mathematics

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Distributed Optimization Control for Heterogeneous Multiagent Systems under Directed Topologies

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Abstract: This paper focuses on the solutions for the distributed optimization coordination problem (DOCP) for heterogeneous multiagent systems under directed topologies. To begin with, a different convex optimization problem is proposed, which implies a weighted average of the objective function of each agent. Sufficient conditions are set to ensure the unique solution for the DOCP. Then, despite the external disruption, a distributed control mechanism is constructed to drive the state of each agent to the auxiliary state in a finite time. Furthermore, it is demonstrated that the outputs of all agents can achieve the optimal value, ensuring global convergence. Moreover, the controller design rule is expanded with event-triggered communication, and there is no Zeno behavior. Finally, to exemplify the usefulness of the theoretical conclusions, a simulation example is offered.

Keywords: distributed optimization coordination; directed topologies; event-triggered; heterogeneous multiagent systems

MSC: 93A14; 90B18; 68T42

Citation: Wang, J.; Liu, D.; Feng, J.; Zhao,Y. Distributed Optimization Control for Heterogeneous Multiagent Systems under Directed Topologies. *Mathematics* **2023**, *11*, 1479. https://doi.org/10.3390/math11061479

Academic Editor: Jean-Charles Pinoli

Received: 11 February 2023
Revised: 10 March 2023
Accepted: 13 March 2023
Published: 17 March 2023

1. Introduction

The distributed cooperative control of multiagent systems (MASs) has received a substantial amount of attention in recent years due to its wide range of applications in consensus [1,2], the rendezvous of mobile vehicles [3], cooperative monitoring [4,5], the synchronous operation of distant generators in the power grid [6], and other domains. In recent years, consensus as a material collective behavior in MASs has received a substantial amount of attention from academics and practitioners alike [7–10]. On account of the conceivable utilization of this technology in various fields of engineering, physics, mathematics, and sociology, substantial effort has been concentrated on designing the distributed protocols to achieve consensus under various considerations.

Amidst the extensive investigation of MASs, the distributed optimization problem (DOP), in which all agents reach the ideal value that minimizes the total of local objective functions [11], is one of the most rudimental and widely studied problems. The DOP is widely used in a variety of real-world applications [12,13]. There have been numerous studies about discrete-time and continuous-time optimization protocols, including the gradient algorithms in [14,15] and the Lagrangian-based algorithms in [16–18], to solve the DOP. In [19], distributed subgradient methods were proposed to solve the DOP. The adaptive optimization problem was studied in [20] based on an innovative distributed adaptive algorithm with irregular gradient gains. A fully distributed optimal algorithm was proposed in [21] for the continuous-time MASs. The optimization problem was turned into a tracking problem by estimating the property of the global optimal state in [22].

The traditional DOP was recently extended to the distributed optimum coordination problem (DOCP) in [23,24] by regarding the conception of the virtual-objective systems. DOCP stands for the collaboration of various continuous-time objective systems for optimum overall performance. As pointed out in [24], the existing distributed optimization algorithms cannot be straightly enforced for the sake of the agent dynamic behaviors. The DOCP can be resolved by constructing an integrated control law that does not require the best solution to be present in the closed-loop system [24–26]. The stability of the agents can be assured due to the characteristics of the complete graph Laplacian matrix, which means that the approaches in [24] may not be used to the directed graph. With different considerations on gradients and local objective functions, Mao et al. [25] devised a distributed ETC for solving DOCP with time-varying networks. In [26], a distributed optimization control strategy for heterogeneous agents with disruption was proposed to solve the DOCP with an undirected graph. Li et al. [27] investigated the distributed optimal output consensus of heterogeneous linear multi-agent systems on unbalanced directed networks, in which the system disregards the impact of outside disturbances. The distributed consensus optimization problem of a multi-agent system with a delay on weighted–balanced networks was studied in [28], which uses a continuous-time distributed optimization algorithm. In [29], a generalized distributed optimization problem for second-order multi-agent systems over a detail-balanced graph was studied based on a centralized event-firing control algorithm. Despite the effectiveness of the aforementioned approaches, there are only a few works that examine the DOCP for heterogeneous MASs with directed topologies, which is still open and remain challenging.

Directed topologies are found in the bulk of practical networks, including industrial transmission lines and quotation systems [30]. Synchronization and consensus in directed networks have received a substantial amount of attention so far [31,32]. It is worthwhile to point out that, in the aforementioned works, many approaches developed for the DOCP can be applied to MASs only under an undirected network. A type of distributed coordination algorithms to work out the DOCP with a weight-balanced digraph is proposed in [33], which contains the global information. According to [24,33,34], since the network topology is undirected, the average state of the multiagent system is the optimal value of the DOCP. This feature may not be guaranteed in directed topologies, necessitating a rethinking of the optimization goal in directed topologies.

Among the aforementioned distributed optimization methods, event-triggered communication (ETC) is particularly appealing since it allows for fewer updating instants while still conserving resources. The ETC strategy, as illustrated in [35,36], can reduce communication and computing overhead in networked coupled systems while maintaining control performance. In ETC, instead of using the continuous state to reach a consensus, the interaction between agents is piecewise-constant. Consequently, many problems, such as data congestion and continuous communication, can be mitigated to a large extent. Many efforts have been concentrated on developing the DOCP for multiagent systems with ETC, which have been influenced by the aforementioned idea of ETC [33,34,37,38]. Wu et al. [34] investigated continuous-time optimization by utilizing adaptive event-based methods that rely solely on neighboring agents' relevant information. Deng et al. [37] proposed a distributed optimization algorithm combining gradient measurement and ETC to guarantee the exponential convergence of the system. The fundamental challenge in developing event-triggered control is avoiding Zeno behavior, which can disrupt the controller's normal operation and result in endless triggers on a limited period. The suggested event-triggered algorithm in [39] can handle a broad range of sensor network problems; however, it is not guaranteed to prevent Zeno behavior. In [33,37], Zeno behavior is avoided by placing an upper constraint on the communication frequency.

In this paper, we continue the previous research by looking at the DOCP for heterogeneous MASs with directed topologies, which is known to be quite challenging. Based on output-regulation techniques, some criteria are proposed to ensure that the DOCP under directed topologies is solved. The following are the primary contributions of this paper. (1) Distinct from [24–26,33,40], our work investigates the DOCP with directed topologies and does not require the networks to be node-balanced. By exploring the properties of the directed network, a different convex optimization problem is proposed, which implies a weighted average of the objective function of each agent. (2) Compared to the previous works [24–26,33,40,41], a novel distributed control law is suggested, which is composed of the solutions of carefully chosen matrix equations. It is demonstrated that all agents' outputs can attain the ideal value that minimizes the sum of local objective functions, ensuring global convergence. (3) The ETC method is added to the proposed control law. Distinct from [33,37], our work designs two different triggering conditions into our ETC strategy, and the Zeno behavior is precluded without a communication frequency upper restriction.

This paper is organized as follows. Some required definitions and lemmas are provided in Section 2. In Section 3, a different convex optimization problem and some assumptions are proposed. In Section 4, the distributed optimization schemes with continuous communication and ETC are designed for heterogeneous MASs over directed networks. A numerical example is provided in Section 5 to demonstrate the usefulness of the theoretical results. Finally, in Section 6, the conclusion is drawn.

