nu(t) & \cdots & 0 & \mu(t) & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & -\nu(t) & \cdots & \vdots & 0 & \mu(t) & \cdots & \vdots & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \nu(t) & \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & \cdots & -\nu(t) & 0 & 0 & \cdots & \mu(t) & 0 & 0 & \cdots & 0 & \cdots \\ \hline 0 & 0 & \cdots & 0 & -\nu(t)-\mu(t) & \nu(t) & \cdots & 0 & \mu(t) & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & -\nu(t)-\mu(t) & \cdots & \vdots & 0 & \mu(t) & \cdots & \vdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \nu(t) & \vdots & \vdots & \ddots & 0 & \cdots \\ \nu(t) & 0 & \cdots & 0 & 0 & 0 & \cdots & -\nu(t)-\mu(t) & 0 & 0 & \cdots & \mu(t) & \cdots \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -\nu(t)-\mu(t) & \nu(t) & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & -\nu(t)-\mu(t) & \cdots & \vdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \nu(t) & \cdots \\ 0 & 0 & \cdots & 0 & \nu(t) & 0 & \cdots & 0 & 0 & 0 & \cdots & -\nu(t)-\mu(t) & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right), \tag{3}$$

Denote the time–dependent probability distribution of the Markov process by $p_{n,j}(t)$ i.e.,

$$p_{n,j}(t) = P\{N(t) = n, \phi(t) = j\},$$

Put $k = n \cdot m + j - 1$ and $p_k = P\{N(t) = n, \phi(t) = j\}$ for any $1 \leq j \leq m$ and $n \geq 0$. Hence, we obtain $\mathbf{p}(t) = (p_0(t), p_1(t), \cdots)^T$.

Now, we have the forward Kolmogorov system for $X(t)$, which has the form

$$\frac{d\mathbf{p}(t)}{dt} = A(t)\mathbf{p}(t). \tag{4}$$

4. Bounds for the General Case

Consider the transposed intensity matrix A, which has the form (3). To obtain estimates for the rate of convergence to the limiting regime, we will use the C-matrix method, which was described in detail in [15].

As in [21], we consider two solutions $\mathbf{p}^*(t)$ and $\mathbf{p}^{**}(t)$ of the forward Kolmogorov system (4) and the corresponding different initial conditions $\mathbf{p}^*(0)$, $\mathbf{p}^{**}(0)$. Hence, their difference $\mathbf{z}(t) = \mathbf{p}^*(t) - \mathbf{p}^{**}(t) = (z_0(t), z_1(t), ...)$ satisfies the equation

$$\frac{d\mathbf{z}}{dt} = A(t)\mathbf{z}(t), \quad t \geq 0. \tag{5}$$

Notice that $\sum_{i=0}^{\infty} z_i(t) = 0$.

Then, one can add $c\sum_{i=0}^{\infty} z_i(t) = 0$ (for any c) to the equation $\frac{dz_0}{dt} = \sum_{j=0}^{\infty} a_{0j} z_j$. Now, rewrite the system (5) in the form

$$\frac{d\mathbf{z}}{dt} = W(t)\mathbf{z}(t), \quad t \geq 0,$$

where $W(t) = A(t) - C(t)$ and $C(t)$ has the form

$$\begin{pmatrix} c(t) & c(t) & c(t) & c(t) & \cdots & c(t) & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and

