mathematics

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Discussion on Fuzzy Integral Inequalities via Aumann Integrable Convex Fuzzy-Number Valued Mappings over Fuzzy Inclusion Relation

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Abstract: Convex bodies are naturally symmetrical. There is also a correlation between the two variables of symmetry and convexity. Their use, in either case, has been feasible in recent years because of their interchangeable and similar properties. The proposed analysis provides information on a new class for a convex function which is known as up and down $(\mathcal{X}_1, \mathcal{X}_2)$-convex fuzzy-Number valued mappings (UD-$(\mathcal{X}_1, \mathcal{X}_2)$-convex $FNVM$). Using this class, we disclosed a number of new versions of integral inequalities. Additionally, we give a number of new related integral inequalities connected to the well-known Hermite-Hadamard-type inequalities. In conclusion, some examples are given to back up and show the value of these new results.

Keywords: fuzzy aumann integral; up and down (X_1, X_2)-convex fuzzy-number valued mappings; Hermite-Hadamard type inequalities

MSC: 26A33; 26A51; 26D10

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Citation: Khan, M.B.; Othman, H.A.; Rakhmangulov, A.; Soliman, M.S.; Alzubaidi, A.M. Discussion on Fuzzy Integral Inequalities via Aumann Integrable Convex Fuzzy-Number Valued Mappings over Fuzzy Inclusion Relation. *Mathematics* **2023**, *11*, 1356. https://doi.org/10.3390/math11061356

Academic Editor: Vilém Novák

Received: 11 February 2023
Revised: 8 March 2023
Accepted: 8 March 2023
Published: 10 March 2023

1. Introduction

The convex function hypothesis provides us with incredible standards and techniques to focus on a wide range of problems in both pure and applied sciences. Numerous important scientific researchers consistently work to enhance and benefit from these initial ideas in order to profit from the convexity hypothesis. This hypothesis is assumed to play a significant and vital role in applied mathematics, notably in nonlinear programming, financial mathematics, mathematical statistics, optimization theory, and functional analysis. The investigation of mathematical inequalities relies heavily on the theory of convexity. Because of how their definitions and characteristics behave, there is a close connection between the theory of inequality, fractional integrals, and convex functions.

In a classical approach, a real-valued mapping $Y : K \to \mathbb{R}$ is called convex if

$$Y(\mu\varkappa + (1-\mu)y) \leq \mu Y(\varkappa) + (1-\mu)Y(y), \tag{1}$$

for all $\varkappa, y \in K, \mu \in [0,1]$, where K is a convex set [1].

Inequalities such as the Hermite-Hadamard (H-H) inequality, the Ostrowski inequality, and Simpson's inequality are produced by convex functions. One of the findings regarding convex functions that have been the subject of the greatest research is the H-H double inequality. We now have both the required and sufficient conditions for a function to be convex [2–10]. One of the most beneficial findings in mathematical analysis is the H-H inequality. This is also known as the classical inequality of the H-H inequality.

The H-H-inequality [11] for convex mapping $Y : K \to \mathbb{R}$ on an interval $K = [\varsigma, \mathfrak{v}]$ is

$$Y\left(\frac{\mathfrak{v}+\varsigma}{2}\right) \leq \frac{1}{\mathfrak{v}-\varsigma}\int_{\varsigma}^{\mathfrak{v}} Y(\mathfrak{x})d\mathfrak{x} \leq \frac{Y(\mathfrak{v})+Y(\varsigma)}{2}, \tag{2}$$

for $\mathfrak{v}, \varsigma \in K$.

Interval analysis, which Moore first proposed in his well-known book [12], is one of the most important methods in numerical analysis. As a result, it has found applications in a wide range of industries, including computer graphics [13,14], differential equations for intervals [15], neural network output optimization [16], and many more. Readers interested in these results and applications should read [17–27].

On the other hand, the generalized convexity of mappings is a powerful tool for solving a wide range of problems in applied analysis and nonlinear analysis, including for many concerns in mathematical physics. Several types of convexity are related to the Hermite-Hadamard inequality; see [28–43] for examples. Iscan [44] presented the concept of harmonic convexity and numerous associated Hermite-Hadamard-type inequalities in 2014. The authors of [45] published the first description of harmonic h-convex functions and several related Hermite-Hadamard inequalities in 2015. Various studies have been conducted in recent years to investigate the relationship between integral inequalities and interval-valued functions, generating numerous noteworthy conclusions. The Minkowski-type inequalities and the Beckenbach-type inequalities were established by Roman-Flores [46], Ostrowski-type inequalities were examined by Chalco-Cano [47] using the extended Hukuhara derivative, and Opial-type inequalities were first introduced by Costa [48]. Zhao et al. [49] recently built on this notion by incorporating interval-valued coordinated convex functions and generating related H-H type inequalities. This was also used to support the H-H- and Fejér-type inequalities for the pre-invex function [50] and the convex interval-valued function for n-polynomials [51]. The idea of interval-valued pre-invex functions was first proposed by Lai et al. [52], and recently this concept has been extended to include interval-valued coordinated pre-invex functions. Similarly, different authors have contributed in this filed to define various type of inequalities, e.g., [53–73] and the references therein.

Moreover, Khan et al. extended these concepts in the field of fuzzy environments such as H-H type inequalities for (h1, h2)-convex fuzzy-number valued mappings [74], (h1, h2)-convex fuzzy-number valued mappings [75], and up and down convex fuzzy-number valued mappings [76]. The H-H inequality was expanded to include interval h-convex functions [77], interval harmonic h-convex functions [78], interval (h1, h2)-convex functions [79], and interval harmonically (h1, h2)-convex functions [80] when interval analysis was combined. The authors of [81] used the definition of the h-Godunova-Levin function to account for this inequality. For more information, see [82–92] and the references therein. On the other hand, Chu et al. defined different classes of convex functions and proposed various types of inequalities. Moreover, they found some error bound inequalities, as well as discussing the validity of the results with the help of well-defined examples; see [93–104] and the references therein.

The following outline covers the remaining section of this manuscript. Several fundamental principles of fractional calculus are covered in Section 2. The new class of convex fuzzy number-valued mappings and related integral inequalities over fuzzy Aunnam integrals are introduced in Section 3, and rely on the techniques that are typically used in inequalities theory. We also provide illustrative examples of the key findings. In Section 4, we discuss conclusions and future planning.

2. Preliminaries

Let X_C be the space of all closed and bounded intervals of $\mathbb{R}$ and $\mathtt{A} \in X_C$ be defined by

$$\mathtt{A} = [\mathtt{A}_*, \mathtt{A}^*] = \{\varkappa \in \mathbb{R} | \mathtt{A}_* \le \varkappa \le \mathtt{A}^*\}, (\mathtt{A}_*, \mathtt{A}^* \in \mathbb{R}) \tag{3}$$

If $\mathtt{A}_* = \mathtt{A}^*$, then $\mathtt{A}$ is referred to as degenerate. In this article, all intervals will be non-degenerate intervals. If $\mathtt{A}_* \ge 0$, then $[\mathtt{A}_*, \mathtt{A}^*]$ is referred to as a positive interval. The set of all positive intervals is denoted by X_C^+ and defined as

$$X_C^+ = \{[\mathtt{A}_*, \mathtt{A}^*] : [\mathtt{A}_*, \mathtt{A}^*] \in X_C \text{ and } \mathtt{A}_* \ge 0\}. \tag{4}$$

Let $i \in \mathbb{R}$ and $i \cdot \mathtt{A}$ be defined by

$$i \cdot \mathtt{A} = \begin{cases} [i\mathtt{A}_*, i\mathtt{A}^*] \text{ if } i > 0, \\ \{0\} \text{ if } i = 0, \\ [i\mathtt{A}^*, i\mathtt{A}_*] \text{ if } i < 0. \end{cases} \tag{5}$$

Then, the Minkowski difference $\mathtt{B} - \mathtt{A}$, addition $\mathtt{A} + \mathtt{B}$ and $\mathtt{A} \times \mathtt{B}$ for $\mathtt{A}, \mathtt{B} \in X_C$ are defined by

$$[\mathtt{B}_*, \mathtt{B}^*] + [\mathtt{A}_*, \mathtt{A}^*] = [\mathtt{B}_* + \mathtt{A}_*, \mathtt{B}^* + \mathtt{A}^*], \tag{6}$$

$$[\mathtt{B}_*, \mathtt{B}^*] \times [\mathtt{A}_*, \mathtt{A}^*] = [\min\{\mathtt{B}_*\mathtt{A}_*, \mathtt{B}^*\mathtt{A}_*, \mathtt{B}_*\mathtt{A}^*, \mathtt{B}^*\mathtt{A}^*\}, \max\{\mathtt{B}_*\mathtt{A}_*, \mathtt{B}^*\mathtt{A}_*, \mathtt{B}_*\mathtt{A}^*, \mathtt{B}^*\mathtt{A}^*\}] \tag{7}$$

$$[\mathtt{B}_*, \mathtt{B}^*] - [\mathtt{A}_*, \mathtt{A}^*] = [\mathtt{B}_* - \mathtt{A}^*, \mathtt{B}^* - \mathtt{A}_*], \tag{8}$$

Remark 1 ([89]). *For given* $[\mathtt{B}_*, \mathtt{B}^*], [\mathtt{A}_*, \mathtt{A}^*] \in \mathbb{R}_I$, *we say that* $[\mathtt{B}_*, \mathtt{B}^*] \le_I [\mathtt{A}_*, \mathtt{A}^*]$ *if, and only if,* $\mathtt{B}_* \le \mathtt{A}_*, \mathtt{B}^* \le \mathtt{A}^*$ *; this is a partial interval or left and right order relation.*

If $[\mathtt{B}_*, \mathtt{B}^*], [\mathtt{A}_*, \mathtt{A}^*] \in \mathbb{R}_I$, *we say that* $[\mathtt{B}_*, \mathtt{B}^*] \subseteq_I [\mathtt{A}_*, \mathtt{A}^*]$ *if, and only if,* $\mathtt{A}_* \le \mathtt{B}_*, \mathtt{B}^* \le \mathtt{A}^*$ *; this is an inclusion interval or up and down (UD) order relation.*

For $[\mathtt{B}_*, \mathtt{B}^*], [\mathtt{A}_*, \mathtt{A}^*] \in X_C$, *the Hausdorff–Pompeiu distance between intervals* $[\mathtt{B}_*, \mathtt{B}^*]$ *, and* $[\mathtt{A}_*, \mathtt{A}^*]$ *is defined by*

$$d_H([\mathtt{B}_*, \mathtt{B}^*], [\mathtt{A}_*, \mathtt{A}^*]) = \max\{|\mathtt{B}_* - \mathtt{A}_*|, |\mathtt{B}^* - \mathtt{A}^*|\}. \tag{9}$$

It is a familiar fact that (X_C, d_H) *is a complete metric space [82,85,86].*

Definition 1 ([82,83]). *A fuzzy subset* $\mathtt{A}$ of $\mathbb{R}$ *is distinguished by a mapping* $f_\mathtt{A} : \mathbb{R} \to [0,1]$, *called the membership mapping of* $\mathtt{A}$ *. That is, a fuzzy subset* $\mathtt{A}$ *of* $\mathbb{R}$ *is a mapping* $f_\mathtt{A} : \mathbb{R} \to [0,1]$ *. So, for further study, we have chosen this notation. We appoint* $F(\mathbb{R})$ *to denote the set of all fuzzy subsets of* $\mathbb{R}$.

Let $\mathtt{A} \in F(\mathbb{R})$ *. Then,* $\mathtt{A}$ *is referred to as a fuzzy number or fuzzy interval if the following properties are satisfied by* $\mathtt{A}$ *:*

(1) $\mathtt{A}$ *should be normal if* $\varkappa \in \mathbb{R}^+$ *and* $\mathtt{A}(\varkappa) = 1$;
(2) $\mathtt{A}$ *should be upper semi-continuous on* $\mathbb{R}^+$ *if, for a given* $\varkappa \in \mathbb{R}^+$, $\varepsilon > 0$ *and* $\delta > 0$, *such that* $\mathtt{A}(\varkappa) - \mathtt{A}(y) < \varepsilon$ *for all* $y \in \mathbb{R}^+$ *with* $|\varkappa - y| < \delta$;
(3) $\mathtt{A}$ *should be fuzzy convex, that is* $\mathtt{A}((1 - \partial)\varkappa + \partial y) \ge min(\mathtt{A}(\varkappa), \mathtt{A}(y))$, *for all* $\varkappa, y \in \mathbb{R}^+$, *and* $\partial \in [0,1]$
(4) $\mathtt{A}$ *should be compactly supported, that is* $cl\{\varkappa \in \mathbb{R}^+ | \mathtt{A}(\varkappa) > 0\}$ *is compact.*

In the next work, we appoint $F(\mathbb{R})$ *to denote the set of all fuzzy numbers of* $\mathbb{R}^+$.