2. Preliminaries

2.1. Notations

Let $R, R^n, R^{n\times m}$ denote sets of real numbers, real n-dimensional vectors, and real $n \times m$-dimensional matrices, respectively. $R_{>0}$ denotes the set of positive real numbers. I_n represents the n-dimensional identity matrix, and $\mathbf{1}_n$ stands for an n-dimensional vector with all components equal to 1. $\|\cdot\|$ is the induced 2-norm of matrices or the Euclidean norm of a vector. Given vectors $x_1,\cdots,x_N, \mathrm{col}(x_1,\cdots,x_N) = \left[x_1^T,\cdots,x_N^T\right]^T$. A diagonal matrix Σ is denoted by $\Sigma = \mathrm{diag}(\sigma_1,\cdots,\sigma_N)$, where $\sigma_i, i \in \{1,2,\cdots,N\}$, is the diagonal element. For a matrix $A \in R^{n\times n}$, A^T is the transpose of A; $A \succ 0$ (or $A \succeq 0$) implies that A is positive definite (or positive semidefinite). The symbol $\otimes$ denotes the Kronecker product of matrices. For a differentiable function $f : R^n \to R$, ∇f is the gradient of f. $\sigma_{min}(A)$ represents the smallest singular value of nonsquare matrix A.

2.2. Graph Theory

For a directed network $\mathcal{G} = (\mathcal{V},\mathcal{E},A)$ with N agents, and $\mathcal{V} = \{1,2,\cdots,N\}$ is the set of agents, $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the set of links. For every pair of nodes i and $j, i,j \in \mathcal{V}$. If a directed path exists between i and j, then the directed network $\mathcal{G}$ is said to be strongly connected. The weighted adjacency matrix $A = \left[a_{ij}\right]_{N\times N}$ of a directed network $\mathcal{G}$ is defined as $a_{ij} > 0$ if $(i,j) \in \mathcal{E}$, and $a_{ij} = 0$ if $(i,j) \notin \mathcal{E}$. The out- and in-degrees of the node $i, i \in \mathcal{V}$ are $\sigma_i^{\mathrm{out}} = \sum_{j=1}^N a_{ji}$ and $\sigma_i^{\mathrm{in}} = \sum_{j=1}^N a_{ij}$, respectively. The degree matrix of the directed network $\mathcal{G}$ is defined as $\Sigma = \mathrm{diag}\{\sigma_1^{\mathrm{in}},\sigma_2^{\mathrm{in}},\cdots,\sigma_N^{\mathrm{in}}\}$. The weighted Laplacian matrix associated with the directed network $\mathcal{G}$ is defined as $\mathcal{L} = \Sigma - A$.

Lemma 1 ([42]). *If $\mathcal{L}$ is irreducible, then $\mathrm{rank}(\mathcal{L}) = N-1$, and $\mathbf{1} = (1,1,\cdots,1)^T$ is the right eigenvector of $\mathcal{L}$ corresponding to eigenvalue 0 with multiplicity 1, i.e., $\mathcal{L}\cdot\mathbf{1} = 0$. Let $\xi^T = (\xi_1,\cdots,\xi_N)^T$ be the left eigenvector of $\mathcal{L}$ corresponding to the eigenvalue 0 , i.e., $\xi^T\mathcal{L} = 0$, and its multiplicity is 1. Then, $\xi_i > 0, i = 1,2,\cdots,N$. In the following, we always assume that $\sum_{i=1}^N \xi_i = 1$. Let $\Xi = diag(\xi_1,\cdots,\xi_N)$, then $\hat{\mathcal{L}} = \frac{1}{2}\left(\Xi\mathcal{L} + \mathcal{L}^T\Xi\right)$ is a symmetric matrix with all row sums equal to zeros, and it has zero eigenvalue with algebraic dimension one.*

Lemma 2 ([43]). *If $Q \in R^{n\times n}$ is such that $q_{ij} = q_{ji}$ and $q_{ii} = -\sum_{j=1,j\neq i}^N q_{ij}, i,j = 1,\cdots,n$, then for all vectors, $x = (x_1,x_2,\cdots,x_n)^T, y = (y_1,y_2,\cdots,y_n)^T$*

$$x^TQy = -\sum_{j>i}^{n} q_{ij}\left(x_i - x_j\right)\left(y_i - y_j\right).$$

Lemma 3 ([44]). *The general algebraic connectivity of the matrix $\mathcal{L}$ under the strongly connected directed network $\mathcal{G}$ is defined as*

$$a_\delta(\mathcal{L}) = \min_{z^T\xi=0, z\neq 0} \frac{z^T\hat{\mathcal{L}}z}{z^T\Xi z}.$$

Lemma 4 ([45]). *The linear matrix inequality K is defined by*

$$K = \begin{pmatrix} K_1 & K_2 \\ K_2^T & K_3 \end{pmatrix},$$

where $K_1 = K_1^T$, and $K_3 = K_3^T$. Therefore, $K \succ 0$ is equal to one of the following conditions:

$$(1)\ K_1 \succ 0, K_3 - K_2^T K_1^{-1} K_2 \succ 0.$$
$$(2)\ K_3 \succ 0, K_1 - K_2 K_3^{-1} K_2^T \succ 0.$$

Lemma 5 ([46]). *Consider a continuous and positive definite function $F(x) : Q \to R$. The system $\dot{x} = f(x)$, $x \subseteq R^m$, $f(\mathbf{0}_n) = \mathbf{0}_n$ is locally finite-time-stable. If there is a neighborhood $Q_0 \subset U$ of the origin which makes $\dot{F}(x) + \alpha_1 F^{\alpha_2}(x) \leq 0, \forall x \in Q_0\backslash 0$, real numbers $\alpha_1 > 0$ and $\alpha_2 \in (0,1)$ hold. If $Q = Q_0 = R^m$ holds, the system is globally finite-time-stable, and the finite convergence time T satisfies $T \leq \frac{V^{1-\alpha_2}(x_0)}{\alpha_1(1-\alpha_2)}$.*

3. Problem Description

Consider the following equations that explain the dynamics of a group of N heterogeneous agents:

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i(u_i(t) + d_i(t)) \\ y_i(t) &= C_i x_i(t) + D_i u_i(t), i = 1, 2, \cdots, N \end{aligned} \tag{1}$$

where $x_i \in R^{n_i}, y_i \in R^q$, and $u_i \in R^{p_i}$ are the sate, the output, and the control input of the ith agent, respectively. $A_i \in R^{n_i \times n_i}$, $B_i \in R^{n_i \times p_i}, C_i \in R^{q \times n_i}$, $D_i \in R^{q \times p_i}$ are constant matrices. The continuous nonlinear time-varying function $d_i(t) \in R^{p_i}$ denotes external disruption.

Assumption 1. *The directed communication topology is strongly connected.*

Assumption 2. *For any $t > 0$, there exists a constant $\overline{d} > 0$, such that max$\|d_i(t)\| \leq \overline{d}, i = 1, 2, \cdots, N$.*

Remark 1. *Compared with [27–29], this paper considers the effects of external disturbances on the system as well as the regulation of control input on the output. Distinct from the network considered in [28,29], the directed network considered in this paper does not require balancing, which renders it more general.*

The major goal of this work is to provide a distributed optimization method for each agent that allows all of the agents' outputs to attain the optimal value $y*$, which solves the following optimization problem:

$$\min_{y\in R^q} \sum_{i=1}^{N} \xi_i g_i(y), \tag{2}$$

where $g_i : R^q \to R$ is a local objective function of the ith agent.