$$W(t) = \left(\begin{array}{cccc|cccc|ccccc}
-\nu(t)-c(t) & \nu(t)-c(t) & \cdots & -c(t) & \mu(t)-c(t) & -c(t) & \cdots & -c(t) & -c(t) & -c(t) & \cdots & -c(t) & \cdots \\
0 & -\nu(t) & \cdots & \vdots & 0 & \mu(t) & \cdots & \vdots & 0 & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \ddots & \nu(t) & \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots & \cdots \\
0 & 0 & \cdots & -\nu(t) & 0 & 0 & \cdots & \mu(t) & 0 & 0 & \cdots & 0 & \cdots \\
\hline
0 & 0 & \cdots & 0 & -\nu(t)-\mu(t) & \nu(t) & \cdots & 0 & \mu(t) & 0 & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & -\nu(t)-\mu(t) & \cdots & \vdots & 0 & \mu(t) & \cdots & \vdots & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \nu(t) & \vdots & \vdots & \ddots & 0 & \cdots \\
\nu(t) & 0 & \cdots & 0 & 0 & 0 & \cdots & -\nu(t)-\mu(t) & 0 & 0 & \cdots & \mu(t) & \cdots \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -\nu(t)-\mu(t) & \nu(t) & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & -\nu(t)-\mu(t) & \cdots & \vdots & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \nu(t) & \cdots \\
0 & 0 & \cdots & 0 & \nu(t) & 0 & \cdots & 0 & 0 & 0 & \cdots & -\nu(t)-\mu(t) & \cdots \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right), \tag{6}$$

Theorem 1. *Let there exist sequence d_k and a function $c(t)$ such that $c(t) \leq \nu(t)$ and $\mu(t) > \nu(t)$ for almost all $t \geq 0$. Then, $X(t)$ is weakly ergodic and*

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq e^{-\int\limits_0^t \beta_*(\tau)d\tau} \|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|$$

for any initial conditions $\mathbf{p}^*(0), \mathbf{p}^{**}(0)$ and all $t \geq 0$ and $\beta_* = \inf a_k(t)$, if all $a_k(t) > 0$, $k \geq 0$.

Proof. Let d_k $(k \geq 0)$ be some sequence. Consider the diagonal matrix $D = diag(d_0, d_1, d_2, \ldots)$. Denote $\|\mathbf{z}\|_{1D} = \|D\mathbf{z}\|_1$.

Hence, DWD^{-1} has the form:

$$\left(\begin{array}{cccc|cccc|ccccc}
-\nu(t)-c(t) & \frac{d_0}{d_1}(\nu(t)-c(t)) & \cdots & -\frac{d_0}{d_{m-1}}c(t) & \frac{d_0}{d_m}(\mu(t)-c(t)) & -\frac{d_0}{d_{m+1}}c(t) & \cdots & -\frac{d_0}{d_{2m-1}}c(t) & -\frac{d_0}{d_{2m}}c(t) & -\frac{d_0}{d_{2m+1}}c(t) & \cdots & -\frac{d_0}{d_{3m-1}}c(t) & \cdots \\
0 & -\nu(t) & \cdots & \vdots & 0 & \frac{d_1}{d_{m+1}}\mu(t) & \cdots & \vdots & 0 & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \ddots & \frac{d_{m-2}}{d_{m-1}}\nu(t) & \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots & \cdots \\
0 & 0 & \cdots & -\nu(t) & 0 & 0 & \cdots & \frac{d_{m-1}}{d_{2m-1}}\mu(t) & 0 & 0 & \cdots & 0 & \cdots \\
\hline
0 & 0 & \cdots & 0 & -\nu(t)-\mu(t) & \frac{d_m}{d_{m+1}}\nu(t) & \cdots & 0 & \frac{d_m}{d_{2m}}\mu(t) & 0 & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & -\nu(t)-\mu(t) & \cdots & \vdots & 0 & \frac{d_{m+1}}{d_{2m+1}}\mu(t) & \cdots & \vdots & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \frac{d_{2m-2}}{d_{2m-1}}\nu(t) & \vdots & \vdots & \ddots & 0 & \cdots \\
\frac{d_{2m-1}}{d_0}\nu(t) & 0 & \cdots & 0 & 0 & 0 & \cdots & -\nu(t)-\mu(t) & 0 & 0 & \cdots & \frac{d_{2m-1}}{d_{3m-1}}\mu(t) & \cdots \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -\nu(t)-\mu(t) & \frac{d_{2m}}{d_{2m+1}}\nu(t) & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & -\nu(t)-\mu(t) & \cdots & \vdots & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \frac{d_{3m-2}}{d_{3m-1}}\nu(t) & \cdots \\
0 & 0 & \cdots & 0 & \frac{d_{3m-1}}{d_m}\nu(t) & 0 & \cdots & 0 & 0 & 0 & \cdots & -\nu(t)-\mu(t) & \cdots \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right), \tag{7}$$

