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On the Practicality of the Analytical Solutions for all Third- and Fourth-Degree Algebraic Equations with Real Coefficients

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Abstract: In order to propose a deeper analysis of the general quartic equation with real coefficients, the analytical solutions for all cubic and quartic equations were reviewed here; then, it was found that there can only be one form of the resolvent cubic that satisfies the following two conditions at the same time: (1) Its discriminant is identical to the discriminant of the general quartic equation. (2) It has at least one positive real root whenever the general quartic equation is non-biquadratic. This unique special form of the resolvent cubic is defined here as the "Standard Form of the Resolvent Cubic", which becomes relevant since it allows us to reveal the relationship between the nature of the roots of the general quartic equation and the nature of the roots of all the forms of the resolvent cubic. Finally, this new analysis is the basis for designing and programming efficient algorithms that analytically solve all algebraic equations of the fourth and lower degree with real coefficients, always avoiding the application of complex arithmetic operations, even when these equations have non-real complex roots.

Keywords: quadratic formula; Tartaglia–Cardano Formulae; Ferrari method; the Standard Form of the Resolvent Cubic; polynomials

MSC: 01-01; 01A40; 12D10; 26-08; 26C05; 26C10

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1. Introduction

For many centuries, the fundamental purpose of algebra was the resolution of algebraic equations; so, before the 16th Century, the most powerful known results of algebra were the analytical solutions given to the Second-Degree Equation (SDE) with real coefficients and real roots; meanwhile, the cases of this equation with non-real roots used to be discarded because they did not make any practical sense at that time.

Afterward, during the first half of the 16th Century, the analytical solutions by radicals for the third- and fourth-degree equations were finally revealed by the legendary renaissance mathematicians Scipione del Ferro, Niccolò Fontana (A.K.A., "Tartaglia"), Girolamo Cardano and his disciple Ludovico Ferrari [1] (p. 403); then, these new results became fundamental for the development of modern mathematics because these motivated, for the first time, the definition and study of non-real complex numbers, which would later give rise to complex analysis, so the cases of the SDE with non-real roots were finally accepted and understood in theoretical terms.

Thus, between the middle of the 16th Century and the late 18th Century, the most important problem in algebra was the search for a general analytical solution for the fifth-degree equations; nevertheless, the nonexistence of general solutions by radicals for algebraic equations of the fifth and higher degree was verified by the proof of the Abel–Ruffini Theorem [2] (p. 51); so, this fact and the appearance of the Galois Theory in the early 19th Century ushered in a new era in the history of algebra.

Therefore, the main purpose of algebra during the last two centuries has been focused on the general study of algebraic structures (groups, fields, rings, etc.), which is generally labeled with names like "Modern Algebra", "Abstract Algebra" or "Higher Algebra"; conversely, the algebra developed before the Abel–Ruffini Theorem is usually labeled as "Basic Algebra" or "Classical Algebra", the analytical solutions of third and fourth-degree equations being the key topics that connect both kinds of algebra, since these topics were the most powerful results given by classical algebra, while solving algebraic equations is not even a goal of modern algebra, which offers a new and more abstract perspective on polynomials and their solvability by radicals.

On the other hand, the appearance and development of calculus and its practical applications gave rise to the modern numerical methods, which have been evolved even faster since the beginning of the computational era; hence, the development achieved by technology today allows numerical methods to approximate with great precision the solutions of almost any kind of equations. It does not matter if they are algebraic or non-algebraic, which is very useful for solving many practical problems; thus, the numerical methods are now the most used tools to "solve" equations in general.

So nowadays, the great art of solving algebraic equations analytically is unappreciated and almost lost, and it has been reduced to high-school basic algebra courses, in which the quadratic formula for solving all the cases of the SDE with real coefficients is typically one of the most advanced topics [3]; meanwhile, the results given by del Ferro, Tartaglia, Cardano and Ferrari are only studied in some college courses for math students and are usually treated superficially, and only as a brief preamble to advanced algebra courses dedicated to Galois Theory and other modern algebra topics.

Additionally, the most well-known and used literature in the mentioned college courses incredibly overlooks or ignores some important facts about the solutions by radicals of third and fourth-degree equations and also tends to expose them as very impractical tools to solve these equations; therefore, it is not surprising that some mathematicians around the world regard them as "absolutely useless old-fashioned stuff" whose only value is historical [4], since at first sight they seem to depend heavily on complex arithmetic, which apparently complicates their application in practical terms. So, in this general scenario, this article has the following four main objectives:

1. Revalue the general analytical solutions of the third and fourth-degree algebraic equations with real coefficients in practical terms for the 21st Century.
2. Present a new analysis of these solutions for these equations to show how these methods can always solve these equations without complex arithmetic operations.
3. Define the "Standard Form of the Resolvent Cubic" (SFRC) in order to expose all the relationships between the nature of the roots of any fourth-degree equation and the nature of the roots of all the forms of the corresponding resolvent cubic.
4. The design of an efficient computing program that solves all these equations, always avoiding the application of numerical methods and complex arithmetic.

In order to achieve these fundamental objectives, in this paper the General Cubic Equation (GCE) will be considered as any equation of the form $ax^3 + bx^2 + cx + d = 0$, with $a, b, c, d \in \mathbb{R}$ and $a \neq 0$; so, according to [5] (pp. 95–98), the Tartaglia–Cardano Formulae allows us to obtain the three roots of the GCE in theoretical terms as follows:

$$x_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \frac{b}{3a} \quad (1)$$

$$x_{2,3} = \left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \left(-\frac{1}{2} \mp \frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \frac{b}{3a}\,; \quad (2)$$

where

$$p = \frac{3ac - b^2}{3a^2} \text{ and } q = \frac{2b^3 - 9abc + 27a^2d}{27a^3}\,. \quad (3)$$

So, it is clear that the parameters p and q given by Equation (3) are also real numbers; additionally, it is important to say that they are also the coefficients of the equation $y^3 + py + q = 0$, which is known as the Depressed Cubic Equation (DCE), whose relationship with the GCE is determined by the change of variable $x = y - \frac{b}{3a}$.

On the other hand, in this paper, the General Quartic Equation (GQE) will be considered as any equation of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$, with $a, b, c, d, e \in \mathbb{R}$ and $a \neq 0$; in addition, the corresponding Depressed Quartic Equation (DQE) will be considered as the equation $y^4 + py^2 + qy + r = 0$, with $p, q, r \in \mathbb{R}$; meanwhile, the relationship between the GQE and the DQE is determined by the change of variable $x = y - \frac{b}{4a}$, so the coefficients of both equations are related as follows:

$$p = \frac{8ac - 3b^2}{8a^2}\,, \quad (4)$$

$$q = \frac{b^3 - 4abc + 8a^2d}{8a^3}\,, \quad (5)$$

$$r = \frac{16ab^2c - 64a^2bd - 3b^4 + 256a^3e}{256a^4}\,; \quad (6)$$

therefore, as stated in [6], Equation (5) allows us to classify the GQE in the following two complementary cases:

1. ***Biquadratic Case*:** If Equation (5) implies $q = 0$, then the DQE is reduced to a biquadratic equation; ergo, the GQE corresponds to this case whenever $b^3 = 4abc - 8a^2d$, and its four roots are given by the following general formulae:

$$x_{1,2} = \pm\sqrt{\frac{-p + \sqrt{p^2 - 4r}}{2}} - \frac{b}{4a} \text{ and } x_{3,4} = \pm\sqrt{\frac{-p - \sqrt{p^2 - 4r}}{2}} - \frac{b}{4a}\,. \quad (7)$$

2. ***Ferrari Case*:** It corresponds to all the non-biquadratic quartic equations. In other words, when Equation (5) implies $q \neq 0$; thus, these equations cannot be solved by the formulae given by Equation (7). So, according to [5] (p. 107), the Ferrari Method was the first known method that could solve this kind of quartic equations; however, this method cannot solve the biquadratic case, because it is undefined when $q = 0$ in the DQE [7] (pp. 23–24).

Since Equations (7) and the Ferrari Method are not complete general solutions of the GQE, a generalized variation of this method was developed in [6], which is a complete general analytical solution for both cases of the GQE; so, this general solution of the GQE depends on the roots of the SFRC, which is the third-degree equation

$s^3 + 2ps^2 + (p^2 - 4r)s - q^2 = 0$. Then, consider the set of all the non-zero roots of the SFRC; that is, $S := \{s_1, s_2, s_3\} - \{0\} \subset \mathbb{C} - \{0\}$, and the parameter α_s defined as follows:

$$\alpha_s := \begin{cases} 0, & \text{if } S = \varnothing \\ -\frac{2q}{\sqrt{s}}, & \text{for } s \in S \neq \varnothing \end{cases}; \tag{8}$$

thus, the actual complete general analytical solution for the GQE is given by the following general formula:

$$x = \frac{\xi\sqrt{s} \pm \sqrt{\xi\alpha_s - 2p - s}}{2} - \frac{b}{4a}; \tag{9}$$

where $s = 0$ whenever $p = q = r = 0$ (in other words, when $S = \varnothing$); otherwise, $s \in S \neq \varnothing$; and $\xi := \pm 1$. Hence, the definitions of S and α_s guarantee that Equation (9) never goes undefined, unlike the original Ferrari Method, which becomes undefined for the biquadratic case; on the other hand, as also stated in [6], the respective discriminants of the GQE, the DQE and the SFRC are all identical to each other, so the most simplified form of the discriminant of these three equations is given in terms of the coefficients of the DQE as follows:

$$\Delta_4 = \frac{4(p^2 + 12r)^3 - [2p(p^2 - 36r) + 27q^2]^2}{27}. \tag{10}$$

In addition, another well-known result that will also be very important in this document is the Viète Theorem, as stated in [8]; because this one reveals how all the roots and all the coefficients of any n-th-degree polynomial equation are related among them.

Likewise, it is also important to point out that the solutions of these equations are relevant because they have several applications in real life problems; for example, some engineering problems can be solved by applying differential equations whose solutions are related to the resolution of third- or fourth-degree polynomial equations; any nonlinear optimization problem with a fourth- or fifth-degree polynomial function as an objective function is solved by finding the roots of the derivative of the objective function, which is a third- or fourth-degree polynomial function in these cases.

Additionally, the solution of quartic equations is related to the solution of some geometric problems, such as the crossed ladders problem, the intersection of two conic sections in $\mathbb{R}^2$, the intersection of a torus and a line, etc. Then, the solutions to these problems are used for the development of software focused on different kinds of applications related to multiple arts and disciplines, such as architecture, astronomy, industrial manufacturing, design, visual effects in modern cinema, computer animation, video game development, etc., among many other practical applications of these algebraic equations in physics and applied sciences.

So, the analysis presented here will begin in Section 2.1, reviewing some important facts about the analytical solutions of the third-degree polynomial equations with real coefficients that are generally overlooked or ignored by the most well-known literature on these topics; these facts will be important to efficiently solve the GCE without numerical methods and while avoiding complex arithmetic, so they will also be fundamental to analyze and solve the SFRC and the GQE without numerical methods and while avoiding complex arithmetic as well.

Afterward, the definition of the SFRC and its importance over the other known forms of the resolvent cubic will be exposed in Section 2.2; also, the SFRC will be fundamental in Section 2.3 to expose all the relationships between the nature of the roots of the GQE and the nature of the roots of all the forms of the resolvent cubic. Meanwhile, Section 2.4 shows how the novelties of the SFRC exposed in the previous subsections are also linked with the criteria to identify a priori the nature of the roots of the GQE. Then, Section 3.1 is dedicated to exposing how the SFRC also helps to analytically solve the GQE, always avoiding the application of complex arithmetic operations in practical terms.

Finally, Section 3 focuses mainly on the development of a computing program that was made using the Wolfram Mathematica programming language to analytically solve and plot all the polynomial equations of the second, third and fourth degree with real coefficients, based on algorithms designed according to all the results exposed in Section 2 and also by applying the formulae that instantly solve the non-biquadratic quartic equations with real coefficients and multiple roots that were also stated in [6].

2. Research Methods

2.1. On the Third-Degree Polynomial Equations with Real Coefficients

2.1.1. The Big Problems with the Analytical Solutions for the General Cubic Equation

In practical terms, the big problem with Equations (1) and (2) to solve the GCE is that these imply some difficulties provoked by the necessity of some annoying complex arithmetic calculations, especially when $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 < 0$, which is known as the "Casus Irreducibilis" [7] (p. 19); likewise, in historical terms, this special case motivated the definition and study of the complex number set $\mathbb{C}$. Nevertheless, according to [9], there exists an alternative trigonometric general formula, based also on the parameters given by Equation (3), to solve the Casus Irreducibilis of the GCE; this is given as follows:

$$x_k = 2\sqrt{-\frac{p}{3}}\cos\left[\frac{1}{3}\arccos\left\{-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}\right\} + \frac{2(k-1)\pi}{3}\right] - \frac{b}{3a}\text{ , where } k = 1, 2, 3. \quad (11)$$

Remark 1. *Note that the inequality* $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 < 0$ *holds whenever the following three inequalities also hold:*

(i) $p < -\sqrt[3]{\frac{27}{4}q^2} \leq 0$,

(ii) $-1 < -\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3} < 1$,

(iii) $0 < \arccos\left\{-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}\right\} < \pi$;

meanwhile, the inequality $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 > 0$ *implies only one of the following three possibilities:*

(iv) $-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3} \in \mathbb{C} - \mathbb{R}$ *whether* $p > 0$,

(v) *the expression* $-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}$ *is undefined whether* $p = 0$,

(vi) $\left|-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}\right| > 1$ *whether* $-\sqrt[3]{\frac{27}{4}q^2} < p < 0$;

hence, the expression $\arccos\left\{-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}\right\}$ *is not defined within set* $\mathbb{R}$ *for possibilities (iv)–(vi).*

Therefore, the problem with Equation (11) is that it is almost always impractical for solving the GCE when this one does not correspond to the Casus Irreducibilis. To illustrate all the previously exposed difficulties in analytically solving cubic equations, consider the equation $x^3 - 21x + 20 = 0$, in which $a = 1$, $b = 0$, $c = p = -21$ and $d = q = 20$; it is clear that the three roots of this equation are $x_1 = 4$, $x_2 = -5$ and $x_3 = 1$, since this equation can also be expressed as $(x-4)[x-(-5)](x-1) = 0$.

However, if Equation (1) is applied to this equation, then $x_1 = \sqrt[3]{-\frac{20}{2} + \sqrt{\left(\frac{20}{2}\right)^2 + \left(\frac{-21}{3}\right)^3}} + \sqrt[3]{-\frac{20}{2} - \sqrt{\left(\frac{20}{2}\right)^2 + \left(\frac{-21}{3}\right)^3}} - \frac{0}{3(1)} = \sqrt[3]{-10 + \sqrt{-243}} + \sqrt[3]{-10 - \sqrt{-243}}$; so, although the relation $\sqrt[3]{-10+\sqrt{-243}}+\sqrt[3]{-10-\sqrt{-243}}=4$ is true, its validity is not obvious, and proving such numerical relations was an almost incomprehensible problem for the renaissance mathematicians because these relations involved square roots of negative real numbers, which were considered absurd at that time.

Nowadays, the resolution of this equation by applying Equations (1) and (2) implies the application of some non-trivial tools of complex arithmetic, such as the De Moivre's Formulae [5] (p. 27), which can greatly complicate the resolution of this equation in practical terms; fortunately, the resolution of this equation by applying Equation (11) just requires verification (with the help of a modern calculator) of the following three relations: $x_1 = 2\sqrt{-\frac{(-21)}{3}}\cos\left[\frac{1}{3}\arccos\left\{-\frac{20}{2}\sqrt{\left(-\frac{3}{-21}\right)^3}\right\}+\frac{2(1-1)\pi}{3}\right]-\frac{0}{3(1)} = 2\sqrt{7}\cos\left[\frac{1}{3}\arccos\left(-\frac{10}{7\sqrt{7}}\right)\right]=4$, $x_2 = 2\sqrt{-\frac{(-21)}{3}}\cos\left[\frac{1}{3}\arccos\left\{-\frac{20}{2}\sqrt{\left(-\frac{3}{-21}\right)^3}\right\}+\frac{2(2-1)\pi}{3}\right]-\frac{0}{3(1)} = 2\sqrt{7}\cos\left[\frac{1}{3}\arccos\left(-\frac{10}{7\sqrt{7}}\right)+\frac{2\pi}{3}\right] = -5$ and $x_3 = 2\sqrt{-\frac{(-21)}{3}}\cos\left[\frac{1}{3}\arccos\left\{-\frac{20}{2}\sqrt{\left(-\frac{3}{-21}\right)^3}\right\}+\frac{2(3-1)\pi}{3}\right]-\frac{0}{3(1)} = 2\sqrt{7}\cos\left[\frac{1}{3}\arccos\left(-\frac{10}{7\sqrt{7}}\right)+\frac{4\pi}{3}\right]=1$.

On the other hand, consider the equation $x^3-9x+28=0$, in which $a=1$, $b=0$, $c=p=-9$ and $d=q=28$; so, if Equations (1) and (2) are applied to this equation, then $x_1 = \sqrt[3]{-\frac{28}{2}+\sqrt{\left(\frac{28}{2}\right)^2+\left(\frac{-9}{3}\right)^3}}+\sqrt[3]{-\frac{28}{2}-\sqrt{\left(\frac{28}{2}\right)^2+\left(\frac{-9}{3}\right)^3}}-\frac{0}{3(1)} = -1-3=-4$ and $x_{2,3} = \left(-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{28}{2}+\sqrt{\left(\frac{28}{2}\right)^2+\left(\frac{-9}{3}\right)^3}}+\left(-\frac{1}{2}\mp\frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{28}{2}-\sqrt{\left(\frac{28}{2}\right)^2+\left(\frac{-9}{3}\right)^3}}-\frac{0}{3(1)} = \left(-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i\right)(-1)+\left(-\frac{1}{2}\mp\frac{\sqrt{3}}{2}i\right)(-3)=2\pm\sqrt{3}i$; thus, it is clear that this equation can be easily solved by Equations (1) and (2).