Definition 2 ([82,83]). *Given* $\mathtt{A} \in F(\mathbb{R})$ *, the level sets or cut sets are given by* $[\mathtt{A}]^i = \{\varkappa \in \mathbb{R}^+ | \mathtt{A}(\varkappa) > i\}$ *for all* $i \in [0,1]$ *and by* $[\mathtt{A}]^0 = \{\varkappa \in \mathbb{R}^+ | \mathtt{A}(\varkappa) > 0\}$ *. These sets are known as* i *-level sets or* i *-cut sets of* $\mathtt{A}$.

Proposition 1 ([87]). *Let* $\mathtt{A}, \mathtt{B} \in F(\mathbb{R})$ *. Then the relation "* $\leq_{\mathbb{F}}$ *" is given on* $F(\mathbb{R})$ *by*

$$\mathtt{A} \leq_{\mathbb{F}} \mathtt{B} \text{ when, and only when, } [\mathtt{A}]^{i} \leq_{I} [\mathtt{B}]^{i}, \text{ for every } i \in [0,1] \tag{10}$$

this is a partial-order or left and right relation.

Proposition 2. ([76]). *Let* $\mathtt{A}, \mathtt{B} \in F(\mathbb{R})$ *. Then the inclusion relation "* $\supseteq_{\mathbb{F}}$ *" is given on* $F(\mathbb{R})$ *by*

$$\mathtt{A} \supseteq_{\mathbb{F}} \mathtt{B} \tag{11}$$

When, and only when, $[\mathtt{A}]^{i} \supseteq_{I} [\mathtt{B}]^{i}$, *for every* $i \in [0,1]$.

This is an up and down fuzzy inclusion relation.

Remember the approaching notions, which are offered in the literature. If $\mathtt{A}, \mathtt{B} \in \mathrm{F}(\mathbb{R})$ and $\partial \in \mathbb{R}$, then, for every $\mathrm{i} \in [0,1]$, the arithmetical operations are defined by

$$[\mathtt{A} \oplus \mathtt{B}]^{i} = [\mathtt{A}]^{i} + [\mathtt{B}]^{i}, \tag{12}$$

$$[\mathtt{A} \otimes \mathtt{B}]^{i} = [\mathtt{A}]^{i} \times [\mathtt{B}]^{i}, \tag{13}$$

$$[\partial \odot \mathtt{A}]^{i} = \partial.[\mathtt{A}]^{i}. \tag{14}$$

These operations follow directly from the Equations (6), (7) and (4), respectively.

Theorem 1 ([82]). *The space* $F(\mathbb{R})$ *dealing with a supremum metric, i.e., for* $\mathtt{A}, \mathtt{B} \in F(\mathbb{R})$

$$\mathrm{d}_{\infty}(\mathtt{A}, \mathtt{B}) = \sup_{0 \leq \mathrm{i} \leq 1} \mathrm{d}_{\mathrm{H}}\left([\mathtt{A}]^{\mathrm{i}}, [\mathtt{B}]^{\mathrm{i}}\right), \tag{15}$$

is a complete metric space, where H denotes the well-known Hausdorff metric on space of intervals.

Now, we define and discuss some properties of Riemann integral operators for interval- and fuzzy-number valued mappings.

Theorem 2 ([82,84]). *If* $Y : [\varsigma, \mathfrak{v}] \subset \mathbb{R} \to X_C$ *is an interval-valued mapping* (IVM) *satisfying that* $Y(\varkappa) = [Y_*(\varkappa), Y^*(\varkappa)]$ *, then* Y *is Aumann integrable (IA-integrable) over* $[\varsigma, \mathfrak{v}]$ *when, and only when,* $Y_*(\varkappa)$ *and* $Y^*(\varkappa)$ *are both integrable over* $[\varsigma, \mathfrak{v}]$ *such that*

$$(\mathrm{IA})\int_{\varsigma}^{\mathfrak{v}} \mathrm{Y}(\varkappa)\mathrm{d}\varkappa = \left[\int_{\varsigma}^{\mathfrak{v}} \mathrm{Y}_*(\varkappa)\mathrm{d}\varkappa, \int_{\varsigma}^{\mathfrak{v}} \mathrm{Y}^*(\varkappa)\mathrm{d}\varkappa\right] \tag{16}$$

Definition 3 ([88]). *Let* $Y : I \subset \mathbb{R} \to F(\mathbb{R})$ *be referred to as fuzzy-number valued mapping* ($FNVM$) *. Then, parametrized form is given by* $Y_i : [\varsigma, \mathfrak{v}] \subset \mathbb{R} \to \mathcal{K}_C$ *and defined as* $Y_i(\varkappa) = [Y_*(\varkappa, i), Y^*(\varkappa, i)]$ *for every* $\varkappa \in I$ *and for every* $i \in [0,1]$ *. Here, for every* $i \in [0,1]$, *the end point real valued mappings* $Y_*(\bullet, i), Y^*(\bullet, i) : I \to \mathbb{R}$ *are called lower and upper mappings of* Y.

Definition 4 ([88]). *Let* $\widetilde{Y} : I \subset \mathbb{R} \to F(\mathbb{R})$ *be a* $FNVM$ *. Then* $\widetilde{Y}(\varkappa)$ *is referred to as continuous at* $\varkappa \in I$, *if for every* $i \in [0,1]$, $Y_i(\varkappa)$ *is continuous when, and only when, both end point mappings* $Y_*(\varkappa, i)$ *and* $Y^*(\varkappa, i)$ *are continuous at* $\varkappa \in I$.

Definition 5 ([84]). *Let* $\widetilde{Y} : [\varsigma, \mathfrak{v}] \subset \mathbb{R} \to F(\mathbb{R})$ *be* $FNVM$ *. The fuzzy Aumann integral* (($\mathcal{FA}$) *-integral) of* $\widetilde{Y}$ *over* $[\varsigma, \mathfrak{v}]$, *denoted by* $(\mathcal{FA})\int_{\varsigma}^{\mathfrak{v}} \widetilde{Y}(\varkappa)d\varkappa$ *, is defined level-wise by*

$$\left[(\mathcal{FA})\int_{\varsigma}^{\vartheta}\widetilde{\mathrm{Y}}(\varkappa)\mathrm{d}\varkappa\right]^{i} \begin{aligned} &= (\mathrm{IA})\int_{\varsigma}^{\vartheta}\mathrm{Y}_{\mathrm{i}}(\varkappa)\mathrm{d}\varkappa \\ &= \left\{\int_{\varsigma}^{\vartheta}\mathrm{Y}(\varkappa,\mathrm{i})\mathrm{d}\varkappa : \mathrm{Y}(\varkappa,\mathrm{i}) \in \mathrm{S}(\mathrm{Y}_{\mathrm{i}})\right\}, \end{aligned} \tag{17}$$

where $S(Y_i) = \{Y(.,i) \to \mathbb{R} : Y(.,i)$ *is integrable and* $Y(\varkappa,i) \in Y_i(\varkappa)\}$, *for every* $i \in [0,1]$. $\widetilde{Y}$ *is* $(\mathcal{FA})$ *-integrable over* $[\varsigma,\vartheta]$ *if* $(\mathcal{FA})\int_{\varsigma}^{\vartheta}\widetilde{Y}(\varkappa)d\varkappa \in F(\mathbb{R})$.

Theorem 3 ([87])**.** *Let* $\mathrm{Y} : [\varsigma,\vartheta] \subset \mathbb{R} \to \mathrm{F}(\mathbb{R})$ *be a* FNVM, *whose parametrized form is given by* $\mathrm{Y}_{\mathrm{i}} : [\varsigma,\vartheta] \subset \mathbb{R} \to \mathcal{K}_C$ *and defined as* $\mathrm{Y}_{\mathrm{i}}(\varkappa) = [\mathrm{Y}_*(\varkappa,\mathrm{i}), \mathrm{Y}^*(\varkappa,\mathrm{i})]$ *for every* $\varkappa \in [\varsigma,\vartheta]$ *and for every* $\mathrm{i} \in [0,1]$. Then Y is $(\mathcal{FA})$*-integrable over* $[\varsigma,\vartheta]$ *when, and only when,* $\mathrm{Y}_*(\varkappa,\mathrm{i})$ and $\mathrm{Y}^*(\varkappa,\mathrm{i})$ *are both integrable over* $[\varsigma,\vartheta]$. *Moreover, if* Y is $(\mathcal{FA})$*-integrable over* $[\varsigma,\vartheta]$, *then*

$$\left[(\mathcal{FA})\int_{\varsigma}^{\vartheta}\mathrm{Y}(\varkappa)\mathrm{d}\varkappa\right]^{\mathrm{i}} \begin{aligned} &= \left[\int_{\varsigma}^{\vartheta}\mathrm{Y}_*(\varkappa,\mathrm{i})\mathrm{d}\varkappa, \int_{\varsigma}^{\vartheta}\mathrm{Y}^*(\varkappa,\mathrm{i})\mathrm{d}\varkappa\right] \\ &= (\mathrm{I}\mathcal{A})\int_{\varsigma}^{\vartheta}\mathrm{Y}_{\mathrm{i}}(\varkappa)\mathrm{d}\varkappa \end{aligned} \tag{18}$$

for every $i \in [0,1]$.

3. Fuzzy Aumann Integral Inequalities for Fuzzy-Number Valued UD-$(\mathcal{X}_1, \mathcal{X}_2)$-Convexity

In this section, we put forward some definitions of UD-$(\mathcal{X}_1, \mathcal{X}_2)$-convex FNVM s and investigate some basic properties using fuzzy Aumann integrals.

Definition 6. *Let* K *be a convex set and* $\mathcal{X}_1, \mathcal{X}_2 : [0,1] \subseteq \mathrm{K} \to \mathbb{R}^+$ *such that* $\mathcal{X}_1, \mathcal{X}_2 \not\equiv 0$. *Then* FNVM $\mathrm{Y} : \mathrm{K} \to \mathrm{F}(\mathbb{R})$ *is said to be* UD-$(\mathcal{X}_1, \mathcal{X}_2)$*-convex* FNVM on K *if*

$$\widetilde{\mathrm{Y}}(\mu\varkappa + (1-\mu)y) \supseteq_{\mathbb{F}} \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu) \odot \widetilde{\mathrm{Y}}(\varkappa) \oplus \mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu) \odot \widetilde{\mathrm{Y}}(y), \tag{19}$$

for all $\varkappa, y \in \mathrm{K}, \mu \in [0,1]$.Y is UD-$(\mathcal{X}_1, \mathcal{X}_2)$*-concave on* K *if inequality* (19) *is reversed.* $\widetilde{\mathrm{Y}}$ *is known as* UD-$(\mathcal{X}_1, \mathcal{X}_2)$*-affine mapping on* K *if*

$$\mathrm{Y}(\mu\varkappa + (1-\mu)y) = \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\odot\mathrm{Y}(\varkappa) \oplus \mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu) \odot \mathrm{Y}(y), \tag{20}$$

for all $\varkappa, y \in \mathrm{K}, \mu \in [0,1]$.

Remark 2. *If* Y *is* UD-$(\mathcal{X}_1, \mathcal{X}_2)$*-convex* FNVM, *then* gY *is also* UD-$(\mathcal{X}_1, \mathcal{X}_2)$*-convex* FNVM for $\mathrm{g} \geq 0$.
If Y and $\mathcal{T}$ *are both* UD-$(\mathcal{X}_1, \mathcal{X}_2)$*-convex* FNVM s, *then* $\max(\mathrm{Y}(\varkappa), \mathcal{T}(\varkappa))$ *is also a* UD-$(\mathcal{X}_1, \mathcal{X}_2)$*-convex* FNVM.
If $\mathcal{X}_2(\mu) \equiv 1$, *then UD -*$(\mathcal{X}_1, \mathcal{X}_2)$ *-convex FNVM becomes UD -*$\mathcal{X}_1$ *-convex FNVM* [81]*:*

$$\mathrm{Y}(\mu\varkappa + (1-\mu)y) \supseteq_{\mathbb{F}} \mathcal{X}_1(\mu) \odot \mathrm{Y}(\varkappa) \oplus \mathcal{X}_1(1-\mu) \odot \mathrm{Y}(y), \forall \varkappa, y \in \mathrm{K}, \mu \in [0,1]. \tag{21}$$

If $\mathcal{X}_1(\mu) = \mu^s$, $\mathcal{X}_2(\mu) \equiv 1$, *then UD -*$(\mathcal{X}_1, \mathcal{X}_2)$ *-convex FNVM becomes UD -s -convex FNVM* [81]*:*

$$\mathrm{Y}(\mu\varkappa + (1-\mu)y) \supseteq_{\mathbb{F}} \mu^{s} \odot \mathrm{Y}(\varkappa) \oplus (1-\mu)^{s} \odot \mathrm{Y}(y), \forall \varkappa, y \in \mathrm{K}, \mu \in [0,1]. \tag{22}$$

If $\mathcal{X}_1(\mu) = \mu$, $\mathcal{X}_2(\mu) \equiv 1$, *then UD -*$(\mathcal{X}_1, \mathcal{X}_2)$ *-convex FNVM becomes UD -convex FNVM* [85]*:*

$$\mathrm{Y}(\mu\varkappa + (1-\mu)y) \supseteq_{\mathbb{F}} \mu \odot \mathrm{Y}(\varkappa) \oplus (1-\mu) \odot \mathrm{Y}(y), \forall \varkappa, y \in \mathrm{K}, \mu \in [0,1]. \tag{23}$$