$$\gamma(W(t))_{1D} = \gamma\left(DW(t)D^{-1}\right) = -\beta_*(t),$$

$\beta_*(t) = \inf \alpha_k$, where

$$\alpha_k(t) = \begin{cases} \left(1 - \frac{d_{2m-1}}{d_0}\right)\nu(t) + c(t),\ k = 0; \\ \left(1 - \frac{d_0}{d_1}\right)\nu(t) + c(t)\frac{d_0}{d_1},\ k = 1 \\ \left(1 - \frac{d_{k-1}}{d_k}\right)\nu(t) - c(t)\frac{d_0}{d_k},\ 2 \le k \le m-1 \\ \left(1 - \frac{d_0}{d_k}\right)\mu(t) + \left(1 - \frac{d_{3k-1}}{d_k}\right)\nu(t) + c(t)\frac{d_0}{d_k},\ k = m \\ \left(1 - \frac{d_{k-m}}{d_k}\right)\mu(t) + \left(1 - \frac{d_{k-1}}{d_k}\right)\nu(t) - c(t)\frac{d_0}{d_k},\ k \equiv r(\text{mod } m); k > m; r \ne 0 \\ \left(1 - \frac{d_{k-m}}{d_k}\right)\mu(t) + \left(1 - \frac{d_{k+2m-1}}{d_k}\right)\nu(t) - c(t)\frac{d_0}{d_k},\ k \equiv 0(\text{mod } m); k > m \end{cases} \tag{8}$$

Therefore, by Theorem 1, see [15] for details, we obtain

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \le e^{-\int\limits_0^t \beta_*(\tau)d\tau} \|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|.$$

□

In order to use Theorem 1, we need to define the sequence d_k. Recall that, by Euclid's division lemma, the number k $(k \ge 0)$ can be represented as

$$k = bq + r, 0 \le r \le b,$$

where r is the remainder of the division by b.

Let $d_0 = 1$ and let there exist positive numbers $\delta_1, \delta_2, ..., \delta_m$ and $(\delta_1\delta_2 \cdot ... \cdot \delta_m) \ge 1$ Therefore, d_k has the following form:

$$d_k = \begin{cases} d_0 = 1, k = 0 \\ d_1 = \delta_1, k = 1 \\ d_2 = \delta_1\delta_2, k = 2 \\ \dots \\ d_k = (\delta_1\delta_2 \cdot ... \cdot \delta_m)^{\frac{k}{m}}, \text{if } k \equiv 0(\text{mod } m); k \ge m \\ d_k = (\delta_1\delta_2 \cdot ... \cdot \delta_r)^{q+1}(\delta_{r+1} \cdot ... \cdot \delta_m)^q, \text{if } k \equiv r(\text{mod } m); r \ne 0; k \ge m \\ \dots \end{cases} \tag{9}$$

Remark 1. *The sequence d_k can be given in another form, but it is important that the condition $a_k(t) > 0$ $(k \ge 0)$ be satisfied.*