However, if Equation (11) is applied to this new equation, then $x_k = 2\sqrt{-\frac{(-9)}{3}}\cos\left[\frac{1}{3}\arccos\left\{-\frac{28}{2}\sqrt{\left(-\frac{3}{-9}\right)^3}\right\}+\frac{2(k-1)\pi}{3}\right]-\frac{0}{3(1)} = 2\sqrt{3}\cos\left[\frac{1}{3}\arccos\left(-\frac{14}{\sqrt{3}}\right)+\frac{2(k-1)\pi}{3}\right]$, for $k=1,2,3$; now, note that $\left|-\frac{14}{\sqrt{3}}\right|>1$, so $\arccos\left(-\frac{14}{\sqrt{3}}\right)$ is not defined in $\mathbb{R}$; in fact, the study of complex analysis allows us to know that $\arccos\left(-\frac{14}{\sqrt{3}}\right)\in\mathbb{C}-\mathbb{R}$. Therefore, the application of Equation (11) in this case requires the use of trigonometric functions with non-real complex arguments, which too extensively complicates in practical terms the resolution of the equation in question.

Additionally, since the analytical solutions of some quartic equations imply the resolution of cubic equations, the problems exposed above extend to those equations as well—this being the main motivation of this research. Finally, a new proof restricted to $\mathbb{R}$ of Equation (11) is included in Appendix A.

2.1.2. The Discriminant and the Nature of the Roots of the General Cubic Equation

According to the theory of polynomial equations [7] (pp. 101–102), the discriminant of any third-degree equation is defined as follows:

$$\Delta_3 := [(u_1-u_2)(u_1-u_3)(u_2-u_3)]^2, \tag{12}$$

where u_1, u_2 and u_3 are the three roots of the equation in question; so, if x_1, x_2 and x_3 are the three roots of the GCE and y_1, y_2 and y_3 are the three roots of the DCE, then all these roots are related as follows: $x_k = y_k - \frac{b}{3a}$ for each $k\in\{1,2,3\}$; thus, it is clear that $x_i - x_j = y_i - y_j$ for all $i, j \in \{1,2,3\}$. Hence, Equation (12) guarantees that the discriminant of the GCE is identical to the discriminant of the DCE. On the other hand, according to [2] (p. 49), the discriminant of the DCE is given as follows:

$$\Delta_3 = -4p^3 - 27q^2\,, \tag{13}$$

where p and q are determined by Equation (3); so, Equation (13) also gives the discriminant of the GCE.

Remark 2. *According to* [7] (p. 103), *Equation (12) also guarantees that the relation between the discriminant and the nature of the roots of the GCE is given as follows:*

(i) *The GCE has three real roots whenever* $\Delta_3 \geq 0$*;*
(ii) *The GCE has only one real root and a couple of non-real complex conjugate roots whenever* $\Delta_3 < 0$.

additionally, $\Delta_3 = 0$ *whenever the GCE has multiple roots; thus,* $\Delta_3 > 0$ *whenever the GCE has three non-multiple real roots, and the GCE never has multiple non-real complex roots.*

In addition, the following proposition states when the GCE has a couple of purely imaginary roots.

Proposition 1. *The GCE has one real root and a couple of purely imaginary conjugate roots if, and only if,* $ad = bc$ *and* $\frac{c}{a} > 0$*; also, these roots are given as follows:* $x_1 = -\frac{b}{a} = -\frac{d}{c}$ *and* $x_{2,3} = \pm\sqrt{\frac{c}{a}}i$.

Proof. Suppose that the three roots of the GCE are given as follows: $x_1 = k$ and $x_{2,3} = \pm mi$, where $k \in \mathbb{R}$ and $m \in \mathbb{R} - \{0\}$; then, this equation can be expressed as follows: $a(x-k)(x-mi)(x+mi) = ax^3 - akx^2 + am^2x - akm^2 = 0$, so $b = -ak$, $c = am^2$ and $d = -akm^2$; thus, $ad = a(-akm^2) = (-ak)(am^2) = bc$ and $\frac{c}{a} = \frac{am^2}{a} = m^2 > 0$ due to $a \neq 0 \neq m$. Now, in order to prove the reciprocal, suppose that $ad = bc$ and $\frac{c}{a} > 0$; then, $c \neq 0$, $b = \frac{ad}{c}$ and $d = \frac{bc}{a}$, so the GCE has the following two forms, simultaneously: $ax^3 + bx^2 + cx + \frac{bc}{a} = ax^2\left(x + \frac{b}{a}\right) + c\left(x + \frac{b}{a}\right) = a\left(x + \frac{b}{a}\right)\left(x^2 + \frac{c}{a}\right) = 0$ and $ax^3 + \frac{ad}{c}x^2 + cx + d = ax^2\left(x + \frac{d}{c}\right) + c\left(x + \frac{d}{c}\right) = a\left(x + \frac{d}{c}\right)\left(x^2 + \frac{c}{a}\right) = 0$; hence, the roots of the GCE are given as follows: $x_1 = -\frac{b}{a} = -\frac{d}{c}$ and $x_{2,3} = \pm\sqrt{-\frac{c}{a}}$; therefore, the roots x_2 and x_3 are a pair of purely imaginary numbers because $-\frac{c}{a} < 0$, thus $x_2 = \sqrt{\frac{c}{a}}i \neq x_3 = -\sqrt{\frac{c}{a}}i$. □

Example 1. *Suppose that* $5x^3 - 3x^2 + 15x - 9 = 0$*. Then* $a = 5$, $b = -3$, $c = 15$ *and* $d = -9$*; so, it is clear that* $ad = bc = -45$ *and* $\frac{c}{a} = 3 > 0$*; therefore, Proposition 1 guarantees* $x_1 = -\frac{(-3)}{5} = -\frac{(-9)}{15} = \frac{3}{5}$ *and* $x_{2,3} = \pm\sqrt{\frac{15}{5}}i = \pm\sqrt{3}i$*. In fact, the given equation can also be expressed as* $5\left(x - \frac{3}{5}\right)\left(x - \sqrt{3}i\right)\left[x - \left(-\sqrt{3}i\right)\right] = 0$.

Finally, consider the following lemma that will also be very important in Section 3.1 in order to avoid complex arithmetic during the resolution of quartic equations.

Lemma 1. *If the GCE has three non-multiple real roots* x_1, x_2 *and* x_3*; then*

(i) $x_1 = \max\{x_1, x_2, x_3\}$
(ii) $x_2 = \min\{x_1, x_2, x_3\}$

Proof. Consider the following definition: $\theta_k := \arccos\left[-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}\right] + 2(k-1)\pi$, then Equation (11) can be rewritten as follows: $x_k = 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\theta_k}{3}\right) - \frac{b}{3a}$, for $k \in \{1, 2, 3\}$; meanwhile, if the GCE has three non-multiple real roots, then Remark 2 guarantees $\Delta_3 > 0$, so Equation (13) implies $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = -\frac{\Delta_3}{108} < 0$; hence, (iii) of Remark 1 guarantees $0 < \theta_1 < \pi$, which implies these other two inequalities: $2\pi < \theta_1 + 2\pi = \theta_2 < 3\pi$ and $4\pi < \theta_1 + 4\pi = \theta_3 < 5\pi$; thus, $0 < \frac{\theta_1}{3} < \frac{\pi}{3}$, $\frac{2\pi}{3} < \frac{\theta_2}{3} < \pi$ and

$\frac{4\pi}{3} < \frac{\theta_3}{3} < \frac{5\pi}{3}$; ergo, the three previous inequalities imply these other ones, respectively: $\frac{1}{2} < \cos\left(\frac{\theta_1}{3}\right) < 1$, $-1 < \cos\left(\frac{\theta_2}{3}\right) < -\frac{1}{2}$ and $-\frac{1}{2} < \cos\left(\frac{\theta_3}{3}\right) < \frac{1}{2}$, which guarantee the following relation:

$$\cos\left(\frac{\theta_2}{3}\right) < \cos\left(\frac{\theta_3}{3}\right) < \cos\left(\frac{\theta_1}{3}\right). \tag{14}$$

On the other hand, (i) of Remark 1 guarantees $p < 0$, which implies $\sqrt{-\frac{p}{3}} \in \mathbb{R}$ and $\sqrt{-\frac{p}{3}} > 0$; hence, this inequality and Equation (14) implies $2\sqrt{-\frac{p}{3}}\cos\left(\frac{\theta_2}{3}\right) < 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\theta_3}{3}\right) < 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\theta_1}{3}\right)$, which also implies $2\sqrt{-\frac{p}{3}}\cos\left(\frac{\theta_2}{3}\right) - \frac{b}{3a} < 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\theta_3}{3}\right) - \frac{b}{3a} < 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\theta_1}{3}\right) - \frac{b}{3a}$; therefore, the previous relation and Equation (11) guarantee $x_2 < x_3 < x_1$, so $x_1 = \max\{x_1, x_2, x_3\}$ and $x_2 = \min\{x_1, x_2, x_3\}$. □

2.1.3. The Cases of the General Cubic Equation with Multiple Roots

Since Remark 2 allows us to know if the GCE has multiple roots, the following proposition allows us to know how many multiple roots it has and how to obtain them.

Proposition 2. *If $\Delta_3 = 0$, then the three roots of the GCE are given as follows:*

(i) *If Equation (3) imply $p \neq 0 \neq q$, then $x_1 = -\sqrt[3]{4q} - \frac{b}{3a} \neq x_2 = x_3 = \sqrt[3]{\frac{q}{2}} - \frac{b}{3a}$.*

(ii) *If Equation (3) imply $p = q = 0$, then $x_1 = x_2 = x_3 = -\frac{b}{3a}$.*

Proof. First of all, note that Equation (13) can be rewritten as $\Delta_3 = -108\left[\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3\right]$; thus, $\Delta_3 = 0$ whenever $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = 0$; therefore,

(i) If $p \neq \Delta_3 = 0 \neq q$, then $\sqrt[3]{-\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} = \sqrt[3]{-\frac{q}{2}} \neq 0$; so, Equations (1) and (2) imply $x_1 = \sqrt[3]{-\frac{q}{2}} + \sqrt[3]{-\frac{q}{2}} - \frac{b}{3a} = 2\sqrt[3]{-\frac{q}{2}} - \frac{b}{3a} = -\sqrt[3]{4q} - \frac{b}{3a}$ and $x_{2,3} = \left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{q}{2}} + \left(-\frac{1}{2} \mp \frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{q}{2}} - \frac{b}{3a} = -\sqrt[3]{-\frac{q}{2}} - \frac{b}{3a} = \sqrt[3]{\frac{q}{2}} - \frac{b}{3a}$; in addition, note that $q \neq 0$ implies $-\sqrt[3]{4q} \neq \sqrt[3]{\frac{q}{2}}$, which finally guarantees $x_1 \neq x_2 = x_3$.

(ii) If $\Delta_3 = p = q = 0$, then $\sqrt[3]{-\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} = 0$; so, Equations (1) and (2) imply $x_1 = 0 + 0 - \frac{b}{3a} = -\frac{b}{3a}$ and $x_{2,3} = \left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right)(0) + \left(-\frac{1}{2} \mp \frac{\sqrt{3}}{2}i\right)(0) - \frac{b}{3a} = -\frac{b}{3a}$. □

Example 2. *Suppose that $2x^3 - 9x^2 + 12x - 4 = 0$, then $a = 2$, $b = -9$, $c = 12$ and $d = -4$; so, Equations (3) and (13) imply $p = -\frac{3}{4} \neq 0$, $q = \frac{1}{4} \neq 0$ and $\Delta_3 = -4\left(-\frac{3}{4}\right)^3 - 27\left(-\frac{1}{4}\right)^2 = 0$; therefore, (i) of Proposition 2 implies $x_1 = -\sqrt[3]{4\left(\frac{1}{4}\right)} - \frac{(-9)}{3(2)} = -1 + \frac{3}{2} = \frac{1}{2}$ and $x_2 = x_3 = \sqrt[3]{\frac{(1/4)}{2}} - \frac{(-9)}{3(2)} = \frac{1}{2} + \frac{3}{2} = 2$. In fact, the given equation can also be expressed as $2\left(x - \frac{1}{2}\right)(x-2)^2 = 0$.*

Example 3. *Suppose that $x^3 + 12x^2 + 48x + 64 = 0$, then $a = 1$, $b = 12$, $c = 48$ and $d = 64$; so, Equations (3) and (13) imply $p = q = \Delta_3 = 0$; therefore, Proposition 2 implies $x_1 = x_2 = x_3 = -\frac{12}{3(1)} = -4$. In fact, the given equation can also be expressed as $[x - (-4)]^3 = 0$.*

Finally, an alternative version of (i) of Proposition 2 is also exposed in Appendix A.

2.1.4. A General Analytical Solution for the Third-Degree Equations in Practical Terms

Now, in order to establish a general and actually practical analytical solution of the GCE, consider the following lemma:

Lemma 2. *(Theorem of Interdependence between the Roots of the GCE) Suppose that x_r is one of the three roots of the GCE, then its other two roots are given as follows:*

$$x = \frac{-(ax_r + b) \pm \sqrt{b^2 - 4ac - (3a^2x_r^2 + 2abx_r)}}{2a}\,. \tag{15}$$

Proof. If x_r is a root of the GCE, then $ax_r^3 + bx_r^2 + cx_r + d = 0$, so $d = -ax_r^3 - bx_r^2 - cx_r$; ergo, the GCE can be expressed as follows:

$$\begin{aligned} ax^3 + bx^2 + cx + d = ax^3 + bx^2 + cx - ax_r^3 - bx_r^2 - cx_r = a\left(x^3 - x_r^3\right) + b\left(x^2 - x_r^2\right) + c(x - x_r) = \\ (x - x_r)\left[a\left(x^2 + x_r x + x_r^2\right) + b(x + x_r) + c\right] = (x - x_r)\left[ax^2 + (ax_r + b)x + \left(ax_r^2 + bx_r + c\right)\right] = 0\,. \end{aligned} \tag{16}$$

So, Equation (16) guarantees that the other two roots of the GCE are the two roots of the SDE given as follows: $ax^2 + (ax_r + b)x + \left(ax_r^2 + bx_r + c\right) = 0$; therefore, Equation (15) is obtained after applying the quadratic Formula to this SDE. □

Since Equation (2) inevitably implies the application of complex arithmetic operations to solve the GCE, Lemma 2 will be very useful for avoiding those operations during the resolution of any third-degree equation; in addition, note that if $x_r = 0$ in Equation (15), then this one is reduced to the quadratic formula because this happens whenever $d = 0$; so, in this case, the GCE is reduced to $ax^3 + bx^2 + cx = x\left(ax^2 + bx + c\right) = 0$.

Theorem 1. *If* p and q *are the coefficients of the DCE, then the three roots of the corresponding GCE can be analytically obtained without complex arithmetic operations as follows:*

(i) *If* $\Delta_3 \le 0$, *then* $x_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{-\frac{\Delta_3}{108}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{-\frac{\Delta_3}{108}}} - \frac{b}{3a}\,.$

(ii) *If* $\Delta_3 > 0$, *then* $x_1 = 2\sqrt{-\frac{p}{3}}\cos\left[\frac{1}{3}\arccos\left\{-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}\right\}\right] - \frac{b}{3a}\,.$

(iii) $x_{2,3} = \frac{-(ax_1+b)\pm\sqrt{b^2-4ac-\left(3a^2x_1^2+2abx_1\right)}}{2a}$ *, for any* $\Delta_3 \in \mathbb{R}$.

Proof. (i) If Equations (3) and (13) imply $\Delta_3 \le 0$, then $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = -\frac{\Delta_3}{108} \ge 0$ and $\sqrt{-\frac{\Delta_3}{108}} \in \mathbb{R}$; so, in this case, Equation (1) allows us to obtain x_1 without complex arithmetic operations as follows: $x_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{-\frac{\Delta_3}{108}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{-\frac{\Delta_3}{108}}} - \frac{b}{3a}\,.$

(ii) If Equations (3) and (13) imply $\Delta_3 > 0$, then, according to (ii) and (iii) of Remark 1, the relation $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = -\frac{\Delta_3}{108} < 0$ guarantees $-1 < -\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3} < 1$ and $0 < \arccos\left\{-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}\right\} < \pi$; so, if $k = 1$ in Equation (11), then this one allows us to obtain x_1 for this case, without complex arithmetic operations.

(iii) Note that (i) and (ii) guarantee $x_1 \in \mathbb{R}$ for any case of the GCEs, so if $x_r = x_1$ in Equation (15), then this one will always allow us to obtain x_2 and x_3 without complex arithmetic, even when the GCE has a couple of non-real complex conjugate roots. □

Remark 3. *Although Remark 1 guarantees that Equation (11) also allows us to obtain* x_2 and x_3 *without complex arithmetic operations whenever* $\Delta_3 > 0$, *(iii) of Theorem 1 will always be easier to apply in practical terms because it works only with real basic arithmetic operations.*

Example 4. *Suppose that* $20x^3 - 76x^2 + 53x - 15 = 0$, *then* $a = 20$, $b = -79$, $c = 53$ and $d = -15$; *so, Equations (3) and (13) imply* $p = -\frac{649}{300}$, $q = -\frac{9841}{6750}$ *and* $\Delta_3 = -4\left(-\frac{649}{300}\right)^3 - 27\left(-\frac{9841}{6750}\right)^2 = -\frac{168921}{10000} < 0$; *therefore, (i) and (iii) of Theorem 1 imply* $x_1 = \sqrt[3]{-\frac{\left(-\frac{9841}{6750}\right)}{2} + \sqrt{-\frac{\left(-\frac{168921}{10000}\right)}{108}}} + \sqrt[3]{-\frac{\left(-\frac{9841}{6750}\right)}{2} - \sqrt{-\frac{\left(-\frac{168921}{10000}\right)}{108}}} - \frac{(-76)}{3(20)} = \sqrt[3]{\frac{9841}{13500} + \frac{137}{200\sqrt{3}}} + \sqrt[3]{\frac{9841}{13500} - \frac{137}{200\sqrt{3}}} + \frac{19}{15} = \frac{26}{15} + \frac{19}{15} = 3$ *and* $x_{2,3} = \frac{-[20(3)+(-76)] \pm \sqrt{(-76)^2 - 4(20)(53) - \left[3(20)^2(3)^2 + 2(20)(-76)(3)\right]}}{2(20)} = \frac{16 \pm \sqrt{-144}}{40} = \frac{2}{5} \pm \frac{3}{10}i$. *In fact, the given equation can also be expressed as* $20(x-3)\left[x - \left(\frac{2}{5} + \frac{3}{10}i\right)\right]\left[x - \left(\frac{2}{5} - \frac{3}{10}i\right)\right] = 0$.

