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$(\alpha-\psi)$ Meir–Keeler Contractions in Bipolar Metric Spaces

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Abstract: In this paper, we introduce the new notion of contravariant $(\alpha-\psi)$ Meir–Keeler contractive mappings by defining α-orbital admissible mappings and covariant Meir–Keeler contraction in bipolar metric spaces. We prove fixed point theorems for these contractions and also provide some corollaries of main results. An example is also be given in support of our main result. In the end, we also solve an integral equation using our result.

Keywords: fixed point; $(\alpha-\psi)$ Meir–Keeler contractive mappings; covariant and contravariant mappings; bipolar metric space

MSC: 47H9; 47H10; 30G35; 46N99; 54H25

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1. Introduction

Fixed point theory is the major branch of non-linear analysis. It has number of applications in other branch of sciences, economics, etc. In 1922, Banach [1] gave a contraction principle to obtain a fixed point theorem in complete metric space. Some other researchers tried to generalize the concept of metric space; see [2–4]. Due to the various applications of the Banach contraction principle, the contraction mapping theorem has been generalized by many researchers in the setting of various topological spaces using different contractive conditions; see [5–14]. In 2012, Samet et al. [15] introduced the new contraction by defining the α-admissible mappings and established fixed point results thereon. In 2013, Kumam et al. [16] extended and generalized the α-admissible mapping of [15], introduced $(\alpha-\psi)$ Meir–Keeler contractive mappings and proved some fixed point theorems in complete metric space. In 2014, Popescu [17] introduced α-orbital admissible mapping to get fixed point theorems.

Recently, in 2016, Mutlu et al. [18] introduced the new type of metric space called bipolar metric space. Since then, researchers have established several fixed point theorems using various contractive conditions in the setting of bipolar metric spaces; see [19–24].

Inspired by this, in the present work, we introduce $(\alpha-\psi)$ Meir–Keeler contractive mappings and establish fixed point theorems in the setting of bipolar metric spaces. The rest of the paper is organized as follows. In Section 2, we review some preliminary definitions and monographs that are required for our main result. In Section 3, we present our main results and establish a fixed point result using $(\alpha-\psi)$ Meir–Keeler contractive mappings in the setting of bipolar metric space. We supplement the derived results with suitable non-trivial examples. In Section 4, we apply the derived fixed point result to find an analytical solution to the integral equation. Finally, we conclude the paper with some open problems for future work.

2. Preliminaries

To prove our main results, we need some basic definitions from the literature as follows:

Definition 1 ([18])**.** *Let X and Y be two non-empty sets and $d : X \times Y \to [0, \infty)$ be a map satisfying the following conditions:*

1. *$d(x, y) = 0$ if and only if $x = y$ for all $(x, y) \in X \times Y$;*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X \cap Y$;*
3. *$d(x_1, y_2) \leq d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$;*
   for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

Then, d is called bipolar metric and (X, Y, d) is called bipolar metric space.

If $X \cap Y = \phi$, then the space is called disjoint; otherwise, it is called joint. The set X is called the left pole and the set Y is called the right pole of (X, Y, d). The elements of X, Y and $X \cap Y$ are called left, right and central elements, respectively.

Definition 2 ([18])**.** *Let (X, Y, d) be a bipolar metric space. Then, any sequence $\{x_n\} \subseteq X$ is called a left sequence and is said to be convergent to the right element; for example, y if $d(x_n, y) \to 0$ as $n \to \infty$. Similarly, a right sequence $\{y_n\} \subseteq Y$ is said to be convergent to a left element; for example, x if $d(x, y_n) \to 0$ as $n \to \infty$.*

Definition 3 ([18])**.** *Let (X, Y, d) be a bipolar metric space.*

1. *A sequence $\{x_n, y_n\}$ on $X \times Y$ is called a bisequence on (X, Y, d).*
2. *If both the sequences $\{x_n\}$ and $\{y_n\}$ converge, then the bisequence $\{x_n, y_n\}$ is said to be convergent. If both sequences $\{x_n\}$ and $\{y_n\}$ converge to the same point $u \in X \cap Y$, then the bisequence $\{x_n, y_n\}$ is called biconvergent.*
3. *A bisequence $\{x_n, y_n\}$ on (X, Y, d) is said to be a Cauchy bisequence if for each $\epsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $d(x_n, y_m) < \epsilon$ for all $n, m \geq N$.*
4. *A bipolar metric space is said to be complete if every Cauchy bisequence is convergent in this space.*

Definition 4 ([18])**.** *Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be two bipolar metric spaces and $T : X_1 \cup Y_1 \to X_2 \cup Y_2$ be a function:*

1. *If $TX_1 \subseteq X_2$ and $TY_1 \subseteq Y_2$, then T is called covariant mapping and is denoted by $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$.*
2. *If $TX_1 \subseteq Y_2$ and $TY_1 \subseteq X_2$, then T is called contravariant mapping and is denoted by $T : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$.*

Definition 5 ([18])**.** *Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be two bipolar metric spaces.*

1. *A map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called left continuous at a point $x_0 \in X$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_2(Tx_0, Ty) < \epsilon$ whenever $d_1(x_0, y) < \delta$.*
2. *A map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called right continuous at a point $y_0 \in Y$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_2(Tx, Ty_0) < \epsilon$ whenever $d_1(x, y_0) < \delta$.*
3. *A map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called continuous if it is left continuous at each $x_0 \in X$ and right continuous at each $y_0 \in Y$.*
4. *A map $T : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$ is called continuous if and only if it is continuous as a covariant map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$*

Definition 6 ([20])**.** *Let $T : (X, Y) \rightrightarrows (X, Y)$ and $\alpha : X \times Y \to [0, \infty)$. Then, T is called α-admissible if*

$$\alpha(x, y) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1, \tag{1}$$

for all $(x, y) \in X \times Y$.

Definition 7 ([20]). *Let $T : (X,Y) \rightleftarrows (X,Y)$ and $\alpha : X \times Y \to [0,\infty)$. Then, T is called α-admissible if*

$$\alpha(x,y) \geq 1 \text{ implies } \alpha(Ty,Tx) \geq 1, \tag{2}$$

for all $(x,y) \in X \times Y$.