5. Bounds for the Case $m = 3$

In this section, we consider a special case of the $PH/M/1$ model when $m = 3$. We will use the results and approach from Section 4. Let the transposed intensity matrix have the following form:

$$A(t) = \begin{pmatrix}
-\nu(t) & \nu(t) & 0 & \mu(t) & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & -\nu(t) & \nu(t) & 0 & \mu(t) & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & -\nu(t) & 0 & 0 & \mu(t) & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -\nu(t)-\mu(t) & \nu(t) & 0 & \mu(t) & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & -\nu(t)-\mu(t) & \nu(t) & 0 & \mu(t) & 0 & \cdots \\
\nu(t) & 0 & 0 & 0 & 0 & -\nu(t)-\mu(t) & 0 & 0 & \mu(t) & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & -\nu(t)-\mu(t) & \nu(t) & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\nu(t)-\mu(t) & \nu(t) & \cdots \\
0 & 0 & 0 & \nu(t) & 0 & 0 & 0 & 0 & -\nu(t)-\mu(t) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad (10)$$

To obtain estimates for the rate of convergence to the limiting regime, we will use the approach from the previous section and Theorem 1.

$$W(t) = \begin{pmatrix}
-\nu(t)-c(t) & \nu(t)-c(t) & -c(t) & \mu(t)-c(t) & -c(t) & -c(t) & -c(t) & -c(t) & -c(t) & \cdots \\
0 & -\nu(t) & \nu(t) & 0 & \mu(t) & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & -\nu(t) & 0 & 0 & \mu(t) & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -\nu(t)-\mu(t) & \nu(t) & 0 & \mu(t) & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & -\nu(t)-\mu(t) & \nu(t) & 0 & \mu(t) & 0 & \cdots \\
\nu(t) & 0 & 0 & 0 & 0 & -\nu(t)-\mu(t) & 0 & 0 & \mu(t) & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & -\nu(t)-\mu(t) & \nu(t) & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\nu(t)-\mu(t) & \nu(t) & \cdots \\
0 & 0 & 0 & \nu(t) & 0 & 0 & 0 & 0 & -\nu(t)-\mu(t) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad (11)$$

Let d_k $(k = 0, 1 \dots)$ be a sequence. Again, consider the diagonal matrix $D = diag(d_0, d_1, d_2, \dots)$. Denote $\|\mathbf{z}\|_{1D} = \|D\mathbf{z}\|_1$.

$$DWD^{-1} = \begin{pmatrix}
-\nu(t)-c(t) & \frac{d_0}{d_1}(\nu(t)-c(t)) & -\frac{d_0}{d_2}c(t) & \frac{d_0}{d_3}(\mu(t)-c(t)) & -\frac{d_0}{d_4}c(t) & -\frac{d_0}{d_5}c(t) & -\frac{d_0}{d_6}c(t) & -\frac{d_0}{d_7}c(t) & -\frac{d_0}{d_8}c(t) & \cdots \\
0 & -\nu(t) & \frac{d_1}{d_2}\nu(t) & 0 & \frac{d_1}{d_4}\mu(t) & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & -\nu(t) & 0 & 0 & \frac{d_2}{d_5}\mu(t) & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -\nu(t)-\mu(t) & \frac{d_3}{d_4}\nu(t) & 0 & \frac{d_3}{d_6}\mu(t) & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & -\nu(t)-\mu(t) & \frac{d_4}{d_5}\nu(t) & 0 & \frac{d_4}{d_7}\mu(t) & 0 & \cdots \\
\frac{d_5}{d_0}\nu(t) & 0 & 0 & 0 & 0 & -\nu(t)-\mu(t) & 0 & 0 & \frac{d_5}{d_8}\mu(t) & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & -\nu(t)-\mu(t) & \frac{d_6}{d_7}\nu(t) & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\nu(t)-\mu(t) & \frac{d_7}{d_8}\nu(t) & \cdots \\
0 & 0 & 0 & \frac{d_8}{d_3}\nu(t) & 0 & 0 & 0 & 0 & -\nu(t)-\mu(t) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad (12)$$

Theorem 2. *Let there exist sequence d_k and a function $c(t)$ such that $c(t) \le \nu(t)$ and $\mu(t) > \nu(t)$ for almost all $t \ge 0$. Then, $X(t)$ is weakly ergodic and*

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \le e^{-\int\limits_0^t \beta_*(\tau)d\tau} \|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|$$

for any initial conditions $\mathbf{p}^*(0), \mathbf{p}^{**}(0)$ and all $t \ge 0$ and $\beta_* = \inf a_k(t)$, if all $a_k(t) > 0$, $k \ge 0$.

