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Van der Pol Equation with a Large Feedback Delay

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Abstract: The well-known Van der Pol equation with delayed feedback is considered. It is assumed that the delay factor is large enough. In the study of the dynamics, the critical cases in the problem of the stability of the zero equilibrium state are identified. It is shown that they have infinite dimension. For such critical cases, special local analysis methods have been developed. The main result is the construction of nonlinear evolutionary boundary value problems, which play the role of normal forms. Such boundary value problems can be equations of the Ginzburg–Landau type, as well as equations with delay and special nonlinearity. The nonlocal dynamics of the constructed equations determines the local behavior of the solutions to the original equation. It is shown that similar normalized boundary value problems also arise for the Van der Pol equation with a large coefficient of the delay equation. The important problem of a small perturbation containing a large delay is considered separately. In addition, the Van der Pol equation, in which the cubic nonlinearity contains a large delay, is considered. One of the general conclusions is that the dynamics of the Van der Pol equation in the presence of a large delay is complex and diverse. It fundamentally differs from the dynamics of the classical Van der Pol equation.

Keywords: delay; bifurcations; stability; normal forms; singular perturbations; dynamics

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1. Introduction

The Van der Pol equation

$$\ddot{u} + a\dot{u} + u + \dot{u}u^2 = 0 \tag{1}$$

is the basic model for many applied problems (see, for example, [1–8]). Under the condition $a > 0$, all solutions (1) tend to the zero equilibrium state as $t \to \infty$, whereas for $a < 0$, there exists a stable cycle. Currently, a lot of research is devoted to problems that use Equation (1) with delayed feedback [9–16]

$$\ddot{u} + a\dot{u} + u + \dot{u}u^2 = \gamma u(t-T), \quad T > 0. \tag{2}$$

Numerical methods are mainly used to study the behavior of solutions to an equation of the form (2).

In this paper, it is assumed that the parameter T is sufficiently large:

$$T \gg 1.$$

This condition opens up the possibility of applying asymptotic methods to the analysis of solutions to Equation (2).

In (2), it is convenient to change $t \to Tt$. As a result, we obtain a singularly perturbed equation

$$\varepsilon^2\ddot{u} + \varepsilon a\dot{u} + u + \varepsilon\dot{u}u^2 = \gamma u(t-1), \tag{3}$$

in which

$$\varepsilon = T^{-1} \ll 1. \tag{4}$$

Note that the degenerate equation for $\varepsilon = 0$ is not informative: if $|\gamma| < 1$, its solutions tend to zero as $t \to \infty$, whereas if $|\gamma| > 1$ the solutions are unbounded as $t \to \infty$.

Here, we will study the local dynamics of Equation (3) under the condition (4), i.e., the behavior as $t \to \infty$ of all solutions with sufficiently small initial conditions in the norm $C_{[0,1]} \times R^2$.

The location of the roots of the characteristic equation

$$\varepsilon^2\lambda^2 + \varepsilon a\lambda + 1 = \gamma \exp(-\lambda) \tag{5}$$

plays an important role for the equation linearized at zero (3)

$$\varepsilon^2 \ddot{u} + \varepsilon a \dot{u} + u = \gamma u(t-1). \tag{6}$$

In the case when all roots (5) have negative real parts and are separated from the imaginary axis at $\varepsilon \to 0$, all solutions (3) from a sufficiently small (and independent of ε) neighborhood of the zero equilibrium tend to zero as $t \to \infty$. If (5) has a root with a positive real part separated from the imaginary axis at $\varepsilon \to 0$, then there cannot be stable solutions in a small neighborhood of the zero solution (5), and hence the problem of dynamics (3) is nonlocal. A necessary condition for locality is

$$a \geq 0.$$

Below, Equation (3) is studied in the critical case when Equation (5) does not have roots with positive real parts separated from the imaginary axis at $\varepsilon \to 0$, but there are roots with their real parts tending to zero as $\varepsilon \to 0$.

The roots of Equation (5) are studied in Section 2. It is shown that the critical cases have infinite dimension, that is, infinitely many roots of (5) tend to the imaginary axis at $\varepsilon \to 0$. In this situation, in the nonlinear analysis of Equation (3), it is neither possible to use the known results on the existence of local invariant integral manifolds [17,18], nor the normal form methods. Therefore, in [19,20], special asymptotic methods for the nonlinear analysis of equations with large delay were developed. These methods allowed in critical cases to construct nonlinear boundary value problems of the Ginzburg–Landau type, which do not contain small parameters, and the nonlocal dynamics of these boundary value problems determines the local behavior of the solutions to the original equation for small ε. It is possible to obtain asymptotic formulas relating the solutions of the original equation and the constructed boundary value problems. This work is based on the results in [19–22]. Note that systematic presentations of studies for the Van der Pol equation with delayed feedback are considered for the first time. All the main results are new. The solutions of the equation with a large delay were also studied analytically and numerically in [23–25]. A number of interesting results in problems of laser physics with a large delay are given in [26].