Definition 8 ([15]). *Let Ψ be the family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:*

1. *ψ is non-decreasing.*
2. *$\sum_{n=1}^{+\infty} \psi^n < \infty$ for all $t > 0$, where ψ^n is the n^{th} iterate of ψ.*

These functions are known as (c)-comparison functions. It can be easily verified that $\psi(t) < t$ for any $t > 0$.

3. Results

Here, we introduce $(\alpha - \psi)$ Meir–Keeler contractions and α-orbital admissible mappings and prove fixed point theorems for these contractions in bipolar metric spaces.

Definition 9. *Let $T : (X,Y) \rightleftarrows (X,Y)$ and $\alpha : X \times Y \to \mathbb{R}$. Then, T is called an α-orbital admissible mapping if*

$$\alpha(x,Tx) \geq 1 \Rightarrow \alpha(T^2x,Tx) \geq 1, \tag{3}$$

and

$$\alpha(Ty,y) \geq 1 \Rightarrow \alpha(Ty,T^2y) \geq 1, \tag{4}$$

For all $(x,y) \in X \times Y$.

Definition 10. *Let (X,Y,d) be a bipolar metric space and $\psi \in \Psi$. Suppose $T : (X,Y) \rightleftarrows (X,Y)$ is an contravariant mapping and if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\epsilon \leq \psi(d(x,y)) < \epsilon + \delta \Rightarrow \alpha(x,Tx)\alpha(Ty,y)\psi(d(Ty,Tx)) < \epsilon, \tag{5}$$

for all $(x,y) \in X \times Y$ and $\alpha : X \times Y \to \mathbb{R}$.

Then, T is said to be contravariant $(\alpha - \psi)$ Meir–Keeler contractive mapping.

Remark 1. *From (5), we get $\alpha(x,Tx)\alpha(Ty,y)\psi(d(Ty,Tx)) < \psi(d(x,y))$, when $x \neq y$. If $x = y$ then $\alpha(x,Tx)\alpha(Ty,y)\psi(d(Ty,Tx)) \leq \psi(d(x,y))$.*

Now, we present our first theorem.

Theorem 1. *Let (X,Y,d) be a complete bipolar metric space. Suppose that $T : (X,Y) \rightleftarrows (X,Y)$ is a contravariant $(\alpha - \psi)$ Meir–Keeler contractive mapping. If the following conditions hold,*

1. *T is α-orbital admissible,*
2. *There exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$,*
3. *T is continuous,*

then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$. Construct the sequences $\{x_n\}$ and $\{y_n\}$ by taking $y_n = Tx_n$ and $x_{n+1} = Ty_n$ for all $n \in \mathbb{N}$. Clearly, $\{x_n,y_n\}$ is a bisequence.

Since T is α-admissible, we obtain

$$\begin{aligned}
\alpha(x_0,y_0) &= \alpha(x_0,Tx_0) \geq 1 \Rightarrow \alpha(T^2x_0,Tx_0) = \alpha(x_1,y_0) \geq 1, \\
\alpha(x_1,y_0) &= \alpha(Ty_0,y_0) \geq 1 \Rightarrow \alpha(Ty_0,T^2y_0) = \alpha(x_1,y_1) \geq 1, \\
\alpha(x_1,y_1) &= \alpha(x_1,Tx_1) \geq 1 \Rightarrow \alpha(T2x_1,Tx_1) = \alpha(x_2,y_1) \geq 1, \\
\alpha(x_2,y_1) &= \alpha(Ty_1,y_1) \Rightarrow \alpha(Ty_1,T^2y_1) = \alpha(x_2,y_2) \geq 1.
\end{aligned}$$

By continuing this process, we get

$$\alpha(x_n, y_n) \geq 1 \; and \; \alpha(x_{n+1}, y_n) \geq 1 \; for \;\; all \; n \in \mathbb{N}. \tag{6}$$

Using Remark 1 and (6), we get

$$\begin{aligned} \psi(d(x_n, y_n)) &= \psi(d(Ty_{n-1}, Tx_n)) \leq \alpha(x_n, y_n)\alpha(x_n, y_{n-1})\psi(d(Ty_{n-1}, Tx_n)), \\ &= \alpha(x_n, Tx_n)\alpha(Ty_{n-1}, y_{n-1})\psi(d(Ty_{n-1}, Tx_n)), \\ &< \psi(d(x_n, y_{n-1})). \end{aligned} \tag{7}$$

Using again Remark 1 and (6), we get

$$\begin{aligned} \psi(d(x_{n+1}, y_n)) &= \psi(d(Ty_n, Tx_n)) \leq \alpha(x_n, y_n)\alpha(x_{n+1}, y_n), \\ &= \alpha(x_n, Tx_n)\alpha(d(Ty_n, y_n))\psi(d(x_n, y_n)), \\ &< \psi(d(x_n, y_n)). \end{aligned} \tag{8}$$

From (7) and (8), using mathematical induction, we have

$$\psi(d(x_n, y_n)) < \psi(d(x_{n-1}, y_{n-1}))\forall n \in \mathbb{N} \tag{9}$$

and

$$\psi(d(x_{n+1}, y_n)) < \psi(d(x_n, y_{n-1}))\forall n \in \mathbb{N}. \tag{10}$$

From (9) and (10), it is clear that $\{\psi(d(x_n, y_n))\}$ and $\{\psi(d(x_{n+1}, y_n))\}$ are monotonically decreasing sequences of positive reals and hence convergent. Let $\{\psi(d(x_n, y_n))\} \to s_1$ and $\{\psi(d(x_{n+1}, y_n))\} \to s_2$ as $n \to \infty$, where $s_1, s_2 \geq 0$.

Now, we prove that $s_1 = 0$ and $s_2 = 0$.

Firstly, suppose if possible that $s_1 > 0$.

Clearly, $\psi(d(x_n, y_n)) \geq s_1 > 0$ for all $n \in \mathbb{N}$.