Proof. $\beta_*(t) = \inf \alpha_k$, where

$$\alpha_k(t) = \begin{cases} \left(1 - \frac{d_5}{d_0}\right)\nu(t) + c(t),\ k = 0 \\ \left(1 - \frac{d_0}{d_1}\right)\nu(t) + c(t)\frac{d_0}{d_1},\ k = 1 \\ \left(1 - \frac{d_1}{d_2}\right)\nu(t) - c(t)\frac{d_0}{d_2},\ k = 2 \\ \left(1 - \frac{d_0}{d_3}\right)\mu(t) + \left(1 - \frac{d_8}{d_3}\right)\nu(t) + c(t)\frac{d_0}{d_3},\ k = 3 \\ \left(1 - \frac{d_{k-3}}{d_k}\right)\mu(t) + \left(1 - \frac{d_{k+5}}{d_k}\right)\nu(t) - c(t)\frac{d_0}{d_k},\ k \equiv 0 (\mathrm{mod}\ 3); k > 3 \\ \left(1 - \frac{d_{k-3}}{d_k}\right)\mu(t) + \left(1 - \frac{d_{k-1}}{d_k}\right)\nu(t) - c(t)\frac{d_0}{d_k},\ k \equiv r (\mathrm{mod}\ 3); r \neq 0; k > 3 \end{cases}$$

Recall that, by Euclid's division lemma, the number k can be represented as

$$k = bq + r, 0 \leq r \leq b,$$

where r is the remainder of division by b.

Let $d_0 = 1$ and let there exist positive number $\delta_1, \delta_2, \delta_3$. Then, the sequence d_k will be in the form

$$d_k = \begin{cases} d_0 = 1, k = 0 \\ d_1 = \delta_1, k = 1 \\ d_2 = \delta_1\delta_2, k = 2 \\ \dots \\ d_k = (\delta_1)^{\frac{k}{3}}(\delta_2)^{\frac{k}{3}}(\delta_3)^{\frac{k}{3}}, \text{if } k \equiv 0 (\mathrm{mod}\ 3); k \geq 3 \\ d_k = (\delta_1)^{q+1}(\delta_2)^{q}(\delta_3)^{q}, \text{if } k \equiv 1 (\mathrm{mod}\ 3); k \geq 4 \\ d_k = (\delta_1)^{q+1}(\delta_2)^{q+1}(\delta_3)^{q}, \text{if } k \equiv 2 (\mathrm{mod}\ 3); k \geq 5 \\ \dots \\ \end{cases} \tag{13}$$

Hence

$$\alpha_k(t) = \begin{cases} (1 - \delta_1^2\delta_2^2\delta_3)\nu(t) + c(t),\ k = 0 \\ \left(1 - \frac{1}{\delta_1}\right)\nu(t) + c(t)\frac{1}{\delta_1},\ k = 1 \\ \left(1 - \frac{1}{\delta_2}\right)\nu(t) - c(t)\frac{1}{\delta_1\delta_2},\ k = 2 \\ \left(1 - \frac{1}{\delta_1\delta_2\delta_3}\right)\mu(t) + (1 - \delta_1^2\delta_2^2\delta_3)\nu(t) + c(t)\frac{1}{\delta_1\delta_2\delta_3},\ k = 3 \\ \left(1 - \frac{1}{\delta_1\delta_2\delta_3}\right)\mu(t) + \left(1 - \frac{1}{\delta_1}\right)\nu(t) - c(t)\frac{1}{(\delta_1)^{q+1}(\delta_2)^{q}(\delta_3)^{q}},\ k \equiv 1 (\mathrm{mod}\ 3); k > 3, q \geq 1 \\ \left(1 - \frac{1}{\delta_1\delta_2\delta_3}\right)\mu(t) + \left(1 - \frac{1}{\delta_2}\right)\nu(t) - c(t)\frac{1}{(\delta_1)^{q+1}(\delta_2)^{q+1}(\delta_3)^{q}},\ k \equiv 2 (\mathrm{mod}\ 3); k > 3, q \geq 1 \\ \left(1 - \frac{1}{\delta_1\delta_2\delta_3}\right)\mu(t) + (1 - \delta_1^2\delta_2^2\delta_3)\nu(t) - c(t)\frac{1}{(\delta_1)^{\frac{k}{3}}(\delta_2)^{\frac{k}{3}}(\delta_3)^{\frac{k}{3}}},\ k \equiv 0 (\mathrm{mod}\ 3); k > 3 \end{cases} \geq$$