Section 3 is devoted to the nonlinear analysis of Equation (3) in critical cases. There are two fundamentally different cases. In the first, we will deal with solutions that slowly oscillate in time, whereas in the second, we study rapidly oscillating solutions. The main result in both cases is the construction of nonlinear boundary value problems which play the role of normal forms. They are called quasi-normal forms.

In Sections 4 and 5, the construction of a normal form for $a = 0$ is considered separately. The corresponding results differ significantly from the case when $a > 0$. In Section 4, the solutions are studied in the neighborhood of the equilibrium state, whereas, in Section 5, in the neighborhood of the cycle. The results of Section 6 are closely related to the results of the previous sections. Specifically, the dynamics of the Van der Pol equation with a large delay control coefficient,

$$\ddot{u} + a\dot{u} + u + \dot{u}u^2 = \gamma\big[\alpha u(t-T) + \beta\dot{u}(t-T)\big], \quad \gamma \gg 1,$$

is studied.

Section 7 is concerned with the construction of a quasi-normal form for the Van der Pol equation with a delay in the nonlinearity

$$\ddot{u} + a\dot{u} + u + \dot{u}(t-T)u^2(t-T) = 0.$$

The following sections share the same structure which we describe here. First, the parameters of the problem are identified for which the critical case is observed in the problem of the stability of the zero equilibrium state. Then, a linearized boundary value problem is considered, and its characteristic equation is given. After that, we study the asymptotic behavior of all those roots of the characteristic equation whose real part tends to zero as the small parameter ε tends to zero. There are infinitely many such roots. Based on them, a set of special solutions is constructed for the linearized problem. Such solutions can be written in a form that allows them to be used to analyze solutions (with unknown amplitudes) of the original nonlinear boundary value problem. It is possible to determine an explicit form for the leading approximation (with respect to the parameter ε) of the corresponding solution. Let us conditionally denote it here by εU_1. Then, we seek solutions of the nonlinear boundary value problem in the form

$$u(t, x, \varepsilon) = \varepsilon U_1 + \varepsilon^3 U_3 + \dots.$$

Note that, here, the absence of coefficients quadratic in ε is due to the fact that there is no quadratic nonlinearity in the original equation. Regarding U_3, it is known in advance that it is periodic in several of its arguments. Substituting the above expression for $u(t, x, \varepsilon)$, to determine U_3, we arrive at a special linear inhomogeneous boundary value problem. The solvability condition for this boundary value problem in the specified class of functions allows us to write an equation for the unknown amplitudes in U_1. Obtaining such equations is the ultimate goal. Such equations are called quasinormal forms. This term arose because, in contrast to normal forms for the finite-dimensional critical case, there is no theorem stating that the normalized equation completely describes the local behavior of solutions. The nonlocal dynamics of these equations, which are also called quasinormal forms, makes it possible to describe the local behavior of solutions to the original boundary value problem. Note that the fulfillment of the conditions for the solvability of the equations for U_3 allows us to define this function explicitly. Below, we will use the U_3 function, but sometimes we will be omitting its formula.

2. Asymptotics of the Roots of the Characteristic Equation

First, we find those values of the parameter γ for which the critical case occurs in (5).

Assume that $\lambda = i\omega\ (\omega > 0)$ is a purely imaginary root of (5). Then, from (5), we arrive at the equation

$$p(\omega) = \gamma \exp\big(-i\omega\varepsilon^{-1}\big),$$

where $p(\omega) = 1 - \omega^2 + ia\omega$. From here, we obtain

$$|p(\omega)| = |\gamma|.$$

By p_0 we denote the smallest value of function $|p(\omega)|$:

$$p_0 = \min_{\omega} |p(\omega)|,$$

and by ω_0 the value ω for which $|p(\omega_0)| = p_0$. The definition of $p(\omega)$ and p_0 immediately implies the following statement.