Remark 2. *Unlike the convex optimization problem considered in [22–26] with undirected networks, the DOCP with heterogeneous agent dynamics of directed networks is considered in our* *work. Combined with the properties of the Laplacian matrix of the directed graph, a new convex optimization problem (2) is proposed, which implies the weighted average of the objective function of each node.*

Remark 3. *The optimization problem (2) considered in this paper has the same form as that in [27–29] when $\xi_i = \frac{1}{N}$. If we take into account how weighted average π_i affects the objective function g_i, then the optimization problem $\min_{y \in R^q} \sum_{i=1}^N \pi_i g_i(y)$ can be transformed into $\min_{y \in R^q} \Pi \sum_{i=1}^N \xi_i g_i(y)$ with $\Pi = \sum_{i=1}^N \pi_i, \xi_i = \frac{\pi_i}{\Pi}$, which has the same form as problem (2) because $\sum_{i=1}^N \xi_i = 1$ and Π is a constant.*

If we define $\tilde{g}(y) = \sum_{i=1}^N \xi_i g_i(y_i)$, $y = \mathrm{col}(y_1, \cdots, y_N)$, the optimization problem (2) can be reformulated as

$$\min_{y \in R^{Nq}} \tilde{g}(y) = \min_{y_i \in R^q} \sum_{i=1}^N \xi_i g_i(y_i), \tag{3a}$$

$$\text{s.t. } (\mathcal{L} \otimes I_q) y = \mathbf{0}_{Nq}, \tag{3b}$$

Assumption 3 ([24]). *For each agent i, the local objective function $g_i : R^q \to R, i = 1, 2, \cdots, N$ is strongly convex, differentiable on R^q. Its gradient $\nabla g_i(z)$ is locally Lipschitz on R^q and satisfies $\|\nabla g_i(x) - \nabla g_j(y)\| \leq \kappa \|x - y\|$, with $\kappa > 0$, for all $x, y \in R^q, i, j = 1, 2, \cdots, N$.*

Remark 4. *To guarantee the unique solution to problem (3), the local objective function $g_i(y)$ must be strictly convex. That means that the control input $u_i(t)$ needs to be redesigned so that the outputs of all agents can attain the optimal value $y^* = \mathrm{argmin}_{y \in R^{Nq}} \tilde{g}(y)$, i.e. $\tilde{g}(y^*) = \min_{y \in R^{Nq}} \tilde{g}(y)$ and $\nabla \tilde{g}(y^*) = \sum_{i=1}^N \xi_i \nabla g_i(y_i^*) = \mathbf{0}$, $y^* = \mathrm{col}(y_1^*, \cdots, y_N^*)$.*

Assumption 4. *(A_i, B_i) is controllable, $\sigma_{min}(B_i^T) > 0$ and*

$$\mathrm{rank} \begin{bmatrix} C_i B_i & D_i \\ A_i B_i & B_i \end{bmatrix} = n_i + q, i = 1, 2, \cdots, N.$$

Lemma 6 ([24]). *Under Assumption 4, the linear matrix equations*

$$B_i \Upsilon_{1i} - \Psi_i = \mathbf{0}_{n_i \times q}, \tag{4a}$$

$$B_i \Upsilon_{2i} + A_i \Psi_i = \mathbf{0}_{n_i \times q}, \tag{4b}$$

$$C_i \Psi_i + D_i \Upsilon_{2i} = \mathbf{I}_{q \times q}, \quad i = 1, \cdots, N \tag{4c}$$

have solution triplets $(\Upsilon_{1,i}, \Upsilon_{2,i}, \Psi_i)$, respectively.

4. Main Results

4.1. DOCP for MASs with Continuous Communications

In this section, by using the output regulation techniques, a distributed control law is proposed to solve the DOCP. According to the solutions of matrices Equation (4), the distributed control method is given by (5)

$$u_i = K_{1i} x_i + \Upsilon_{1i} \dot{\eta}_i + (\Upsilon_{2i} - K_{1i} \Psi_i) \eta_i - c_1 \,\mathrm{sgn}(K_{2i}(x_i - \Psi_i \eta_i)), \tag{5a}$$

$$\dot{\eta}_i = - \nabla g_i(y_i) - \alpha_1 \sum_{j=1}^N \mathcal{L}_{ij} y_j - \alpha_2 \sum_{j=1}^N \mathcal{L}_{ij} \int_0^t y_j(\sigma) d\sigma, \tag{5b}$$

where $\eta_i \in R^q$ is the auxiliary state of agent i; $c_1 \in R_{>0}, \alpha_1 \in R_{>0}, \alpha_2 \in R_{>0}, K_{1i} \in R^{p_i \times n_i}$, and $K_{2i} \in R^{p_i \times n_i}$ are matrices to be determined; and $\Upsilon_{1i}, \Upsilon_{2i}, \Psi_i$ are the solutions of (4).

Remark 5. *The first three items in the controller u_i are designed for ensuring that the state x_i approaches the auxiliary state $\eta_i(t)$; $-c_1\,\mathrm{sgn}(K_{2i}(x_i - \Psi_i\eta_i))$ is the term to remove the influence of external disruption $d_i(t)$; $-\nabla g_i(y_i)$ is the gradient term to guide the agents for optimization; $-\alpha_1 \sum_{j=1}^{N} \mathcal{L}_{ij} y_j - \alpha_2 \sum_{j=1}^{N} \mathcal{L}_{ij} \int_0^t y_j(\sigma)d\sigma$ is the consensus term to ensure that all the agents converge to the optimal state.*

The closed-loop system can be stated by inputting the control input (5) into system (1)

$$\begin{aligned}\dot{x}_i(t) =&(A_i + B_iK_{1i})x_i(t) + B_i\Upsilon_{1i}\dot{\eta}_i + (B_i\Upsilon_{2i} - B_iK_{1i}\Psi_i)\eta_i \\ &- c_1B_i\,\mathrm{sgn}(K_{2i}(x_i(t) - \Psi_i\eta_i)) + B_id_i(t).\end{aligned} \tag{6}$$

Substituting (4a) and (4b) into (6), the following compact form of system (6) can be obtained:

$$\dot{x} - \Psi\dot{\eta} =(A + BK_1)(x - \Psi\eta) - c_1B\,\mathrm{sgn}(K_2(x - \Omega\eta)) + Bd, \tag{7}$$

where $x = \mathrm{col}(x_1, \cdots, x_N)$, $\eta = \mathrm{col}(\eta_1, \cdots, \eta_N)$, $d = \mathrm{col}(d_1, \cdots, d_N)$, $A = \mathrm{diag}(A_1, \cdots, A_N)$, $B = \mathrm{diag}(B_1, \cdots, B_N)$, $K_1 = \mathrm{diag}(K_{11}, \cdots, K_{1N})$, $K_2 = \mathrm{diag}(K_{21}, \cdots, K_{2N})$, $\Psi = \mathrm{diag}(\Psi_1, \cdots, \Psi_N)$, $\nabla\tilde{g}(y) = \mathrm{col}(\nabla g_1(y_1), \cdots, \nabla g_N(y_N))$.

Let $\delta_x = x - \Psi\eta$, $A_c = (A + BK_1)$. Then, (7) can be written as

$$\dot{\delta_x} = A_c\delta_x - c_1B\,\mathrm{sgn}(K_2\delta_x) + Bd, \tag{8}$$

where $K_2 = B^TP$ is the feedback control with matrix P to be determined later. Therefore, we have the following theorem.

Theorem 1. *Suppose Assumptions 1–4 hold. $A_i + B_iK_{1i}$ is Hurwitz by choosing the appropriate matrices $K_{1i}, i = 1, 2, \cdots, N$, and $\Upsilon_{1i}, \Upsilon_{2i}, \Psi_i$ that are given in (4). Then, δ_x can converge to zero in a finite time T_1 if there exists a positive matrix P and constant $c_1 > 0$ satisfying*

$$\begin{aligned}c_1 > \bar{d}; (H_1)\\ PA_c + (A_c)^TP \prec 0. (H_2)\end{aligned}$$

Proof. Define the Lyapunov function candidate as follows:

$$V_1 = \frac{1}{2}\delta_x^TP\delta_x.$$

Taking the derivative of V_1 along with (8) yields

$$\begin{aligned}\dot{V}_1 =&\delta_x^TP(A_c\delta_x - c_1B\,\mathrm{sgn}(K_2\delta_x) + Bd) \\ =&\frac{1}{2}\delta_x^T\Big(PA_c + (A_c)^TP\Big)\delta_{x_i} + \Big(\delta_x^TPBd - c_1\delta_x^TPB\,\mathrm{sgn}\Big(B^TP\delta_x\Big)\Big) \\ \leq&\|B^TP\delta_x\|\|d\| - c_1\|B^TP\delta_x\| \\ \leq&-\sqrt{2}(c_1 - \bar{d})\sigma_{min}(B^T)\sigma_{min}(P^{\frac{1}{2}})V_1^{\frac{1}{2}}.\end{aligned}$$