$$\geq \begin{cases} (1 - \delta_1^2\delta_2^2\delta_3)\nu(t) + c(t),\ k = 0 \\ \left(1 - \frac{1}{\delta_1}\right)\nu(t) + c(t)\frac{1}{\delta_1},\ k = 1 \\ \left(1 - \frac{1}{\delta_2}\right)\nu(t) - c(t)\frac{1}{\delta_1\delta_2},\ k = 2 \\ \left(1 - \frac{1}{\delta_1\delta_2\delta_3}\right)\mu(t) + (1 - \delta_1^2\delta_2^2\delta_3)\nu(t) + c(t)\frac{1}{\delta_1\delta_2\delta_3},\ k = 3 \\ \left(1 - \frac{1}{\delta_1\delta_2\delta_3}\right)\mu(t) + \left(1 - \frac{1}{\delta_1}\right)\nu(t) - c(t)\frac{1}{\delta_1^2\delta_2\delta_3},\ k \equiv 1 (\mathrm{mod}\ 3); k > 3 \\ \left(1 - \frac{1}{\delta_1\delta_2\delta_3}\right)\mu(t) + \left(1 - \frac{1}{\delta_2}\right)\nu(t) - c(t)\frac{1}{\delta_1^2\delta_2^2\delta_3},\ k \equiv 2 (\mathrm{mod}\ 3); k > 3 \\ \left(1 - \frac{1}{\delta_1\delta_2\delta_3}\right)\mu(t) + (1 - \delta_1^2\delta_2^2\delta_3)\nu(t) - c(t)\frac{1}{\delta_1^2\delta_2^2\delta_3^2},\ k \equiv 0 (\mathrm{mod}\ 3); k > 3 \end{cases}$$

□

6. Example 1

Let d_k have the form (13) and

$\delta_1 = 0.6,$
$\delta_2 = 2,$
$\delta_3 = 1,$
$\nu(t) = 1 + 0.1\sin t,$
$\mu(t) = 30 + \cos t,$
$c(t) = 0.5.$

To obtain an estimate of the rate of convergence to the limiting regime, we use Theorem 2. We obtain following system:

$$\alpha_k(t) = \begin{cases} (1 - 0.6^2 \cdot 2^2 \cdot 1)(1 + 0.1\sin t) + 0.5, \ k = 0 \\ \left(1 - \frac{1}{0.6}\right)(1 + 0.1\sin t) + \frac{0.5}{2}, \ k = 1 \\ \left(1 - \frac{1}{2}\right)(1 + 0.1\sin t) - \frac{0.5}{0.6 \cdot 2}, \ k = 2 \\ \left(1 - \frac{1}{0.6\cdot 2\cdot 1}\right)(30 + \cos t) + (1 - 0.6^2 \cdot 2^2 \cdot 1)(1 + 0.1\sin t) + \frac{0.5}{0.6\cdot 2\cdot 1}, \ k = 3 \\ \left(1 - \frac{1}{0.6\cdot 2\cdot 1}\right)(30 + \cos t) + \left(1 - \frac{1}{0.6}\right)(1 + 0.1\sin t) - \frac{0.5}{0.6^2\cdot 2\cdot 1}, \ k \equiv 1 (\mathrm{mod}\ 3); k > 3 \\ \left(1 - \frac{1}{0.6\cdot 2\cdot 1}\right)(30 + \cos t) + \left(1 - \frac{1}{2}\right)(1 + 0.1\sin t) - \frac{0.5}{0.6^2\cdot 2^2\cdot 1}, \ k \equiv 2 (\mathrm{mod}\ 3); k > 3 \\ \left(1 - \frac{1}{0.6\cdot 2\cdot 1}\right)(30 + \cos t) + (1 - 0.6^2 \cdot 2^2 \cdot 1)(1 + 0.1\sin t) - \frac{0.5}{0.6^2\cdot 2^2\cdot 1^2}, \ k \equiv 0 (\mathrm{mod}\ 3); k > 3 \end{cases} \tag{14}$$