Let $\epsilon = s_1$. Then, by hypothesis, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\epsilon \;\; \leq \;\; \psi(d(x_{n_0}, y_{n_0})) < \epsilon + \delta. \tag{11}$$

From (5), we have

$$\begin{aligned} \psi(d(x_{n_0+1}, y_{n_0+1})) &\leq \alpha(x_{n_0+1}, y_{n_0+1})\alpha(x_{n_0+1}, y_{n_0})\psi(d(x_{n_0+1}, y_{n_0+1})), \\ &= \alpha(x_{n_0+1}, Tx_{n_0+1})\alpha(Ty_{n_0}, y_{n_0})\psi(d(Ty_{n_0}, Tx_{n_0+1})) < \epsilon = s_1, \end{aligned}$$

a contradiction.

So, $s_1 = 0$.

Similarly, one can prove easily that $s_2 = 0$.

Hence, $\psi(d(x_n, y_n)) \to 0$ and $\psi(d(x_{n+1}, y_n)) \to 0$ as $n \to \infty$. By using the definition of continuity of ψ at $t = 0$, we can say that

$$d(x_n, y_n) \to 0 \; and \; d(x_{n+1}, y_n) \to 0 \; as \; n \to \infty. \tag{12}$$

For a given $\epsilon > 0$, by the hypothesis, there exists $\delta > 0$ such that (5) holds. Without loss of generality, let us assume that $\delta < \epsilon$.

Since $\psi(d(x_n, y_n)) \to 0$ and $\psi(d(x_{n+1}, y_n)) \to 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\psi(d(x_{n-1}, y_{n-1})) < \frac{\delta}{3} \; for \;\; all \; n \geq N_1, \tag{13}$$

$$\psi(d(x_n, y_{n-1})) < \frac{\delta}{3} \; for \; all \; n \geq N_2. \tag{14}$$

Now, we shall prove that

$$\psi(d(x_{n+l}, y_n)) < \epsilon \tag{15}$$

and

$$\psi(d(x_n, y_{n+l})) < \epsilon, \ for \ \ all \ n \geq N. \tag{16}$$

where $N = \max\{N_1, N_2\}$.

Firstly, using mathematical induction, we prove (15), that is $\psi(d(x_{n+l}, y_n)) < \epsilon$. From (14), clearly the inequality holds for $l = 1$.

Suppose that the result is true for some $l = k$, that is

$$\psi(d(x_{n+k}, y_n)) < \epsilon, \ for \ \ all \ n \geq N. \tag{17}$$

Now, by using the definition of bipolar metric space, (13), (14) and (17), we get

$$\begin{aligned} \psi(d(x_{n+k}, y_{n-1})) &\leq \psi(d(x_{n+k}, y_n) + d(x_n, y_n) + d(x_n, y_{n-1})) \\ &\leq \psi(d(x_{n+k}, y_n)) + \psi(d(x_n, y_n)) + \psi(d(x_n, y_{n-1})) \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \epsilon = \frac{2\delta}{3} + \epsilon < \epsilon + \delta. \end{aligned} \tag{18}$$

If $\psi(d(x_{n+k}, y_{n-1})) \geq \epsilon$, then by (5), we have

$$\begin{aligned} \psi(d(x_{n+k+1}, y_n)) &\leq \alpha(x_n, y_n)\alpha(x_{n+k+1}, y_{n+k})\psi(d(x_{n+k+1}, y_n)), \\ &= \alpha(x_n, Tx_n)\alpha(Ty_{n+k}, y_{n+k})\psi(d(Ty_{n+k}, Tx_n)), \\ &< \epsilon. \end{aligned}$$

Hence, (15) holds.

If $\psi(d(x_{n+k}, y_{n-1})) \leq \epsilon$, then by Remark 1, we have

$$\begin{aligned} \psi(d(x_{n+k+1}, y_n)) &\leq \alpha(x_n, y_n)\alpha(x_{n+k+1}, y_{n+k})\psi(d(x_{n+k+1}, y_n)), \\ &= \alpha(x_n, Tx_n)\alpha(Ty_{n+k}, y_{n+k})\psi(d(Ty_{n+k}, x_n)), \\ &< \psi(d(x_n, y_{n+k})) < \epsilon. \end{aligned}$$

So, (15) holds for $l = k + 1$.

Hence,

$$d(x_n, y_m) < \epsilon \ for \ \ all \ n > m \geq N. \tag{19}$$

Again, using mathematical induction, we prove (16).

Using the definition of bipolar metric space, (13) and (14), we get

$$\begin{aligned} \psi(d(x_n, y_{n+1})) &\leq \psi(d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_{n+1}, y_{n+1})) \\ &\leq \psi(d(x_{n+1}, y_{n+1})) + \psi(d(x_{n+1}, y_n)) + \psi(d(x_n, y_n)) \\ &\leq \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta < \epsilon. \end{aligned}$$

So, (16) holds for $l = 1$.

Now, let us suppose that the result is true for some $l = k$, that is,

$$\psi(d(x_n, y_{n+k})) < \epsilon \ for \ \ all \ n \geq N. \tag{20}$$

Now, by using the definition of bipolar metric space, (13), (14) and (20), we get

$$\begin{aligned} \psi(d(x_{n-1}, y_{n+k})) &\leq \psi(d(x_{n-1}, y_{n-1}) + d(x_n, y_{n-1}) + d(x_n, y_{n+k})), \\ &\leq \psi(d(x_{n-1}, y_{n-1}) + \psi(d(x_n, y_{n-1}) + \psi(d(x_n, y_{n+k})) \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \epsilon = \frac{2\delta}{3} + \epsilon < \epsilon + \delta. \end{aligned} \tag{21}$$

If $\psi(d(x_{n-1}, y_{n+k})) \geq \epsilon$, then by (5), we have

$$\begin{aligned} \psi(d(x_n, y_{n+k+1})) &\leq \alpha(x_{n+k}, y_{n+k})\alpha(x_{n+1}, y_n)\psi(d(x_n, y_{n+k+1})), \\ &= \alpha(x_{n+k}, Tx_{n+k})\alpha(Ty_n, y_n)\psi(d(Tx_{n+k}, Ty_n)), \\ &< \epsilon. \end{aligned}$$