Thus, on the basis of the condition of the theorem , one has

$$\dot{V}_1 + \sqrt{2}(c_1 - \bar{d})\sigma_{min}(B^T)\sigma_{min}(P^{\frac{1}{2}})V_1^{\frac{1}{2}} \leq 0. \tag{9}$$

By Lemma 5, one has $\lim_{t\to T_1} V_1 \to 0$. Therefore, δ_x converges to zero in a finite time T_1. According to (9), the convergence time has the following form:

$$T_1 = \frac{V^{\frac{1}{2}}(0)}{\sqrt{2}(c_1 - \bar{d})\sigma_{min}(B^T)\sigma_{min}(P^{\frac{1}{2}})}.$$

By (4c), one has $y_i = C_i x_i + D_i u_i = \eta_i, t > T_1, \forall i \in \mathcal{V}$. The control law (5b) can be further obtained as follows:

$$\dot{y}_i = -\nabla g_i(y_i) - \alpha_1 \sum_{j=1}^{N} \mathcal{L}_{ij} y_j - \alpha_2 \sum_{j=1}^{N} \mathcal{L}_{ij} \int_0^t y_j(\sigma) d\sigma. \tag{10}$$

□

Remark 6. *It is clear from the foregoing analysis that when $t > T_1, x_i = \Psi_i \eta_i, y_i = \eta_i, \forall i \in \mathcal{V}$ hold. In fact, the solutions of carefully designed matrix Equation (4) ensures that the output of all agents can be tracked to the auxiliary variable, thereby facilitating the solving of the DOCP (3). To ensure the solvability of the matrix equations, sufficient conditions in terms of constant matrix are constructed. The matrices Equation (4) is an important part of the control law (5).*

Let $\bar{y} = \sum_{i=1}^{N} \xi_i y_i$ be the average state and $\delta_{y_i} = y_i - \bar{y}$ be the consensus error for the sake of convenience later in this study. Given that $\sum_{j=1}^{N} \mathcal{L}_{ij} \bar{y} = 0$, (10) is converted into the following form

$$\dot{y}_i = -\nabla g_i(y_i) - \alpha_1 \sum_{j=1}^{N} \mathcal{L}_{ij} \delta_{y_j} - \alpha_2 \sum_{j=1}^{N} \mathcal{L}_{ij} \int_0^t \delta_{y_j}(\sigma) d\sigma. \tag{11}$$

If we let $\varphi_j(t) = \int_0^t \delta_{y_j}(\sigma) d\sigma$, the control law (5) can be written as

$$u_i = K_{1i} x_i + \Upsilon_{1i} \pi_i + (\Upsilon_{2i} - K_{1i} \Psi_i) \eta_i - c_1 \operatorname{sgn}(K_{2i}(x_i - \Psi_i \eta_i)); \tag{12a}$$

$$\dot{y}_i = -\nabla g_i(y_i) - \alpha_1 \sum_{j=1}^{N} \mathcal{L}_{ij} \delta_{y_j} - \alpha_2 \sum_{j=1}^{N} \mathcal{L}_{ij} \varphi_j(t); \tag{12b}$$

$$\dot{\varphi}_i = \delta_{y_i}. \tag{12c}$$

Let $\varphi(t) = (\varphi_1^T(t), \cdots, \varphi_N^T(t))^T$, $y = (y_1^T, \cdots, y_N^T)^T$, $\delta_y(t) = (\delta_{y_1}^T(t), \delta_{y_2}^T(t), \cdots, \delta_{y_N}^T(t))^T$. Then, (12) can be rewritten in a compact form as

$$\dot{y}(t) = -\nabla \tilde{g}(y) - \alpha_1 (\mathcal{L} \otimes I_q) \delta_y(t) - \alpha_2 (\mathcal{L} \otimes I_q) \varphi(t); \tag{13a}$$

$$\dot{\varphi} = \delta_y(t). \tag{13b}$$

From (13), the equilibrium point $\bar{y}$ satisfies

$$\mathbf{0} = -\nabla \tilde{g}(\bar{y}) - \alpha_1 (\mathcal{L} \otimes I_q) \delta_{\bar{y}}(t) - \alpha_2 (\mathcal{L} \otimes I_q) \bar{\varphi}(t). \tag{14}$$

Then, by multiplying both sides of (14) by $(\xi^T \otimes I_q)$, and because $\xi^T \mathcal{L} = 0$, one obtains $\sum_{i=1}^{N} \xi_i \nabla g_i(\bar{y}) = 0$, which means that $\bar{y}$ is an optimal solution of the DOCP (3).

If $M = I_N - \mathbf{1}_N \xi^T$, it is easy to check that $\mathcal{L}M = M\mathcal{L} = \mathcal{L}$. Then, the error system can be obtained as follows:

$$\begin{aligned} \dot{\delta}_y(t) &= -(M \otimes I_q) \nabla \tilde{g}(y) - \alpha_1 (\mathcal{L} \otimes I_q) \delta_y(t) - \alpha_2 (\mathcal{L} \otimes I_q) \varphi(t); \\ \dot{\varphi} &= \delta_y(t). \end{aligned}$$

The compact matrix form can be recast as

$$\begin{pmatrix} \dot{\delta}_y(t) \\ \dot{\varphi}(t) \end{pmatrix} = Q \begin{pmatrix} \delta_y(t) \\ \varphi(t) \end{pmatrix} + \begin{pmatrix} -(M \otimes I_q) \nabla \tilde{g}(y) \\ \mathbf{0}_{Nq} \end{pmatrix}, \tag{15}$$

where

$$Q = \begin{pmatrix} -\alpha_1(\mathcal{L} \otimes I_q) & -\alpha_2(\mathcal{L} \otimes I_q) \\ I_{Nq} & \mathbf{0}_{Nq} \end{pmatrix}.$$

We are now in a position to present our major result on closed-loop system (13) convergence.

Theorem 2. *Suppose Assumptions 1–4 hold. A_i^c is Hurwitz by choosing the appropriate matrices $K_{1i}, i = 1, 2, \cdots, N$, and $\Upsilon_{1i}, \Upsilon_{2i}, \Psi_i$ that are given in (4). The control law stated in (12) can be used to solve the DOCP for multiagent system (1) if*

$$\alpha_1 > \frac{\sqrt{2(\kappa+2)\alpha_2 a_\delta(\mathcal{L}) + \kappa^2} + \kappa}{2a_\delta(\mathcal{L})}; \tag{16}$$

$$\alpha_2 > \frac{\kappa}{2a_\delta(\mathcal{L})}. \tag{17}$$

Proof. Consider the following Lyapunov function

$$V_2 = \frac{1}{2}\begin{pmatrix} \delta_y^T(t) & \varphi^T(t) \end{pmatrix} R \begin{pmatrix} \delta_y(t) \\ \varphi(t) \end{pmatrix},$$

where

$$R = \begin{pmatrix} \Xi \otimes I_q & \frac{\alpha_2}{\alpha_1}(\Xi \otimes I_q) \\ \frac{\alpha_2}{\alpha_1}(\Xi \otimes I_q) & 2\alpha_2(\hat{\mathcal{L}} \otimes I_q) \end{pmatrix}.$$

The proof is completed in two parts. In the first part, we show that $V_2 \geq 0$ and $V_2 = 0$ only when $\delta_y = 0$ and $\varphi(t) = 0$. In the second part, we construct criteria in which $\dot{V}_2 \leq 0$ and $\dot{V}_2 = 0$ only if $\delta_y = 0$ and $\varphi(t) = 0$.