Recall that it is necessary that all $a_k > 0$

$$\inf_k(a_k(t)) \geq \begin{cases} 0.016, \ k = 0 \\ 0.1, \ k = 1 \\ 0.033, \ k = 2 \\ 4.804, \ k = 3 \\ 3.459, \ k \equiv 1 (\mathrm{mod}\ 3); k > 3 \\ 4.978, \ k \equiv 2 (\mathrm{mod}\ 3); k > 3 \\ 4.040, \ k \equiv 0 (\mathrm{mod}\ 3); k > 3 \end{cases} \tag{15}$$

$$\beta_*(t) = \inf_k(a_k(t)) = 0.016$$

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq e^{-0.016t}\|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|$$

Now, we will build graphs (Figures 2–5) by choosing the dimension of the system equal to 90

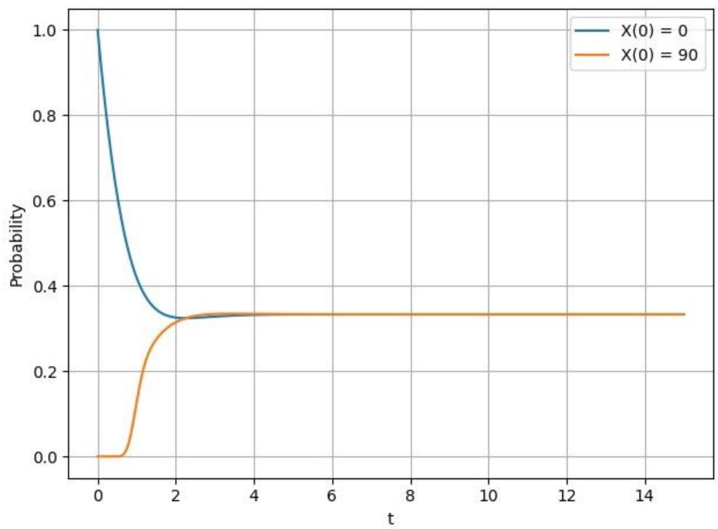

Figure 2. Probability of an empty queuing system for $t \in [0, 14]$.