If $\mathcal{X}_1(\mu) = \mathcal{X}_2(\mu) \equiv 1$, *then UD -*$(\mathcal{X}_1, \mathcal{X}_2)$ *-convex FNVM becomes UD -P -convex FNV* [81]*:*

$$\mathrm{Y}(\mu\varkappa + (1-\mu)y) \supseteq_{\mathbb{F}} \mathrm{Y}(\varkappa) \oplus \mathrm{Y}(y), \forall \varkappa, y \in \mathrm{K}, \mu \in [0,1]. \tag{24}$$

Theorem 4. *Let* K *be a convex set,* $\mathcal{X}_1, \mathcal{X}_2 : [0,1] \subseteq \mathrm{K} \to \mathbb{R}^+$ *such that* $\mathcal{X}_1, \mathcal{X}_2 \not\equiv 0$, *and let* $\mathrm{Y} : \mathrm{K} \to \mathrm{F}(\mathbb{R})$ *be a* FNVM *whose parametrized form is given by* $\mathrm{Y}_{\mathrm{i}} : [\varsigma, \mathfrak{v}] \subset \mathbb{R} \to \mathcal{K}_C$ *and defined as*

$$\mathrm{Y}_{\mathrm{i}}(\varkappa) = [\mathrm{Y}_*(\varkappa, \mathrm{i}), \mathrm{Y}^*(\varkappa, \mathrm{i})], \forall \varkappa \in \mathrm{K}. \tag{25}$$

for all $\varkappa \in [\varsigma, \mathfrak{v}]$ *and for all* $\mathrm{i} \in [0,1]$. *Then* Y is UD-$(\mathcal{X}_1, \mathcal{X}_2)$*-convex on* K, *if, and only if, for all* $\mathrm{i} \in [0,1]$,$\mathrm{Y}_*(\varkappa, \mathrm{i})$ *and* $\mathrm{Y}^*(\varkappa, \mathrm{i})$ *are* $(\mathcal{X}_1, \mathcal{X}_2)$*-convex and concave mappings, respectively.*

Proof. Assume that for each $\mathrm{i} \in [0,1]$,$\mathrm{Y}_*(\varkappa, \mathrm{i})$ and $\mathrm{Y}^*(\varkappa, \mathrm{i})$ are $(\mathcal{X}_1, \mathcal{X}_2)$-convex mappings on K. Then from (11), we have

$$\mathrm{Y}_*(\mu\varkappa + (1-\mu)y, \mathrm{i}) \leq \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\mathrm{Y}_*(\varkappa, \mathrm{i}) + \mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\mathrm{Y}_*(y, \mathrm{i})$$

$$\forall \varkappa, y \in \mathrm{K}, \mu \in [0,1],$$

and

$$\mathrm{Y}^*(\mu\varkappa + (1-\mu)y, \mathrm{i}) \geq \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\mathrm{Y}^*(\varkappa, \mathrm{i}) + \mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\mathrm{Y}^*(y, \mathrm{i}),$$

$$\forall \varkappa, y \in \mathrm{K}, \mu \in [0,1].$$

Then by (25), (4) and (5), we obtain

$$\begin{aligned}\mathrm{Y}_{\mathrm{i}}(\mu\varkappa + (1-\mu)y) &= [\mathrm{Y}_*(\mu\varkappa + (1-\mu)y, \mathrm{i}), \mathrm{Y}^*(\mu\varkappa + (1-\mu)y, \mathrm{i})],\\ &\supseteq_{\mathrm{I}} \left[\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\mathrm{Y}_*(\varkappa, \mathrm{i}), \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\mathrm{Y}^*(\varkappa, \mathrm{i})\right]\\ &+ \left[\mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\mathrm{Y}_*(y, \mathrm{i}), \mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\mathrm{Y}^*(y, \mathrm{i})\right],\end{aligned}$$

that is

$$\mathrm{Y}(\mu\varkappa + (1-\mu)y) \supseteq_{\mathbb{F}} \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu) \odot \mathrm{Y}(\varkappa) \oplus \mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu) \odot \mathrm{Y}(y),$$

$\forall \varkappa, y \in \mathrm{K}, \mu \in [0,1]$. Hence, Y is UD-$(\mathcal{X}_1, \mathcal{X}_2)$-convex FNVM on K.

Conversely, let Y be UD-$(\mathcal{X}_1, \mathcal{X}_2)$-convex FNVM on K. Then for all $\varkappa, y \in \mathrm{K}$ and $\mu \in [0,1]$, we have $\mathrm{Y}(\mu\varkappa + (1-\mu)y) \supseteq_{\mathbb{F}} \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu) \odot \mathrm{Y}(\varkappa) \oplus \mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu) \odot \mathrm{Y}(y)$. Therefore, from (25), we have

$$\mathrm{Y}_{\mathrm{i}}(\mu\varkappa + (1-\mu)y) = [\mathrm{Y}_*(\mu\varkappa + (1-\mu)y, \mathrm{i}), \mathrm{Y}^*(\mu\varkappa + (1-\mu)y, \mathrm{i})].$$

Again, from (25), (4) and (5), we obtain

$$\begin{aligned}&\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\mathrm{Y}_{\mathrm{i}}(\varkappa) + \mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\mathrm{Y}_{\mathrm{i}}(\varkappa)\\ &= \left[\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\mathrm{Y}_*(\varkappa, \mathrm{i}), \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\mathrm{Y}^*(\varkappa, \mathrm{i})\right]\\ &+ \left[\mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\mathrm{Y}_*(y, \mathrm{i}), \mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\mathrm{Y}^*(y, \mathrm{i})\right],\end{aligned}$$

for all $\varkappa, y \in \mathrm{K}$ and $\mu \in [0,1]$. Then by UD-$(\mathcal{X}_1, \mathcal{X}_2)$-convexity of Y, we have for all $\varkappa, y \in \mathrm{K}$ and $\mu \in [0,1]$ such that

$$\mathrm{Y}_*(\mu\varkappa + (1-\mu)y, \mathrm{i}) \leq \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\mathrm{Y}_*(\varkappa, \mathrm{i}) + \mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\mathrm{Y}_*(y, \mathrm{i}),$$

and

$$\mathrm{Y}^*(\mu\varkappa + (1-\mu)y, \mathrm{i}) \geq \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\mathrm{Y}^*(\varkappa, \mathrm{i}) + \mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\mathrm{Y}^*(y, \mathrm{i}),$$

for each $\mathrm{i} \in [0,1]$. Hence, the result follows. □

Example 1. *We consider* $\mathcal{X}_1(\mu) = \mu, \mathcal{X}_2(\mu) \equiv 1$, *for* $\mu \in [0,1]$ *and the FNVMs* $Y : [0,1] \to F(\mathbb{R})$ *defined by*

$$\mathrm{Y}(\varkappa)(\sigma) = \begin{cases} \frac{\sigma}{2\varkappa^2} & \sigma \in [0, 2\varkappa] \\ \frac{4\varkappa^2 - \sigma}{2\varkappa^2} & \sigma \in (2\varkappa, 4\varkappa] \\ 0 & otherwise, \end{cases} \tag{26}$$

Then, for each $i \in [0,1]$, *we have* $Y_i(\varkappa) = [2i\varkappa, (4-2i)\varkappa]$. *Since end point functions* $Y_*(\varkappa, i)$, $Y^*(\varkappa, i)$ *are* $(\mathcal{X}_1, \mathcal{X}_2)$ *-convex functions for each* $i \in [0,1]$. *Hence* $Y(\varkappa)$ *is UD* -$(\mathcal{X}_1, \mathcal{X}_2)$ *-convex FNVM.*

Definition 7. *Let K be convex set,* $\mathcal{X}_1, \mathcal{X}_2 : [0,1] \subseteq K \to \mathbb{R}^+$ *such that* $\mathcal{X}_1, \mathcal{X}_2 \not\equiv 0$ *and let* $Y : K \to F(\mathbb{R})$ *be a FNVM whose parametrized form is given by* $Y_i : [\varsigma, \mathfrak{v}] \subset \mathbb{R} \to \mathcal{K}_C$ *and defined as*

$$\mathrm{Y_i}(\varkappa) = [\mathrm{Y}_*(\varkappa, \mathrm{i}), \mathrm{Y}^*(\varkappa, \mathrm{i})], \forall \varkappa \in \mathrm{K},$$

for all $\varkappa \in [\varsigma, \mathfrak{v}]$ *and for all* $i \in [0,1]$. *Then Y is lower UD* -$(\mathcal{X}_1, \mathcal{X}_2)$ *-convex (concave) on K, if, and only if, for all* $i \in [0,1]$,

$$\mathrm{Y}_*(\mu\varkappa + (1-\mu)y, \mathrm{i}) \leq (\geq) \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\mathrm{Y}_*(\varkappa, \mathrm{i}) + \mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\mathrm{Y}_*(y, \mathrm{i}),$$

and

$$\mathrm{Y}^*(\mu\varkappa + (1-\mu)y, \mathrm{i}) = \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\mathrm{Y}^*(\varkappa, \mathrm{i}) + \mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\mathrm{Y}^*(y, \mathrm{i}).$$

Definition 8. *Let K be convex set,* $\mathcal{X}_1, \mathcal{X}_2 : [0,1] \subseteq K \to \mathbb{R}^+$ *such that* $\mathcal{X}_1, \mathcal{X}_2 \not\equiv 0$ *and let* $Y : K \to F(\mathbb{R})$ *be a FNVM whose parametrized form is given by* $Y_i : [\varsigma, \mathfrak{v}] \subset \mathbb{R} \to \mathcal{K}_C$ *and defined as*

$$\mathrm{Y_i}(\varkappa) = [\mathrm{Y}_*(\varkappa, \mathrm{i}), \mathrm{Y}^*(\varkappa, \mathrm{i})], \forall \varkappa \in \mathrm{K}.$$

for all $\varkappa \in [\varsigma, \mathfrak{v}]$ *and for all* $i \in [0,1]$. *Then Y is upper UD* -$(\mathcal{X}_1, \mathcal{X}_2)$ *-convex (concave) on K, if, and only if, for all* $i \in [0,1]$,

$$\mathrm{Y}_*(\mu\varkappa + (1-\mu)y, \mathrm{i}) = \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\mathrm{Y}_*(\varkappa, \mathrm{i}) + \mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\mathrm{Y}_*(y, \mathrm{i}),$$

and

$$\mathrm{Y}^*(\mu\varkappa + (1-\mu)y, \mathrm{i}) \leq (\geq) \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\mathrm{Y}^*(\varkappa, \mathrm{i}) + \mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\mathrm{Y}^*(y, \mathrm{i}).$$

Now, we present some new estimates of fuzzy Annum integral inequalities over UD -$(\mathcal{X}_1, \mathcal{X}_2)$ *-convex FNVM with* $\mathcal{X}_1, \mathcal{X}_2 : [0,1] \to \mathbb{R}^+$.

Remark 3. *Now we obtain some special results, applying mild restrictions over the definition of UD* -$(\mathcal{X}_1, \mathcal{X}_2)$ *-convex FNVMs, such that* [74]*:*

If $\mathcal{X}_2(\mu) \equiv 1$, *then lower UD* -$(\mathcal{X}_1, \mathcal{X}_2)$ *-convex FNVM becomes lower* $\mathcal{X}_1$ *-convex FNVM, that is*

$$\mathrm{Y}(\mu\varkappa + (1-\mu)y) \leq_{\mathbb{F}} \mathcal{X}_1(\mu) \odot \mathrm{Y}(\varkappa) \oplus \mathcal{X}_1(1-\mu) \odot \mathrm{Y}(y), \forall \varkappa, y \in \mathrm{K}, \mu \in [0,1]. \tag{27}$$

If $\mathcal{X}_1(\mu) = \mu^s$, $\mathcal{X}_2(\mu) \equiv 1$, *then lower UD* -$(\mathcal{X}_1, \mathcal{X}_2)$ *-convex FNVM becomes lower s -convex FNVM, that is*

$$\mathrm{Y}(\mu\varkappa + (1-\mu)y) \leq_{\mathbb{F}} \mu^s \odot \mathrm{Y}(\varkappa) \oplus (1-\mu)^s \odot \mathrm{Y}(y), \forall \varkappa, y \in \mathrm{K}, \mu \in [0,1]. \tag{28}$$

If $\mathcal{X}_1(\mu) = \mu, \mathcal{X}_2(\mu) \equiv 1$, then lower UD -$(\mathcal{X}_1, \mathcal{X}_2)$ -convex FNVM becomes lower convex FNVM , that is

$$\Upsilon(\mu\varkappa + (1-\mu)y) \leq_{\mathbb{F}} \mu \odot \Upsilon(\varkappa) \oplus (1-\mu) \odot \Upsilon(y), \forall \varkappa, y \in \mathrm{K}, \mu \in [0,1]. \tag{29}$$

If $\mathcal{X}_1(\mu) = \mathcal{X}_2(\mu) \equiv 1$, then lower UD -$(\mathcal{X}_1, \mathcal{X}_2)$ -convex FNVM becomes lower P -convex FNVM , that is

$$\Upsilon(\mu\varkappa + (1-\mu)y) \leq_{\mathbb{F}} \Upsilon(\varkappa) \oplus \Upsilon(y), \forall \varkappa, y \in \mathrm{K}, \mu \in [0,1]. \tag{30}$$

Here is our first main result, which depends upon the fuzzy Auman integral and UD -$(\mathcal{X}_1, \mathcal{X}_2)$ -convex FNVM with $\mathcal{X}_1, \mathcal{X}_2 : [0,1] \to \mathbb{R}^+$.