Hence, (16) holds.
If $\psi(d(x_{n-1}, y_{n+k})) < \epsilon$, then by Remark 1, we have

$$\begin{aligned} \psi(d(x_n, y_{n+k+1})) &\leq \alpha(x_{n+k}, y_{n+k})\alpha(x_{n+1}, y_n)\psi(d(x_n, y_{n+k+1})), \\ &= \alpha(x_{n+k}, Tx_{n+k})\alpha(Ty_n, y_n)\psi(d(Tx_{n+k}, y_n)), \\ &< \psi(d(x_{n+k}, y_n)) < \epsilon. \end{aligned}$$

So, (16) holds for $l = k + 1$.
Hence,

$$d(x_n, y_m) < \epsilon \; for \;\; all \; m > n \geq N. \tag{22}$$

From (19) and (22), we can say that $\{x_n, y_n\}$ is a Cauchy bisequence. Since (X, Y, d) is a complete bipolar metric space, then $\{x_n, y_n\}$ biconverges. That is, there exists $u \in X \cap Y$ such that $\{x_n\} \to u$ and $\{y_n\} \to u$ as $n \to \infty$. As T is a continuous map, one has

$$(x_n) \to u \; implies \;\; that \; y_n = Tx_n \to Tu.$$

Combining $y_n = Tx_n \to Tu$ with $(y_n) \to u$, we get $Tu = u$. □

In the next theorem, we omit continuity and give a new condition to get the fixed point.

Theorem 2. *Let (X, Y, d) be a complete bipolar metric space. Suppose that $T : (X, Y) \rightleftarrows (X, Y)$ is a contravariant $(\alpha - \psi)$ Meir–Keeler contractive mapping. If the following conditions hold,*

1. *T is α-orbital admissible,*
2. *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*
3. *If $\{x_n, y_n\}$ is a bisequence such that $\alpha(x_n, y_n) \geq 1$ for all n and $y_n \to u \in X \cap Y$ as $n \to \infty$, then $\alpha(Tu, u) \geq 1$,*

then T has a fixed point.

Proof. From the proof of Theorem 1, we conclude that $\{x_n, y_n\}$ is a Cauchy bisequence. Since (X, Y, d) is a complete bipolar metric space, then $\{x_n, y_n\}$ is biconvergent. Hence, there exist $u \in X \cap Y$ such that $x_n \to u$, $y_n \to u$.

From condition (3), we get $\alpha(Tu, u) \geq 1$.

By applying the definition of bipolar metric space, ψ, Remark 1, (6) and the above inequality, we get

$$\begin{aligned} \psi(d(Tu, u)) &\leq \psi(d(Tu, Tx_n) + d(Ty_n, Tx_n) + d(Ty_n, u)), \\ &\leq \psi(d(Tu, Tx_n)) + \psi(d(Ty_n, Tx_n)) + \psi(d(Ty_n, u)), \\ &\leq \alpha(x_n, y_n)\alpha(Tu, u)\psi(d(Tu, Tx_n)), \\ &+ \alpha(x_n, Tx_n)\alpha(Ty_n, y_n)\psi(d(Ty_n, Tx_n)), \\ &+ \alpha(x_n, Tx_n)\alpha(Ty_n, y_n)\psi(d(Ty_n, Tx_n)) + \psi(d(Ty_n, u)), \\ &\leq \psi(d(x_n, u)) + \psi(d(x_n, y_n)) + \psi(d(u, y_n)). \end{aligned}$$

Letting $n \to \infty$ in the above inequality and using (22), we get

$$\psi(d(Tu, u)) \leq 0.$$

That is, $d(Tu, u) = 0$.

Hence, $Tu = u$. $\square$

Now, we introduce generalized $(\alpha - \psi)$ Meir–Keeler contractive mappings and prove fixed point theorem for these mappings.

Definition 11. *Let (X, Y, d) be a bipolar metric space and $\psi \in \Psi$. Suppose $T : (X, Y) \rightleftarrows (X, Y)$ be an contravariant mapping and that for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\epsilon \leq \psi(M(x, y)) < \epsilon + \delta \Rightarrow \alpha(x, Tx)\alpha(Ty, y)\psi(d(Ty, Tx)) < \epsilon, \tag{23}$$

where $M(x, y) = max\{d(x, y), d(x, Tx), d(Ty, y), \frac{d(x,Tx)+d(Ty,y)}{2}\}$; for all $(x, y) \in X \times Y$.
Then, T is said to be a generalized contravariant $(\alpha - \psi)$ Meir–Keeler contractive mapping.

Remark 2. *From (23), we get $\alpha(x, Tx)\alpha(Ty, y)\psi(d(Ty, Tx)) < \psi(M(x, y))$, when $x \neq y$. If $x = y$ then $\alpha(x, y)\psi(d(Ty, Tx)) \leq \psi(M(x, y))$.*

Theorem 3. *Let (X, Y, d) be a complete bipolar metric space. Suppose that $T : (X, Y) \rightleftarrows (X, Y)$ is a generalized contravariant $(\alpha - \psi)$ Meir–Keeler contractive mapping. If the following conditions hold,*

1. *T is α-orbital admissible,*
2. *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*
3. *T is orbital continuous,*

then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Construct sequences $\{x_n\}$ and $\{y_n\}$ by taking $y_n = Tx_n$ and $x_{n+1} = Ty_n$ for all $n \in \mathbb{N}$. Clearly $\{x_n, y_n\}$ is a bisequence.
Since T is α-orbital admissible, from Theorem 1, we get

$$\alpha(x_n, y_n) \geq 1 \ and \ \alpha(x_{n+1}, y_n) \geq 1 \ for \ \ all \ n \in \mathbb{N}. \tag{24}$$

Using Remark 2 and (24), we get

$$\begin{aligned}
\psi(d(x_n, y_n)) &= \psi(d(Ty_{n-1}, Tx_n)) \leq \alpha(x_n, Tx_n)\alpha(Ty_{n-1}, y_{n-1})\psi(d(Ty_{n-1}, Tx_n)), \\
&< \psi(M(x_n, y_{n-1})), \\
&= \psi(max\{d(x_n, y_{n-1}), d(x_n, Tx_n), d(Ty_{n-1}, y_{n-1}), \frac{d(x_n, Tx_n) + d(Ty_{n-1}, y_{n-1})}{2}\}), \\
&= \psi(max\{d(x_n, y_{n-1}), d(x_n, y_n), d(x_n, y_{n-1}), \frac{d(x_n, y_n) + d(x_n, y_{n-1})}{2}\}), \\
&\leq \psi(max\{d(x_n, y_n), d(x_n, y_{n-1})\}).
\end{aligned}$$