Example 5. *Suppose that* $15x^3 + 94x^2 - 75x + 14 = 0$, then $a = 15$, $b = 94$, $c = -75$ and $d = 14$; *so, Equations (3) and (13) imply* $p = -\frac{12211}{675}$, $q = \frac{2697968}{91125}$ and $\Delta_3 = -4\left(-\frac{12211}{675}\right)^3 - 27\left(\frac{2697968}{91125}\right)^2 = \frac{662596}{50625} > 0$; *therefore, (ii) and (iii) of Theorem 1 imply* $x_1 = 2\sqrt{-\frac{\left(-\frac{12211}{675}\right)}{3}} \cos\left[\frac{1}{3}\arccos\left\{-\frac{\left(\frac{2697968}{91125}\right)}{2}\sqrt{\left(-\frac{3}{\left\{-\frac{12211}{675}\right\}}\right)^3}\right\}\right] - \frac{94}{3(15)} = 2\left(\frac{\sqrt{12211}}{45}\right)\cos\left[\frac{1}{3}\arccos\left(-\frac{1348984}{12211\sqrt{12211}}\right)\right] - \frac{94}{45} = \frac{2\sqrt{12211}}{45}\left(\frac{56}{\sqrt{12211}}\right) - \frac{94}{45} = \frac{112}{45} - \frac{94}{45} = \frac{2}{5}$ and $x_{2,3} = \frac{-\left[15\left(\frac{2}{5}\right)+94\right] \pm \sqrt{94^2 - 4(15)(-75) - \left[3(15)^2\left(\frac{2}{5}\right)^2 + 2(15)(94)\left(\frac{2}{5}\right)\right]}}{2(15)} = \frac{-100 \pm 110}{30}$; *thus, according to Lemma 1,* $x_2 = \frac{-100-110}{30} = -7$ and $x_3 = \frac{-100+110}{30} = \frac{1}{3}$. *In fact, the given equation can also be expressed as* $15\left(x - \frac{2}{5}\right)[x - (-7)]\left(x - \frac{1}{3}\right) = 0$.

Finally, note that (i) of Theorem 1 coincides with Proposition 1 whenever $\Delta_3 = 0$; however, the equivalence between (iii) of Theorem 1 and Proposition 1 when $\Delta_3 = 0$ is not so obvious; therefore, this equivalence is proven in Appendix B.

2.2. The Definition and Relevance of the Standard Form of the Resolvent Cubic

2.2.1. The Difference between "Resolvent Cubic" and "SFRC"

First of all, here, it is important to say that one of the main purposes of this work is to expose the need to establish the SFRC as the most relevant form of the resolvent cubic over any other known form of this third-degree equation; so, in this paper, "SFRC" is always referred to this specific form of the resolvent cubic, which is given as follows:

$$R_C(s) = 0, \text{ where } R_C(s) := s^3 + 2ps^2 + \left(p^2 - 4r\right)s - q^2\,, \tag{17}$$

being p, q and r the coefficients of the DQE; meanwhile, "Resolvent Cubic" is here referred to any form of this equation, which can be expressed in general as follows:

$$a_2 R_C(a_1 t + a_0) = 0, \text{ for some } a_0 \in \mathbb{R} \text{ and } a_1,\ a_2 \in \mathbb{R} - \{0\}; \tag{18}$$

so, it is clear that the roots of the resolvent cubic and the roots of the SFRC are related in general as follows:

$$t_k = \frac{s_k - a_0}{a_1}\,, \text{ for each } k \in \{1,2,3\}. \tag{19}$$

Likewise, according to [6], the forms of the resolvent cubic with $a_0 \neq 0$ in Equation (18) are considered as "the translated forms of the Resolven Cubic", whereas the forms with $a_0 = 0$ are considered as "the non-translated forms of the Resolvent Cubic"; at first sight, all these considerations might seem superfluous, but they become relevant since the SFRC always has at least one positive real root for any Ferrari Case, as also stated in [6]; so, according to Equation (19), this fact also holds for the non-translated forms of the resolvent cubic whenever $a_1 > 0$; however, Equation (19) also implies that this fact does not necessarily hold for the translated forms of this equation.

On the other hand, note that the coefficients p and q of the DCE given by Equation (3) are different from the coefficients of the DQE given by Equations (4) and (5); hence, to avoid confusions and ambiguities related to the SFRC, from now on, the relations stated in Equation (3) will be laid out again according to the coefficients of the SFRC as follows:

$$p^* := -\frac{p^2 + 12r}{3} \text{ and } q^* := -\frac{2p^3 - 72pr + 27q^2}{27}, \tag{20}$$

where p, q and r are given by Equations (4)–(6).

So, Equations (13) and (20) imply Equation (10), when p and q are respectively substituted by p^* and q^* in Equation (13); in addition, Propositions 1 and 2, Lemmas 1 and 2, Theorem 1 and Remarks 1 and 2 also hold for the SFRC according to these changes.

2.2.2. The Discriminant of the GQE vs. the Discriminant of the Resolvent Cubic

In order to expose the general relationship between the discriminant of the GQE and the discriminant of the resolvent cubic, consider the following theorem.

Theorem 2. *If Δ_{RC} is the discriminant of the resolvent cubic, then this quantity and the discriminant of the GQE are related in general as follows: $\Delta_{RC} = \frac{\Delta_4}{a_1^6}$*, for some $a_1 \in \mathbb{R} - \{0\}$.

Proof. Since Equation (10) guarantees that the discriminant of the SFRC is identical to the discriminant of the GQE, Equation (12) implies that the discriminant of the GQE and the discriminant of the resolvent cubic are respectively given as follows: $\Delta_4 = [(s_1 - s_2)(s_1 - s_3)(s_2 - s_3)]^2$ and $\Delta_{RC} = [(t_1 - t_2)(t_1 - t_3)(t_2 - t_3)]^2$; additionally, note that Equation (19) guarantees $\frac{s_i - s_j}{a_1} = t_i - t_j$, for all $i, j \in \{1, 2, 3\}$ and some $a_1 \in \mathbb{R} - \{0\}$; which finally implies $\Delta_{RC} = [(t_1 - t_2)(t_1 - t_3)(t_2 - t_3)]^2 = \left[\left(\frac{s_1 - s_2}{a_1}\right)\left(\frac{s_1 - s_3}{a_1}\right)\left(\frac{s_2 - s_3}{a_1}\right)\right]^2$ $= \frac{[(s_1 - s_2)(s_1 - s_3)(s_2 - s_3)]^2}{\left(a_1^3\right)^2} = \frac{\Delta_4}{a_1^6}$. □

Note that Theorem 2 guarantees that the discriminant of the Resolvent Cubic and the discriminant of the GQE coincide only when at least one of the two following possibilities occurs: $\Delta_{RC} = \Delta_4 = 0$ or $a_1 = \pm 1$; thus, it is clear that the first possibility happens for any form of the resolvent cubic whenever this one and the GQE have multiple roots, as previously known [6]; on the other hand, the second possibility occurs in general only for some specific forms of the Resolvent Cubic.

2.2.3. Defining the Standard Form of the Resolvent Cubic

According to [6,10,11], all the known forms of the resolvent cubic are listed as follows:

(i) The SFRC given by Equation (17), where $a_0 = a_1 = 1$, $a_2 = 0$ and $\Delta_{RC} \equiv \Delta_4$.
(ii) $R_C(t - p) = t^3 - pt^2 - 4rt + \left(4pr - q^2\right) = 0$, so $a_0 = -p$ and $a_1 = a_2 = 1$; thus, Theorem 2 also guarantees $\Delta_{RC} \equiv \Delta_4$, whereas Equation (19) implies that its roots and the roots of the SFRC are related as follows: $t_k = s_k + p$, for each $k \in \{1, 2, 3\}$.
(iii) $R_C(2t) = 8t^3 + 8pt^2 + \left(2p^2 - 8r\right)t - q^2 = 0$, so $a_0 = 0$, $a_1 = 2$ and $a_2 = 1$; thus, Theorem 2 and Equation (10) imply $\Delta_{RC} = \frac{\Delta_4}{2^6} = \frac{4\left(p^2 + 12r\right)^3 - \left[2p\left(p^2 - 36r\right) + 27q^2\right]^2}{1728} \not\equiv \Delta_4$, whereas Equation (19) implies that its roots and the roots of the SFRC are related as follows: $t_k = \frac{s_k}{2}$, for each $k \in \{1, 2, 3\}$.
(iv) $R_C(2t - p) = 8t^3 - 4pt^2 - 8rt + \left(4pr - q^2\right) = 0$, so $a_0 = -p$, $a_1 = 2$ and $a_2 = 1$; thus, Theorem 2 and Equation (10) also imply $\Delta_{RC} = \frac{\Delta_4}{2^6} = \frac{4\left(p^2 + 12r\right)^3 - \left[2p\left(p^2 - 36r\right) + 27q^2\right]^2}{1728} \not\equiv \Delta_4$, whereas Equation (19) implies that its roots and the roots of the SFRC are related as follows: $t_k = \frac{s_k + p}{2}$, for each $k \in \{1, 2, 3\}$

(v) $\frac{1}{64}R_C(4t) = t^3 + \frac{p}{2}t^2 + \left(\frac{p^2}{16} - \frac{r}{4}\right)t - \frac{q^2}{64} = 0$, so $a_0 = 0$, $a_1 = 4$ and $a_2 = \frac{1}{64}$; thus, Theorem 2 and Equation (10) imply $\Delta_{RC} = \frac{\Delta_4}{4^6} = \frac{4\left(p^2+12r\right)^3 - \left[2p\left(p^2-36r\right)+27q^2\right]^2}{110592} \not\equiv \Delta_4$, whereas Equation (19) implies that its roots and the roots of the SFRC are related as follows: $t_k = \frac{s_k}{4}$, for each $k \in \{1,2,3\}$.

(vi) $-R_C(-t) = t^3 - 2pt^2 + \left(p^2 - 4r\right)t + q^2 = 0$, so $a_0 = 0$ and $a_1 = a_2 = -1$; thus, Theorem 2 also implies $\Delta_{RC} \equiv \Delta_4$, whereas Equation (19) implies that its roots and the roots of the SFRC are related as follows: $t_k = -s_k$, for each $k \in \{1,2,3\}$.

Hence, Theorem 2 guarantees that the forms (i), (ii) and (vi) of the resolvent cubic are the only ones in which $\Delta_{RC} \equiv \Delta_4$; therefore, the SFRC can be defined as the only form of the resolvent cubic that has the following two properties at the same time:

- **Property I.** Its discriminant is identical to the discriminant of the GQE.
- **Property II.** It always has at least one positive real root when the GQE corresponds to the Ferrari Case.

Although the forms (ii) and (vi) also have Property I, Property II does not necessarily hold for form (ii) because Equation (19) guarantees $t_k > p$ whenever $s_k > 0$, and p can be any real number; on the other hand, for form (vi), Equation (19) guarantees $t_k < 0$ whenever $s_k > 0$; in other words, if the SFRC has Property II, then form (vi) always has at least one negative real root whenever the GQE corresponds to the Ferrari Case, whereas the Viète Theorem guarantees the relation $q^2 = -s_1s_2s_3 > 0$ in form (vi); hence, the law of signs and the properties of non-real complex numbers guarantee that the other two roots of this form of the resolvent cubic can be real with the same signs (both positive or both negative) or a couple of non-real complex conjugate numbers, because the product of this kind of couples are always a positive real number [12] (p. 99), so form (vi) cannot have Property II either.

So, Equation (17) is the only one that has Properties I and II simultaneously, and this simultaneity guarantees the unicity of the SFRC; finally, the strongest justifications to consider the SFRC as the most relevant form of the resolvent cubic are listed as follows:

- All the known forms of the resolvent cubic can be expressed in terms of the function R_C.
- As stated in [6], Property II guarantees that the GQE can always be analytically solved without complex arithmetic, even when this equation has non-real roots.
- As it will be exposed in the next subsection, the SFRC is fundamental to determine all the relationships between the nature of the roots of the GQE and the nature of the roots of all the forms of the resolvent cubic.

2.3. The Relationship between the Nature of the Roots of the Resolvent Cubic and the Nature of the Roots of the General Quartic Equation

2.3.1. The Configuration of the GQE According to the Nature of its Roots

According to [13], there are only three possibilities in general for the nature of the roots of the GQE in terms of real and non-real complex roots, which are listed as follows:

1. The four roots of the GQE are all real.
2. Only two of the four roots of the GQE are real, and the other two are a couple of non-real complex conjugate numbers.
3. The four roots of the GQE conform a pair of couples of non-real complex conjugate numbers.

However, the resolvent cubic is not even mentioned in [13] because the analysis stated in that document is purely geometrical, unlike the analysis exposed here, which is mainly algebraic, and it will reveal that some of the relationships between the nature of the roots of the GQE and the nature of the roots of its resolvent cubic are determined by the signs of the roots of the SFRC, when this one has three real roots. Now consider the following lemma that will be very useful for the subsequent subsubsections.

Lemma 3. *If the GQE corresponds to the Ferrari Case and $s_1 > 0$ is one of the three roots of its SFRC, then the roots of this third-degree equation are related as follows:*

$$s_{2,3} = \frac{-(2p + s_1) \pm \sqrt{(2p + s_1)^2 - \alpha_s^2}}{2}\text{, with } \alpha_s = -\frac{2q}{\sqrt{s_1}}\,. \tag{21}$$

Proof. If the GQE corresponds to the Ferrari Case and $s_1 > 0$ is a root of the corresponding SFRC with $\alpha_s = -\frac{2q}{\sqrt{s_1}}$ in agreement with Equation (8), then $s_1^3 + 2ps_1^2 + (p^2 - 4r)s_1 = q^2 \neq 0$ and $\alpha_s \in \mathbb{R} - \{0\}$; thus, these facts imply $\alpha_s^2 = \left(-\frac{2q}{\sqrt{s_1}}\right)^2 = \frac{4q^2}{s_1} = \frac{4\left[s_1^3 + 2ps_1^2 + (p^2 - 4r)s_1\right]}{s_1} = 4(s_1^2 + 2ps_1 + p^2 - 4r) > 0$; therefore,

$$(2p + s_1)^2 - \alpha_s^2 = (2p + s_1)^2 - 4\left(s_1^2 + 2ps_1 + p^2 - 4r\right) = 16r - 4ps_1 - 3s_1^2\,. \tag{22}$$

On the other hand, if Lemma 2 is applied to the SFRC with $a = 1$, $b = 2p$, $c = p^2 - 4r$ and $x_r = s_1$, then:

$$s_{2,3} = \frac{-[(1)s_1 + 2p] \pm \sqrt{(2p)^2 - 4(1)(p^2 - 4r) - \left[3(1)^2 s_1^2 + 2(1)(2p)s_1\right]}}{2(1)} = \frac{-(2p + s_1) \pm \sqrt{16r - 4ps_1 - 3s_1^2}}{2}\,; \tag{23}$$

hence, Equations (22) and (23) imply Equation (21). □

Since Property II guarantees that Lemma 3 holds for all the non-biquadratic quartic equations, this lemma will be very useful to determine the relationships between the nature of the roots of the GQE and the nature of the roots of the resolvent cubic.

2.3.2. How Do the Roots of the Resolvent Cubic Determine Whether All the Roots of the GQE Are of the Same Nature?

Lemma 4. *If the GQE does not have multiple roots, then it has four real roots or four non-real complex roots if, and only if, its resolvent cubic has three non-multiple real roots.*

Proof. Let k, l, m and n be real numbers such that the GQE can be expressed as follows:

$$a\left(x^2 - kx + l\right)\left(x^2 - mx + n\right) = ax^4 - a(k + m)x^3 + a(km + l + n)x^2 - a(kn + lm)x + aln = 0; \tag{24}$$

thus, Equation (24) and the quadratic formula imply that the four roots of the GQE are given as follows:

$$x_{1,2} = \frac{k \pm \sqrt{k^2 - 4l}}{2} \text{ and } x_{3,4} = \frac{m \pm \sqrt{m^2 - 4n}}{2}\,. \tag{25}$$

Meanwhile, the coefficients of Equation (24) imply the following relations: $b = -a(k + m)$, $c = a(km + l + n)$, $d = -a(kn + lm)$ and $e = aln$; hence, these relations and Equations (4)–(6) imply these other relations:

$$p = \frac{8a[a(km + l + n)] - 3[-a(k + m)]^2}{8a^2} = -\frac{1}{4}\left[\frac{(k - m)^2}{2} + \left(k^2 - 4l\right) + \left(m^2 - 4n\right)\right], \tag{26}$$

$$q = \frac{[-a(k+m)]^3 - 4a[-a(k+m)][a(km+l+n)] + 8a^2[-a(kn+lm)]}{8a^3} = -\frac{(k-m)\left[\left(k^2-4l\right)-\left(m^2-4n\right)\right]}{8}, \quad (27)$$

$$r = \frac{16a[-a(k+m)]^2[a(km+l+n)] - 64a^2[-a(k+m)][-a(kn+lm)] - 3[-a(k+m)]^4 + 256a^3(aln)}{256a^4} = \frac{\left[(k-m)^2-4\left(k^2-4l\right)\right]\left[(k-m)^2-4\left(m^2-4n\right)\right]}{256}. \quad (28)$$