Theorem 5. *Let $\Upsilon : [\varsigma, \mathfrak{v}] \to F(\mathbb{R})$ be a UD -$(\mathcal{X}_1, \mathcal{X}_2)$ -convex FNVM with $\mathcal{X}_1, \mathcal{X}_2 : [0,1] \to \mathbb{R}^+$ and $\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right) \neq 0$, whose parametrized form is given by $\Upsilon_i : [\varsigma, \mathfrak{v}] \subset \mathbb{R} \to \mathcal{K}_C$ and defined as $\Upsilon_i(\varkappa) = [\Upsilon_*(\varkappa, i), \Upsilon^*(\varkappa, i)]$ for all $\varkappa \in [\varsigma, \mathfrak{v}]$ and for all $i \in [0,1]$. If $\Upsilon \in \mathcal{IA}_{([\varsigma,\mathfrak{v}],i)}$, then*

$$\begin{aligned} \frac{1}{2\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)} \odot \Upsilon\left(\frac{\varsigma+\mathfrak{v}}{2}\right) &\supseteq_{\mathbb{F}} \frac{1}{\mathfrak{v}-\varsigma} \odot (\mathcal{FA})\int_{\varsigma}^{\mathfrak{v}} \Upsilon(\varkappa)d\varkappa \\ &\supseteq_{\mathbb{F}} [\Upsilon(\varsigma) \oplus \Upsilon(\mathfrak{v})] \odot \int_0^1 \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu. \end{aligned} \tag{31}$$

Proof. Let $\Upsilon : [\varsigma, \mathfrak{v}] \to \mathrm{F}(\mathbb{R})$ be a UD-$(\mathcal{X}_1, \mathcal{X}_2)$-convex FNVM. Then, by hypothesis, we have

$$\frac{1}{\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)} \odot \Upsilon\left(\frac{\varsigma+\mathfrak{v}}{2}\right) \supseteq_{\mathbb{F}} \Upsilon(\mu\varsigma + (1-\mu)\mathfrak{v}) \oplus \Upsilon((1-\mu)\varsigma + \mu\mathfrak{v}).$$

Therefore, for every $i \in [0,1]$, we have

$$\begin{aligned} \frac{1}{\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)}\Upsilon_*\left(\frac{\varsigma+\mathfrak{v}}{2}, i\right) &\leq \Upsilon_*(\mu\varsigma + (1-\mu)\mathfrak{v}, i) + \Upsilon_*((1-\mu)\varsigma + \mu\mathfrak{v}, i), \\ \frac{1}{\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)}\Upsilon^*\left(\frac{\varsigma+\mathfrak{v}}{2}, i\right) &\geq \Upsilon^*(\mu\varsigma + (1-\mu)\mathfrak{v}, i) + \Upsilon^*((1-\mu)\varsigma + \mu\mathfrak{v}, i). \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)}\int_0^1 \Upsilon_*\left(\frac{\varsigma+\mathfrak{v}}{2}, i\right)d\mu &\leq \int_0^1 \Upsilon_*(\mu\varsigma + (1-\mu)\mathfrak{v}, i)d\mu + \int_0^1 \Upsilon_*((1-\mu)\varsigma + \mu\mathfrak{v}, i)d\mu, \\ \frac{1}{\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)}\int_0^1 \Upsilon^*\left(\frac{\varsigma+\mathfrak{v}}{2}, i\right)d\mu &\geq \int_0^1 \Upsilon^*(\mu\varsigma + (1-\mu)\mathfrak{v}, i)d\mu + \int_0^1 \Upsilon^*((1-\mu)\varsigma + \mu\mathfrak{v}, i)d\mu. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)}\Upsilon_*\left(\frac{\varsigma+\mathfrak{v}}{2}, i\right) &\leq \frac{2}{\mathfrak{v}-\varsigma}\int_{\varsigma}^{\mathfrak{v}} \Upsilon_*(\varkappa, i)d\varkappa, \\ \frac{1}{\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)}\Upsilon^*\left(\frac{\varsigma+\mathfrak{v}}{2}, i\right) &\geq \frac{2}{\mathfrak{v}-\varsigma}\int_{\varsigma}^{\mathfrak{v}} \Upsilon^*(\varkappa, i)d\varkappa. \end{aligned}$$

That is

$$\frac{1}{\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)}\left[\Upsilon_*\left(\frac{\varsigma+\mathfrak{v}}{2}, i\right), \Upsilon^*\left(\frac{\varsigma+\mathfrak{v}}{2}, i\right)\right] \supseteq_{I} \frac{2}{\mathfrak{v}-\varsigma}\left[\int_{\varsigma}^{\mathfrak{v}} \Upsilon_*(\varkappa, i)d\varkappa, \int_{\varsigma}^{\mathfrak{v}} \Upsilon^*(\varkappa, i)d\varkappa\right].$$

Thus,

$$\frac{1}{2\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)} \odot \Upsilon\left(\frac{\varsigma+\mathfrak{v}}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\mathfrak{v}-\varsigma} \odot (\mathcal{FA})\int_{\varsigma}^{\mathfrak{v}} \Upsilon(\varkappa)d\varkappa. \tag{32}$$

In a similar way as the above, we have

$$\frac{1}{\mathfrak{v}-\varsigma} \odot (\mathcal{FA})\int_{\varsigma}^{\mathfrak{v}} \Upsilon(\varkappa)d\varkappa \supseteq_{\mathbb{F}} [\Upsilon(\varsigma) \oplus \Upsilon(\mathfrak{v})] \odot \int_0^1 \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu. \tag{33}$$

Combining (32) and (33), we have

$$\frac{1}{2\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)} \odot \Upsilon\left(\frac{\varsigma+\mathfrak{v}}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\mathfrak{v}-\varsigma} \odot (\mathcal{FA})\int_{\varsigma}^{\mathfrak{v}} \Upsilon(\varkappa)d\varkappa \supseteq_{\mathbb{F}} [\Upsilon(\varsigma) \oplus \Upsilon(\mathfrak{v})] \odot \int_0^1 \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu$$

hence, the result follows. □

Remark 4. *If $\mathcal{X}_2(\mu) \equiv 1$, then Theorem 5 reduces to the result for UD -$\mathcal{X}_1$ -convex FNV* [81]*:*

$$\frac{1}{2\mathcal{X}_1\left(\frac{1}{2}\right)} \odot \Upsilon\left(\frac{\varsigma+\mathfrak{v}}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\mathfrak{v}-\varsigma} \odot (\mathcal{FA})\int_{\varsigma}^{\mathfrak{v}} \Upsilon(\varkappa)d\varkappa \supseteq_{\mathbb{F}} [\Upsilon(\varsigma) \oplus \Upsilon(\mathfrak{v})] \odot \int_0^1 \mathcal{X}_1(\mu)d\mu. \tag{34}$$

If $\mathcal{X}_1(\mu) = \mu^s$ and $\mathcal{X}_2(\mu) \equiv 1$, then Theorem 5 reduces to the result for UD -s -convex FNVM :

$$2^{s-1} \odot \Upsilon\left(\frac{\varsigma+\mathfrak{v}}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\mathfrak{v}-\varsigma} \odot (\mathcal{FA})\int_{\varsigma}^{\mathfrak{v}} \Upsilon(\varkappa)d\varkappa \supseteq_{\mathbb{F}} \frac{1}{s+1} \odot [\Upsilon(\varsigma) \oplus \Upsilon(\mathfrak{v})]. \tag{35}$$

If $\mathcal{X}_1(\mu) = \mu$ and $\mathcal{X}_2(\mu) \equiv 1$, then Theorem 5 reduces to the result for UD -convex FNVM [76]*:*

$$\Upsilon\left(\frac{\varsigma+\mathfrak{v}}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\mathfrak{v}-\varsigma} \odot (\mathcal{FA})\int_{\varsigma}^{\mathfrak{v}} \Upsilon(\varkappa)d\varkappa \supseteq_{\mathbb{F}} \frac{\Upsilon(\varsigma) \oplus \Upsilon(\mathfrak{v})}{2}. \tag{36}$$

If $\mathcal{X}_1(\mu) = \mathcal{X}_2(\mu) \equiv 1$, then Theorem 5 reduces to the result for UD -P -convex FNV M [81]*:*

$$\frac{1}{2} \odot \Upsilon\left(\frac{\varsigma+\mathfrak{v}}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\mathfrak{v}-\varsigma} \odot (\mathcal{FA})\int_{\varsigma}^{\mathfrak{v}} \Upsilon(\varkappa)d\varkappa \supseteq_{\mathbb{F}} \Upsilon(\varsigma) \oplus \Upsilon(\mathfrak{v}). \tag{37}$$

If $\Upsilon_*(\varsigma, i) = \Upsilon^*(\mathfrak{v}, i)$ then Theorem 5 reduces to the result for the classical $(\mathcal{X}_1, \mathcal{X}_2)$ -convex function,

$$\frac{1}{2\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)} \Upsilon\left(\frac{\varsigma+\mathfrak{v}}{2}\right) \leq \frac{1}{\mathfrak{v}-\varsigma}\int_{\varsigma}^{\mathfrak{v}} \Upsilon(\varkappa)d\varkappa \leq [\Upsilon(\varsigma) + \Upsilon(\mathfrak{v})]\int_0^1 \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu. \tag{38}$$

Example 2: *We consider the fuzzy-number valued mapping $\Upsilon : [\varsigma, \mathfrak{v}] = [2, 3] \to F(\mathbb{R})$ defined by*

$$\Upsilon(\varkappa)(\theta) = \begin{cases} \frac{\theta-2+\varkappa^{\frac{1}{2}}}{1-\varkappa^{\frac{1}{2}}} \theta \in \left[2-\varkappa^{\frac{1}{2}}, 3\right] \\ \frac{2+\varkappa^{\frac{1}{2}}-\theta}{\varkappa^{\frac{1}{2}}-1} \theta \in \left(3, 2+\varkappa^{\frac{1}{2}}\right] \\ 0 \; otherwise. \end{cases} \tag{39}$$

Then, for each $i \in [0,1]$, we have $\Upsilon_i(\varkappa) = \left[(1-i)\left(2-\varkappa^{\frac{1}{2}}\right)+3i, (1-i)\left(2+\varkappa^{\frac{1}{2}}\right)+3i\right]$. Since left and right end point mappings $\Upsilon_*(\varkappa, i) = (1-i)\left(2-\varkappa^{\frac{1}{2}}\right)+3i, \Upsilon^*(\varkappa, i) = (1-i)\left(2+\varkappa^{\frac{1}{2}}\right)+3i$, are $(\mathcal{X}_1, \mathcal{X}_2)$ -convex and concave mappings with $\mathcal{X}_1(\mu) = \mu, \mathcal{X}_2(\mu) \equiv 1$, for $\mu \in [0,1]$ and for each $i \in [0,1]$, then $\Upsilon(\varkappa)$ is UD -$(\mathcal{X}_1, \mathcal{X}_2)$ -convex FNVM with $\mathcal{X}_1(\mu) = \mu, \mathcal{X}_2(\mu) \equiv 1$, for $\mu \in [0,1]$. We clearly see that $\widetilde{\Upsilon} \in L([\varsigma, \mathfrak{v}], F(\mathbb{R}))$. Now, computing the following

$$\frac{1}{2\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)}\mathrm{Y}_*\left(\frac{\varsigma+\mathfrak{v}}{2},\mathrm{i}\right) \le \frac{1}{\mathfrak{v}-\varsigma}\int_\varsigma^{\mathfrak{v}}\mathrm{Y}_*(\varkappa,\mathrm{i})d\varkappa \le [\mathrm{Y}_*(\varsigma,\mathrm{i})+\mathrm{Y}_*(\mathfrak{v},\mathrm{i})]\int_0^1\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu.$$

$$\frac{1}{2\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)}\mathrm{Y}_*\left(\frac{\varsigma+\mathfrak{v}}{2},\mathrm{i}\right) = \mathrm{Y}_*\left(\frac{5}{2},\mathrm{i}\right) = \frac{4-\sqrt{10}}{2}(1-\mathrm{i})+3\mathrm{i},$$

$$\frac{1}{\mathfrak{v}-\varsigma}\int_\varsigma^{\mathfrak{v}}\mathrm{Y}_*(\varkappa,\mathrm{i})d\varkappa = \int_2^3\left((1-\mathrm{i})\left(2-\varkappa^{\frac{1}{2}}\right)+3\mathrm{i}\right)d\varkappa = \frac{\left(6+4\sqrt{2}-6\sqrt{3}\right)}{3}(1-\mathrm{i})+3\mathrm{i},$$

$$[\mathrm{Y}_*(\varsigma,\mathrm{i})+\mathrm{Y}_*(\mathfrak{v},\mathrm{i})]\int_0^1\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu = \frac{\left(4-\sqrt{2}-\sqrt{3}\right)}{2}(1-\mathrm{i})+3\mathrm{i},$$

for all $i \in [0,1]$. That means

$$\frac{4-\sqrt{10}}{2}(1-\mathrm{i})+3\mathrm{i} \le \frac{\left(6+4\sqrt{2}-6\sqrt{3}\right)}{3}(1-\mathrm{i})+3\mathrm{i} \le \frac{\left(4-\sqrt{2}-\sqrt{3}\right)}{2}(1-\mathrm{i})+3\mathrm{i}.$$