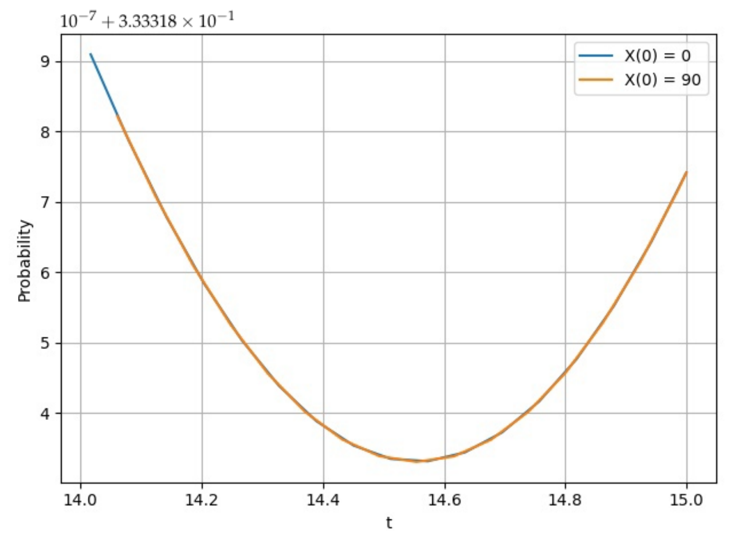

Figure 3. Approximation of the limiting probability of an empty queuing system for $t \in [14, 15]$.

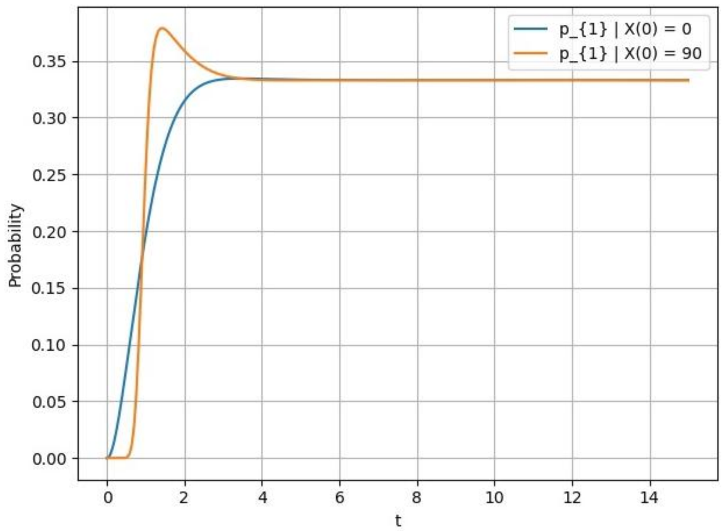

Figure 4. Probability $p_1(t)$ for $t \in [0, 14]$; this figure shows the rate of convergence.

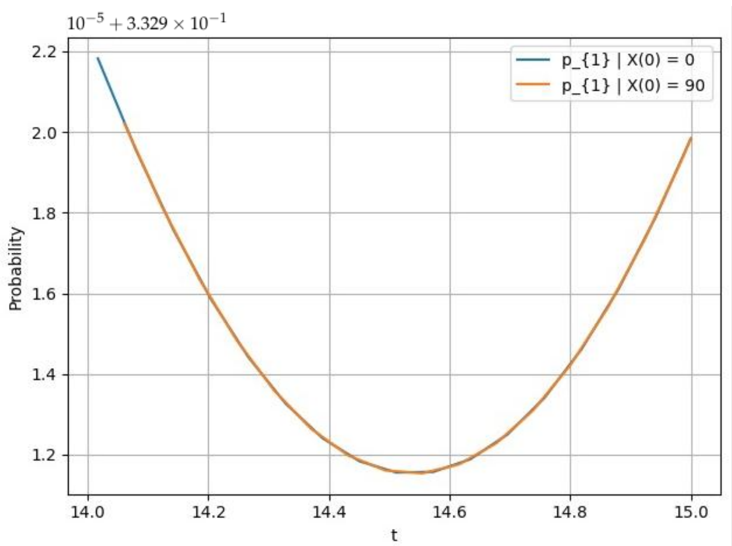

Figure 5. Approximation of the limiting probability $p_1(t)$ for $t \in [14, 15]$.

7. Conclusions

We managed to obtain exhaustive estimates of the rate of convergence to the limiting regime for the $PH/M/1$ model using the C-matrix method and considered a numerical example where we solved the forward Kolmogorov system using the fourth-order Runge–Kutta method. Theorem 1 shows that the accuracy of the main estimate depends on the choice of the sequence d_k, for $k \geq 0$.

This method can probably be applied to the $M/PH/c$ or $PH/PH/c$ models as well. This may be an area for further research.

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