Now, consider the following three definitions:

$$\delta_1 := k-m,\ \delta_2 := k^2-4l \text{ and } \delta_3 := m^2-4n; \quad (29)$$

therefore, Equations (26)–(29) imply the following relations:

$$p = -\frac{1}{4}\left(\frac{\delta_1^2}{2}+\delta_2+\delta_3\right),\ q = -\frac{\delta_1(\delta_2-\delta_3)}{8} \text{ and } r = \frac{\left(\delta_1^2-4\delta_2\right)\left(\delta_1^2-4\delta_3\right)}{256} \quad (30)$$

so, Equations (10) and (30) imply that the discriminant of the GQE and of the SFRC is given as follows:

$$\Delta_4 = \frac{4\left(p^2+12r\right)^3 - \left[2p\left(p^2-36r\right)+27q^2\right]^2}{27} = \frac{1}{27}\left(4\left[\left\{-\frac{1}{4}\left(\frac{\delta_1^2}{2}+\delta_2+\delta_3\right)\right\}^2 + 12\left\{\frac{\left(\delta_1^2-4\delta_2\right)\left(\delta_1^2-4\delta_3\right)}{256}\right\}^3 - \left[2\left\{-\frac{1}{4}\left(\frac{\delta_1^2}{2}+\delta_2+\delta_3\right)\right\}\left(\left\{-\frac{1}{4}\left(\frac{\delta_1^2}{2}+\delta_2+\delta_3\right)\right\}^2 - 36\left\{\frac{\left(\delta_1^2-4\delta_2\right)\left(\delta_1^2-4\delta_3\right)}{256}\right\}\right) + 27\left\{-\frac{\delta_1(\delta_2-\delta_3)}{8}\right\}^2\right]^2\right) =$$
$$\frac{\delta_1^8\delta_2\delta_3 - 4\delta_1^6\delta_2^2\delta_3 + 6\delta_1^4\delta_2^3\delta_3 - 4\delta_1^2\delta_2^4\delta_3 + \delta_2^5\delta_3 - 4\delta_1^6\delta_2\delta_3^2 + 4\delta_1^4\delta_2^2\delta_3^2 + 4\delta_1^2\delta_2^3\delta_3^2 - 4\delta_2^4\delta_3^2 + 6\delta_1^4\delta_2\delta_3^3 + 4\delta_1^2\delta_2^2\delta_3^3 + 6\delta_2^3\delta_3^3 - 4\delta_1^2\delta_2\delta_3^4 - 4\delta_2^2\delta_3^4 + \delta_2\delta_3^5}{256} =$$
$$\frac{\delta_2\delta_3\left[\left\{\delta_1^2-(\delta_2+\delta_3)\right\}^2 - 4\delta_2\delta_3\right]^2}{256}. \quad (31)$$

Now, note that Theorem 2 and Remark 2 guarantee that the resolvent cubic has three non-multiple real roots whenever $\Delta_4 > 0$; on the other hand, it is clear that the three definitions given by Equation (29) guarantee $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$; additionally, $\left[\left\{\delta_1^2 - (\delta_2+\delta_3)\right\}^2 - 4\delta_2\delta_3\right]^2 > 0$ and $\delta_2\delta_3 \neq 0$ in Equation (31) whenever $\Delta_4 \neq 0$. Therefore, $\Delta_4 > 0$ whenever $\delta_2\delta_3 > 0$, that is, when $k^2-4l > 0$ and $m^2-4n > 0$ or when $k^2-4l < 0$ and $m^2-4n < 0$; so, Equation (25) guarantee that this happens whenever $x_1, x_2, x_3, x_4 \in \mathbb{R}$ or $x_1, x_2, x_3, x_4 \in \mathbb{C}-\mathbb{R}$. □

Theorem 3. *If the GQE has four real roots or four non-real complex roots, then its resolvent cubic has three real roots.*

Proof. Lemma 4 guarantees this for the cases without multiple roots; in addition, this fact was proven for the cases with multiple roots in [6]. □

Remark 4. *According to* [6], *the reciprocal of Theorem 3 does not hold in general because for all the quartic equations of the form* $a(x-l)^2(x-m)(x-n) = 0$, *with* $l \in \mathbb{R}$ and $m, n \in \mathbb{C}-\mathbb{R}$, such that $m = \overline{n}$, the SFRC has the form $(s-v)(s-w)^2 = 0$, with $v, w \in \mathbb{R}$, such that $v \geq 0$ and $w < 0$, *being that this is the only case of the GQE with multiple roots that has this characteristic. So, Equation (19) guarantees that this is the only case of the GQE where its resolvent cubic has three real roots, but not all the roots of the GQE are of the same nature.*

Corollary 1. *If the SFRC has three real roots and these ones are not negative multiple roots, then the corresponding GQE has four real roots or four non-real complex roots.*

Proof. This is an immediate consequence of Lemma 4 and Remark 4. □

Theorem 4. *If $\Delta_4 > 0$, then:*

(i) *The GQE has four non-multiple real roots if, and only if, the SFRC has three different non-negative real roots.*

(ii) *The four roots of the GQE are two different couples of non-real complex conjugate numbers if, and only if, the SFRC has three different real roots and at least one of them is negative.*

Proof. Biquadratic Case. If $q = 0$, then the SFRC is reduced to the equation $s^3 + 2ps^2 + (p^2 - 4r)s = s\left[s^2 + 2ps + (p^2 - 4r)\right] = 0$, so the quadratic formula implies that the three roots of the SFRC are $s_1 = 0$ and $s_{2,3} = -p \pm 2\sqrt{r}$; thus, Remark 2 guarantees that $\Delta_4 > 0$ implies $s_2, s_3 \in \mathbb{R} - \{0\}$ and $r > 0$ because $s_2 \neq s_3$; in addition, the GQE does not have multiple roots. Therefore, Equations (8) and (9) imply that the four roots of the GQE are given as follows:

$$x_{1,3} = \frac{\sqrt{-p + 2\sqrt{r}} \pm \sqrt{-p - 2\sqrt{r}}}{2} - \frac{b}{4a} = \frac{\sqrt{s_2} \pm \sqrt{s_3}}{2} - \frac{b}{4a} \text{ and } x_{2,4} = \frac{-\sqrt{-p + 2\sqrt{r}} \pm \sqrt{-p - 2\sqrt{r}}}{2} - \frac{b}{4a} = \frac{-\sqrt{s_2} \pm \sqrt{s_3}}{2} - \frac{b}{4a}\,; \quad (32)$$

now consider the following three subcases:

- The SFRC has three non-negative real roots whenever $s_2 > 0$ and $s_3 > 0$; thus, Equation (32) guarantees that this occurs if, and only if, the four roots of the GQE are all real.
- The SFRC has only one negative real root whenever $s_2 = -p + 2\sqrt{r} > 0 > -p - 2\sqrt{r} = s_3$; so, according to Equation (32), this happens whenever the four roots of the GQE are two different couples of non-real complex conjugate numbers with the same imaginary parts.
- If the SFRC has two different negative real roots, then $0 > s_2 = -p + 2\sqrt{r} > -p - 2\sqrt{r} = s_3$, thus $0 \neq \sqrt{|s_2|} - \sqrt{|s_3|} \neq \sqrt{|s_2|} + \sqrt{|s_3|} > 0$; hence, Equation (32) implies $x_{1,4} = -\frac{b}{4a} \pm \frac{\sqrt{|s_2|}+\sqrt{|s_3|}}{2}i$ and $x_{2,3} = -\frac{b}{4a} \pm \frac{\sqrt{|s_2|}-\sqrt{|s_3|}}{2}i$, so $x_1, x_2, x_3, x_4 \in \mathbb{C} - \mathbb{R}$. Therefore, the SFRC has two different negative real roots whenever the four roots of the GQE are two different couples of non-real complex conjugate numbers with the same real parts.

Ferrari Case. According to Property II, the SFRC has at least one positive real root s_1; thus, $\sqrt{s_1} \in \mathbb{R}$, and the other two roots of the SFRC are given by Equation (21); now consider the following two definitions: $\beta_1 := \alpha_s - 2p - s_1$ and $\beta_2 := -\alpha_s - 2p - s_1$. Hence, $\beta_1\beta_2 = [-(2p + s_1) + \alpha_s][-(2p + s_1) - \alpha_s] = (2p + s_1)^2 - \alpha_s^2$, so Equation (21) can be rewritten as follows:

$$s_{2,3} = \frac{-(2p + s_1) \pm \sqrt{\beta_1\beta_2}}{2}\,; \quad (33)$$

in addition, Equation (9) implies that the four roots of the GQE are given as follows:

$$x_{1,3} = \frac{\sqrt{s_1} \pm \sqrt{\beta_1}}{2} - \frac{b}{4a} \text{ and } x_{2,4} = \frac{-\sqrt{s_1} \pm \sqrt{\beta_2}}{2} - \frac{b}{4a}\,. \quad (34)$$

So, if $\Delta_4 > 0$, then Property I and Remark 2 guarantee that the SFRC has three non-multiple real roots, thus $\beta_1\beta_2 > 0$ in Equation (33); hence, this happens whenever only one of the following two possibilities occurs:

(i) $\beta_1 > 0$ and $\beta_2 > 0$, simultaneously.

(ii) $\beta_1 < 0$ and $\beta_2 < 0$, simultaneously.

Therefore, Equation (34) guarantees $x_1, x_2, x_3, x_4 \in \mathbb{R}$ for possibility (i), and $x_1, x_2, x_3, x_4 \in \mathbb{C} - \mathbb{R}$ for possibility (ii); now, the Viète Theorem guarantees $q^2 = s_1s_2s_3 > 0$; ergo, $s_1 > 0$ and the law of signs implies these other two possibilities:

(iii) $s_2 > 0$ and $s_3 > 0$, simultaneously.

(iv) $s_2 < 0$ and $s_3 < 0$, simultaneously.

So, possibility (i) implies $0 < \frac{\beta_1+\beta_2}{2} = \frac{(\alpha_s-2p-s_1)+(-\alpha_s-2p-s_1)}{2} = -(2p+s_1)$; hence, this inequality and $\beta_1\beta_2 > 0$ guarantee $-(2p+s_1)+\sqrt{\beta_1\beta_2} > 0$, so Equation (33) implies $s_2 > 0$, whereas possibility (iii) guarantees $s_3 > 0$ as well. On the other hand, possibility (ii) implies $0 > \frac{\beta_1+\beta_2}{2} = -(2p+s_1)$; hence, this inequality and $\beta_1\beta_2 > 0$ guarantee $-(2p+s_1)-\sqrt{\beta_1\beta_2} < 0$, so Equation (33) implies $s_3 < 0$, whereas possibility (iv) guarantees $s_2 > 0$ as well. Therefore, $\Delta_4 \neq 0$, and Equation (34) with possibilities (i) and (iii) guarantees $x_1, x_2, x_3, x_4 \in \mathbb{R}$ all different from each other whenever $s_2 > 0$ and $s_3 > 0$; whereas $\Delta_4 \neq 0$ and Equation (34) with possibilities (ii) and (iv) guarantee that x_1, x_2, x_3, x_4 are two different couples of non-real complex conjugate numbers whenever $s_2 < 0$ and $s_3 < 0$. □

Corollary 2. *The GQE has four real roots if, and only if, the SFRC has three non-negative real roots.*

Proof. Theorem 4 guarantees this fact for the cases without multiple roots; in addition, this fact was also proven for all the cases with multiple roots in [6]. □

2.3.3. How Do the Roots of the Resolvent Cubic Determine Whether the GQE Has Only Two Real Roots?

Remark 4 gave a partial answer to this question, so in order to state a complete answer, the following theorem is exposed.

Theorem 5. *The GQE has two different real roots and two non-real complex conjugate roots if, and only if, its resolvent cubic has one real root and two non-real complex conjugate roots.*

Proof. If the GQE has two different real roots and two non-real complex conjugate roots, then it does not have multiple roots; that is, $\Delta_4 \neq 0$. On the other hand, Theorem 2 and Remark 2 guarantee that $\Delta_4 < 0$ whenever the resolvent cubic has one real root and two non-real complex conjugate roots, whereas $\delta_2\delta_3 < 0$ in Equation (31); in other words, when only one of the following two possibilities occurs in Equation (25),

(i) $k^2 - 4l > 0$ and $m^2 - 4n < 0$, simultaneously.
(ii) $k^2 - 4l < 0$ and $m^2 - 4n > 0$, simultaneously.

So, possibility (i) occurs when $x_1, x_2 \in \mathbb{R}$ and $x_3, x_4 \in \mathbb{C} - \mathbb{R}$, such that $x_1 \neq x_2$ and $x_3 = \overline{x_4}$; and possibility (ii) occurs when $x_1, x_2 \in \mathbb{C} - \mathbb{R}$ and $x_3, x_4 \in \mathbb{R}$, such that $x_1 = \overline{x_2}$ and $x_3 \neq x_4$. □

Note that Remark 4 also guarantees that Theorem 5 never holds for the case where the two real roots of the GQE are multiple; however, the following corollary finally gives a complete answer to the question of the title of this subsubsection.

Corollary 3. *The GQE has two real roots and two non-real complex conjugate roots if, and only if, only one of the following two possibilities occurs:*

(i) *The resolvent cubic has one real root and two non-real complex conjugate roots.*
(ii) *The SFRC has two multiple negative real roots.*

Proof. This is an immediate consequence of Theorem 5 and Remark 4. □

2.4. How Does One Identify a Priori the Nature of the Roots of the GQE?

This question has been answered for all the cases of the GQE with multiple roots in [6]. However, the following results will help to give a complete answer to this question for any case of the GQE.

Theorem 6. *If the GQE does not have multiple roots, then*

(i) *It has four real roots if, and only if, the following three inequalities hold simultaneously:* $\left(p^2+12r\right)^3 > \left[p\left(p^2-36r\right)+\frac{27}{2}q^2\right]^2$*,* $p<0$ *and* $p^2>4r$*.*

(ii) *It has two real roots and two non-real complex conjugate roots if, and only if,* $\left(p^2+12r\right)^3 < \left[p\left(p^2-36r\right)+\frac{27}{2}q^2\right]^2$*.*

(iii) *It has four non-real complex roots if, and only if,* $\left(p^2+12r\right)^3 > \left[p\left(p^2-36r\right)+\frac{27}{2}q^2\right]^2$ *with* $p \geq 0$ *or with* $p^2 \leq 4r$*.*

Proof. (i) First of all, note that Equation (10) guarantees $\Delta_4 > 0$ whenever $\left(p^2+12r\right)^3 > \left[p\left(p^2-36r\right)+\frac{27}{2}q^2\right]^2$, so Theorem 3 implies that this inequality also guarantees that all the roots of the GQE are of the same nature, whereas the SFRC has three real roots.

Likewise, if the GQE has four real roots, then (i) of Theorem 4 guarantees that the three real roots of the SFRC are non-negative; since the GQE does not have multiple roots, at most one of these roots is equal to zero, whereas the other two roots must be strictly positive. So, if these roots are $s_1 \geq 0$, $s_2 > 0$ and $s_3 > 0$; then, according to the Viète Theorem, the following two inequalities hold: $s_1+s_2+s_2 = -2p > 0$ and $s_1s_2+s_1s_3+s_2s_3 = p^2-4r > 0$; therefore, $p<0$ and $p^2>4r$. Now, in order to prove the reciprocal, consider the following two cases:

Case 1. Suppose that $p \geq 0$; then the relation $s_1+s_2+s_3 = -2p \leq 0$ implies $s_1 \leq -(s_2+s_3)$; however, if (i) of Theorem 4 guarantees $s_1 \geq 0$, $s_2 > 0$ and $s_3 > 0$, then $0 \leq s_1 \leq -(s_2+s_3) < 0$, which is an obvious contradiction.

Case 2. Suppose that $p^2 \leq 4r$; then the relation $s_1s_2+s_1s_3+s_2s_3 = p^2-4r \leq 0$ implies $s_1(s_2+s_3) \leq -s_2s_3$; however, if (i) of Theorem 4 guarantees $s_1 \geq 0$, $s_2 > 0$ and $s_3 > 0$, then $0 \leq s_1(s_2+s_3) \leq -s_2s_3 < 0$, so again, this is another obvious contradiction.

(ii) First of all, consider that Equation (10) guarantees $\Delta_4 < 0$ whenever $\left(p^2+12r\right)^3 < \left[p\left(p^2-36r\right)+\frac{27}{2}q^2\right]^2$, so the result in question is a consequence of (ii) of Remark 2 and Theorem 5.

(iii) This is a consequence of (ii) of Theorem 4 and (i) of this theorem. □

Corollary 4. *If the GQE does not have multiple roots, then*

(i) *It has four real roots if, and only if, the following three inequalities hold simultaneously:* $\Delta_4 > 0$*,* $p<0$ *and* $p^2>4r$*.*

(ii) *It has two real roots and two non-real complex conjugate roots if, and only if,* $\Delta_4 < 0$*.*

(iii) *It has four non-real complex roots if, and only if,* $\Delta_4 > 0$ *with* $p \geq 0$ *or with* $p^2 \leq 4r$*.*

Proof. This holds because of Theorem 6 and the validity of Equation (10). □

Corollary 5. *If* $\Delta_4 > 0$*, then*

(i) $p<0$ *and* $p^2>4r$ *if, and only if, the SFRC has three different non-negative real roots.*

(ii) $p \geq 0$ *or* $p^2 \leq 4r$ *if, and only if, the SFRC has three different real roots and at least one of them is negative.*

Proof. All these are consequences of Theorem 4 and Corollary 4. □

Corollary 6. If the GQE corresponds to the Ferrari Case and $\Delta_4 > 0$; then

(i) $p<0$ and $p^2>4r$ *if, and only if, the SFRC has three different positive real roots.*

(i) $p \geq 0$ *or* $p^2 \leq 4r$ *if, and only if, the SFRC has one positive real root and two different negative real roots.*

Proof. Since the Viète Theorem guarantees $q^2 = s_1s_2s_3 > 0$ in Equation (17) whenever the GQE corresponds to the Ferrari Case, the law of signs guarantees that (i) of Corollary 5 holds whenever the SFRC has three different positive real roots; meanwhile, (ii) of Corollary 5 holds whenever the SFRC has one positive real root and two different negative real roots. □

Proposition 3. If $p = 0$ and $p^2 = 4r$, then $\Delta_4 \leq 0$.

Proof. If $p = 0$ and $p^2 = 4r$, then $r = 0$ as well; so, Equation (10) implies $\Delta_4 = \frac{4[0+12(0)]^3 - \left[(0)\left\{0^2 - 36(0)\right\} + 27q^2\right]^2}{27} = -27q^4 \leq 0$. □

Remark 5. *Note that Proposition 3 guarantees the impossibility of* $p = 0$ and $p^2 = 4r$ *at the same time in (iii) of Theorem 6 and (iii) of Corollary 4.*

Finally, the following theorem gives a complete and slightly more precise answer to the question in the title of this subsection than what is stated in [13].