Lemma 1. *The following holds:*

$$\omega_0 = \begin{cases} 0, & \text{if } a^2 > 2, \\ \left(1 - \dfrac{a^2}{2}\right)^{1/2}, & \text{if } a^2 < 2. \end{cases}$$

The smallest value $|\gamma| = \gamma_0$ for which Equation (5) has a purely imaginary root is determined by

$$\gamma_0 = |p(\omega_0)| = \begin{cases} 1, & \text{if } a^2 \geq 2, \\ \dfrac{a}{2}(4 - a^2)^{1/2}, & \text{if } a^2 < 2. \end{cases}$$

Note that $\gamma_0 = 0$ for $a = 0$.

Let us formulate the main statements about the asymptotic behavior in critical cases of those roots of (5) whose real parts tend to zero as $\varepsilon \to 0$. We omit simple justifications.

Lemma 2. *Let*

$$|\gamma| < \gamma_0.$$

Then, for all sufficiently small ε, all roots (5) have negative real parts separated from zero as $\varepsilon \to 0$.

Lemma 3. *Let*

$$|\gamma| > \gamma_0.$$

Then, for all sufficiently small ε, Equation (5) has a root with a positive real part separated from zero as $\varepsilon \to 0$.

In order to justify Lemma 2, it suffices to note that for $\gamma = 0$, and hence for all $0 < |\gamma| < \gamma_0$ the real parts of all (5) roots are negative. Same for Lemma 3.

In this paper, we study critical cases when it is assumed that for arbitrary fixed γ_1, the parameter γ either satisfies the equality

$$\gamma = \gamma_0 + \varepsilon^2\gamma_1, \tag{7}$$

or

$$\gamma = -\gamma_0 - \varepsilon^2\gamma_1. \tag{8}$$

In these cases, there is no root of the characteristic equation with a positive real part separated from zero, but there are roots whose real parts tend to zero at $\varepsilon \to 0$.

Consider the asymptotic behavior of all the roots of Equation (5) whose real parts tend to zero as $\varepsilon \to 0$. The corresponding asymptotic expansions are fundamentally different for the cases when $a_0^2 > 2$ and when $a_0^2 < 2$. The justification of the corresponding expansions is based on the standard perturbation theory.

Lemma 4. *Let*

$$a_0^2 > 2. \tag{9}$$

Then, $\gamma_0 = 1$, and for those roots $\lambda_k(\varepsilon)$, $k = 0, \pm1, \pm2, \dots$, of Equation (5) whose real parts tend to to zero as $\varepsilon \to 0$, the following asymptotic equalities hold:

1. For

$$\gamma = 1 + \varepsilon^2\gamma_1 \tag{10}$$

we have

$$\lambda_k(\varepsilon) = 2\pi ik + \varepsilon\lambda_{k1} + \varepsilon^2\lambda_{k2} + O(\varepsilon^3), \quad k = 0, \pm1, \pm2, \dots,$$

where

$$\lambda_{k1} = -2\pi ika, \quad \lambda_{k2} = -2\pi^2k^2(a^2 - 2) + 2\pi ika^2 + \gamma_1.$$

2. *For*

$$\gamma = -1 - \varepsilon^2 \gamma_1 \tag{11}$$

we have

$$\lambda_k(\varepsilon) = i\pi(2k+1) + \varepsilon\lambda_{k1} + \varepsilon^2\lambda_{k2} + O(\varepsilon^3), \quad k = 0, \pm 1, \pm 2, \dots,$$

where

$$\lambda_{k1} = -\pi(2k+1)a, \quad \lambda_{k2} = -\frac{1}{2}\pi^2(2k+1)^2(a^2-2) + i\pi(2k+1)a^2 + \gamma_1.$$

Next, consider the case when

$$0 < a^2 < 2. \tag{12}$$

Let us introduce some notation. Let $\theta = \theta(\varepsilon) \in [0, 1)$ be the value that completes the expression $\omega_0\varepsilon^{-1}$ to an integer. Set

$$R_1 = (ia - 2\omega_0)\gamma_0^{-1}\exp(-i\Omega_0),$$
$$R_2 = \frac{1}{2}R_1^2 + \gamma_0^{-1}\exp(-i\Omega_0), \quad R_3 = (a - 2i\omega_0)R_1\gamma_0^{-1}\exp(-i\Omega_0).$$