Now, since ψ is a non-decreasing function, one has $d(x_n, y_n) \leq max\{d(x_n, y_n), d(x_n, y_{n-1})\}$.
If possible, suppose that $d(x_n, y_n) > d(x_n, y_{n-1})$, then $d(x_n, y_n) < d(x_n, y_n)$, a contradiction.
Hence,

$$d(x_n, y_n) \leq d(x_n, y_{n-1}), \ for \ \ all \ n \in \mathbb{N}. \tag{25}$$

Similarly, by using Remark 2 and (24), one can easily obtain

$$d(x_{n+1}, y_n) \leq d(x_n, y_n), \ for \ \ all \ n \in \mathbb{N}. \tag{26}$$

From (25) and (26), it is clear that $\{d(x_n, y_n)\}$ and $\{d(x_{n+1}, y_n)\}$ are monotonically decreasing sequences of positive reals and hence convergent. Let $\{d(x_n, y_n)\} \to s_1$ and $\{d(x_{n+1}, y_n)\} \to s_2$ as $n \to \infty$, where $s_1, s_2 \geq 0$. This implies that

$$\lim\{psid(x_n, y_n)\} = \lim\{\psi(M(x_n, y_n))\} = \psi(s_1) \ as \ n \to \infty. \tag{27}$$

and

$$\lim\{\psi(d(x_{n+1},y_n))\} = \lim\{\psi(M(x_{n+1},y_n))\} = \psi(s_2) \ as \ n \to \infty. \tag{28}$$

Now, we prove that $s_1 = 0$ and $s_2 = 0$.

Firstly, suppose that $s_1 > 0$.

Clearly, $d(x_n, y_n) \geq s_1 > 0$ for all $n \in \mathbb{N}$.

Let $\epsilon = s_1$. Then, by hypothesis, there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\psi(\epsilon) \quad \leq \quad \psi(M(x_{n_0}, y_{n_0})) < \psi(\epsilon) + \delta. \tag{29}$$

From (23), we have

$$\begin{aligned}\psi(d(x_{n_0+1}, y_{n_0+1})) &\leq \alpha(x_{n_0+1}, y_{n_0+1})\alpha(x_{n_0+1}, y_{n_0})\psi(d(x_{n_0+1}, y_{n_0+1})),\\ &= \alpha(Ty_{n_0}, y_{n_0})\alpha(x_{n_0+1}, Tx_{n_0+1})\psi(d(Ty_{n_0}, Tx_{n_0+1})) < \psi(\epsilon).\end{aligned}$$

Using non-decreasing nature of ψ, we get

$$d(x_{n_0+1}, y_{n_0+1}) < \epsilon = s_1. \tag{30}$$

a contradiction. So, $s_1 = 0$.

Similarly, one can prove easily that $s_2 = 0$.

Now, we prove that $\{x_n, y_n\}$ is a Cauchy bisequence; that is, $\lim_{n,m\to\infty} d(x_n, y_m) = 0$

Indeed, if we suppose that $\{x_n, y_n\}$ is not a Cauchy bisequence, then there exists $\epsilon > 0$ and subsequences $\{n(i)\}$ and $\{n(i+1)\}$ of natural numbers such that

$$d(x_{n(i)}, y_{n(i+1)}) > 2\epsilon, \tag{31}$$

for all $i \in \mathbb{N}$. For this $\epsilon > 0$ there exists $\delta > 0$ such that $\epsilon \leq \psi(M(x,y)) < \epsilon + \delta$ implies that $\alpha(x, Tx)\alpha(Ty, y)\psi(d(Ty, Tx)) < \epsilon$.

Set $r = min\{\epsilon, \delta\}$. Since $d(x_n, y_n)$ and $d(x_{n+1}, y_n) \to 0$ as $n \to \infty$, there exists $n_1, n_2 \in \mathbb{N}$ such that

$$d(x_n, y_n) \quad < \quad \frac{r}{8} \ for \ \ all \ n \geq n_1, \ and \tag{32}$$

$$d(x_{n+1}, y_n) \quad < \quad \frac{r}{8} \ for \ \ all \ n \geq n_2. \tag{33}$$

Choose $N = max\{n_1, n_2\}$. Then, the above inequalities still hold for all $n \geq N$.

Let $n(i) > N$. We get $n(i) \leq n(i+1) - 1$. If $d(x_{n(i)}, y_{n(i+1)-1}) \leq \epsilon + \frac{r}{2}$; then, using the definition of bipolar metric space, (32) and (33), we have

$$\begin{aligned}d(x_{n(i)}, y_{n(i+1)}) &\leq d(x_{n(i)}, y_{n(i+1)-1}) + d(x_{n(i+1)}, y_{n(i+1)-1}) + d(x_{n(i+1)}, y_{n(i+1)}),\\ &< \epsilon + \frac{r}{2} + \frac{r}{8} + \frac{r}{8},\\ &= \epsilon + \frac{3}{4}r < 2\epsilon,\end{aligned}$$

a contradiction. So, there exists k such that $n(i) \leq k \leq n(i+1)$ and $d(x_{n(i)}, y_k) > \epsilon + \frac{r}{2}$. Now if $d(x_{n(i)+1}, y_{n(i)}) \geq \epsilon + \frac{r}{2}$, then by (35), $d(x_{n(i)+1}, y_{n(i)}) \geq \epsilon + \frac{r}{2} > r + \frac{r}{2} > \frac{r}{8}$, a contradiction.

So, there exist values of k such that $n(i) \leq k \leq n(i+1)$ such that $d(x_{n(i)}, y_k) < \epsilon + \frac{r}{2}$. Choose the smallest integer k with $k \geq n(i)$ such that $d(x_{n(i)}, y_k) \geq \epsilon + \frac{r}{2}$.Thus, $d(x_{n(i)}, y_{k-1}) < \epsilon + \frac{r}{2}$.