Theorem 7. *The nature of the roots of the GQE can be identified a priori as follows:*

The GQE has four real roots if, and only if, only one of the following two possibilities happens:
(i) $\Delta_4 \geq 0$*,* $p < 0$ *and* $p^2 > 4r$.
(ii) $p \leq 0$ *and* $p^2 = 4r \geq q = 0$.
The GQE has two real roots and two non-real complex conjugate roots if, and only if, only one of the following four possibilities happens:
(iii) $\Delta_4 < 0$.
(iv) $\Delta_4 = 0$*,* $p \geq q \neq 0$ *and* $r > 0$.
(v) $\Delta_4 = 0$*,* $p < q \neq 0$ *and* $p^2 \leq 4r$.
(vi) $p > q = r = 0$.
The GQE has four non-real complex roots if, and only if, only one of the following four possibilities happens:
(vii) $\Delta_4 > 0$*,* $p \geq 0$ *and* $p^2 > 4r$.
(viii) $\Delta_4 > 0$*,* $p \geq 0$ *and* $p^2 < 4r$.
(ix) $\Delta_4 > 0$*,* $p < 0$ *and* $p^2 \leq 4r$.
(x) $p > 0$ *and* $p^2 = 4r > q = 0$.

Proof. (i) Theorem 6 and Corollary 4 guarantee this for $\Delta_4 > 0$, whereas $\Delta_4 = 0$ corresponds to the cases with multiple roots; so, according to [6], the GQE has the form $a(x-l)^2(x-m)(x-n) = 0$ with $l, m, n \in \mathbb{R}$, such that all of them are different from each other whenever $p < q = r = 0$ or $p < 0 \neq q$ with $p^2 > 4r$; thus, possibility (i) also holds with these conditions; in addition, the GQE has the form $a(x-l)^3(x-m) = 0$ with l, $m \in \mathbb{R}$, such that $l \neq m$, whenever $p^2 = -12r > 0$ and $27q^2 = -8p^3 > 0$; so, it is clear that possibility (i) also holds with these conditions.

(ii) According to [6], the GQE has the form $a(x-l)^2(x-m)^2 = 0$ with $l, m \in \mathbb{R}$; whenever $p \leq 0$ and $p^2 = 4r \geq q = 0$, it does not matter whether $l \neq m$ or $l = m$; and these conditions do not correspond to any other possibility of this theorem.

(iii) This is guaranteed by (ii) of Corollary 4.

(iv)–(vi) According to [6], each of these different possibilities occurs whenever the GQE has the form $a(x-l)^2(x-m)(x-n) = 0$ with $l \in \mathbb{R}$ and $m, n \in \mathbb{C} - \mathbb{R}$, such that $m = \overline{n}$.

(vii)–(ix) These possibilities are guaranteed by (iii) of Corollary 4 and Proposition 3.

(x) According to [6], the GQE has the form $a(x-l)^2(x-m)^2 = 0$ with l
, $m \in \mathbb{C} - \mathbb{R}$, such that $l = \overline{m}$, whenever $p > 0$ and $p^2 = 4r > q = 0$; and these conditions do not correspond to any other possibility of this theorem. □

Remark 6. *Note that Equation (10) guarantees* $\Delta_4 = 0$ *in possibilities (ii), (vi) and (x) of Theorem 7, so this condition was not included in the mentioned possibilities due to its redundancy.*

3. Development of the Program and Results

3.1. How to Avoid Complex Arithmetic during the Resolution of the GQE?

In Section 2.1, it was exposed how to avoid complex arithmetic during the resolution of the GCE; in addition, it is important to know how to avoid in general the application of these kinds of operations during the resolution of the GQE in order to design an efficient program that can solve all these equations without complex arithmetic. So, according to [6], the following three facts related to the coefficients of the DQE guarantee that complex arithmetic is also completely unneeded to analytically solve the GQE:

1. Equation (7) solves the GQE without dealing with the square roots of non-real complex numbers whenever $q = 0$ with $p^2 \geq 4r$.
2. Equation (32) solves the GQE without dealing with the square roots of non-real complex numbers whenever $q = 0$ with $r \geq 0$.
3. Property II of the SFRC guarantees that Equation (9) can solve the GQE without dealing with the square roots of non-real complex roots whenever $q \neq 0$.

Note that facts 1 and 2 guarantee that the GQE can always be solved while avoiding complex arithmetic operations whenever this equation corresponds to the biquadratic case; meanwhile, fact 3 guarantees this for the Ferrari Case; however, although the existence of the SFRC's root referred to in Property II was proven in [6], it was not explained how to obtain in practical terms that root in that document; so, the following theorem finally reveals how to always obtain this root effectively.

Theorem 8. *Since the SFRC has at least one positive real root $s_1$whenever the GQE corresponds to the Ferrari Case, this root can always be obtained without complex arithmetic, as follows:*

(i) *If $\Delta_4 \leq 0$, then*$s_1 = \sqrt[3]{-\frac{q^*}{2} + \sqrt{-\frac{\Delta_4}{108}}} + \sqrt[3]{-\frac{q^*}{2} - \sqrt{-\frac{\Delta_4}{108}}} - \frac{2p}{3} > 0.$

(ii) *If $\Delta_4 > 0$, then*$s_1 = 2\sqrt{-\frac{p^*}{3}} \cos\left[\frac{1}{3}\arccos\left\{-\frac{q^*}{2}\sqrt{\left(-\frac{3}{p^*}\right)^3}\right\}\right] - \frac{2p}{3} > 0.$

Proof. Since Property I guarantees that the discriminant of the GQE is identical to the discriminant of the SFRC, consider the following three possible cases:

Case 1. If $\Delta_4 < 0$, then (ii) of Remark 2 guarantees that the SFRC has only one real root and a couple of non-real complex conjugate roots; thus, Property II guarantees that the only real root of the SFRC must be strictly positive; hence, Equations (4)–(6) and (20) and (i) of Theorem 1 allow us to obtain this root without complex arithmetic operations as follows: $s_1 = \sqrt[3]{-\frac{q^*}{2} + \sqrt{-\frac{\Delta_4}{108}}} + \sqrt[3]{-\frac{q^*}{2} - \sqrt{-\frac{\Delta_4}{108}}} - \frac{2p}{3} > 0.$

Case 2. If $\Delta_4 = 0$, then the GQE and the SFRC have multiple roots, whereas Remark 2 guarantees that the three roots of the SFRC are all real; thus, according to [6], if the GQE corresponds to the Ferrari Case, then there are only the following two possibilities:

- The three roots of the SFRC are all positive, so for this possibility the root s_1 can be obtained without complex arithmetic operations as in Case 1.
- Only one of the three roots of the SFRC is positive, whereas the other two are negative and multiple; hence, Equations (4)–(6) and (20) and Proposition 2 imply that the roots of the SFRC are given as follows for this possibility: $s_1 = -\sqrt[3]{4q^*} - \frac{2p}{3} > 0$ and $s_2 = s_3 = \sqrt[3]{\frac{q^*}{2}} - \frac{2p}{3} < 0$; now note that the first of these relations is equivalent to the formula of Case 1 depending on whether (i) of Theorem 1 is applied to the SFRC with $\Delta_3 = \Delta_4 = 0$.

Finally, note that the possibility of two positive multiple roots and only one negative root is disregarded here because the Viète Theorem and the law of signs would imply the following contradictory relation in Equation (17): $q^2 = s_1 s_2 s_3 < 0$.

Case 3. If $\Delta_4 > 0$, then the SFRC corresponds to the Casus Irreducibilis; so, according to Remark 2, this equation has three non-multiple real roots. Additionally, all these roots are other than zero depending on whether Equation (5) implies $q \neq 0$—that is, when the GQE corresponds to the Ferrari Case. Thus, Equations (4)–(6), (11) and (20) imply that the three roots of the SFRC can be obtained without complex arithmetic in the case as follows:

$$s_k = 2\sqrt{-\frac{p^*}{3}}\cos\left(\frac{\theta_k}{3}\right) - \frac{2p}{3} \neq 0, \text{ where } \theta_k := \arccos\left[-\frac{q^*}{2}\sqrt{\left(-\frac{3}{p^*}\right)^3}\right] + 2(k-1)\pi, \text{ for } k = 1,2,3. \quad (35)$$

On the other hand, if Property II guarantees that the SFRC has at least one positive real root for this case, then at least the greatest of the three real roots of the SFRC must always be positive; that is, $\max\{s_1, s_2, s_3\} > 0$. Thus, (i) of Lemma 1 guarantees that this root is given in general by Equation (35) whenever $k = 1$. □

Therefore, Theorem 8 and Equations (7)–(9) and (32) allow us to solve the GQE without complex arithmetic operations, even when the GQE has non-real complex roots.

Example 6. *Suppose that* $2x^4 - 12x^3 + 15x^2 + 6x - 8 = 0$, *then* $a = 2$, $b = -12$, $c = 15$, $d = 6$ *and* $e = -8$; *hence, Equations (4)–(6) imply ,*$p = -6$, $q = -\frac{3}{2} \neq 0$ *and* $r = \frac{35}{16}$ *, so Equation (10) implies* $\Delta_4 = \frac{4\left[(-6)^2+12\left(\frac{35}{16}\right)\right]^3 - \left[2(-6)\left\{(-6)^2-36\left(\frac{35}{16}\right)\right\}+27\left(-\frac{3}{2}\right)^2\right]^2}{27} = \frac{47089}{2} > 0$. *On the other hand, note that* $p < 0$ *and* $p^2 = 36 > \frac{35}{4} = 4r$; *so, (i) of Theorem 7 guarantees that the given equation has four real roots; meanwhile, Equation (20) implies* $p^* = -\frac{(-6)^2+12\left(\frac{35}{16}\right)}{3} = -\frac{83}{4}$ *and* $q^* = -\frac{2(-6)^3-72(-6)\left(\frac{35}{16}\right)+27\left(-\frac{3}{2}\right)^2}{27} = -\frac{85}{4}$; *thus, (ii) of Theorem 8 implies* $s_1 = 2\sqrt{-\frac{\left(-\frac{83}{4}\right)}{3}}\cos\left[\frac{1}{3}\arccos\left\{-\frac{\left(-\frac{85}{4}\right)}{2}\sqrt{\left(-\frac{3}{\left\{-\frac{83}{4}\right\}}\right)^3}\right\}\right] - \frac{2(-6)}{3} = 2\left(\frac{1}{2}\sqrt{\frac{83}{3}}\right)\cos\left[\frac{1}{3}\arccos\left(\frac{255}{83}\sqrt{\frac{3}{83}}\right)\right] + 4 = \sqrt{\frac{83}{3}}\left(\frac{5\sqrt{3}}{\sqrt{83}}\right) + 4 = 5 + 4 = 9 > 0$; *therefore, Equation (8) implies*$\alpha_s = -\frac{2(-3/2)}{\sqrt{9}} = 1$, *so Equation (9) implies*$x = \frac{\xi\sqrt{9} \pm \sqrt{\xi(1)-2(-6)-9}}{2} - \frac{(-12)}{4(2)} = \frac{3\xi+3\pm\sqrt{\xi+3}}{2}$; *hence,* $x_1 = \frac{3+3+\sqrt{1+3}}{2} = 4$, $x_2 = \frac{3-3+\sqrt{1+3}}{2} = 2$ *and* $x_{3,4} = \frac{-3+3\pm\sqrt{-1+3}}{2} = \pm\frac{1}{\sqrt{2}}$. *In fact, the given equation can also be expressed as* $2(x-4)(x-2)\left(x - \frac{1}{\sqrt{2}}\right)\left[x - \left(-\frac{1}{\sqrt{2}}\right)\right] = 0$; *finally, note that Lemma 3 implies that the other two roots of the SFRC are* $s_{2,3} = \frac{3}{2} \pm \sqrt{2} > 0$, *which agrees with (i) of Theorem 4.*

Example 7. *Suppose that* $5x^4 - 57x^3 + 216x^2 + 2x - 816 = 0$, then $a = 5$, $b = -57$, $c = 216$, $d = 2$ and $e = -816$; *hence, Equations (4)–(6) imply* $p = -\frac{1107}{200}$, $q = \frac{61447}{1000} \neq 0$ and $r = -\frac{1454883}{160000}$, *so Equation (10) implies* $\Delta_4 = \frac{4\left[\left(-\frac{1107}{200}\right)^2+12\left(-\frac{1454883}{160000}\right)\right]^3 - \left[2\left(-\frac{1107}{200}\right)\left\{\left(-\frac{1107}{200}\right)^2-36\left(-\frac{1454883}{160000}\right)\right\}+27\left(\frac{61447}{1000}\right)^2\right]^2}{27} = -\frac{5556938723856}{15625} < 0$; *thus, (iii) of Theorem 7 guarantees that the given equation has only two real roots and a couple of non-real complex conjugate roots. On the other hand, Equation (20) implies* $q^* = -\frac{2\left(-\frac{1107}{200}\right)^3-72\left(-\frac{1107}{200}\right)\left(-\frac{1454883}{160000}\right)+27\left(\frac{61447}{1000}\right)^2}{27} = -\frac{90724}{25}$; *thus, (i) of Theorem 8 implies* $s_1 = \sqrt[3]{-\frac{\left(-\frac{90724}{25}\right)}{2} + \sqrt{-\frac{\left(-\frac{5556938723856}{15625}\right)}{108}}} + \sqrt[3]{-\frac{\left(-\frac{90724}{25}\right)}{2} - \sqrt{-\frac{\left(-\frac{5556938723856}{15625}\right)}{108}}} - \frac{2\left(-\frac{1107}{200}\right)}{3} = \sqrt[3]{\frac{45362}{25} + \frac{130962\sqrt{3}}{125}} + \sqrt[3]{\frac{45362}{25} - \frac{130962\sqrt{3}}{125}} + \frac{369}{100} = \frac{74}{5} + \frac{369}{100} = \frac{1849}{100} > 0$; *therefore, Equation (8) implies* $\alpha_s = -\frac{2(61447/1000)}{\sqrt{1849/100}} = -\frac{1429}{50}$, *so Equation (9) implies* $x = \frac{\xi\sqrt{\frac{1849}{100}} \pm \sqrt{\xi\left(-\frac{1429}{50}\right)-2\left(-\frac{1107}{200}\right)-\frac{1849}{100}}}{2} - \frac{(-57)}{4(5)} = \frac{43\xi+57}{20} \pm \frac{1}{2}\sqrt{\frac{-1429\xi-371}{50}}$; *hence,* $x_{1,2} = \frac{43+57}{20} \pm \frac{1}{2}\sqrt{\frac{-1429-371}{50}} = 5 \pm \frac{\sqrt{-36}}{2} = 5 \pm 3i$, $x_3 = \frac{-43+57}{20} + \frac{1}{2}\sqrt{\frac{1429-371}{50}} = \frac{7}{10} + \frac{1}{2}\left(\frac{23}{5}\right) = 3$ *and*

$x_4 = \frac{-43+57}{20} - \frac{1}{2}\sqrt{\frac{1429-371}{50}} = \frac{7}{10} - \frac{1}{2}\left(\frac{23}{5}\right) = -\frac{8}{5}$. *In fact, the given equation can also be expressed as* $5[x-(5+3i)][x-(5-3i)](x-3)\left[x-\left(-\frac{8}{5}\right)\right] = 0$; *finally, note that Lemma 3 implies that the other two roots of the SFRC are* $s_{2,3} = -\frac{371}{100} \pm \frac{69}{5}i \in \mathbb{C}-\mathbb{R}$, *which agree with Theorem 5.*

Example 8. *Suppose that* $4x^4+4x^3+17x^2+4x+13 = 0$, *then* $a = b = d = 4$, $c = 17$, $d = 6$ *and* $e = 13$; *hence, Equations (4)–(6) imply* $p = \frac{31}{8}$, $q = -1 \neq 0$ *and* $r = \frac{833}{256}$, *so Equation (10) implies* $\Delta_4 = \frac{4\left[\left(\frac{31}{8}\right)^2+12\left(\frac{833}{256}\right)\right]^3-\left[2\left(\frac{31}{8}\right)\left\{\left(\frac{31}{8}\right)^2-36\left(\frac{833}{256}\right)\right\}+27(-1)^2\right]^2}{27} = \frac{28227}{16} > 0$. *On the other hand, note that* $p > 0$ *and* $p^2 = \frac{961}{64} > \frac{833}{64} = 4r$; *so, (vii) of Theorem 7 guarantees that the given equation has four non-real complex roots; meanwhile, Equation (20) implies* $p^* = -\frac{\left(\frac{31}{8}\right)^2+12\left(\frac{833}{256}\right)}{3} = -\frac{865}{48}$ *and* $q^* = -\frac{2\left(\frac{31}{8}\right)^3-72\left(\frac{31}{8}\right)\left(\frac{833}{256}\right)+27(-1)^2}{27} = \frac{24463}{864}$; *thus, (ii) of Theorem 8 implies* $s_1 = 2\sqrt{-\frac{\left(-\frac{865}{48}\right)}{3}}\cos\left[\frac{1}{3}\arccos\left\{-\frac{\left(\frac{24463}{864}\right)}{2}\sqrt{\left(-\frac{3}{\left\{-\frac{865}{48}\right\}}\right)^3}\right\}\right] - \frac{2\left(\frac{31}{8}\right)}{3}$ $= 2\left(\frac{\sqrt{865}}{12}\right)\cos\left[\frac{1}{3}\arccos\left(-\frac{24463}{865\sqrt{865}}\right)\right] - \frac{31}{12} = \frac{\sqrt{865}}{6}\left(\frac{17}{\sqrt{865}}\right) - \frac{31}{12} = \frac{17}{6} - \frac{31}{12} = \frac{1}{4} > 0$; *therefore, Equation (8) implies* $\alpha_s = -\frac{2(-1)}{\sqrt{1/4}} = 4$, *so Equation (9) implies* $x = \frac{\xi\sqrt{\frac{1}{4}}\pm\sqrt{\xi(4)-2\left(\frac{31}{8}\right)-\frac{1}{4}}}{2} - \frac{4}{4(4)} = \frac{\xi-1}{4} \pm \frac{\sqrt{4\xi-8}}{2}$; *hence,* $x_{1,2} = \frac{1-1}{4} \pm \frac{\sqrt{4-8}}{2} = \pm i$ *and* $x_{3,4} = \frac{-1-1}{4} \pm \frac{\sqrt{-4-8}}{2} = -\frac{1}{2} \pm \sqrt{3}i$. *In fact, the given equation can also be expressed as* $4(x-i)[x-(-i)]\left[x-\left(-\frac{1}{2}+\sqrt{3}i\right)\right]\left[x-\left(-\frac{1}{2}-\sqrt{3}i\right)\right] = 0$; *finally, note that Lemma 3 guarantees that the other two roots of the SFRC are* $s_{2,3} = -4 \pm 2\sqrt{3} < 0$, *which agrees with (ii) of Theorem 4.*

Remark 7. *Note that if the GQE corresponds to the Ferrari Case, then the knowledge of* Δ_4 *given by Equation (10) implies that the application of Equation (20) to obtain* p^* *is sometimes unneeded to solve the GQE; that is, if* $\Delta_4 \leq 0$, *then (i) of Theorem 8 requires only the obtention of* q^* *to solve the GQE (Example 7); meanwhile, if* $\Delta_4 > 0$, *then (ii) of Theorem 8 demands the obtention of* p^* *and* q^* *to solve the GQE (Examples 6 and 8).*

Finally, it is important to point out that Equation (8) also theoretically works to solve the GQE when s is a negative real number or when $s \in \mathbb{C}-\mathbb{R}$; however, if this happens, then the resolution of the GQE implies the application of some annoying operations with non-real complex numbers, such as divisions and square roots whose obtaining requires at least the application of the result exposed in [5] (pp. 16–18), which complicates in practical terms the resolution of the GQE. Thus, the importance of Theorem 8 lies in ensuring that these uneasy complex arithmetic operations are always avoidable when the GQE corresponds to the Ferrari Case.