Part 1: From Lemma 3, one can obtain

$$\begin{aligned} V_2(t) \geq& \frac{1}{2}\delta_y^T(t)(\Xi \otimes I_q)\delta_y(t) + \alpha_2 a_\delta(\mathcal{L})\varphi^T(t)(\Xi \otimes I_q)\varphi(t) + \frac{1}{2}\frac{\alpha_2}{\alpha_1}\varphi^T(t)(\Xi \otimes I_q)\delta_y(t) \\ &+ \frac{1}{2}\frac{\alpha_2}{\alpha_1}\varphi^T(t)(\Xi \otimes I_q)\delta_y(t) \\ =& \frac{1}{2}\begin{pmatrix} \delta_y^T(t) & \varphi^T(t) \end{pmatrix} \tilde{R} \begin{pmatrix} \delta_y(t) \\ \varphi(t) \end{pmatrix}, \end{aligned}$$

where

$$\tilde{R} = \begin{pmatrix} \Xi \otimes I_q & \frac{\alpha_2}{\alpha_1}(\Xi \otimes I_q) \\ \frac{\alpha_2}{\alpha_1}(\Xi \otimes I_q) & 2\alpha_2 a_\delta(\mathcal{L})(\Xi \otimes I_q) \end{pmatrix}.$$

By Lemma 4 and conditions (16), (17), we have $a_\delta(\mathcal{L}) \geq \frac{\alpha_2}{2\alpha_1^2}$. Therefore, $\tilde{R} \succ 0$, and one can conclude that $V_2 \geq 0$ and $V_2 = 0$ only when $\delta_y(t) = 0$ and $\varphi(t) = 0$.

Part 2: Taking the derivative of V_2 along (15) yields

$$\begin{aligned} \dot{V}_2 =& \begin{pmatrix} \delta_y^T(t) & \varphi^T(t) \end{pmatrix} R \begin{pmatrix} -(M \otimes I_q)\nabla\tilde{g}(y) \\ \mathbf{0}_{Nq} \end{pmatrix} + \begin{pmatrix} \delta_y^T(t) & \varphi^T(t) \end{pmatrix} RQ \begin{pmatrix} \delta_y(t) \\ \varphi(t) \end{pmatrix} \\ =& W_1(t) + W_2(t), \end{aligned} \tag{18}$$

where

$$\begin{aligned} W_1(t) =& \begin{pmatrix} \delta_y^T(t) & \varphi^T(t) \end{pmatrix} R \begin{pmatrix} -(M \otimes I_q)\nabla\tilde{g}(y) \\ \mathbf{0}_{Nq} \end{pmatrix} \\ =& -\delta_y^T(t)(\Xi M \otimes I_q)\nabla\tilde{g}(y) - \frac{\alpha_2}{\alpha_1}\varphi^T(t)(\Xi M \otimes I_q)\nabla\tilde{g}(y). \end{aligned} \tag{19}$$

By Lemma 2 and Assumption 3, one can obtain

$$-\delta_y^T(t)(\Xi M \otimes I_q)\nabla\tilde{g}(y) \leq \frac{\kappa}{2}\sum_{i=1}^{N}\sum_{j\neq i}\xi_i\xi_j\|y_i - y_j\|^2 \leq \kappa\sum_{i=1}^{N}\xi_i\|\delta_{y_i}(t)\|^2. \tag{20}$$

Similarly, we can obtain that

$$-\varphi^T(t)(\Xi M \otimes I_q)\nabla\tilde{g}(y) \leq \frac{\kappa}{2}\sum_{i=1}^{N}\xi_i\|\varphi_i\|^2 + \frac{\kappa}{2}\sum_{i=1}^{N}\xi_i\|\delta_{y_i}(t)\|^2. \tag{21}$$

Then, by substituting (20), (21) into (19), one obtains that

$$W_1(t) \leq \frac{\kappa}{2}\frac{\alpha_2}{\alpha_1}\sum_{i=1}^{N}\xi_i\|\varphi_i(t)\|^2 + \left(\frac{\kappa}{2}\frac{\alpha_2}{\alpha_1} + \kappa\right)\sum_{i=1}^{N}\xi_i\|\delta_{y_i}(t)\|^2. \tag{22}$$

In addition,

$$\frac{1}{2}(RQ + Q^{\mathrm{T}}R) = \begin{pmatrix} -\alpha_1(\hat{\mathcal{L}} \otimes I_q) + \frac{\alpha_2}{\alpha_1}(\Xi \otimes I_q) & 0 \\ 0 & -\frac{\alpha_2^2}{\alpha_1}(\hat{\mathcal{L}} \otimes I_q) \end{pmatrix}.$$

By the definition of $a_\delta(\mathcal{L})$, it yields that

$$\begin{aligned}
W_2(t) =& \begin{pmatrix} \delta_y^T(t) & \varphi^T(t) \end{pmatrix}\left[\frac{1}{2}(RQ + Q^T R)\right]\begin{pmatrix} \delta_y(t) \\ \varphi(t) \end{pmatrix} \\
=& \delta_y^T(t)\left[\frac{\alpha_2}{\alpha_1}(\Xi \otimes I_q) - \alpha_1(\hat{\mathcal{L}} \otimes I_q)\right]\delta_y(t) - \frac{\alpha_2^2}{\alpha_1}\varphi^T(t)(\hat{\mathcal{L}} \otimes I_q)\varphi(t) \\
\leq& \delta_y^T(t)\left[\frac{\alpha_2}{\alpha_1} - \alpha_1 a_\delta(\mathcal{L})\right](\Xi \otimes I_q)\delta_y(t) - \frac{\alpha_2^2}{\alpha_1}a_\delta(\mathcal{L})\varphi^T(t)(\Xi \otimes I_q)\varphi(t) \\
=& \left[\frac{\alpha_2}{\alpha_1} - \alpha_1 a_\delta(\mathcal{L})\right]\sum_{i=1}^{N}\xi_i\|\delta_{y_i}(t)\|^2 - \frac{\alpha_2^2}{\alpha_1}a_\delta(\mathcal{L})\sum_{i=1}^{N}\xi_i\|\varphi_i(t)\|^2.
\end{aligned} \tag{23}$$

Then, by substituting (22), (23) into (18) and using (16), (17), one obtains that

$$\dot{V}_2 \leq \left[\left(\frac{\kappa}{2}\frac{\alpha_2}{\alpha_1} + \kappa\right) + \frac{\alpha_2}{\alpha_1} - \alpha_1 a_\delta(\mathcal{L})\right]\sum_{i=1}^{N}\xi_i\|\delta_{y_i}(t)\|^2 + \left[\frac{\kappa}{2}\frac{\alpha_2}{\alpha_1} - \frac{\alpha_2^2}{\alpha_1}a_\delta(\mathcal{L})\right]\sum_{i=1}^{N}\xi_i\|\varphi_i(t)\|^2 \leq 0.$$

Thus, $\dot{V}_2 \leq 0$ and $\dot{V}_2 = 0$ only when $\delta_y(t) = 0$ and $\varphi(t) = 0$. Therefore, the multiagent system (1) reaches global consensus. Finally, all of the agents' outputs reach the equilibrium point $\bar{y}$, and the DOCP (3) is solved.
□

Remark 7. *Control protocol design for the DOCP under a directed network is acknowledged to be tough. In this paper, we put the proportional integral control protocols into the control strategy (5) and use the output regulation techniques so that all of the agents' outputs reach the equilibrium point, and the DOCP (3) is solved. Theorem 2 shows that the perturbations of parameters α_1 and α_2 do not change the consensus performance as long as condition (16) is satisfied.*

4.2. DOCP for MASs with ETC

Due to the control law's communication structure (12), agents must constantly collect local messages and adjust their control signals, which is extravagant and would result in a waste of resources. To this end, to tackle problem (3) for heterogeneous MASs (1), an ETC strategy is developed, in which communications are required only at specific time instants, and no Zeno behavior is seen. The ETC law is proposed as

$$u_i = K_{1i}x_i + \Upsilon_{1i}\pi_i + (\Upsilon_{2i} - K_{1i}\Psi_i)\eta_i - c_1\,\mathrm{sgn}(K_{2i}(x_i - \Psi_i\eta_i)); \tag{24a}$$

$$\dot{y}_i = -\nabla g_i(y_i) - \alpha_1 \sum_{j=1}^{N} \mathcal{L}_{ij}\delta_{y_j}\left(t^i_{k_i(t)}\right) - \alpha_2 \sum_{j=1}^{N} \mathcal{L}_{ij}\varphi_j\left(t^i_{k_i(t)}\right); \tag{24b}$$

$$\dot{\varphi}_i = \delta_{y_i}. \tag{24c}$$

where t^i_k is the kth triggering instant of the ith agent, and $k_i(t) = argmax_k\{t^i_k \leq t\}$. The triggering criteria, which will be defined later, decide the triggering instants of the ith agent $t^i_1, \cdots, t^i_k, \cdots$.