Similarly, it can be easily shown that

$$\frac{1}{2\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)}\mathrm{Y}^*\left(\frac{\varsigma+\mathfrak{v}}{2},\mathrm{i}\right) \ge \frac{1}{\mathfrak{v}-\varsigma}\int_\varsigma^{\mathfrak{v}}\mathrm{Y}^*(\varkappa,\mathrm{i})d\varkappa \ge [\mathrm{Y}^*(\varsigma,\mathrm{i})+\mathrm{Y}^*(\mathfrak{v},\mathrm{i})]\int_0^1\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu$$

for all $i \in [0,1]$, such that

$$\frac{1}{2\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)}\mathrm{Y}^*\left(\frac{\varsigma+\mathfrak{v}}{2},\mathrm{i}\right) = \mathrm{Y}^*\left(\frac{5}{2},\mathrm{i}\right) = \frac{4+\sqrt{10}}{2}(1-\mathrm{i})+3\mathrm{i},$$

$$\frac{1}{\mathfrak{v}-\varsigma}\int_\varsigma^{\mathfrak{v}}\mathrm{Y}^*(\varkappa,\mathrm{i})d\varkappa = \int_2^3\left((1-\mathrm{i})\left(2+\varkappa^{\frac{1}{2}}\right)+3\mathrm{i}\right)d\varkappa = \frac{\left(6-4\sqrt{2}+6\sqrt{3}\right)}{3}(1-\mathrm{i})+3\mathrm{i},$$

$$[\mathrm{Y}^*(\varsigma,\mathrm{i})+\mathrm{Y}^*(\mathfrak{v},\mathrm{i})][\mathrm{Y}_*(\varsigma,\mathrm{i})+\mathrm{Y}_*(\mathfrak{v},\mathrm{i})]\int_0^1\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu = \frac{\left(4+\sqrt{2}+\sqrt{3}\right)}{2}(1-\mathrm{i})+3\mathrm{i}.$$

From which, we have

$$\frac{4+\sqrt{10}}{2}(1-\mathrm{i})+3\mathrm{i} \ge \frac{\left(6-4\sqrt{2}+6\sqrt{3}\right)}{3}(1-\mathrm{i})+3\mathrm{i} \ge \frac{\left(4+\sqrt{2}+\sqrt{3}\right)}{2}(1-\mathrm{i})+3\mathrm{i},$$

that is

$$\begin{aligned}&\left[\frac{4-\sqrt{10}}{2}(1-\mathrm{i})+3\mathrm{i}, \frac{4+\sqrt{10}}{2}(1-\mathrm{i})+3\mathrm{i}\right]\\ &\supseteq_{\mathrm{I}}\left[\frac{\left(6+4\sqrt{2}-6\sqrt{3}\right)}{3}(1-\mathrm{i})+3\mathrm{i}, \frac{\left(6-4\sqrt{2}+6\sqrt{3}\right)}{3}(1-\mathrm{i})+3\mathrm{i}\right]\\ &\supseteq_{\mathrm{I}}\left[\frac{\left(4-\sqrt{2}-\sqrt{3}\right)}{2}(1-\mathrm{i}), \frac{\left(4+\sqrt{2}+\sqrt{3}\right)}{2}(1-\mathrm{i})+3\mathrm{i}\right]\end{aligned}$$

for all $i \in [0,1]$.
Hence,

$$\begin{aligned}\frac{1}{2\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)} \odot \mathrm{Y}\left(\frac{\varsigma+\mathfrak{v}}{2}\right) &\supseteq_{\mathbb{F}} \frac{1}{\mathfrak{v}-\varsigma} \odot (\mathcal{FA})\int_\varsigma^{\mathfrak{v}}\mathrm{Y}(\varkappa)d\varkappa\\ &\supseteq_{\mathbb{F}} [\mathrm{Y}(\varsigma) \oplus \mathrm{Y}(\mathfrak{v})] \odot \int_0^1\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu.\end{aligned}$$

Theorem 6. *Let $Y : [\varsigma, \mathfrak{v}] \to F(\mathbb{R})$ be a UD -$(\mathcal{X}_1, \mathcal{X}_2)$ -convex FNVM with $\mathcal{X}_1, \mathcal{X}_2 : [0,1] \to \mathbb{R}^+$ and $\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right) \neq 0$, whose parametrized form is given by $Y_i : [\varsigma, \mathfrak{v}] \subset \mathbb{R} \to \mathcal{K}_C$ and defined as $Y_i(\varkappa) = [Y_*(\varkappa, i), Y^*(\varkappa, i)]$ for all $\varkappa \in [\varsigma, \mathfrak{v}]$ and for all $i \in [0,1]$. If $Y \in \mathcal{IA}_{([\varsigma,\mathfrak{v}],i)}$, then*

$$\begin{aligned}\frac{1}{4\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]^2} \odot \mathrm{Y}\left(\frac{\varsigma+\mathfrak{v}}{2}\right) \supseteq_{\mathbb{F}} \rhd_2 \supseteq_{\mathbb{F}} \frac{1}{\mathfrak{v}-\varsigma} \odot (\mathcal{FA})\int_\varsigma^{\mathfrak{v}}\mathrm{Y}(\varkappa)d\varkappa \supseteq_{\mathbb{F}} \rhd_1\\ \supseteq_{\mathbb{F}} [\mathrm{Y}(\varsigma) \oplus \mathrm{Y}(\mathfrak{v})] \odot \left[\frac{1}{2}+\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]\int_0^1\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu, \quad (40)\end{aligned}$$

where

$$\rhd_1 = \left[\frac{\Upsilon(\varsigma)\oplus\Upsilon(\mathfrak{v})}{2} \oplus \Upsilon\left(\frac{\varsigma+\mathfrak{v}}{2}\right)\right] \odot \int_0^1 \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu,$$
$$\rhd_2 = \frac{1}{4\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)} \odot \left[\Upsilon\left(\frac{3\varsigma+\mathfrak{v}}{4}\right) \oplus \Upsilon\left(\frac{\varsigma+3\mathfrak{v}}{4}\right)\right],$$

and $\rhd_1 = [\rhd_{1*}, \rhd_1{}^*]$, $\rhd_2 = [\rhd_{2*}, \rhd_2{}^*]$.

Proof. Take $\left[\varsigma, \frac{\varsigma+\mathfrak{v}}{2}\right]$, we have

$$\frac{1}{\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)} \odot \Upsilon\left(\frac{\mu\varsigma+(1-\mu)\frac{\varsigma+\mathfrak{v}}{2}}{2} + \frac{\mu\varsigma+(1-\mu)\frac{\varsigma+\mathfrak{v}}{2}}{2}\right) \supseteq_{\mathbb{F}} \Upsilon\left(\mu\varsigma+(1-\mu)\frac{\varsigma+\mathfrak{v}}{2}\right) \oplus \Upsilon\left((1-\mu)\varsigma+\mu\frac{\varsigma+\mathfrak{v}}{2}\right).$$

Therefore, for every $\mathrm{i} \in [0,1]$, we have

$$\frac{1}{\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)}\Upsilon_*\left(\frac{\mu\varsigma+(1-\mu)\frac{\varsigma+\mathfrak{v}}{2}}{2} + \frac{(1-\mu)\varsigma+\mu\frac{\varsigma+\mathfrak{v}}{2}}{2}, \mathrm{i}\right) \le \Upsilon_*\left(\mu\varsigma+(1-\mu)\frac{\varsigma+\mathfrak{v}}{2}, \mathrm{i}\right) + \Upsilon_*\left((1-\mu)\varsigma+\mu\frac{\varsigma+\mathfrak{v}}{2}, \mathrm{i}\right),$$
$$\frac{1}{\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)}\Upsilon^*\left(\frac{\mu\varsigma+(1-\mu)\frac{\varsigma+\mathfrak{v}}{2}}{2} + \frac{(1-\mu)\varsigma+\mu\frac{\varsigma+\mathfrak{v}}{2}}{2}, \mathrm{i}\right) \ge \Upsilon^*\left(\mu\varsigma+(1-\mu)\frac{\varsigma+\mathfrak{v}}{2}, \mathrm{i}\right) + \Upsilon^*\left((1-\mu)\varsigma+\mu\frac{\varsigma+\mathfrak{v}}{2}, \mathrm{i}\right).$$

In consequence, we obtain

$$\frac{1}{4\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)}\Upsilon_*\left(\frac{3\varsigma+\mathfrak{v}}{4}, \mathrm{i}\right) \le \frac{1}{\mathfrak{v}-\varsigma}\int_\varsigma^{\frac{\varsigma+\mathfrak{v}}{2}} \Upsilon_*(\varkappa, \mathrm{i})d\varkappa,$$
$$\frac{1}{4\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)}\Upsilon^*\left(\frac{3\varsigma+\mathfrak{v}}{4}, \mathrm{i}\right) \ge \frac{1}{\mathfrak{v}-\varsigma}\int_\varsigma^{\frac{\varsigma+\mathfrak{v}}{2}} \Upsilon^*(\varkappa, \mathrm{i})d\varkappa.$$

That is

$$\frac{1}{4X_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)}\left[\Upsilon_*\left(\frac{3\varsigma+\mathfrak{v}}{4}, \mathrm{i}\right), \Upsilon^*\left(\frac{3\varsigma+\mathfrak{v}}{4}, \mathrm{i}\right)\right] \supseteq_{\mathrm{I}} \frac{1}{\mathfrak{v}-\varsigma}\left[\int_\varsigma^{\frac{\varsigma+\mathfrak{v}}{2}} \Upsilon_*(\varkappa, \mathrm{i})d\varkappa, \int_\varsigma^{\frac{\varsigma+\mathfrak{v}}{2}} \Upsilon^*(\varkappa, \mathrm{i})d\varkappa\right].$$

It follows that

$$\frac{1}{4\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)} \odot \Upsilon\left(\frac{3\varsigma+\mathfrak{v}}{4}\right) \supseteq_{\mathbb{F}} \frac{1}{\mathfrak{v}-\varsigma} \odot \int_\varsigma^{\frac{\varsigma+\mathfrak{v}}{2}} \Upsilon(\varkappa)d\varkappa. \tag{41}$$

In a similar way as above, we have

$$\frac{1}{4X_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)} \odot \Upsilon\left(\frac{\varsigma+3\mathfrak{v}}{4}\right) \supseteq_{\mathbb{F}} \frac{1}{\mathfrak{v}-\varsigma} \odot \int_{\frac{\varsigma+\mathfrak{v}}{2}}^{\mathfrak{v}} \Upsilon(\varkappa)d\varkappa. \tag{42}$$

Combining (41) and (42), we have

$$\frac{1}{4\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)} \odot \left[\Upsilon\left(\frac{3\varsigma+\mathfrak{v}}{4}\right) \oplus \Upsilon\left(\frac{\varsigma+3\mathfrak{v}}{4}\right)\right] \supseteq_{\mathbb{F}} \frac{1}{\mathfrak{v}-\varsigma} \odot \int_\varsigma^{\mathfrak{v}} \Upsilon(\varkappa)d\varkappa.$$

By using Theorem 5, we have

$$\frac{1}{4\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]^2} \odot \Upsilon\left(\frac{\varsigma+\mathfrak{v}}{2}\right) = \frac{1}{4\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]^2} \odot \Upsilon\left(\frac{1}{2}.\frac{3\varsigma+\mathfrak{v}}{4} + \frac{1}{2}.\frac{\varsigma+3\mathfrak{v}}{4}\right).$$