3.2. Designing a Program That Analytically Solves the SDE, the GCE and the GQE without Complex Arithmetic

It is clear that Proposition 1 gives an easy analytical solution to the GCE without complex arithmetic operations when this equation has purely imaginary roots; in addition, the formulae of Proposition 2 give an easy analytical solution to any case of the GCE with multiple roots applying only real arithmetic operations; meanwhile, the formulae of Theorem 1 can analytically solve all the other cases of the GCE without complex arithmetic operations. On the other hand, the previous subsection exposes how to analytically solve any case of the GQE without complex arithmetic operations; however, in order to make it simpler, consider the following theorem.

Theorem 9. *If the GQE corresponds to the Ferrari Case, then the resolvent cubic and the GQE always have the same number of multiple roots; in addition,*

(i) *These equations have exactly three multiple roots if, and only if,* $\Delta_4 = 0$ *and* $p^2 = -12r$*; meanwhile, the four roots of the GQE are given as follows:* $x_1 = x_2 = x_3 = \sqrt{-\frac{p}{6}} - \frac{b}{4a} \neq x_4 = -\sqrt{-\frac{3p}{2}} - \frac{b}{4a}$ *whether* $q > 0$*, and* $x_1 = x_2 = x_3 = -\sqrt{-\frac{p}{6}} - \frac{b}{4a} \neq x_4 = \sqrt{-\frac{3p}{2}} - \frac{b}{4a}$ *whether* $q < 0$.

(ii) *These equations have only two multiple roots if, and only if,* $\Delta_4 = 0$ *and* $p^2 > -12r$*; meanwhile, the four roots of the GQE are given by the following general formula:* $x = \frac{1}{2}\left[\xi\sqrt{s_1} \pm \sqrt{2\left(s_2 - \frac{\xi q}{\sqrt{s_1}}\right)}\right] - \frac{b}{4a}$, *where* $s_1 = \frac{9q^2-32pr}{p^2+12r} > 0$ *and* $s_2 = -\frac{2p\left(p^2-4r\right)+9q^2}{2(p^2+12r)} \neq 0$.

This theorem does not require any proof here, because all its affirmations have already been proven in [6]; additionally, Theorem 9 and Equations (7) and (32) allow us to affirm that the non-biquadratic quartic equations with non-multiple roots other than zero are the only cases of the GQE in which the resolvent cubic is actually indispensable to analytically solve the GQE; hence, Theorem 8 and Equations (8) and (9) will only be used here to solve the GQE when this one corresponds to the Ferrari Case with non-multiple roots other than zero; otherwise, the resolvent cubic will always be overlooked.

Now, in order to design a program that can analytically solve all algebraic equations with real coefficients of the second, third and fourth degree, it is important to consider here the SDE as any equation of the form $ax^2 + bx + c = 0$, with $a, b, c \in \mathbb{R}$ and $a \neq 0$; so, it is well known that the general solution for this equation is given by the quadratic formula [3], which guarantees the following three cases of the SDE:

1. The quadratic equations with two different real roots (whenever $b^2 - 4ac > 0$).
2. The quadratic equations with two multiple real roots (whenever $b^2 - 4ac = 0$).
3. The quadratic equations with a couple of non-real complex conjugate roots (whenever $b^2 - 4ac < 0$).

Meanwhile, the GCE will be classified in the following four main cases:

1. The equations with multiple roots, which are easily solved by Proposition 2.
2. The equations with purely imaginary roots, which can be solved by Proposition 1.
3. The cubic equations that have zero as a non-multiple root so that these equations are essentially quadratic equations; thus, these ones can be solved by Lemma 2 with $x_r = x_1 = 0$, which reduces this lemma to the quadratic formula.
4. The equations with non-multiple roots different from zero that are also non-purely imaginary roots, so these equations can be solved in general by Theorem 1.

In addition, the GQE will also be classified in four main cases as follows:

1. All the biquadratic equations, which can be easily solved by Equation (7) when $p^2 \geq 4r$; otherwise, they will be solved by Equation (32).
2. The non-biquadratic quartic equations with multiple roots, which can be easily solved by Theorem 9.
3. The non-biquadratic quartic equations that have zero as a non-multiple root, which are essentially cubic equations because $e = x_1 = 0 \neq d$ in these equations, so these ones are reduced to the equation $ax^4 + bx^3 + cx^2 + dx = x\left(ax^3 + bx^2 + cx + d\right) = 0$; thus, the other three roots of this kind of quartic equations can be obtained by applying Proposition 1 or Theorem 1 to the cubic equation $ax^3 + b^2 + cx + d = 0$.
4. The non-biquadratic quartic equations with non-multiple roots different from zero, which can only be solved by applying Theorem 8 and Equations (8) and (9).

Finally, Figure 1 shows how all the procedures described above are implemented in a single program.

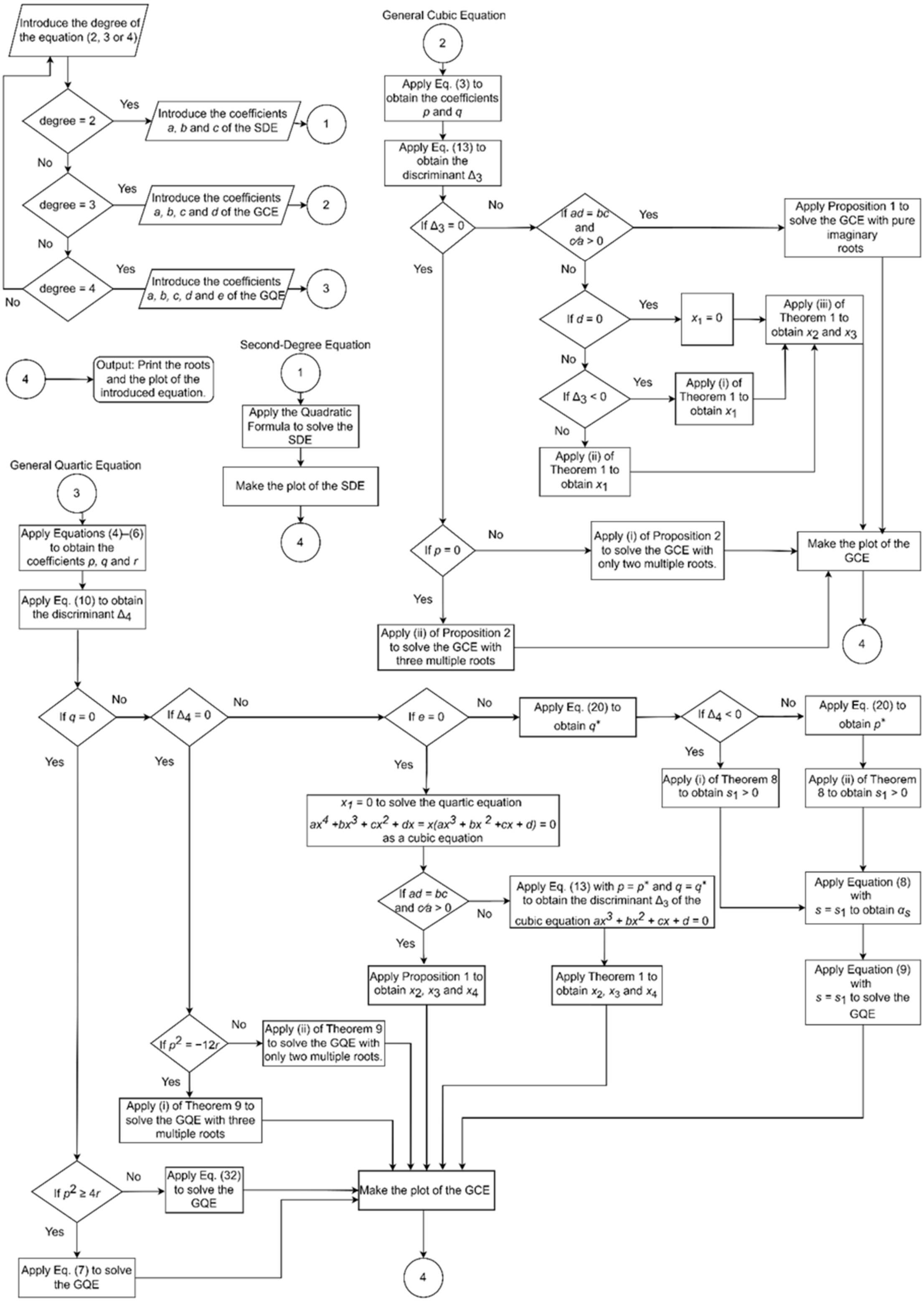
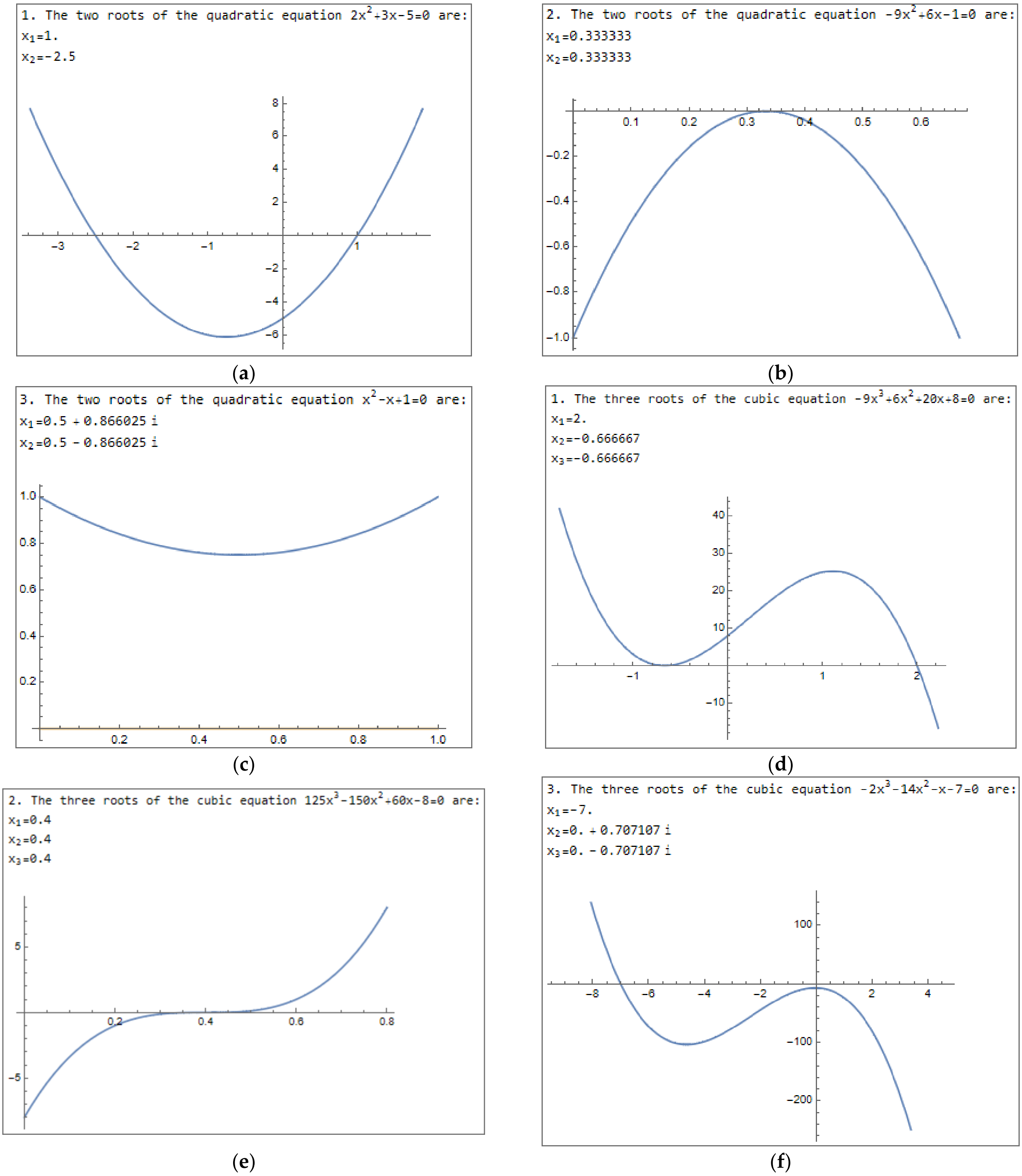


Figure 1. Flowchart of a program that analytically solves the SDE, the GCE and the GQE, without complex arithmetic.

3.3. Results

The procedures described in the previous subsection have been programmed in Wolfram Mathematica, so all the panels in Figure 2 are screenshots of the output given by this program.



(**a**) (**b**) (**c**) (**d**) (**e**) (**f**)