Using the definition of bipolar metric space and (33), we get

$$\begin{aligned}d(x_{n(i)}, y_k) &\leq d(x_{n(i)}, y_{k-1}) + d(x_k, y_{k-1}) + d(x_k, y_k)\\ &\leq \epsilon + \frac{r}{2} + \frac{r}{8} + \frac{r}{8} = \epsilon + \frac{3}{4}r.\end{aligned}$$

Now, we can choose a natural number k satisfying $n(i) \leq l \leq n(i+1)$ such that

$$\epsilon + \frac{r}{2} \leq d(x_{n(i)}, y_k) < \epsilon + \frac{3}{4}r. \tag{34}$$

Therefore,

$$d(x_{n(i)}, y_k) \leq \epsilon + \frac{3}{4}r < \epsilon + r, \tag{35}$$

$$d(x_{n(i)}, y_{n(i)}) \leq \frac{r}{8} < \epsilon + r, \tag{36}$$

$$d(x_{k+1}, y_k) \leq \frac{r}{8} < \epsilon + r. \tag{37}$$

Now, (35)–(37) imply that $\epsilon \leq M(x_{n(i)}, y_k) < \epsilon + r \leq \epsilon + \delta$ and so $\psi(\epsilon) \leq \psi(M(x_{n(i)}, y_k)) < \psi(\epsilon + r) \leq \psi(\epsilon + \delta) \leq \psi(\epsilon) + \psi(\delta)$.
Since T is a generalized $(\alpha - \psi)$ Meir–Keeler contractive mapping,

$$\psi(d(x_{k+1}, y_{n(i)})) \leq \alpha(x_{n(i)}, Tx_{n(i)})\alpha(Ty_k, y_k)\psi(d(Ty_k, Tx_{n(i)})) < \psi(\epsilon).$$

This implies that

$$d(x_{k+1}, y_{n(i)}) < \epsilon. \tag{38}$$

Using the definition of bipolar metric space, we get

$$d(x_{n(i)}, y_k) \leq d(x_{n(i)}, y_{n(i)}) + d(x_{k+1}, y_{n(i)}) + d(x_{k+1}, y_k),$$

which implies that

$$\begin{aligned} d(x_{n(i),y_k}) - d(x_{n(i)}, y_{n(i)}) - d(x_{k+1}, y_k) &\leq d(x_{k+1}, y_{n(i)}) \\ \epsilon + \frac{r}{2} - \frac{r}{8} - \frac{r}{8} &< d(x_{k+1}, y_{n(i)}). \end{aligned}$$

This shows that

$$\epsilon < d(x_{k+1}, y_{n(i)}). \tag{39}$$

This contradicts (38).

So, $\{x_n, y_n\}$ is a Cauchy bisequence. Since (X, Y, d) is a complete bipolar metric space, then $\{x_n, y_n\}$ biconverges. That is, there exists $u \in X \cap Y$ such that $\{x_n\} \to u$ and $\{y_n\} \to u$ as $n \to \infty$. As T is an orbital continuous map,

$$\{x_n\} \to u \textit{ implies that } y_n = Tx_n \to Tu.$$

Combining $y_n = Tx_n \to Tu$ with $y_n \to u$, we have $Tu = u$. □

In the next theorem, we add a condition to get a unique fixed point.

Theorem 4. *If in Theorems 1–3 we add the following hypothesis (H), then we get the unique fixed point.*
(H) If $Tx = x$ then $\alpha(x, Tx) \geq 1$.

Proof. If possible, let us suppose that T has two distinct fixed points u and v. Then, from the hypothesis (H),
$\alpha(u, Tu), \alpha(v, Tv) \geq 1$.
Now, by Remark 1,

$$d(u, v) = d(Tu, Tv) \leq \alpha(u, Tu)\alpha(v, Tv)d(Tu, Tv) < d(u, v)$$

which is a contradiction and so $u = v$. In a similar way, one can prove Theorems 2 and 3. □

Definition 12. *Let (X, Y, d) be a bipolar metric space. Suppose $T : (X, Y) \rightrightarrows (X, Y)$ be a covariant mapping and for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\epsilon \leq d(x, y) < \epsilon + \delta \Rightarrow d(Tx, Ty) < \epsilon, \tag{40}$$

for all $(x, y) \in X \times Y$.

Then, T is said to be a covariant Meir–Keeler contractive mapping.

Remark 3. *From (40), we get $d(Tx, Ty) < d(x, y)$, whenever $x \neq y$. If $x = y$ then $d(Tx, Ty) \leq d(x, y)$.*

Theorem 5. *Let (X, Y, d) be a complete bipolar metric space. Suppose that $T : (X, Y) \rightrightarrows (X, Y)$ is a covariant Meir–Keeler contractive mapping. Then, T has a unique fixed point.*

Proof. Using Remark 3 and (40), we get

$$d(x_n, y_n) = d(Tx_{n-1}, Ty_{n-1}) \leq d(x_{n-1}, y_{n-1}) \tag{41}$$

Again, using Remark 3 and (40), we get

$$d(x_n, y_n) = d(Tx_{n-1}, Ty_{n-1}) \leq d(x_{n-1}, y_{n-1}) \tag{42}$$

From (41) and (42), it is clear that $\{d(x_n, y_n)\}$ and $\{d(x_n, y_{n+1})\}$ are monotonically decreasing sequences of positive reals and hence convergent. Let $\{d(x_n, y_n)\} \to s_1$ and $\{d(x_n, y_{n+1})\} \to s_2$ as $n \to \infty$, where $s_1, s_2 \geq 0$.

Now, we prove that $s_1 = 0$ and $s_2 = 0$.

Firstly, suppose, if possible that $s_1 > 0$.

Clearly, $d(x_n, y_n) \geq s_1 > 0$ for all $n \in \mathbb{N}$.