For simplicity for $\forall t \in [t^i_k, t^i_{k+1}), k = 1, 2, \cdots$, we denote $\theta_i(t) = -\sum_{j=1}^{N} L_{ij}\delta_{y_j}(t), \vartheta_i(t) = -\sum_{j=1}^{N} L_{ij}\varphi_j(t), e_{\theta_i}(t) = \theta_i(t^i_k) - \theta_i(t), e_{\vartheta_i}(t) = \vartheta_i(t^i_k) - \vartheta_i(t), e_\theta(t) = [e_{\theta_1}(t), \cdots, e_{\theta_N}(t)]^T$, $e_\vartheta(t) = [e_{\vartheta_1}(t), \cdots, e_{\vartheta_N}(t)]^T$.

Then, one can obtain

$$\begin{aligned} \dot{\delta}_y(t) =& -(M \otimes I_q)\nabla\tilde{g}(y) - \alpha_1(\mathcal{L} \otimes I_q)\delta_y(t) - \alpha_2(\mathcal{L} \otimes I_q)\varphi(t) + \alpha_1(M \otimes I_q)e_\theta(t) \\ &+ \alpha_2(M \otimes I_q)e_\vartheta(t); \\ \dot{\varphi}(t) =& \delta_y(t). \end{aligned}$$

Similar to (15) is the following:

$$\begin{pmatrix} \dot{\delta}_y(t) \\ \dot{\varphi}(t) \end{pmatrix} = Q\begin{pmatrix} \delta_y(t) \\ \varphi(t) \end{pmatrix} + \begin{pmatrix} Z \\ \mathbf{0}_{Nq}. \end{pmatrix}, \tag{25}$$

where $Z = -(M \otimes I_q)\nabla\tilde{g}(y) + \alpha_1(M \otimes I_q)e_\theta(t) + \alpha_2(M \otimes I_q)e_\vartheta(t)$.

When we say that agent i triggers at time $t^i_{k_i}$, we imply that agent i recommences its control value at time $t^i_{k_i+1}$ and sends its current state to its out-neighbors. The measurement errors $e_{\theta_i}(t)$ and $e_{\vartheta_i}(t)$ are both reset to zero meanwhile. Given that agent i is triggered at the instant $t^i_{k_i}$, the following triggering condition can be used to calculate its next triggering instant.

$$t^i_{k_i+1} \triangleq inf\{t : t > t^i_{k_i}, \Pi_{\theta_i}(t) \geq 0 \cup \Pi_{\vartheta_i}(t) \geq 0\}, \tag{26}$$

where

$$\Pi_{\theta_i}(t) = \|e_{\theta_i}(t)\|^2 - \frac{1}{\mu_1}e^{-\sigma_1 t}; \Pi_{\vartheta_i}(t) = \|e_{\vartheta_i}(t)\|^2 - \frac{1}{\mu_2}e^{-\sigma_2 t},$$

where μ_1,μ_2, σ_1, σ_2 are positive constants. An event is triggered for agent i when one of the triggering conditions of $\|e_{\theta_i}(t)\|^2 \geq \frac{1}{\mu_1}e^{-\sigma_1 t}$ and $\|e_{\vartheta_i}(t)\|^2 \geq \frac{1}{\mu_2}e^{-\sigma_2 t}$ is fulfilled.

Remark 8. *Compared with the centralized event-triggering mechanism designed in [29], which needs global information, the distributed event-triggering mechanism designed in this needs only the information of neighbor nodes, which effectively reduces the communication burden.*

We are now in a position to provide our second main result on system (25) convergence.

Theorem 3. *Suppose Assumptions 1–4 hold. $A_i + B_i K_{1i}$ is Hurwitz by choosing the appropriate matrices $K_{1i}, i = 1, 2, \cdots, N$, and $\Upsilon_{1i}, \Upsilon_{2i}, \Psi_i$ that are given in (4). With the triggering condition (26), the DOCP for the multiagent system (1) is solved by the event-triggered control law (24) if*

$$\alpha_1 > \frac{\sqrt{2(\kappa+6)\alpha_2 a_\delta(\mathcal{L}) + \kappa^2} + \kappa}{2a_\delta(\mathcal{L})}; \tag{27}$$

$$\alpha_2 > \frac{\kappa+4}{2a_\delta(\mathcal{L})}. \tag{28}$$

Furthermore, the closed-loop system (25) does exhibit the Zeno behavior.

Proof. The time derivative of V_2 along the trajectory of (25) can be calculated using the same method as in the demonstration of Theorem 2.

$$\begin{aligned}\dot{V}_2 =& \begin{pmatrix} \delta_y^T(t) & \varphi^T(t) \end{pmatrix} R \begin{pmatrix} Z \\ \mathbf{0}_{Nq} \end{pmatrix} + \begin{pmatrix} \delta_y^T(t) & \varphi^T(t) \end{pmatrix} RQ \begin{pmatrix} \delta_y(t) \\ \varphi(t) \end{pmatrix} \\ =& W_1(t) + W_2(t) + W_3(t).\end{aligned} \tag{29}$$

where $W_1(t)$,$W_2(t)$ are the same as in Theorem (2).

Let $U = EM$, and its eigenvalues (counting multiplicities) are as follows: $0 = \chi_1 \leq \chi_2 \leq \cdots \leq \chi_q$. Let $\mu_1 = (\frac{\alpha_1^3}{\alpha_2} + \alpha_1\alpha_2)\chi_q^2$, $\mu_2 = (\frac{\alpha_2^3}{\alpha_1} + \alpha_1\alpha_2)\chi_q^2$. Thus, we have

$$\begin{aligned}W_3(t) =& \alpha_1\delta_y^T(t)(U \otimes I_q)e_\theta(t) + \alpha_2\delta_y^T(t)(U \otimes I_q)e_\vartheta(t) + \alpha_2\varphi^T(t)(U \otimes I_q)e_\theta(t) \\ &+ \frac{\alpha_2^2}{\alpha_1}\varphi^T(t)(U \otimes I_q)e_\vartheta(t) \\ \leq& \frac{2\alpha_2}{\alpha_1}\sum_{i=1}^N \xi_i\|\delta_{y_i}(t)\|^2 + \mu_1\sum_{i=1}^N \|e_{\theta_i}(t)\|^2 + \frac{2\alpha_2}{\alpha_1}\sum_{i=1}^N \xi_i\|\varphi_i(t)\|^2 + \mu_2\sum_{i=1}^N \|e_{\vartheta_i}(t)\|^2.\end{aligned} \tag{30}$$