Lemma 5. *Let (12) hold. Then, for those roots $\lambda_k(\varepsilon)$, $k = 0, \pm 1, \pm 2, \dots$, of (5) whose real parts tend to zero as $\varepsilon \to 0$, the following asymptotic equalities hold:*

$$\lambda_k(\varepsilon) = i(\omega_0\varepsilon^{-1} + \theta) + i(\Omega + 2\pi k) + \varepsilon\lambda_{k1} + \varepsilon^2\lambda_{k2} + O(\varepsilon^3),$$

where

$$\lambda_{k1} = R_1(\theta - \Omega_0 + 2\pi k),$$
$$\lambda_{k2} = R_2(\theta - \Omega_0 + 2\pi k)^2 + R_3(\theta - \Omega_0 + 2\pi k) + \gamma_1\exp(-i\Omega_0),$$

and, moreover,

$$\Re R_1 = 0, \quad \Re R_2 < 0. \tag{13}$$

3. Nonlinear Analysis

We study cases (9) and (12) separately. Note that the corresponding results for them are essentially different.

3.1. Slowly Oscillating Solutions

Here, we assume that (9) is satisfied, i.e., $a_0^2 > 2$ and $\gamma_0 = 1$. First, let us formulate the final result. To do this, we introduce the equation

$$\frac{\partial\xi}{\partial\tau} = \left(\frac{a^2}{2} - 1\right)\frac{\partial^2\xi}{\partial x^2} - a^2\frac{\partial\xi}{\partial x} + \gamma_1\xi - \xi^2\frac{\partial\xi}{\partial x} \tag{14}$$

with two types of boundary conditions: (1) the periodicity condition

$$\xi(\tau, x+1) \equiv \xi(\tau, x). \tag{15}$$

and the (2) the antiperiodicity condition

$$\xi(\tau, x+1) \equiv -\xi(\tau, x). \tag{16}$$

Theorem 1. *Let conditions (9) and (10) ((9) and (11)) hold, and let function $\xi(\tau,x)$ be a bounded solution for $\tau\to\infty$, $x\in[0,1]$ of the boundary value problem (14) and (15), for $\gamma=1$, and (14) and (16), for $\gamma=-1$. Then, function*

$$u(t,x,\varepsilon)=\varepsilon^{1/2}\xi(\tau,x) \text{ for } \tau=\varepsilon^2 t,\ x=(1-\varepsilon a)t \tag{17}$$

satisfies Equation (3) up to $o(\varepsilon^{5/2})$.

Justification. We notice that the roots $\lambda_k(\varepsilon)$ discussed in Lemma 4 correspond to the solutions $v_k(t,\varepsilon)$ of the linear Equation (6) and $v_k(t,\varepsilon)=\exp(\lambda_k(\varepsilon)t)$, which means that the solutions (6) are

$$v(t,\varepsilon)=\sum_{k=-\infty}^{\infty}\xi_k\exp(\lambda_k(\varepsilon)t),$$

where ξ_k are arbitrary complex constants. For the cases (7) and (8), this expression can be represented, respectively, as

$$v(t,\varepsilon)=\sum_{k=-\infty}^{\infty}\xi_k(\tau)\exp(2\pi ikx)$$

and

$$v(t,\varepsilon)=\sum_{k=-\infty}^{\infty}\xi_k(\tau)\exp(\pi i(2k+1)x),$$

where $\tau=\varepsilon^2 t$, $x=(1-\varepsilon a)t$, and the Fourier coefficients of $\xi(\tau,x)$ are defined by $\xi_k(\tau)=\xi_k\exp\big((\lambda_{k2}+O(\varepsilon))\tau\big)$.

We seek solutions $u(t,x,\varepsilon)$ of the nonlinear Equation (3), namely 'close to critical' solutions $v(t,\varepsilon)$ of the linear Equation (6), in the form

$$u(t,x,\varepsilon)=\varepsilon^{1/2}\xi(\tau,x)+\varepsilon^{3/2}u_3(\tau,x)+\varepsilon^{5/2}u_5(\tau,x)+O(\varepsilon^3), \tag{18}$$

where $\xi(\tau,x)$ is an unknown real function for which:

1. In the case (7), the periodicity condition in x

$$\xi(\tau,x+1)\equiv\xi(\tau,x).$$

2. In the case (8), the periodicity condition in x

$$\xi(\tau,x+1)\equiv-\xi(\tau,x).$$

We substitute the formal series (18) into Equation (3) and equate the coefficients of the various powers of ε.