Therefore, for every $\mathrm{i} \in [0,1]$, we have

$$\begin{aligned}
\frac{1}{4[\mathcal{X}_1(\frac{1}{2})\mathcal{X}_2(\frac{1}{2})]^2}\mathrm{Y}_*\left(\frac{\varsigma+\mathfrak{v}}{2},\mathrm{i}\right) &= \frac{1}{4[\mathcal{X}_1(\frac{1}{2})\mathcal{X}_2(\frac{1}{2})]^2}\mathrm{Y}_*\left(\frac{1}{2}.\frac{3\varsigma+\mathfrak{v}}{4}+\frac{1}{2}.\frac{\varsigma+3\mathfrak{v}}{4},\mathrm{i}\right),\\
\frac{1}{4[\mathcal{X}_1(\frac{1}{2})\mathcal{X}_2(\frac{1}{2})]^2}\mathrm{Y}^*\left(\frac{\varsigma+\mathfrak{v}}{2},\mathrm{i}\right) &= \frac{1}{4[\mathcal{X}_1(\frac{1}{2})\mathcal{X}_2(\frac{1}{2})]^2}\mathrm{Y}^*\left(\frac{1}{2}.\frac{3\varsigma+\mathfrak{v}}{4}+\frac{1}{2}.\frac{\varsigma+3\mathfrak{v}}{4},\mathrm{i}\right),
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{4[\mathcal{X}_1(\frac{1}{2})\mathcal{X}_2(\frac{1}{2})]^2}\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\mathrm{Y}_*\left(\frac{3\varsigma+\mathfrak{v}}{4},\mathrm{i}\right)+\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\mathrm{Y}_*\left(\frac{\varsigma+3\mathfrak{v}}{4},\mathrm{i}\right)\right],\\
&\geq \frac{1}{4[\mathcal{X}_1(\frac{1}{2})\mathcal{X}_2(\frac{1}{2})]^2}\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\mathrm{Y}^*\left(\frac{3\varsigma+\mathfrak{v}}{4},\mathrm{i}\right)+\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\mathrm{Y}^*\left(\frac{\varsigma+3\mathfrak{v}}{4},\mathrm{i}\right)\right],
\end{aligned}$$

$$\begin{aligned}
&= \rhd_{2*},\\
&= \rhd_2{}^*,
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\mathfrak{v}-\varsigma}\int_\varsigma^{\mathfrak{v}}\mathrm{Y}_*(\varkappa,\mathrm{i})d\varkappa,\\
&\geq \frac{1}{\mathfrak{v}-\varsigma}\int_\varsigma^{\mathfrak{v}}\mathrm{Y}^*(\varkappa,\mathrm{i})d\varkappa,
\end{aligned}$$

$$\begin{aligned}
&\leq \left[\frac{\mathrm{Y}_*(\varsigma,\mathrm{i})+\mathrm{Y}_*(\mathfrak{v},\mathrm{i})}{2}+\mathrm{Y}_*\left(\frac{\varsigma+\mathfrak{v}}{2},\mathrm{i}\right)\right]\int_0^1\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu,\\
&\geq \left[\frac{\mathrm{Y}^*(\varsigma,\mathrm{i})+\mathrm{Y}^*(\mathfrak{v},\mathrm{i})}{2}+\mathrm{Y}^*\left(\frac{\varsigma+\mathfrak{v}}{2},\mathrm{i}\right)\right]\int_0^1\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu,
\end{aligned}$$

$$\begin{aligned}
&= \rhd_{1*},\\
&= \rhd_1{}^*,
\end{aligned}$$

$$\begin{aligned}
&\leq \left[\frac{\mathrm{Y}_*(\varsigma,\mathrm{i})+\mathrm{Y}_*(\mathfrak{v},\mathrm{i})}{2}+\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)(\mathrm{Y}_*(\varsigma,\mathrm{i})+\mathrm{Y}_*(\mathfrak{v},\mathrm{i}))\right]\\
&\quad .\int_0^1\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu,\\
&\geq \left[\frac{\mathrm{Y}^*(\varsigma,\mathrm{i})+\mathrm{Y}^*(\mathfrak{v},\mathrm{i})}{2}+\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)(\mathrm{Y}^*(\varsigma,\mathrm{i})+\mathrm{Y}^*(\mathfrak{v},\mathrm{i}))\right]\\
&\quad .\int_0^1\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu,
\end{aligned}$$

$$\begin{aligned}
&= [\mathrm{Y}_*(\varsigma,\mathrm{i})+\mathrm{Y}_*(\mathfrak{v},\mathrm{i})]\left[\frac{1}{2}+\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]\int_0^1\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu,\\
&= [\mathrm{Y}^*(\varsigma,\mathrm{i})+\mathrm{Y}^*(\mathfrak{v},\mathrm{i})]\left[\frac{1}{2}+\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]\int_0^1\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu,
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{1}{4\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]^2}\mathrm{Y}\left(\frac{\varsigma+\mathfrak{v}}{2},\mathrm{i}\right) &\supseteq_I \rhd_2 \supseteq_I \frac{1}{\mathfrak{v}-\varsigma}(\mathfrak{T}\mathcal{A})\int_\varsigma^{\mathfrak{v}}\mathrm{Y}(\varkappa,\mathrm{i})d\varkappa \supseteq_I \rhd_1\\
&\supseteq_I [\mathrm{Y}(\varsigma,\mathrm{i})+\mathrm{Y}(\mathfrak{v},\mathrm{i})]\left[\frac{1}{2}+\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]\int_0^1\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu,
\end{aligned}$$

that is

$$\begin{aligned}
\frac{1}{4\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]^2}\odot\mathrm{Y}\left(\frac{\varsigma+\mathfrak{v}}{2}\right) &\supseteq_{\mathbb{F}} \rhd_2 \supseteq_{\mathbb{F}} \frac{1}{\mathfrak{v}-\varsigma}\odot(\mathcal{FA})\int_\varsigma^{\mathfrak{v}}\mathrm{Y}(\varkappa)d\varkappa \supseteq_{\mathbb{F}} \rhd_1\\
&\supseteq_{\mathbb{F}} [\mathrm{Y}(\varsigma)\oplus\mathrm{Y}(\mathfrak{v})]\odot\left[\frac{1}{2}+\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]\int_0^1\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu,
\end{aligned}$$

hence, the result follows. □

Example 3. *We consider* $\mathcal{X}_1(\mu)=\mu,\mathcal{X}_2(\mu)\equiv 1$, *for* $\mu\in[0,1]$, *and the FNVMY* $:[\varsigma,\mathfrak{v}]=[2,3]\to F(\mathbb{R})$ *defined by,* $Y_i(\varkappa)=\left[(1-i)\left(2-\varkappa^{\frac{1}{2}}\right)+3i,(1-i)\left(2+\varkappa^{\frac{1}{2}}\right)+3i\right]$, *as in Example 2, then*

$Y(\varkappa)$ is UD -$(\mathcal{X}_1, \mathcal{X}_2)$ -convex FNVM . We have $Y_*(\varkappa, i) = (1-i)\left(2 - \varkappa^{\frac{1}{2}}\right) + 3i$ and $Y^*(\varkappa, i) = (1-i)\left(2 + \varkappa^{\frac{1}{2}}\right) + 3i$. We now compute the following:

$$\frac{1}{4\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]^2}\mathrm{Y}_*\left(\frac{\varsigma+\mathfrak{v}}{2}, \mathrm{i}\right) = \mathrm{Y}_*\left(\frac{5}{2}, \mathrm{i}\right) = \frac{4-\sqrt{10}}{2}(1-\mathrm{i}) + 3i,$$
$$\frac{1}{4\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]^2}\mathrm{Y}^*\left(\frac{\varsigma+\mathfrak{v}}{2}, \mathrm{i}\right) = \mathrm{Y}^*\left(\frac{5}{2}, \mathrm{i}\right) = \frac{4+\sqrt{10}}{2}(1-\mathrm{i}) + 3i,$$

$$\rhd_{2*} = \frac{1}{4\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]^2}\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\mathrm{Y}_*\left(\frac{3\varsigma+\mathfrak{v}}{4}, \mathrm{i}\right) + \mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\mathrm{Y}_*\left(\frac{\varsigma+3\mathfrak{v}}{4}, \mathrm{i}\right)\right] = \frac{5-\sqrt{11}}{4}(1-\mathrm{i}) + 3i,$$
$$\rhd_2{}^* = \frac{1}{4\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]^2}\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\mathrm{Y}^*\left(\frac{3\varsigma+\mathfrak{v}}{4}, \mathrm{i}\right) + \mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\mathrm{Y}^*\left(\frac{\varsigma+3\mathfrak{v}}{4}, \mathrm{i}\right)\right] = \frac{7+\sqrt{11}}{4}(1-\mathrm{i}) + 3i.$$

$$\rhd_{1*} = \left[\frac{\mathrm{Y}_*(\varsigma,\mathrm{i})+\mathrm{Y}_*(\mathfrak{v},\mathrm{i})}{2} + \mathrm{Y}_*\left(\frac{\varsigma+\mathfrak{v}}{2}, \mathrm{i}\right)\right]\int_0^1 \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu = \frac{\left(8-\sqrt{2}-\sqrt{3}-\sqrt{10}\right)}{4}(1-\mathrm{i}) + 3i,$$
$$\rhd_1{}^* = \left[\frac{\mathrm{Y}^*(\varsigma,\mathrm{i})+\mathrm{Y}^*(\mathfrak{v},\mathrm{i})}{2} + \mathrm{Y}^*\left(\frac{\varsigma+\mathfrak{v}}{2}, \mathrm{i}\right)\right]\int_0^1 \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu = \frac{\left(8+\sqrt{2}+\sqrt{3}+\sqrt{10}\right)}{4}(1-\mathrm{i}) + 3i,$$

$$[\mathrm{Y}_*(\varsigma,\mathrm{i}) + \mathrm{Y}_*(\mathfrak{v},\mathrm{i})]\left[\frac{1}{2} + \mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]\int_0^1 \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu = \frac{\left(4-\sqrt{2}-\sqrt{3}\right)}{2}(1-\mathrm{i}) + 3i,$$
$$[\mathrm{Y}^*(\varsigma,\mathrm{i}) + \mathrm{Y}^*(\mathfrak{v},\mathrm{i})]\left[\frac{1}{2} + \mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]\int_0^1 \mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)d\mu = \frac{\left(4+\sqrt{2}+\sqrt{3}\right)}{2}(1-\mathrm{i}) + 3i,$$

then we obtain that

$$\begin{aligned}(1-\mathrm{i})\frac{4-\sqrt{10}}{2} + 3i &\le \frac{5-\sqrt{11}}{4}(1-\mathrm{i}) + 3i \le \frac{\left(6+4\sqrt{2}-6\sqrt{3}\right)}{3}(1-\mathrm{i}) + 3i\\ &\le \frac{\left(8-\sqrt{2}-\sqrt{3}-\sqrt{10}\right)}{4}(1-\mathrm{i}) + 3i \le \frac{(1-\mathrm{i})\left(4-\sqrt{2}-\sqrt{3}\right)}{2} + 3i\\ (1-\mathrm{i})\frac{4+\sqrt{10}}{2} + 3i &\ge \frac{7+\sqrt{11}}{4}(1-\mathrm{i}) + 3i \ge \frac{\left(6-4\sqrt{2}+6\sqrt{3}\right)}{3}(1-\mathrm{i}) + 3i\\ &\ge \frac{\left(8+\sqrt{2}+\sqrt{3}+\sqrt{10}\right)}{4}(1-\mathrm{i}) + 3i \ge \frac{(1-\mathrm{i})\left(4+\sqrt{2}+\sqrt{3}\right)}{2} + 3i.\end{aligned}$$

Hence, Theorem 6 is verified.

Theorem 7. *Let $Y, \mathfrak{S} : [\varsigma, \mathfrak{v}] \to F(\mathbb{R})$ be two UD -$(\mathcal{X}_1, \mathcal{X}_2)$ -convex FNVM s with $\mathcal{X}_1, \mathcal{X}_2 : [0,1] \to \mathbb{R}^+$ and $\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right) \neq 0$, whose parametrized form is given by $Y_i, \mathfrak{S}_i : [\varsigma, \mathfrak{v}] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ and defined as $Y_i(\varkappa) = [Y_*(\varkappa, i), Y^*(\varkappa, i)]$ and $\mathfrak{S}_i(\varkappa) = [\mathfrak{S}_*(\varkappa, i), \mathfrak{S}^*(\varkappa, i)]$ for all $\varkappa \in [\varsigma, \mathfrak{v}]$ and for all $i \in [0,1]$. If $Y \otimes \mathfrak{S} \in \mathcal{IA}_{([\varsigma,\mathfrak{v}],i)}$, then*

$$\begin{aligned}\frac{1}{\mathfrak{v}-\varsigma} \odot (\mathcal{FA})\int_\varsigma^{\mathfrak{v}} \mathrm{Y}(\varkappa) \quad &\otimes \mathfrak{S}(\varkappa)d\varkappa \supseteq_{\mathbb{F}} \mathfrak{Y}(\varsigma, \mathfrak{v}) \odot \int_0^1 [\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)]^2 d\mu\\ &\oplus \mathfrak{H}(\varsigma, \mathfrak{v}) \odot \int_0^1 \mathcal{X}_1(\mu)\mathcal{X}_2(\mu)\mathcal{X}_1(1-\mu)\mathcal{X}_2(1-\mu)d\mu,\end{aligned} \tag{43}$$

where $\mathfrak{Y}(\varsigma, \mathfrak{v}) = Y(\varsigma) \otimes \mathfrak{S}(\varsigma) \oplus Y(\mathfrak{v}) \otimes \mathfrak{S}(\mathfrak{v})$,$\mathfrak{H}(\varsigma, \mathfrak{v}) = Y(\varsigma) \otimes \mathfrak{S}(\mathfrak{v}) \oplus Y(\mathfrak{v}) \otimes \mathfrak{S}(\varsigma)$, and $\mathfrak{Y}_i(\varsigma, \mathfrak{v}) = [\mathfrak{Y}_*((\varsigma, \mathfrak{v}), i), \mathfrak{Y}^*((\varsigma, \mathfrak{v}), i)]$ and $\mathfrak{H}_i(\varsigma, \mathfrak{v}) = [\mathfrak{H}_*((\varsigma, \mathfrak{v}), i), \mathfrak{H}^*((\varsigma, \mathfrak{v}), i)]$.