Let $\epsilon = s_1$. Then, by hypothesis, there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\epsilon \quad \leq \quad d(x_{n_0}, y_{n_0}) < \epsilon + \delta. \tag{43}$$

From (40), we have

$$\begin{aligned} d(x_{n_0+1}, y_{n_0+1}) &\leq d(x_{n_0+1}, y_{n_0+1}), \\ &= d(Tx_{n_0}, Ty_{n_0}) < \epsilon = s_1. \end{aligned}$$

a contradiction. So $s_1 = 0$.

Similarly, one can prove easily that $s_2 = 0$.

Hence,

$$d(x_n, y_n) \to 0 \text{ and } d(x_n, y_{n+1}) \to 0 \text{ as } n \to \infty. \tag{44}$$

For given $\epsilon > 0$, by the hypothesis, there exists $\delta > 0$ such that (40) holds. Without loss of generality, let us assume that $\delta < \epsilon$.

Since $d(x_n, y_n) \to 0$ and $d(x_n, y_{n+1}) \to 0$, then there exists $N_1, N_2 \in \mathbb{N}$ such that

$$d(x_{n-1}, y_{n-1}) < \frac{\delta}{3} \; for \;\; all \; n \geq N_1, \tag{45}$$

$$d(x_{n-1}, y_n) < \frac{\delta}{3} \; for \; all \; n \geq N_2. \tag{46}$$

Now, we shall prove that

$$d(x_n, y_{n+l}) < \epsilon \tag{47}$$

and

$$d(x_{n+l}, y_n) < \epsilon, \; for \;\; all \; n \geq N. \tag{48}$$

where $N = \max\{N_1, N_2\}$.

Firstly, using mathematical induction, we prove (47), that is $d(x_n, y_{n+l}) < \epsilon$.

From (44), the inequality clearly holds for $l = 1$.

Suppose that it is true for some $l = k$, that is

$$d(x_n, y_{n+k}) < \epsilon, \ for \ \ all \ n \geq N. \tag{49}$$

Now, by using the definition of bipolar metric space, (45), (46) and (49), we get

$$\begin{aligned} d(x_{n-1}, y_{n+k}) &\leq d(x_{n-1}, y_n) + d(x_n, y_n) + d(x_n, y_{n+k}) \\ &\leq d(x_{n-1}, y_n) + d(x_n, y_n) + d(x_n, y_{n+k}) \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \epsilon = \frac{2\delta}{3} + \epsilon < \epsilon + \delta. \end{aligned} \tag{50}$$

If $d(x_{n-1}, y_{n+k}) \geq \epsilon$, then by (40), we have

$$d(x_n, y_{n+k+1}) < \epsilon.$$

Hence, (47) holds.

If $d(x_{n+k}, y_{n-1}) \leq \epsilon$, then by Remark 3, we have

$$d(x_{n+k+1}, y_n) < d(x_{n+k}, y_{n-1}) < \epsilon$$

So, Equation (47) holds for $l = k + 1$.

Hence,

$$d(x_n, y_m) < \epsilon \ for \ \ all \ n > m \geq N. \tag{51}$$

Similarly, one can prove Equation (48), from which we conclude that

$$d(x_n, y_m) < \epsilon \ for \ \ all \ m > n \geq N. \tag{52}$$

From (51) and (52), we can say that $\{x_n, y_n\}$ is a Cauchy bisequence. Since (X, Y, d) is a complete bipolar metric space, then $\{x_n, y_n\}$ biconverges. That is, there exists $u \in X \cap Y$ such that $\{x_n\} \to u$ and $\{y_n\} \to u$ as $n \to \infty$. Since,T is continuous,

$$\{x_n\} \to u \ implies \ \ that \ x_{n+1} = Tx_n \to Tu,$$

We get $Tu = u$.

Uniqueness: If possible, suppose that u and v are two different fixed points of T. Then, by Remark 3,

$$d(u, v) = d(Tu, Tv) < d(u, v),$$

which holds only when $u = v$. □

Example 1. *Let $X = (-\infty, 0]$, $Y = [0, \infty)$ and $d : (-\infty, 0] \times [0, \infty) \to [0, \infty)$ as $d(x, y) = |x - y|$. Then, (X, Y, d) is a complete bipolar metric space. Define $T : (-\infty, 0] \cup [0, \infty) \rightleftarrows (-\infty, 0] \cup [0, \infty)$ by $Tx = \frac{-x}{3}$, for all $x \in (-\infty, 0] \cup [0, \infty)$, and $\psi(t) = \frac{t}{2}$, $\alpha(x, y) = 1$ for all $(x, y) \in X \times Y$. $T((-\infty, 0]) \subset [0, \infty)$ and $T([0, \infty)) \subset (-\infty, 0]$. It is clear that T is a continuous contravariant mapping.*

As $x \in (-\infty, 0]$, there exists $a \in [0, \infty)$ such that $x = -a$. Now,

$$\psi(d(x, y)) = \psi(|x - y|) = \psi(| - a - y|) = \psi(a + y) = \frac{a + y}{2},$$

$$\psi(d(Ty, Tx)) = \psi(d(\frac{-y}{3}, \frac{-x}{3})) = \psi(|\frac{-y}{3} + (\frac{-x}{3})|) = \frac{a + y}{6}.$$

Clearly, by taking $\delta = 2\epsilon$, (5) is satisfied. So, all the conditions of Theorem 1 hold and T has a fixed point. Clearly, 0 is the fixed point of T.

4. Consequences

The following are the consequences of our main results.

Corollary 1. *Let (X, Y, d) be a bipolar metric space and $\psi \in \Psi$. Suppose $T : (X, Y) \rightleftarrows (X, Y)$ be a contravariant mapping and if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\epsilon \leq \psi(d(x, y)) < \epsilon + \delta \Rightarrow \psi(d(Ty, Tx)) < \frac{\epsilon}{L}. \tag{53}$$

where $\psi \in \Psi$ and $L \geq 1$. Then, T has a fixed point.

Proof. Taking $\alpha(x, y) = \sqrt{L}$ in Theorem 1, one can obtain the proof. □

Corollary 2. *Let (X, Y, d) be a bipolar metric space and $\psi \in \Psi$. Suppose $T : (X, Y) \rightleftarrows (X, Y)$ be a contravariant mapping and if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\epsilon \leq M(d(x, y)) < \epsilon + \delta \Rightarrow \psi(d(Ty, Tx)) < \frac{\epsilon}{L}. \tag{54}$$

where $\psi \in \Psi$ and $L \geq 1$. Then, T has a fixed point.