Figure 2. *Cont.*

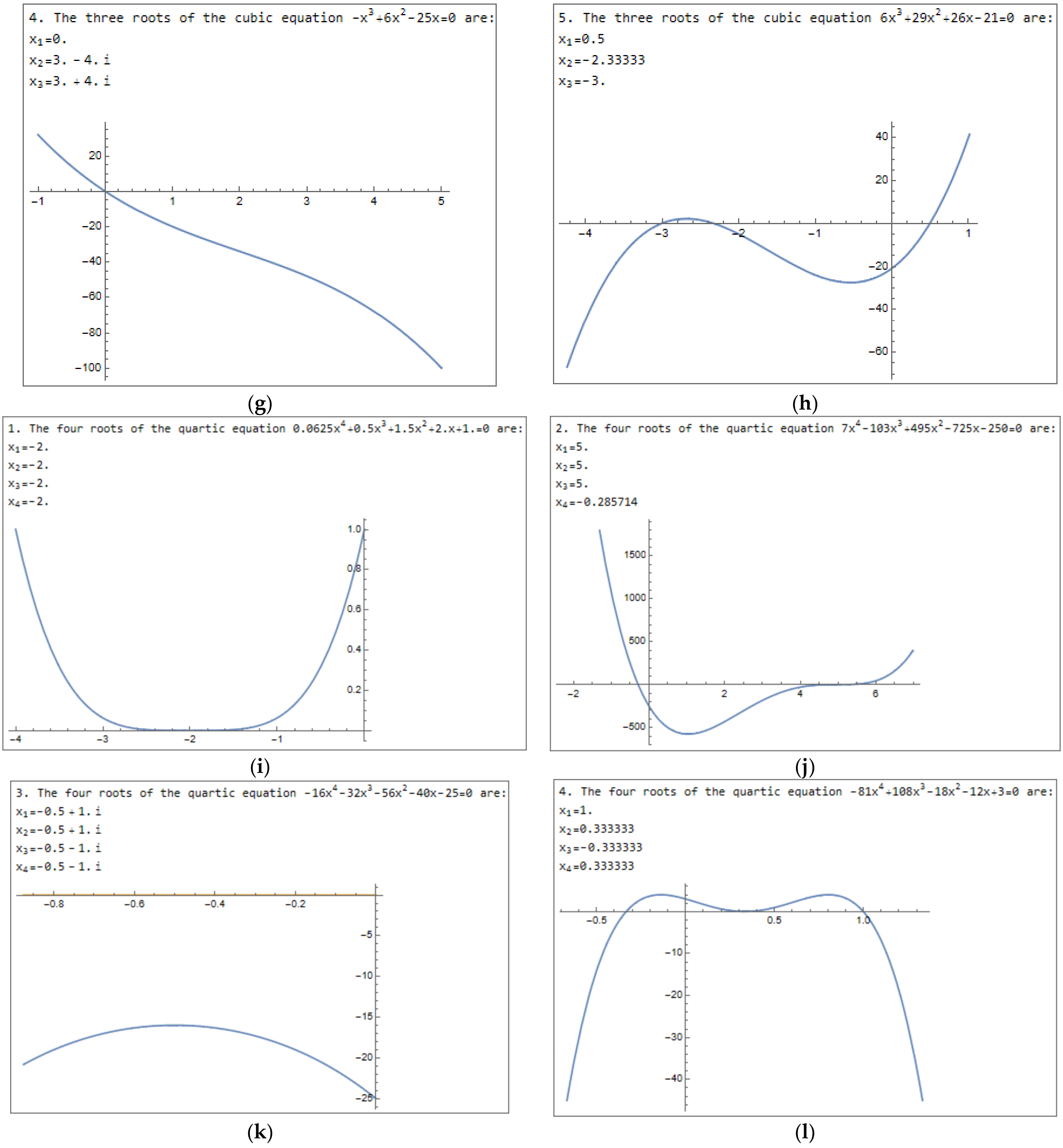


Figure 2. *Cont.*

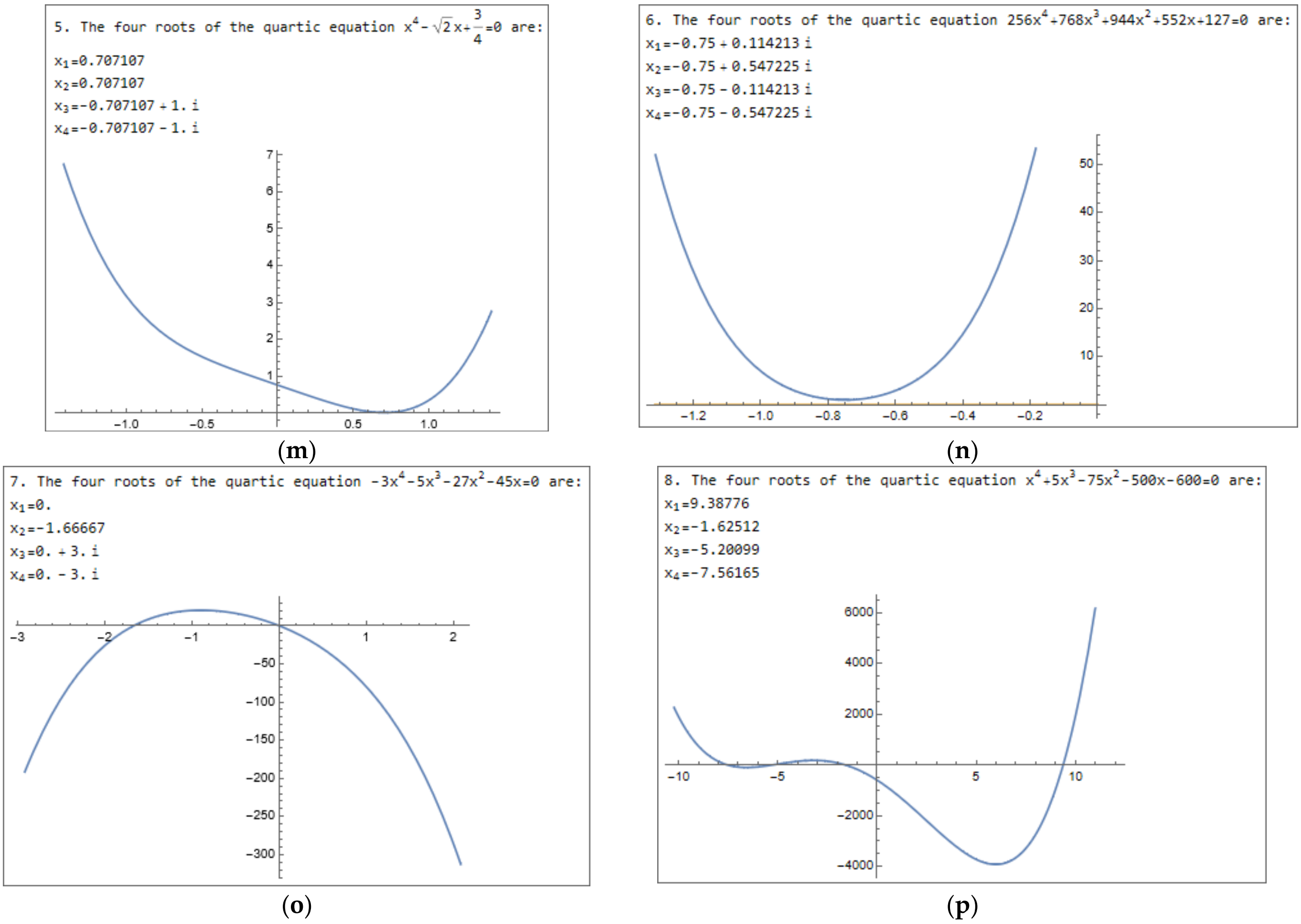


Figure 2. Examples of different cases of the SDE, the GCE and the GQE: (**a**) Case of the SDE with $b^2 - 4ac > 0$. (**b**) Case of the SDE with $b^2 - 4ac = 0$. (**c**) Case of the SDE with $b^2 - 4ac < 0$. (**d**) Case of the GCE with only two multiple roots. (**e**) Case of the GCE with three multiple roots. (**f**) Case of the GCE with purely imaginary roots. (**g**) Case of the GCE with zero as a non-multiple root. (**h**) Case of the GCE with non-multiple roots other than zero. (**i**) Case of the GQE with four multiple roots. (**j**) Case of the GQE with three multiple roots. (**k**) Case of the GQE with a pair of two different couples of multiple roots. (**l**) Case of a biquadratic equation with only two multiple roots. (**m**) Case of a non-biquadratic equation with only two multiple roots. (**n**) Case of a biquadratic equation without multiple roots. (**o**) Case of the GQE with zero as a non-multiple root and a couple of purely imaginary roots. (**p**) Case a non-biquadratic quartic equation with non-multiple roots other than zero.

Remark 8. *As stated in the Introduction of this paper, the coefficient q of the DQE determines if the GQE corresponds to the biquadratic case or to the Ferrari Case; in other words, if Equation (5) implies* $q = 0$, *then the GQE and the DQE are biquadratic equations, but if Equation (5) implies* $q \neq 0$, *then the GQE and the DQE are non-biquadratic equations; however, in graphical terms, the difference between biquadratic and non-biquadratic quartic equations is that the plots of the biquadratic equations are symmetrical with respect to the vertical line in* $\mathbb{R}^2$ *whose equation is* $x = -\frac{b}{4a}$ *(thus, the plot of the DQE is always symmetrical with respect to the ordinate axis), while the plots of the non-biquadratic quartic equations do not have any symmetry; so, all of this can be observed in all panels from (i) to (p) of Figure 2.*

Finally, since the plots are not substantial for the fundamental objectives of this paper, all the criteria applied in this program to make them are presented in Appendix C; however, it is also important to say here that some of these criteria are perfect examples of the practicality and usefulness of the analytical solution for the GCE that has been erroneously described as being less practical than the iterative Newton–Raphson Method applied to third-degree algebraic equations [2] (p. 51).

4. Discussion

The main objection to solving the third- and fourth-degree equations by applying analytical methods has always been the apparently inevitable application of complex arithmetic operations and all the difficulties that this implies; however, the results exposed here have proven that complex arithmetic is actually not needed in order to analytically solve the GCE and the GQE in practical terms. In addition, another arguable common objection to using these tools is that they are not as quick and easy to apply as the quadratic formula, and some of the examples exposed here seem to validate that argument, such as Examples 4, 5 or 7, whose solutions inevitably involved fractions with huge integer numbers.

Despite this, the analytical methods to solve these equations in particular, as exposed here, are more efficient than the numerical methods because the analytical ones go straight to the precise solution right away; meanwhile, the numerical methods work by approximating the solution by iterating, which can imply arithmetic calculations even more tiresome than the ones applied in every example of this paper. So, if the same powerful computational tools that are used to apply the numerical methods are used to apply the analytical methods to solve these equations, then all their solutions can be obtained faster without iterating, as the program presented in Section 3 shows.

Additionally, the accuracy of the obtained results will depend on the computing power of the computational tools used; for example, the program presented in Section 3 was made using the Wolfram Mathematica programming language, and it was observed that the precision error in the obtained results is generally less than 10^{-15}, which can be considered very accurate in solving a practical problem.

Likewise, it is certainly somewhat ironic that this document presents how to use the methods that were discovered half a millennium ago to analytically solve the GCE and the GQE, always avoiding the use of complex arithmetic, since the development of these methods gave rise to the definition and study of complex numbers; thus, in historical terms, the Casus Irreducibilis of third-degree equations caused this because this case of the GCE occurs whenever it has three different real roots, while the solutions given by Equations (1) and (2) to this case inevitably imply the appearance of imaginary quantities.

In this sense, a new proof of the trigonometric solution given by Equation (11) to the Casus Irreducibilis is also included in Appendix A, without alluding to the existence of imaginary or non-real complex numbers; so curiously, if everything exposed here about how to avoid complex arithmetic during the resolution of cubic and quartic equations with real coefficients had been known 500 years ago, then the definition and study of set $\mathbb{C}$ would have been delayed.

5. Conclusions

The most relevant results exposed in this paper are listed as follows:

1. The results that guarantee that the GCE and the GQE can always be solved analytically without complex arithmetic (Theorems 1 and 8).
2. The general relationship between the discriminant of the GQE and the discriminant of all forms of the resolvent cubic (Theorem 2).
3. The definition and relevance of the SFRC (Section 2.2).
4. All the relationships between the nature of the roots of the GQC and the nature of the roots of the corresponding resolvent cubic (Section 2.3).
5. The program developed in Section 3.

In addition, the results shown in Figure 2 are perfect examples of the effectiveness of the analytical methods to solve the GCE and the GQE, which can be applied to give a more efficient solution to practical problems related to these kinds of algebraic equations; nevertheless, it does not mean that the program presented in Section 3 cannot be improved.

In this sense, note that all the known forms of the resolvent cubic that are characterized by Equation (18) were listed in Section 2.2.3; however, another special form of the resolvent cubic that is not characterized by Equation (18) is known, hence it was not included in that list; this special form is obtained by the Descartes Method to solve quartic equations, as exposed in [7] (p. 66), and is given as follows:

$$R_C\left(t^2\right) = t^6 + 2pt^4 + \left(p^2 - 4r\right)t^2 - q^2 = 0\,; \tag{36}$$

this equation is typically considered another form of the resolvent cubic, although it is a sixth-degree equation; thus, the Fundamental Theorem of Algebra guarantees that it has six complex roots [1] (pp. 399–400); in addition, its discriminant is different from Δ_{RC} and Δ_4, so Equation (10) and Theorem 2 do not work for this equation.

In spite of these important facts, in practical terms, to solve the GQE and the DQE, it is obvious that Equation (36) is essentially the SFRC given by Equation (17); therefore, Equation (36) can also be defined here as the "Bicubic SFRC", whose properties are material for a subsequent article because the relationship between the nature of the roots of the GQE and the nature of the roots of Equation (36) in terms of rationality can be very useful for designing more sophisticated and accurate algorithms to analyze and solve the GQE without numerical methods or complex arithmetic.

Finally, the authors of this paper hope that this will contribute to tearing down all the great prejudices that currently exist regarding the practical use of the invaluable results given by del Ferro, Tartaglia, Cardano and Ferrari five centuries ago with the help of modern computational tools.

Author Contributions: Main ideas, introduction, all proofs, examples, programming, discussion, results, conclusions, Appendix C and English—translation, M.C.-P.; introduction, Sections 3.2 and 3.3, Figure 1, programming–debugging, discussion, conclusions and Appendix C, M.A.M.-C.; introduction, examples, Figure 2, debugging, discussion, results and conclusions, A.T.-M.; epistemological and methodological advise, references and writing—review, A.B.V.-C.; writing—review and English—translation, T.A.-R. All authors have read and agreed to the published version of the manuscript.

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Appendix A

This appendix is dedicated to presenting a proof of Equation (11) without appealing to the existence of non-real complex numbers. So, first of all, consider that the relationship between the GCE and the DCE is determined by the following equation:

$$x = y - \frac{b}{3a}\,; \tag{A1}$$

in addition, consider the following well-known trigonometric identities that hold for any α, β, $\theta \in \mathbb{R}$ [12] (pp. 417, 523, 524, 526), and the subsequent propositions:

$$\sin(\alpha \pm \beta) \equiv \sin\alpha\cos\beta \pm \sin\beta\cos\alpha, \tag{A2}$$

$$\cos(\alpha \pm \beta) \equiv \cos\alpha\cos\beta \mp \sin\alpha\sin\beta, \tag{A3}$$

$$\sin^2\theta + \cos^2\theta \equiv 1. \tag{A4}$$

Proposition A1. *The relation* $\cos^3\theta \equiv \frac{\cos 3\theta + 3\cos\theta}{4}$ *holds for any* $\theta \in \mathbb{R}$.

Proof. If $\theta = \alpha = \beta$, then Equations (A2) and (A3) guarantee $\sin 2\theta \equiv 2\sin\theta\cos\theta$ and $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$ for any $\theta \in \mathbb{R}$; so, these identities and Equations (A2)–(A4) imply the following relations for any $\theta \in \mathbb{R}$: $\cos 3\theta \equiv \cos(2\theta + \theta) \equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta \equiv \left(\cos^2\theta - \sin^2\theta\right)\cos\theta - (2\sin\theta\cos\theta)\sin\theta \equiv \cos^3\theta - 3\sin^2\theta\cos\theta \equiv \cos^3\theta - 3\left(1 - \cos^2\theta\right)$ $\cos\theta \equiv 4\cos^3\theta - 3\cos\theta$; therefore, $\cos^3\theta \equiv \frac{\cos 3\theta + 3\cos\theta}{4}$ for any $\theta \in \mathbb{R}$. □

Proposition A2. *If* $\theta \in \mathbb{R}$, *then the relation* $\cos\theta \equiv \cos(\theta + 2n\pi)$ *holds for any* $n \in \mathbb{Z}$.

Proof. First of all, consider that $\cos 2n\pi \equiv \cos 2\pi \equiv 1$ and $\sin 2n\pi \equiv \sin 2\pi \equiv 0$ for all $n \in \mathbb{Z}$ [12] (pp. 456–457); then, Equation (A3) implies $\cos(\theta + 2n\pi) \equiv \cos\theta\cos 2n\pi - \sin\theta\sin 2n\pi \equiv \cos\theta \cdot 1 - \sin\theta \cdot 0 \equiv \cos\theta$. □

Now, consider the following theorem.

Theorem A1. *If* $\Delta_3 > 0$, *then the three non-multiple real roots of the DCE are given as follows:* $y_k = 2\sqrt{-\frac{p}{3}}\cos\left[\frac{1}{3}\arccos\left\{-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}\right\} + \frac{2(k-1)\pi}{3}\right]$, *for* $k = 1, 2, 3$.

Proof. Suppose that there exist $\alpha, \theta \in \mathbb{R}$, with $\alpha > 0$, such that the expression $\alpha\cos\theta \in \mathbb{R}$ is a root of the DCE; then, $\alpha^3\cos^3\theta + p\alpha\cos\theta + q = 0$, so this equality and Proposition A1 imply the following equation: $\alpha^3\left(\frac{\cos 3\theta + 3\cos\theta}{4}\right) + p\alpha\cos\theta + q = \frac{\alpha^3}{4}\cos 3\theta + \left(\frac{3\alpha^2}{4} + p\right)\alpha\cos\theta + q = 0$; now note that the previous equation holds whether the following two equations also hold simultaneously:

$$\frac{3\alpha^2}{4} + p = 0, \tag{A5}$$

$$\frac{\alpha^3}{4}\cos 3\theta + q = 0. \tag{A6}$$

On the other hand, if $\Delta_3 > 0$, then Equation (13) implies $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = -\frac{\Delta_3}{108} < 0$; thus, (i) of Remark 1 guarantees $p < 0$, which implies the following three facts: $\sqrt{-\frac{p}{3}} \in \mathbb{R}$, $\sqrt{\left(-\frac{3}{p}\right)^3} \in \mathbb{R}$ and $\sqrt{-\frac{p}{3}} > 0$; so, Equation (A5) implies the following relation:

$$\alpha = \sqrt{-\frac{4p}{3}} = 2\sqrt{-\frac{p}{3}} > 0; \tag{A7}$$

meanwhile, Equations (A6) and (A7) imply the following equality:

$$\cos 3\theta = -\frac{4q}{\alpha^3} = -\frac{4q}{\left(2\sqrt{-p/3}\right)^3} = -\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}, \tag{A8}$$

which also involves real numbers, only; hence, Equation (A8) and Proposition A2 guarantee that the relation $\cos 3\theta = \cos(3\theta - 2n\pi) = -\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}$ holds for any $n \in \mathbb{Z}$; then, there can exist an infinite number of real values of angle θ that satisfy Equation (A8), which are given as follows: $\theta_n = \frac{1}{3}arccos\left\{-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}\right\} + \frac{2n\pi}{3}$, for any $n \in \mathbb{Z}$. Therefore, θ_n and Equation (A7) guarantee that all the roots of the DCE are given as follows:

$$y = \alpha\cos\theta_n = 2\sqrt{-\frac{p}{3}}\cos\left[\frac{1}{3}\arccos\left\{-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}\right\} + \frac{2n\pi}{3}\right], \text{ for all } n \in \mathbb{Z}; \quad \text{(A9)}$$

likewise, it is clear that $\theta_0 = \frac{1}{3}\arccos\left\{-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}\right\}$, so the following relation also holds for any $n \in \mathbb{Z}$:

$$\theta_n = \theta_0 + \frac{2n\pi}{3}\,. \quad \text{(A10)}$$

On the other hand, consider the definitions of the following three sets: $A_1 := \{l \in \mathbb{Z} : l = 3m \text{ with } m \in \mathbb{Z}\}$, $A_2 := \{l \in \mathbb{Z} : l = 3m+1 \text{ with } m \in \mathbb{Z}\}$ and $A_3 := \{l \in \mathbb{Z} : l = 3m+2 \text{ with } m \in \mathbb{Z}\}$; thus, it is clear that $A_1 \cup A_2 \cup A_3 = \mathbb{Z}$ whereas $A_i \cap A_j = \varnothing$ whenever $i \neq j$. Now consider that Equation (A10) and Proposition A2 guarantee the following three relations for all $m \in \mathbb{Z}$:

$$\begin{aligned}\cos\theta_{3m} &= \cos\left[\theta_0 + \tfrac{2(3m)\pi}{3}\right] = \cos(\theta_0 + 2m\pi) = \cos\theta_0\,,\\ \cos\theta_{3m+1} &= \cos\left[\theta_0 + \tfrac{2(3m+1)\pi}{3}\right] = \cos\left[\left\{\theta_0 + \tfrac{2(1)\pi}{3}\right\} + 2m\pi\right] = \cos(\theta_1 + 2m\pi) = \cos\theta_1\,,\\ \cos\theta_{3m+2} &= \cos\left[\theta_0 + \tfrac{2(3m+2)\pi}{3}\right] = \cos\left[\left\{\theta_0 + \tfrac{2(2)\pi}{3}\right\} + 2m\pi\right] = \cos(\theta_2 + 2m\pi) = \cos\theta_2\,.\end{aligned} \quad \text{(A11)}$$

Finally, $\Delta_3 > 0$ and Equations (A11) guarantee that there is a unique non-multiple real root y_k of the DCE related by Equation (A9) with all the elements of set A_k, for each $k \in \{1,2,3\}$; which is given as follows: $y_k = \alpha\cos\theta_{k-1} = 2\sqrt{-\frac{p}{3}}\cos\left[\frac{1}{3}\arccos\left\{-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}\right\} + \frac{2(k-1)\pi}{3}\right]$. □

Corollary A1. *If $\Delta_3 > 0$, then the three non-multiple real roots of the GCE are given as follows:* $x_k = 2\sqrt{-\frac{p}{3}}\cos\left[\frac{1}{3}\arccos\left\{-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}\right\} + \frac{2(k-1)\pi}{3}\right] - \frac{b}{3a}$, for $k = 1,2,3$.