Example 4. *We consider $\mathcal{X}_1(\mu) = \mu$,$\mathcal{X}_2(\mu) \equiv 1$, for$\mu \in [0,1]$, and the fuzzy-number valued mappings $Y, \mathfrak{S} : [\varsigma, \mathfrak{v}] = [0,2] \to E_C$ defined by*

$$\mathrm{Y}(\varkappa)(\sigma) = \begin{cases} \frac{\theta}{\varkappa}\theta \in [0, \varkappa] \\ \frac{2\varkappa-\theta}{\varkappa}\theta \in (\varkappa, 2\varkappa] \\ 0 \; otherwise, \end{cases} \tag{44}$$

$$\mathfrak{S}(\varkappa)(\sigma) = \begin{cases} \frac{\theta-\varkappa}{2-\varkappa}\theta \in [\varkappa, 2] \\ \frac{8-e^{\varkappa}-\theta}{8-e^{\varkappa}-2}\theta \in (2, 8-e^{\varkappa}] \\ 0 \; otherwise. \end{cases} \tag{45}$$

Then, for each $i \in [0,1]$, we have $Y_i(\varkappa) = [i\varkappa, (2-i)\varkappa]$ and $\mathfrak{S}_i(\varkappa) = [(1-i)\varkappa + 2i, (1-i)(8-e^{\varkappa}) + 2i]$. Since end point functions $Y_*(\varkappa, i) = i\varkappa$, $\mathfrak{S}_*(\varkappa, i) = (1-i)\varkappa + 2i$, and $Y^*(\varkappa, i) = (2-i)\varkappa$, $\mathfrak{S}^*(\varkappa, i) = (1-i)(8-e^{\varkappa}) + 2i(\mathcal{X}_1, \mathcal{X}_2)$

-convex and concave functions for each $i \in [0,1]$, respectively. Hence Y,$\mathfrak{S}$ both are UD -$(\mathcal{X}_1, \mathcal{X}_2)$ -convex FNVM. We now compute the following

$$\frac{1}{\mathfrak{v}-\varsigma}\int_{\varsigma}^{\mathfrak{v}} \mathrm{Y}_*(\varkappa,\mathrm{i}) \times \mathfrak{S}_*(\varkappa,\mathrm{i})\mathrm{d}\varkappa = \frac{1}{2}\int_0^2 \left(\mathrm{i}(1-\mathrm{i})\varkappa^2 + 2\mathrm{i}^2\varkappa\right)\mathrm{d}\varkappa = \frac{2}{3}i(2+\mathrm{i})$$

$$\frac{1}{\mathfrak{v}-\varsigma}\int_{\varsigma}^{\mathfrak{v}} \mathrm{Y}^*(\varkappa,\mathrm{i}) \times \mathfrak{S}^*(\varkappa,\mathrm{i})\mathrm{d}\varkappa = \frac{1}{2}\int_0^2 ((1-\mathrm{i})(2-\mathrm{i})\varkappa(8-\mathrm{e}^{\varkappa}) + 2\mathrm{i}(2-\mathrm{i})\varkappa)\mathrm{d}\varkappa$$
$$= \frac{(2-\mathrm{i})}{2}\left[15 - 11\mathrm{i} + (\mathrm{i}-1)\mathrm{e}^2\right],$$

$$\mathfrak{Y}_*((\varsigma,\mathfrak{v}),\mathrm{i})\int_0^1 [\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)]^2\mathrm{d}\mu = \frac{4\mathrm{i}}{3}$$
$$\mathfrak{Y}^*((\varsigma,\mathfrak{v}),\mathrm{i})\int_0^1 [\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)]^2\mathrm{d}\mu = \frac{2(2-\mathrm{i})\left[(1-\mathrm{i})\left(8-\mathrm{e}^2\right)+2\mathrm{i}\right]}{3},$$

$$\mathfrak{H}_*((\varsigma,\mathfrak{v}),\mathrm{i})\int_0^1 \mathcal{X}_1(\mu)\mathcal{X}_2(\mu)\mathcal{X}_1(1-\mu)\mathcal{X}_2(1-\mu)\mathrm{d}\mu = \frac{2\mathrm{i}^2}{3}$$
$$\mathfrak{H}^*((\varsigma,\mathfrak{v}),\mathrm{i})\int_0^1 \mathcal{X}_1(\mu)\mathcal{X}_2(\mu)\mathcal{X}_1(1-\mu)\mathcal{X}_2(1-\mu)\mathrm{d}\mu = \frac{(2-\mathrm{i})(7-5\mathrm{i})}{3},$$

for each $i \in [0,1]$, that means

$$\left[\frac{2}{3}\mathrm{i}(1+2\mathrm{i}), \frac{(2-\mathrm{i})}{2}\left[15-11\mathrm{i}+(\mathrm{i}-1)\mathrm{e}^2\right]\right] \supseteq_{\mathrm{I}} \frac{1}{3}\left[2\mathrm{i}(2+\mathrm{i}), (2-\mathrm{i})\left[2(1-\mathrm{i})\left(8-\mathrm{e}^2\right) - \mathrm{i}+7\right]\right]$$

hence, Theorem 7 is illustrated.

Theorem 8. *Let $Y, \mathfrak{S} : [\varsigma, \mathfrak{v}] \to F(\mathbb{R})$ be two UD -$(\mathcal{X}_1, \mathcal{X}_2)$ -convex FNVM s with $\mathcal{X}_1, \mathcal{X}_2 : [0,1] \to \mathbb{R}^+$ and $\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right) \neq 0$, whose parametrized form is given by $Y_i, \mathfrak{S}_i : [\varsigma, \mathfrak{v}] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ and defined as $Y_i(\varkappa) = [\mathrm{Y}_*(\varkappa, i), Y^*(\varkappa, i)]$ and $\mathfrak{S}_i(\varkappa) = [\mathfrak{S}_*(\varkappa, i), \mathfrak{S}^*(\varkappa, i)]$ for all $\varkappa \in [\varsigma, \mathfrak{v}]$ and for all $i \in [0,1]$. If $Y \otimes \mathfrak{S} \in \mathcal{IA}_{([\varsigma,\mathfrak{v}],i)}$, then*

$$\frac{1}{2\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]^2} \odot \mathrm{Y}\left(\frac{\varsigma+\mathfrak{v}}{2}\right) \otimes \mathfrak{S}\left(\frac{\varsigma+\mathfrak{v}}{2}\right)$$
$$\supseteq_{\mathbb{F}} \frac{1}{\mathfrak{v}-\varsigma} \odot (\mathcal{FA})\int_{\varsigma}^{\mathfrak{v}} \mathrm{Y}(\varkappa) \otimes \mathfrak{S}(\varkappa)\mathrm{d}\varkappa \oplus \mathfrak{H}(\varsigma,\mathfrak{v})\int_0^1 [\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)]^2\mathrm{d}\mu$$
$$\oplus\, \mathfrak{Y}(\varsigma,\mathfrak{v}) \odot \int_0^1 \mathcal{X}_1(\mu)\mathcal{X}_2(\mu)\mathcal{X}_1(1-\mu)\mathcal{X}_2(1-\mu)\mathrm{d}\mu, \tag{46}$$

where$\mathfrak{Y}(\varsigma, \mathfrak{v}) = Y(\varsigma) \otimes \mathfrak{S}(\varsigma) \oplus Y(\mathfrak{v}) \otimes \mathfrak{S}(\mathfrak{v})$,$\mathfrak{H}(\varsigma, \mathfrak{v}) = Y(\varsigma) \otimes \mathfrak{S}(\mathfrak{v}) \oplus Y(\mathfrak{v}) \otimes \mathfrak{S}(\varsigma)$, and $\mathfrak{Y}_i(\varsigma, \mathfrak{v}) = [\mathfrak{Y}_*((\varsigma, \mathfrak{v}), i), \mathfrak{Y}^*((\varsigma, \mathfrak{v}), i)]$ and $\mathfrak{H}_i(\varsigma, \mathfrak{v}) = [\mathfrak{H}_*((\varsigma, \mathfrak{v}), i), \mathfrak{H}^*((\varsigma, \mathfrak{v}), i)]$.

Proof. By hypothesis, for each $\mathrm{i} \in [0,1]$, we have

$$\mathrm{Y}_*\left(\frac{\varsigma+\mathfrak{v}}{2},\mathrm{i}\right) \times \mathfrak{S}_*\left(\frac{\varsigma+\mathfrak{v}}{2},\mathrm{i}\right)$$
$$\mathrm{Y}^*\left(\frac{\varsigma+\mathfrak{v}}{2},\mathrm{i}\right) \times \mathfrak{S}^*\left(\frac{\varsigma+\mathfrak{v}}{2},\mathrm{i}\right)$$

$$\leq \left[\mathcal{X}_1\left(\tfrac{1}{2}\right)\mathcal{X}_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix} \mathrm{Y}_*(\mu\varsigma+(1-\mu)\mathfrak{v},\mathrm{i}) \times \mathfrak{S}_*(\mu\varsigma+(1-\mu)\mathfrak{v},\mathrm{i}) \\ +\mathrm{Y}_*(\mu\varsigma+(1-\mu)\mathfrak{v},\mathrm{i}) \times \mathfrak{S}_*((1-\mu)\varsigma+\mu\mathfrak{v},\mathrm{i}) \end{bmatrix}$$
$$+\left[\mathcal{X}_1\left(\tfrac{1}{2}\right)\mathcal{X}_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix} \mathrm{Y}_*((1-\mu)\varsigma+\mu\mathfrak{v},\mathrm{i}) \times \mathfrak{S}_*(\mu\varsigma+(1-\mu)\mathfrak{v},\mathrm{i}) \\ +\mathrm{Y}_*((1-\mu)\varsigma+\mu\mathfrak{v},\mathrm{i}) \times \mathfrak{S}_*((1-\mu)\varsigma+\mu\mathfrak{v},\mathrm{i}) \end{bmatrix},$$
$$\geq \left[\mathcal{X}_1\left(\tfrac{1}{2}\right)\mathcal{X}_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix} \mathrm{Y}^*(\mu\varsigma+(1-\mu)\mathfrak{v},\mathrm{i}) \times \mathfrak{S}^*(\mu\varsigma+(1-\mu)\mathfrak{v},\mathrm{i}) \\ +\mathrm{Y}^*(\mu\varsigma+(1-\mu)\mathfrak{v},\mathrm{i}) \times \mathfrak{S}^*((1-\mu)\varsigma+\mu\mathfrak{v},\mathrm{i}) \end{bmatrix}$$
$$+\left[\mathcal{X}_1\left(\tfrac{1}{2}\right)\mathcal{X}_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix} \mathrm{Y}^*((1-\mu)\varsigma+\mu\mathfrak{v},\mathrm{i}) \times \mathfrak{S}^*(\mu\varsigma+(1-\mu)\mathfrak{v},\mathrm{i}) \\ +\mathrm{Y}^*((1-\mu)\varsigma+\mu\mathfrak{v},\mathrm{i}) \times \mathfrak{S}^*((1-\mu)\varsigma+\mu\mathfrak{v},\mathrm{i}) \end{bmatrix},$$

$$\leq \left[\mathcal{X}_1\left(\tfrac{1}{2}\right)\mathcal{X}_2\left(\tfrac{1}{2}\right)\right]^2\left[\begin{array}{c}\Upsilon_*(\mu\varsigma+(1-\mu)\mathfrak{v},\mathrm{i})\times\mathfrak{S}_*(\mu\varsigma+(1-\mu)\mathfrak{v},\mathrm{i})\\+\Upsilon_*((1-\mu)\varsigma+\mu\mathfrak{v},\mathrm{i})\times\mathfrak{S}_*((1-\mu)\varsigma+\mu\mathfrak{v},\mathrm{i})\end{array}\right]$$

$$+\left[\mathcal{X}_1\left(\tfrac{1}{2}\right)\mathcal{X}_2\left(\tfrac{1}{2}\right)\right]^2\left[\begin{array}{c}(\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\Upsilon_*(\varsigma,\mathrm{i})+\mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\Upsilon_*(\mathfrak{v},\mathrm{i}))\\(\mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\mathfrak{S}_*(\varsigma,\mathrm{i})+\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\mathfrak{S}_*(\mathfrak{v},\mathrm{i}))\\+(\mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\Upsilon_*(\varsigma,\mathrm{i})+\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\Upsilon_*(\mathfrak{v},\mathrm{i}))\\(\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\mathfrak{S}_*(\varsigma,\mathrm{i})+\mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\mathfrak{S}_*(\mathfrak{v},\mathrm{i}))\end{array}\right],$$