Then, by substituting (22), (23), (30) into (29), one obtains that

$$\begin{aligned}\dot{V}_2 \leq& \left(\frac{\kappa}{2}\frac{\alpha_2}{\alpha_1} + \kappa + \frac{3\alpha_2}{\alpha_1} - \alpha_1 a_\delta(\mathcal{L})\right)\sum_{i=1}^N \xi_i\|\delta_{y_i}(t)\|^2 + \left(\frac{\kappa}{2}\frac{\alpha_2}{\alpha_1} - \frac{\alpha_2^2}{\alpha_1}a_\delta(\mathcal{L}) + \frac{2\alpha_2}{\alpha_1}\right)\sum_{i=1}^N \xi_i\|\varphi_i(t)\|^2 \\ &+ \mu_1\sum_{i=1}^N \|e_{\theta_i}(t)\|^2 + \mu_2\sum_{i=1}^N \|e_{\vartheta_i}(t)\|^2.\end{aligned}$$

Combining condition (26), we deduce that

$$\begin{aligned}\dot{V}_2 \leq& \left(\frac{\kappa}{2}\frac{\alpha_2}{\alpha_1} + \kappa + \frac{3\alpha_2}{\alpha_1} - \alpha_1 a_\delta(\mathcal{L})\right)\sum_{i=1}^N \xi_i\|\delta_{y_i}(t)\|^2 + \left(\frac{\kappa}{2}\frac{\alpha_2}{\alpha_1} - \frac{\alpha_2^2}{\alpha_1}a_\delta(\mathcal{L}) + \frac{2\alpha_2}{\alpha_1}\right)\sum_{i=1}^N \xi_i\|\varphi_i(t)\|^2 \\ &+ Ne^{-\sigma_1 t} + Ne^{-\sigma_2 t}.\end{aligned}$$

Let $V_3 = V_2 + \frac{N}{\sigma_1}e^{-\sigma_1 t} + \frac{N}{\sigma_2}e^{-\sigma_2 t}$. By using (27), (28), one has

$$\begin{aligned}\dot{V}_3 =& \dot{V}_2 - Ne^{-\sigma_1 t} - Ne^{-\sigma_2 t} \\ \leq& \left(\frac{\kappa}{2}\frac{\alpha_2}{\alpha_1} + \kappa + \frac{3\alpha_2}{\alpha_1} - \alpha_1 a_\delta(\mathcal{L})\right)\sum_{i=1}^N \xi_i\|\delta_{y_i}(t)\|^2 + \left(\frac{\kappa}{2}\frac{\alpha_2}{\alpha_1} - \frac{\alpha_2^2}{\alpha_1}a_\delta(\mathcal{L}) + \frac{2\alpha_2}{\alpha_1}\right)\sum_{i=1}^N \xi_i\|\varphi_i(t)\|^2 \\ \leq& 0.\end{aligned} \tag{31}$$

Thus, the multiagent system (1) reaches global consensus, and the DOCP (3) is solved.

Then, we prove that the Zeno behavior could be excluded. From (24), over the interval $[t_k^i, t_{k+1}^i)$, the upper right-hand Dini derivative of $e_{\theta_i}(t)$ can be expressed as

$$\mathrm{D}^+ e_{\theta_i}(t) = -\sum_{j=1}^{N} \mathcal{L}_{ij} \nabla g_i(y_i) + \alpha_1 \sum_{j=1}^{N} \mathcal{L}_{ij} \theta_j \left(t_k^i\right) + \alpha_2 \sum_{j=1}^{N} \mathcal{L}_{ij} \vartheta_j \left(t_k^i\right).$$

By noting that $e_{\theta_i}(t_k^i) = 0$, the solution of $e_{\theta_i}(t)$ is given as

$$e_{\theta_i}(t) = \int_{t_k^i}^{t} (-\sum_{j=1}^{N} \mathcal{L}_{ij} \nabla g_i(y_i(\tau)) + \alpha_1 \sum_{j=1}^{N} \mathcal{L}_{ij} \theta_j \left(t_k^i\right) + \alpha_2 \sum_{j=1}^{N} \mathcal{L}_{ij} \vartheta_j \left(t_k^i\right)) \mathrm{d}\tau, .$$

Define t_{k+1}^i as the next triggering instant of agent i. Since $d_i(t), i = 1, 2, \cdots, N$, is bounded, $\lim_{t\to\infty} \delta_y(t) = 0$, and $\lim_{t\to\infty} \varphi(t) = 0$, one can obtain the boundedness of the variables $\nabla g_i(y_i), \theta_j(t), \vartheta_j(t)$. Define $h_0, h_1, h_2, \zeta \in R_{>0}$ such that $\|\nabla g_i(y_i)\| \leq h_0$, $\|\theta_j(t)\| \leq h_1$, $\|\vartheta_j(t)\| \leq h_2$, $\forall i = 1, \cdots, N$, and denote $max\{L_{ii}\} \leq \zeta, i = 1, 2, \cdots, N$. Thus, it is concluded that

$$\|e_{\theta_i}(t)\| \leq (2\zeta h_0 + 2\zeta\alpha_1 h_1 + 2\zeta\alpha_2 h_2)(t - t_k^i).$$

By noting the triggering condition (26), one obtains

$$\|e_{\theta_i}(t)\|^2 \leq \frac{1}{\mu_1} e^{-\sigma_1 t}, \forall t \in [t_k^i, t_{k+1}^i).$$

Then, a lower bound τ_k^i of $t_{k+1}^i - t_k^i$ can be obtained by solving the following inequality

$$\sqrt{\frac{1}{\mu_1} e^{-\sigma_1 \left(\tau_k^i + t_k^i\right)}} \leq (2\zeta h_0 + 2\zeta\alpha_1 h_1 + 2\zeta\alpha_2 h_2)\tau_k^i. \tag{32}$$

It should be noticed that τ_k^i always exists and is strictly positive in a finite time by (32). Consequently, no Zeno behavior is exhibited, and this completes the proof. □

Remark 9. *By carefully designing the ETC parameters μ_1, μ_2, σ_1, σ_2, it can be ensured that the DOCP can be solved and global convergence can be guaranteed. The exponential functions in the triggering condition (26) play a significant role in restricting measurement errors and eliminating Zeno behavior without placing an upper bound on the communications frequency.*

5. Illustrative Examples

In this section, a numerical example is given to visualize the theoretical results in Section 4. Consider a heterogeneous multiagent with seven agents described by (1), where $A_1 = [0, 1; 0, 0]$, $B_1 = [0, 1; 1, -2]$, $C_1 = [1, 1]$, $A_2 = [0, 1; 1, 0]$, $B_2 = [0, -2; 1, 1]$, $C_2 = [1, 1]$, $A_3 = [0, -1; 1, -2]$, $B_3 = [1, 0; 3, -1]$, $C_3 = [-1, 1]$, $A_4 = [-1, 0; -2, 2]$, $B_4 = [1, 2; 3, 4]$, $C_4 = [-1, 1]$, $A_5 = [0, 0; -1, 0]$, $B_5 = [1, 2; 1, 0]$, $C_5 = [1, -1]$, $A_{6,7} = [0, 0; 0, 1]$, $B_{6,7} = [1, -2; 1, -4]$, $C_{6,7} = [1, -1]$, $D_{1,2,3} = [-1, 1]$, $D_{4,5} = [1, -1]$, $D_{6,7} = [1, 1]$. The Laplacian matrix $\mathcal{L}$ of the directed and strongly connected network described in Figure 1 is

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

The local objective function of the ith agent is given by $g_i(y) = sin(\frac{i}{10}y), i = 1, 2 \cdots 7$. As an example, take the bounded external disturbance as follows:

$$d_{1,2} = [\sin(t), \cos(t)]^T, d_{3,4} = [2\cos(t), \sin(t)]^T, d_{5,6,7} = [2\cos(t), \cos(t)]^T.$$