Proof. Taking $\alpha(x, y) = \sqrt{L}$ in Theorem 3, one can obtain the proof. □

5. Application

Theorem 6. *Let us consider the following integral equation*

$$\mathfrak{w}(\beta) = \mathfrak{m}(\beta) + \lambda_1 \int \mathfrak{P}_1(\beta, \xi, \mathfrak{w}(\xi))d\xi + \lambda_2 \int \mathfrak{P}_2(\beta, \xi, \mathfrak{w}(\xi))d\xi \tag{55}$$

$\beta \in F_1 \cup F_2$, $F_1 \cup F_2$ is a Lebesgue measurable set with finite measure and λ_1, λ_2 are constants.

Suppose that $\mathfrak{P}_1 : F_1^2 \cup F_2^2 \times [0, \infty) \to [0, \infty)$ and $\mathfrak{P}_2 : F_1^2 \cup F_2^2 \times [0, \infty) \to [0, \infty)$.

There is a continuous function $\zeta : F_1^2 \cup F_2^2 \to [0, \infty)$ and $k \in (0, 1)$ such that for all $(\beta, \xi) \in F_1^2 \cup F_2^2$ and $\mathfrak{m}(\beta) \in L^\infty(F_1) \cup L^\infty(F_2)$

$$|\lambda_i(\mathfrak{P}_i(\beta, \xi, \mathfrak{w}(\xi))) - \lambda_i(\mathfrak{P}_i(\beta, \xi, \mathfrak{y}(\xi)))| \leq \frac{k}{4}\zeta(\beta, \xi)|\mathfrak{w}(\xi) - \mathfrak{y}(\xi)|$$

for all $i = 1, 2$ and $||\int \zeta(\beta, \xi)d\xi|| \leq 1$ that is $\sup_{\beta \in F_1 \cup F_2} \int |\zeta(\beta, \xi)d\xi| \leq 1$.

Then, (55) has a unique solution in $L^\infty(F_1) \cup L^\infty(F_2)$.

Proof. Let $X = L^\infty(F_1)$ and $Y = L^\infty(F_2)$ be two normed linear spaces, where F_1 and F_2 are two Lebesgue measurable sets with $m(F_1 \cup F_2) < \infty$.

Consider $d : X \times Y \to [0, \infty)$ as $d(x, y) = ||x - y||_\infty$. (X, Y, d) is a complete bipolar metric space. Define a covariant mapping as $T(\mathfrak{w}(\beta)) = \mathfrak{m}(\beta) + \lambda_1 \int \mathfrak{P}_1(\beta, \xi, \mathfrak{w}(\xi))d\xi + \lambda_2 \int \mathfrak{P}_2(\beta, \xi, \mathfrak{w}(\xi))d\xi$.

Now, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\epsilon \leq d(\mathfrak{w}(\xi), \mathfrak{y}(\xi)) < \epsilon + \delta$.

$$
\begin{aligned}
d(T\mathfrak{w}(\xi), T\mathfrak{y}(\xi)) &= ||T\mathfrak{w}(\xi) - T\mathfrak{y}(\xi)|| \\
&= ||\mathfrak{m}(\xi) + \lambda_1 \int \mathfrak{P}_1(\beta, \xi, \mathfrak{w}(\xi))d\xi + \lambda_2 \int \mathfrak{P}_2(\beta, \xi, \mathfrak{w}(\xi))d\xi \\
&- \mathfrak{m}(\xi) - \lambda_1 \int \mathfrak{P}_1(\beta, \xi, \mathfrak{y}(\xi))d\xi - \lambda_2 \int \mathfrak{P}_2(\beta, \xi, \mathfrak{y}(\xi))d\xi|| \\
&\leq \frac{1}{2}\zeta(\beta, \xi)|\mathfrak{w}(\xi) - \mathfrak{y}(\xi)| \\
&\leq \frac{1}{2}d(\mathfrak{w}(\xi), \mathfrak{y}(\xi)) \\
&< \frac{1}{2}(\epsilon + \delta) \\
&< \epsilon.
\end{aligned}
$$

Hence, all the conditions of Theorem 5 are satisfied. So, T has a unique fixed point, and (55) has a unique solution. □

Example 2. *Consider the following integral equation:*

$$
\mathfrak{w}(\beta) = 0.01\beta + 0.2\int_0^\beta \left(\frac{\xi}{4} - 0.2\beta\right)\mathfrak{w}(\beta)\, d\xi + \sin(0.1)\left(\int_0^\beta \left(-\beta + \frac{\xi}{3} + 1\right)\mathfrak{w}(\beta)\, d\xi\right).
$$

It can be verified that the solution of the above integral equation is given by

$$
\mathfrak{w}(\beta) = \frac{0.01\beta}{0.0982\beta^2 - 0.0998\beta + 1}.
$$

This solution is depicted in Figure 1.

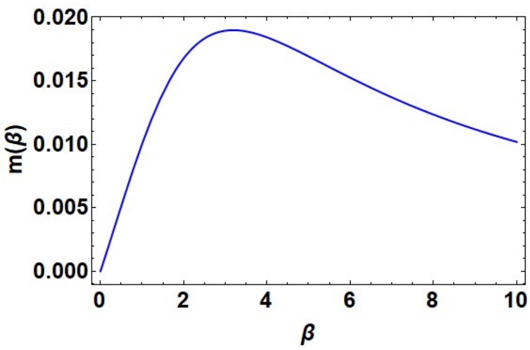

Figure 1. Solution of the integral equation in the example of Section 5.

6. Conclusions

In this paper, we have introduced a new notion of α-orbital admissible mappings, and using this we have defined $(\alpha - \psi)$ Meir–Keeler Contractive mappings and established fixed point results. Our results have generalized some proven results in the past. The derived results have been supported with non-trivial examples. The results have been applied to find analytical solutions of integral equation. It is an open problem to extend/generalize our results in the setting of other topological spaces such as bipolar controlled metric space, neutrosophic metric spaces, etc.

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