$$\geq \left[\mathcal{X}_1\left(\tfrac{1}{2}\right)\mathcal{X}_2\left(\tfrac{1}{2}\right)\right]^2\left[\begin{array}{c}\Upsilon^*(\mu\varsigma+(1-\mu)\mathfrak{v},\mathrm{i})\times\mathfrak{S}^*(\mu\varsigma+(1-\mu)\mathfrak{v},\mathrm{i})\\+\Upsilon^*((1-\mu)\varsigma+\mu\mathfrak{v},\mathrm{i})\times\mathfrak{S}^*((1-\mu)\varsigma+\mu\mathfrak{v},\mathrm{i})\end{array}\right]$$

$$+\left[\mathcal{X}_1\left(\tfrac{1}{2}\right)\mathcal{X}_2\left(\tfrac{1}{2}\right)\right]^2\left[\begin{array}{c}(\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\Upsilon^*(\varsigma,\mathrm{i})+\mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\Upsilon^*(\mathfrak{v},\mathrm{i}))\\(\mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\mathfrak{S}^*(\varsigma,\mathrm{i})+\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\mathfrak{S}^*(\mathfrak{v},\mathrm{i}))\\+(\mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\Upsilon^*(\varsigma,\mathrm{i})+\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\Upsilon^*(\mathfrak{v},\mathrm{i}))\\(\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)\mathfrak{S}^*(\varsigma,\mathrm{i})+\mathcal{X}_1(1-\mu)\mathcal{X}_2(\mu)\mathfrak{S}^*(\mathfrak{v},\mathrm{i}))\end{array}\right],$$

$$= \left[\mathcal{X}_1\left(\tfrac{1}{2}\right)\mathcal{X}_2\left(\tfrac{1}{2}\right)\right]^2\left[\begin{array}{c}\Upsilon_*(\mu\varsigma+(1-\mu)\mathfrak{v},\mathrm{i})\times\mathfrak{S}_*(\mu\varsigma+(1-\mu)\mathfrak{v},\mathrm{i})\\+\Upsilon_*((1-\mu)\varsigma+\mu\mathfrak{v},\mathrm{i})\times\mathfrak{S}_*((1-\mu)\varsigma+\mu\mathfrak{v},\mathrm{i})\end{array}\right]$$

$$+2\left[\mathcal{X}_1\left(\tfrac{1}{2}\right)\mathcal{X}_2\left(\tfrac{1}{2}\right)\right]^2\left[\begin{array}{c}\mathcal{X}_1(\mu)\mathcal{X}_2(\mu)\mathcal{X}_1(1-\mu)\mathcal{X}_2(1-\mu)\mathfrak{Y}_*((\varsigma,\mathfrak{v}),\mathrm{i})\\+[\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)]^2\mathfrak{H}_*((\varsigma,\mathfrak{v}),\mathrm{i})\end{array}\right],$$

$$= \left[\mathcal{X}_1\left(\tfrac{1}{2}\right)\mathcal{X}_2\left(\tfrac{1}{2}\right)\right]^2\left[\begin{array}{c}\Upsilon^*(\mu\varsigma+(1-\mu)\mathfrak{v},\mathrm{i})\times\mathfrak{S}^*(\mu\varsigma+(1-\mu)\mathfrak{v},\mathrm{i})\\+\Upsilon^*((1-\mu)\varsigma+\mu\mathfrak{v},\mathrm{i})\times\mathfrak{S}^*((1-\mu)\varsigma+\mu\mathfrak{v},\mathrm{i})\end{array}\right]$$

$$+2\left[\mathcal{X}_1\left(\tfrac{1}{2}\right)\mathcal{X}_2\left(\tfrac{1}{2}\right)\right]^2\left[\begin{array}{c}\mathcal{X}_1(\mu)\mathcal{X}_2(\mu)\mathcal{X}_1(1-\mu)\mathcal{X}_2(1-\mu)\mathfrak{Y}^*((\varsigma,\mathfrak{v}),\mathrm{i})\\+[\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)]^2\mathfrak{H}^*((\varsigma,\mathfrak{v}),\mathrm{i})\end{array}\right],$$

that is

$$= \left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]^2\left[\begin{array}{c}Y(\mu\varsigma+(1-\mu)\mathfrak{v})\times S(\mu\varsigma+(1-\mu)\mathfrak{v})\\+Y((1-\mu)\varsigma+\mu\mathfrak{v})\times S((1-\mu)\varsigma+\mu\mathfrak{v})\end{array}\right]$$

$$+2\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]^2\left[\begin{array}{c}\mathcal{X}_1(\mu)\mathcal{X}_2(\mu)\mathcal{X}_1(1-\mu)\mathcal{X}_2(1-\mu)Y(\varsigma,\mathfrak{v})\\+[\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)]^2H(\varsigma,\mathfrak{v})\end{array}\right],$$

$\mathcal{FA}$-Integrating over $[0,1]$, we have

$$\frac{1}{2\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]^2}\odot\Upsilon\left(\frac{\varsigma+\mathfrak{v}}{2}\right)\otimes\mathfrak{S}\left(\frac{\varsigma+\mathfrak{v}}{2}\right)$$

$$\supseteq_{\mathbb{F}}\frac{1}{\mathfrak{v}-\varsigma}\odot(\mathcal{FA})\int_{\varsigma}^{\mathfrak{v}}\Upsilon(\varkappa)\otimes\mathfrak{S}(\varkappa)d\varkappa\oplus\mathfrak{H}(\varsigma,\mathfrak{v})\odot\int_0^1[\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)]^2d\mu$$

$$\oplus\,\mathfrak{Y}(\varsigma,\mathfrak{v})\odot\int_0^1\mathcal{X}_1(\mu)\mathcal{X}_2(\mu)\mathcal{X}_1(1-\mu)\mathcal{X}_2(1-\mu)d\mu,$$

hence, the required result. □

Example 5. *We consider* $\mathcal{X}_1(\mu)=\mu$,$\mathcal{X}_2(\mu)\equiv 1$, *for* $\mu\in[0,1]$ *, and the FNVMsY,* $\mathfrak{S}:[\varsigma,\mathfrak{v}]=[0,2]\to E_C$, *as in Example 4. Then, for each* $i\in[0,1]$, *we have* $Y_i(\varkappa)=[i\varkappa,(2-i)\varkappa]$ *and* $\mathfrak{S}_i(\varkappa)=[(1-i)\varkappa+2i,(1-i)(8-e^{\varkappa})+2i]$ *and,* $Y(\varkappa),\mathfrak{S}(\varkappa)$ *are UD -*$(\mathcal{X}_1,\mathcal{X}_2)$ *-convex FNVM s, respectively. We have* $Y_*(\varkappa,i)=i\varkappa$,$Y^*(\varkappa,i)=(2-i)\varkappa$ *and* $\mathfrak{S}_*(\varkappa,i)=(1-i)\varkappa+2i$, $\mathfrak{S}^*(\varkappa,i)=(1-i)(8-e^{\varkappa})+2i$ *. We now compute the following*

$$\frac{1}{2\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]^2}\Upsilon_*\left(\frac{\varsigma+\mathfrak{v}}{2},\mathrm{i}\right)\times\mathfrak{S}_*\left(\frac{\varsigma+\mathfrak{v}}{2},\mathrm{i}\right)=2i(1+\mathrm{i}),$$

$$\frac{1}{2\left[\mathcal{X}_1\left(\frac{1}{2}\right)\mathcal{X}_2\left(\frac{1}{2}\right)\right]^2}\Upsilon^*\left(\frac{\varsigma+\mathfrak{v}}{2},\mathrm{i}\right)\times\mathfrak{S}^*\left(\frac{\varsigma+\mathfrak{v}}{2},\mathrm{i}\right)=2(2-\mathrm{i})[8-6\mathrm{i}+(\mathrm{i}-1)\mathrm{e}].$$

$$\frac{1}{\mathfrak{v}-\varsigma}\int_{\varsigma}^{\mathfrak{v}} \Upsilon_*(\varkappa,\mathrm{i}) \times \mathfrak{S}_*(\varkappa,\mathrm{i})d\varkappa = \frac{1}{2}\int_0^2 \left(\mathrm{i}(1-\mathrm{i})\varkappa^2 + 2\mathrm{i}^2\varkappa\right)d\varkappa = \frac{2}{3}i(2+\mathrm{i}),$$
$$\frac{1}{\mathfrak{v}-\varsigma}\int_{\varsigma}^{\mathfrak{v}} \Upsilon^*(\varkappa,\mathrm{i}) \times \mathfrak{S}^*(\varkappa,\mathrm{i})d\varkappa = \frac{1}{2}\int_0^2 ((2-\mathrm{i})\varkappa)((1-\mathrm{i})(8-e^{\varkappa}) + 2\mathrm{i})d\varkappa$$
$$= \frac{(2-\mathrm{i})}{2}\left[15 - 11\mathrm{i} + (\mathrm{i}-1)e^2\right].$$
$$\mathfrak{Y}_*((\varsigma,\mathfrak{v}),\mathrm{i})\int_0^1 [\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)]^2 d\mu = \frac{2\mathrm{i}}{3},$$
$$\mathfrak{Y}^*((\varsigma,\mathfrak{v}),\mathrm{i})\int_0^1 [\mathcal{X}_1(\mu)\mathcal{X}_2(1-\mu)]^2 d\mu = \frac{(2-\mathrm{i})\left[8-6\mathrm{i}+(\mathrm{i}-1)e^2\right]}{3},$$
$$\mathfrak{H}_*((\varsigma,\mathfrak{v}),\mathrm{i})\int_0^1 \mathcal{X}_1(\mu)\mathcal{X}_2(\mu)\mathcal{X}_1(1-\mu)\mathcal{X}_2(1-\mu)d\mu = \frac{4\mathrm{i}^2}{3},$$
$$\mathfrak{H}^*((\varsigma,\mathfrak{v}),\mathrm{i})\int_0^1 \mathcal{X}_1(\mu)\mathcal{X}_2(\mu)\mathcal{X}_1(1-\mu)\mathcal{X}_2(1-\mu)d\mu = \frac{2(2-\mathrm{i})(7-5\mathrm{i})}{3},$$

for each $i \in [0,1]$, *that means*

$$[2\mathrm{i}(1+\mathrm{i}), 2(2-\mathrm{i})[8-6\mathrm{i}+(\mathrm{i}-1)e]] \supseteq_I \left[\frac{2}{3}\mathrm{i}(2+\mathrm{i}), \frac{(2-\mathrm{i})}{2}\left[15-11\mathrm{i}+(\mathrm{i}-1)e^2\right]\right]$$
$$+ \frac{1}{3}\left[2\mathrm{i}(1+2\mathrm{i}), (2-\mathrm{i})\left[2(7-5\mathrm{i}) + (1-\mathrm{i})\left(8-e^2\right) - 8\mathrm{i} + 14\right]\right]$$

hence, Theorem 8 is illustrated.

4. Conclusions

The discovery of the Hermite-Hadamard type inequality is the primary topic of this paper. We keep a UD-($\mathcal{X}_1, \mathcal{X}_2$)-convex FNVM as the starting point for our analysis. We are also able to obtain new generalized inequalities. In order to evaluate the efficacy and accuracy of our results, we make some pertinent parameter choices. The particular choices change our main results and give new bounds for the recently found fuzzy Hermite-Hadamard inequality. In remarks, we present the new exceptional instances. The results of the aforementioned remarks provide the inequality's lower and upper bounds. We provide compelling examples to back up our conclusions. We believe that the present results have implications for various domains of the pure and practical sciences, including optimization and convex analysis. Regardless, we think that these discoveries add to our knowledge of the behavior, traits, and numerous real-world uses of fuzzy calculus. In later works, we will employ the Hermite-Hadamard-type and Iscan-Holder-type inequality to create a variety of novel, fascinating inequalities for various classes of convex and pre-invex functions.

Author Contributions: Conceptualization, M.B.K.; methodology, M.B.K.; validation, M.B.K. and M.S.S.; formal analysis, H.A.O.; investigation, M.B.K.; resources, M.B.K. and A.R.; data curation, M.B.K. and A.M.A.; writing—original draft preparation, M.B.K.; writing—review and editing, M.B.K.; visualization, M.S.S., H.A.O., A.R. and A.M.A.; supervision, M.B.K.; project administration, M.S.S.; funding acquisition, M.B.K. and H.A.O. All authors have read and agreed to the published version of the manuscript.

Funding: The authors would like to thank the Deanship of Scientific Research at Umm Al-Qura University for supporting this work by Grant Code: 22UQU4390021DSR05.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the Rector, COMSATS University Islamabad, Islamabad, Pakistan, for providing excellent research. The authors would like to thank the Deanship of Scientific Research at Umm Al-Qura University for supporting this work by Grant Code: 22UQU4390021DSR05.

Conflicts of Interest: The authors declare no conflict of interest.

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