The control law (5) and triggering function (26) parameters are chosen as follows: $K_{11} = [-2, -3; -1, -1]$, $\Upsilon_{11} = [0; -0.2]$, $\Upsilon_{21} = [-0.8; -0.4]$, $\Psi_1 = [-0.4; 0.4]$; $K_{12} = [-1.5, -1.5; 0.5, 0.5]$,$\Upsilon_{12} = [0.25; -0.25]$, $\Upsilon_{22} = [-0.5; 0]$, $\Psi_2 = [0.5; 0]$; $K_{13} = [-1, 1; -2, 2]$, $\Upsilon_{13} = [0.333; 0]$, $\Upsilon_{23} = [1; 1.333]$, $\Psi_3 = [0.333; 1]$; $K_{14} = [2, -3; -1, 1.5]$, $\Upsilon_{14} = [0; -0.09]$, $\Upsilon_{24} = [0.73; -0.45]$, $\Psi_4 = [-0.18; 0.36]$; $K_{15} = [1, -1; -1, 0.5]$, $\Upsilon_{15} = [0; 0.2]$, $\Upsilon_{25} = [0.4; -0.2]$,$\Psi_5 = [0.4; 0]$; $K_{16} = [-2, 2; -0.5, 1]$, $\Upsilon_{16} = [0; -0.25]$, $\Upsilon_{26} = [1; 0.5]$, $\Psi_6 = [0.5; 1]$; $K_{17} = [-2, 2; -0.5, 1]$, $\Upsilon_{17} = [0; -0.25]$, $\Upsilon_{27} = [1; 0.5]$, $\Psi_7 = [0.5; 1]$, $c_1 = 5$, $\alpha_1 = 5$, $\alpha_2 = 1.5$, $\sigma_1 = \sigma_2 = 0.2$, $a_\delta(\mathcal{L}) = 0.3765$. By Theorem 3, we select $\alpha_1 = 10$ and $\alpha_2 = 7$ in the control protocols (24). The initial values $x(0)$ are randomly given, and $\varphi(0) = 0$.

It is clear from Figure 1 that the network is strongly connected. Figure 2 displays that the consensus errors $\delta_{y_i}(t), i = 1, 2, \cdots, 7$ converge to 0 in a finite time, which means the output of all agents $y_i(t)$ achieves an optimal value of $y^* = -0.4101$. The evolution of the optimization goal is depicted in Figure 3, where we can see that the global objective function $\sum_{i=1}^N \xi_i g_i(y_i)$ converges to the global optimal solution $\sum_{i=1}^N \xi_i g_i(y^*) = -0.1684$. Therefore, the optimization problem (2) is solved. Figure 4 depicts the triggering instants of all agents, which reveal that the communication is discrete. Thus, no Zeno behavior is displayed.

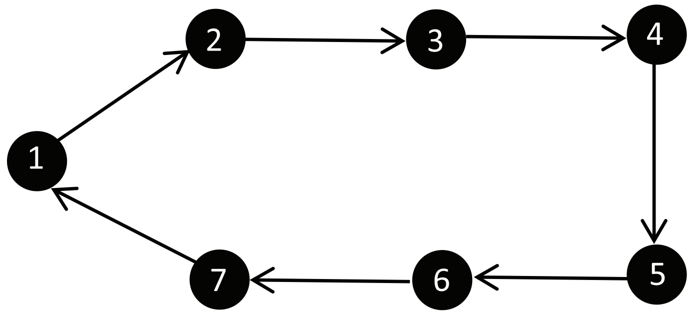

Figure 1. Topologies of a strongly connected network.

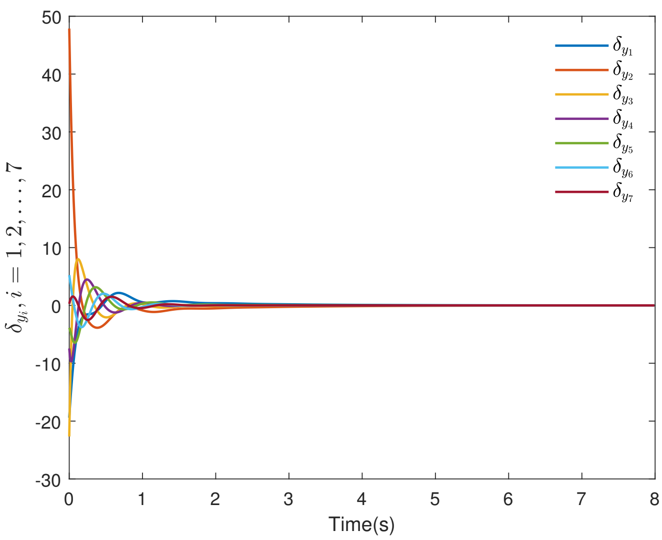

Figure 2. Trajectory of consensus error $\delta_{y_i}(t), i = 1, 2, \ldots, 7$.

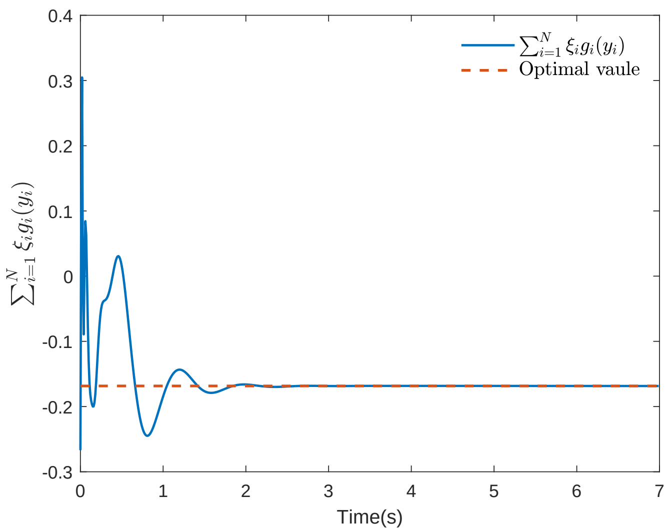

Figure 3. The sum of local objection functions $\sum_{i=1}^{N} \xi_i g_i(y_i)$.

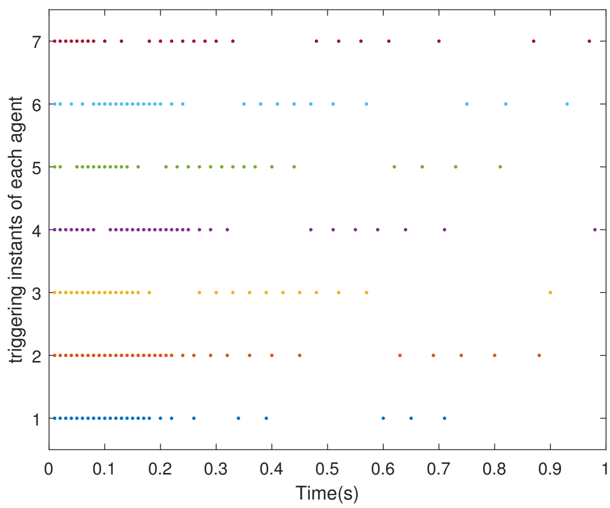

Figure 4. Triggering instants of each agent.

6. Conclusions

Firstly, a control rule for the DOCP of heterogeneous MASs with directed topology is provided. By using the output regulation techniques, the proposed control law can solve the DOCP and ensure that global convergence despite the existence of bounded external disturbances. The proposed control law is then extended to ETC methods, which enable agents to avoid continuous communication. It is demonstrated that with this algorithm, consensus can be achieved, and the Zeno behavior can be avoided. One numerical example validates the efficiency of the theoretical conclusions. The future study will focus on extending this paper to directed networks with a switching topology.

Author Contributions: D.L. performed the data analyses and wrote the manuscript; J.F. contributed to the conception of the study; Y.Z. and J.F. helped perform the analysis with constructive discussions; J.W. performed the experiment. All authors have read and agreed to the published version of the manuscript.

Funding: This research is supported by the National Science Foundation of China under Grant 62006159.

Data Availability Statement: The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest: The authors declare no conflict of interest.

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