Proof. This is an immediate consequence of Theorem A1 and Equation (A1). □

Remark A1. *Note that Equation (11) can also solve the GCE when this one has only two multiple real roots because in this case, $p \neq \Delta_3 = 0 \neq q$ guarantees* $\left|-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}\right| = 1$; so, $\arccos\left\{-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}\right\} = \arccos(-1) = \pi$ *whether* $q > 0$ *and* $\arccos\left\{-\frac{q}{2}\sqrt{\left(-\frac{3}{p}\right)^3}\right\} = \arccos 1 = 0$ *whether* $q < 0$; *which imply the following alternative formulae that are equivalent to (i) of Proposition 2:*

(i) $x_1 = -\sqrt{-\frac{4p}{3}} - \frac{b}{3a} \neq x_2 = x_3 = \sqrt{-\frac{p}{3}} - \frac{b}{3a}$ *whether* $q > 0$,

(ii) $x_1 = \sqrt{-\frac{4p}{3}} - \frac{b}{3a} \neq x_2 = x_3 = -\sqrt{-\frac{p}{3}} - \frac{b}{3a}$ *whether* $q < 0$;

meanwhile, if the GCE has three multiple real roots, then Equation (11) becomes undefined because (ii) of Proposition 2 guarantees $\Delta_3 = p = q = 0$ for this case.

Appendix B

This appendix is mainly focused on exposing the equivalence between Proposition 2 and Theorem 1 when $\Delta_3 = 0$. For this purpose, it is clear that Proposition 2 is a particular case of (i) of Theorem 1 for x_1 when $\Delta_3 = 0$; now, in order to prove the equivalence between Proposition 2 and (iii) of Theorem 1 for x_2 and x_3 when $\Delta_3 = 0$, consider the following two possible cases:

Case 1: Supposing that the GCE has only two multiple real roots, then Proposition 2 guarantees that the three roots of this equation are given as follows: $x_1 = -\sqrt[3]{4q} - \frac{b}{3a} \neq x_2 = x_3 = \sqrt[3]{\frac{q}{2}} - \frac{b}{3a}$; hence, if $l := -\sqrt[3]{4q} - \frac{b}{3a}$ and $m := \sqrt[3]{\frac{q}{2}} - \frac{b}{3a}$, then the GCE is given as follows: $a(x-l)(x-m)^2 = ax^3 - a(l+2m)x^2 + a(2l+m)mx - alm^2 = 0$; so $b = -a(l+2m)$ and $c = a(2l+m)m$; ergo, $x_1 = l = -\frac{b}{a} - 2m$ and $c = a\left[2\left(-\frac{b}{a} - 2m\right) + m\right]m = -2bm - 3am^2$. Therefore: $\frac{-(ax_1+b) \pm \sqrt{b^2 - 4ac - \left(3a^2x_1^2 + 2abx_1\right)}}{2a} = \frac{-\left[a\left(-\frac{b}{a} - 2m\right) + b\right] \pm \sqrt{b^2 - 4a(-2bm - 3am^2) - \left[3a^2\left(-\frac{b}{a} - 2m\right)^2 + 2ab\left(-\frac{b}{a} - 2m\right)\right]}}{2a} = \frac{b + 2am - b \pm \sqrt{b^2 + 8abm + 12a^2m^2 - 3b^2 - 12abm - 12a^2m^2 + 2b^2 + 4abm}}{2a} = \frac{2am \pm \sqrt{0}}{2a} = m = \sqrt[3]{\frac{q}{2}} - \frac{b}{3a} = x_2 = x_3$, so (i) and (iii) of Theorem 1 are equivalent to Proposition 1 whether $\Delta_3 = 0$ in this case.

Case 2: Supposing that the GCE has three multiple real roots, then Proposition 2 guarantees that these roots are given as follows: $x_1 = x_2 = x_3 = -\frac{b}{3a}$; hence, the GCE is given as follows: $a\left(x + \frac{b}{3a}\right)^3 = ax^3 + bx^2 + \frac{b^2}{3a}x + \frac{b^3}{27a^2} = 0$; so, $c = \frac{b^2}{3a}$. Therefore: $\frac{-(ax_1+b) \pm \sqrt{b^2 - 4ac - \left(3a^2x_1^2 + 2abx_1\right)}}{2a} = \frac{-\left[a\left(-\frac{b}{3a}\right) + b\right] \pm \sqrt{b^2 - 4a\left(\frac{b^2}{3a}\right) - \left[3a^2\left(-\frac{b}{3a}\right)^2 + 2ab\left(-\frac{b}{3a}\right)\right]}}{2a} = \frac{\frac{b}{3} - b \pm \sqrt{b^2 - \frac{4b^2}{3} - \frac{b^2}{3} + \frac{2b^2}{3}}}{2a} = \frac{-\frac{2b}{3} \pm \sqrt{0}}{2a} = -\frac{b}{3a} = x_2 = x_3$, so (i) and (iii) of Theorem 1 are also equivalent to Proposition 1 whether $\Delta_3 = 0$ in this case.

Remark A2. *Note that the equivalence between (iii) of Theorem 1 and Equation (2) for* $\Delta_3 < 0$ *and the equivalence between (iii) of Theorem 1 and Equation (11) for* $\Delta_3 > 0$ *are guaranteed by Lemma 2.*

Remark A3. *Note that Lemma 1 also holds when the GCE has three multiple real roots because* $x_1 = x_2 = x_3 = \max\{x_1, x_2, x_3\} = \min\{x_1, x_2, x_3\}$ *in this case; however, this lemma does not hold in general when the GCE has only two multiple real roots because (i) of Proposition 2 and Remark A1 guarantee the following facts for that case:*

(i) $x_1 = \min\{x_1, x_2, x_3\} \neq x_2 = x_3 = \max\{x_1, x_2, x_3\}$ *whenever* $q > 0$
(ii) $x_1 = \max\{x_1, x_2, x_3\} \neq x_2 = x_3 = \min\{x_1, x_2, x_3\}$ *whenever* $q < 0$

therefore, if $\Delta_3 = 0$, *then Lemma 1 holds only when* $q \leq 0$.

Appendix C

In order to make the plots of the program exposed in Section 3, consider the "middle point of the polynomial", defined as $x_m := -\frac{b}{na} \in \mathbb{R}$, where n is the polynomial's degree, so $n \in \{2, 3, 4\}$; in addition, x_{min} and x_{max} are the respective lower and upper limits of the interval of the plot given in the output of the program, so $x_{min} < x_{max}$ and they also satisfy $x_{min} \leq 0 \leq x_{max}$ in order to show the origin on each plot of the output.

Second-Degree Equation: Define $f(x) := ax^2 + bx + c$, whose two roots are given by the Quadratic Formula: $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, whereas $x_m = -\frac{b}{2a}$; then, consider the following cases:

Case 1. If $ac \geq 0 \neq b$, then $x_m \neq 0$, so define $l := |x_m| > 0$.
Case 2. If $ac < 0$, then $x_1, x_2 \in \mathbb{R}$ and $x_1 \neq x_2$, so define $l := \frac{3|x_2 - x_1|}{4} > 0$.
Case 3. If $ac = b = 0$, then $x_1 = x_2 = x_m = 0$, so define $l := 2|a| > 0$.

Case 4. If $b^2 - 4ac < 0 = b$, then $x_m = \mathrm{Re}(x_{1,2}) = 0 \neq \mathrm{Im}(x_{1,2})$, so define $l := |\mathrm{Im}(x_1)| = |\mathrm{Im}(x_2)| > 0$.

Finally, put $x_{max} = x_m + l$ and $x_{min} = x_m - l$, for all Cases 1–4.

General Cubic Equation: Define $f(x) := ax^3 + bx^2 + cx + d$, whose three roots are x_1, x_2 and x_3, whereas $x_m = -\frac{b}{3a}$; then, consider the following cases:

Case 1. If $x_m = x_1$, then there are the following subcases:

(i) If $b = p = q = 0$, then $x_1 = x_2 = x_3 = x_m = 0$, so define $l := 3|a| > 0$.

(ii) If $p > 0 = b = q$, then $\Delta_3 < 0$, so $x_1 = x_m = 0$ and $x_2, x_3 \in \mathbb{C} - \mathbb{R}$; therefore, define $l := \sqrt[3]{p} > 0$.

(iii) If $p \geq q = 0 \neq b$, then $\Delta_3 = p = q = 0 \neq x_m = x_1 = x_2 = x_3$ or $\Delta_3 < 0 \neq x_m = x_1$ with $x_2, x_3 \in \mathbb{C} - \mathbb{R}$, so define $l := |x_m| > 0$.

Thus, put $x_{max} = x_m + l$ and $x_{min} = x_m - l$, for all subcases (i)–(iii).

Case 2. If $\Delta_3 < 0$ with $x_m \neq x_1 \in \mathbb{R}$, then define $l := |x_m - x_1| > 0$; finally, if $l < |x_m|$; then, put $x_{max} = x_m + |x_m|$ and $x_{min} = x_m - |x_m|$; otherwise, put $x_{max} = x_m + \frac{3l}{2}$ and $x_{min} = x_m - \frac{3l}{2}$.

Case 3. If $\Delta_3 \geq 0$ and $x_m \neq x_1$, then the three roots of the GCE are all real and at most two of them are multiple; thus, define $l := \max\{x_1, x_2, x_3\} - \min\{x_1, x_2, x_3\} > 0$. Finally, if $\max\{x_1, x_2, x_3\} \geq 0$, then put $x_{max} = x_m + \frac{3l}{4}$, otherwise put $x_{max} = 0$; and if $\min\{x_1, x_2, x_3\} \leq 0$, then put $x_{min} = x_m - \frac{3l}{4}$, otherwise put $x_{min} = 0$.

General Quartic Equation: Define $f(x) := ax^4 + bx^3 + cx^2 + dx + e$, whose four roots are x_1, x_2, x_3 and x_4, whereas $x_m = -\frac{b}{4a}$; then, consider the following cases:

Case 1. According to Theorem 7, the GQE has four real roots when $\Delta_4 \geq 0$ with $p < 0$ and $p^2 > 4r$, or when $p \leq 0$ with $p^2 = 4r \geq q = 0$; so, there are the following subcases:

(i) According to [6], if $b = p = q = r = 0$, then $x_1 = x_2 = x_3 = x_4 = x_m = 0$, so define $l := 4|a| > 0$.

(ii) If $b \neq 0 = p = q = r$, then $x_1 = x_2 = x_3 = x_4 = x_m \neq 0$, so define $l := 4|x_m| > 0$.

(iii) If at least one of the three main coefficients of the DQE are different from zero, then $\max\{x_1, x_2, x_3, x_4\} \neq \min\{x_1, x_2, x_3, x_4\}$; so, if $\Delta_4 = 0$, then define $l := 2\max\{|x_m - \max\{x_1, x_2, x_3, x_4\}|, |x_m - \min\{x_1, x_2, x_3, x_4\}|\} > 0$, otherwise define $l := \max\{|x_m - \max\{x_1, x_2, x_3, x_4\}|, |x_m - \min\{x_1, x_2, x_3, x_4\}|\} > 0$.

Finally, if $\max\{x_1, x_2, x_3, x_4\} \geq 0$, then put $x_{max} = \max\{x_1, x_2, x_3 x_4\} + \frac{l}{4}$, otherwise put $x_{max} = 0$; and if $\min\{x_1, x_2, x_3, x_4\} \leq 0$, then put $x_{min} = \min\{x_1, x_2, x_3, x_4\} - \frac{l}{4}$, otherwise put $x_{min} = 0$; for all subcases (i)–(iii).

Case 2. According to Theorem 7, the GQE has four non-real complex roots when $\Delta_4 > 0$ with $p \geq 0$ or $p^2 \leq 4r$, or when $\Delta_4 = q = 0 < p$ with $p^2 = 4r$; then, consider the derivative of f given as follows: $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$, thus, all the critical points of function f are determined by the real roots of the following cubic equation:

$$4ax^3 + 3bx^2 + 2cx + d = 0; \tag{A12}$$

hence, Equation (3) applied to Equation (A12) gives $p^* = \frac{8ac - 3b^2}{16a}$ and $q^* = \frac{27b^3 - 108abc + 216a^2 d}{864a^3}$, so Equation (13) implies that the discriminant of Equation (A12) is $\Delta_3 = -4p^{*3} - 27q^{*2}$. Now, consider the following subcases:

(i) If $\Delta_3 > 0$, then Equation (A12) has three different real roots, so f has three different critical points; in addition, Lemma 1 guarantees that Equation (11) applied to Equation (A12) with $k = 1$ will always give the largest of the critical points of f, whereas the smallest of these points is obtained with $k = 2$. Thus, if c_{p1} and c_{p2} are respectively the largest and the smallest of the critical points of f, then define $l := c_{p1} - c_{p2} > 0$; finally, if $c_{p1} \geq 0$, then put $x_{max} = x_m + \frac{3l}{4}$, otherwise put $x_{min} = 0$; and if $c_{p2} \leq 0$, then put $x_{min} = x_m - \frac{3l}{4}$, otherwise put $x_{min} = 0$.

(ii) If $\Delta_3 = 0 > p^*$, then Equation (A12) has three real roots and two of them are multiple; thus, f has only two different critical points, so define $l := \sqrt[3]{-\frac{4p^*}{3}} + \sqrt[3]{-\frac{p^*}{3}} > 0$. In addition, if c_{p1} and c_{p2} are respectively the largest and the smallest of the two critical points of f; then, (i) of Proposition 2 applied to Equation (A12) guarantees $c_{p1} = \sqrt[3]{\frac{q^*}{2}} + x_m$ and $c_{p2} = -\sqrt[3]{4q^*} + x_m$ whether $q^* > 0$, otherwise $c_{p1} = -\sqrt[3]{4q^*} + x_m$ and $c_{p2} = \sqrt[3]{\frac{q^*}{2}} + x_m$ (note that Remark A1 guarantees that the definition of l in this subcase is also equivalent to $l = c_{p1} - c_{p2}$). Finally, put x_{max} and x_{min} as in subcase (i).

(iii) If $\Delta_3 = p^* = q^* = 0$ or $\Delta_3 < 0 = f'(x_m)$, then there are two possibilities: Equation (A12) has three multiple real roots, which are all x_m; or x_m is the only real root of Equation (A12), which does not have multiple roots. In any possibility, x_m is the only critical point of f; therefore, define $l := 4|a| > 0$ whether $x_m = b = 0$, otherwise define $l := |x_m| > 0$. Finally, if $x_m \geq 0$, then put $x_{max} = x_m + \frac{3l}{4}$, otherwise put $x_{max} = 0$; and if $x_m \leq 0$, then put $x_{min} = x_m - \frac{3l}{4}$, otherwise put $x_{min} = 0$.

(iv) If $\Delta_3 < 0 \neq f'(x_m)$, then Equation (A12) has only one real root, which is different from x_m, and a couple of non-real complex conjugate roots; so, f has only one critical point, which is $c_p = \sqrt[3]{-\frac{q^*}{2} + \sqrt{-\frac{\Delta_3}{108}}} + \sqrt[3]{-\frac{q^*}{2} - \sqrt{-\frac{\Delta_3}{108}}} + x_m \neq x_m$; therefore, define $l := |x_m - c_p| > 0$. Finally, if $2l \geq |x_m|$, then put $x_{max} = x_m + 2l$ and $x_{min} = x_m - 2l$; if $2l < |x_m|$ and $x_m > 0$, then put $x_{max} = x_m + 2l$ and $x_{min} = 0$; and if $2l < |x_m|$ and $x_m < 0$, then put $x_{max} = 0$ and $x_{min} = x_m - 2l$.

Case 3. According to Theorem 7, the GQE has only two real roots when $\Delta_4 < 0$ or $p \geq q \neq \Delta_4 = 0 < r$ or $p < q \neq \Delta_4 = 0 < p^2 \leq 4r$ or $p > q = r = \Delta_4 = 0$; now, consider the following subcases:

(i) If $b = d = e = 0 < c/a$, then zero is a multiple root of the GQE and it also has two purely imaginary roots; in addition, Equation (A12) guarantees that f has only one critical point, which is $x_m = 0$; finally, put $x_{max} = |a| > 0$ and $x_{min} = -x_{max} < 0$.

(ii) If $\Delta_4 = f(x_m) = 0 \neq x_m$, then x_m is a multiple real root of the GQE, so define $l := 2|x_m| > 0$; finally, put $x_{max} = x_m + l$ and $x_{min} = x_m - l$.

(iii) If $\Delta_4 < 0$, then the GQE do not have multiple roots and only two of them are real, so at least one of this real roots is different from x_m; therefore, put $l_k = |x_m - x_k| \geq 0$ whether $\mathrm{Im}(x_k) = 0$ (that is, when $x_k \in \mathbb{R}$), otherwise put $l_k = 0$, for each $k \in \{1,2,3,4\}$; so, define $l := 2\max\{l_1, l_2, l_3, l_4\} > 0$. Additionally, if $l \geq -2x_m$, then put $x_{max} = x_m + l$, otherwise put $x_{max} = 0$; and if $l \geq 2x_m$, then put $x_{min} = x_m - l$, otherwise put $x_{min} = 0$.

Finally, it is important to say that in the cases of the SDE with two non-real roots and the GQE with four non-real roots, the plot of the function $g(x) := 0$ was also included to emphasize the fact that these equations do not have any real root (see panels (c), (k) and (n) of Figure 2).

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