Note that the expression $\xi(\tau,x)$ when t is replaced by $t-1$ in (18) corresponds, up to $O(\varepsilon^2)$, to the following:

$$\xi(\tau-\varepsilon^2,x-1+\varepsilon a)=\xi(\tau,x)-\varepsilon^2\frac{\partial\xi(\tau,x)}{\partial\tau}+\varepsilon a\frac{\partial\xi(\tau,x)}{\partial x}+\frac{1}{2}\varepsilon^2a^2\frac{\partial^2\ xi(\tau,x)}{\partial x^2},$$

for $\gamma=1$, and

$$\xi(\tau-\varepsilon^2,x-1+\varepsilon a)=-\left(\xi(\tau,x)-\varepsilon^2\frac{\partial\xi(\tau,x)}{\partial\tau}+\varepsilon a\frac{\partial\xi(\tau,x)}{\partial x}+\frac{1}{2}\varepsilon^2a^2\frac{\partial^2\xi(\tau,x)}{\partial x^2}\right)$$

for $\gamma=-1$.

In the cases, $\gamma=1$ and $\gamma=-1$, the calculations are the same, so, here, we restrict ourselves to the case $\gamma=-1$.

At the first and second steps, collecting the coefficients of $\varepsilon^{1/2}$ and $\varepsilon^{3/2}$ respectively, we obtain $\xi(\tau,x)\equiv-\xi(\tau,x+1)$, $u_3(\tau,x)\equiv-u(\tau,x-1)$, which are true. Thus, $\xi(\tau,x)$ is not yet defined, and for u_3 one can set $u_3 \equiv 0$. Then, collecting the coefficients at $\varepsilon^{5/2}$, we arrive at the expression

$$u_5(\tau, x) + u_5(\tau, x-1) = -\frac{\partial \xi}{\partial \tau} + \left(\frac{a^2}{2} - 1\right)\frac{\partial^2 \xi}{\partial x^2} - a^2 \frac{\partial \xi}{\partial x} + \gamma_1 \xi - \xi^2 \frac{\partial \xi}{\partial x}.$$

From the condition that the function $u_5(\tau, x)$ is antiperiodic in x, from this equality, we immediately arrive at the equality (14) and we can set $u_5 \equiv 0$. The theorem is proven.

It follows from this theorem that, under the formulated conditions, the constructed boundary value problems (14)–(16) play the role of normal forms for Equation (3).

Note that the solutions of the boundary value problems (14) and (15) for $\gamma_1 < 0$ tend to zero as $t \to \infty$; for $\gamma_1 = 0$, all solutions tend to an equilibrium state; and for $\gamma_1 > 0$, there are solutions unbounded as $\varepsilon \to \infty$ (for example, $\xi_0 \exp(i\gamma_1 \tau)$, where ξ_0 is arbitrary). For the boundary value problem (14) and (16), the situation is similar. For $\gamma_1 < (a^2/2 - 1)\pi^2$, all solutions tend to zero as $\tau \to \infty$; for $\gamma_1 = (a^2/2 - 1/)\pi^2$, the zero solution is neutrally stable; and for $\gamma_1 > (a^2/2 - 1\Big)\pi^2$, there are solutions unbounded as $\tau \to \infty$ (for sufficiently small initial conditions). Thus, we conclude that cases (7) and (9) are not interesting.

Remark 1. *Consider an equation of the form* (1)*, but with the nonlinearity u^3 (instead of $\dot{u}u^2$). Repeating all the previous constructions, we obtain as a quasi-normal form the boundary value problem*

$$\frac{\partial \xi}{\partial \tau} = \left(\frac{a^2}{2} - 1\right)\frac{\partial^2 \xi}{\partial x^2} - a^2 \frac{\partial \xi}{\partial x} + \gamma_1 \xi - \xi^3$$

with boundary condition (15) *or* (16)*. The properties of the solutions to this boundary value problem for $\gamma_1 > 0$ are less trivial: in the case of* (15)*, positive and negative homogeneous equilibrium states are stable, and in the case of* (16)*, as γ_1 increases, starting from the value $(a^2/2 - 1)\pi^2$, a stable 2-periodic in x equilibrium state bifurcates from zero. In addition, in the analogue of* (17)*, which relates the solutions of the quasi-normal form and the original equation, the factor $\varepsilon^{1/2}$ is replaced by ε.*

3.2. Rapidly Oscillating Solutions

Let (12) hold, i.e., $a^2 < 2$. The roots $\lambda_k(\varepsilon)$, $k = 0, \pm 1, \pm 2, \ldots$, which were mentioned in Lemma 5, correspond to the solution of the linear Equation (6), $v_k(t, \varepsilon) = \exp(\lambda_k(\varepsilon)t)$, and hence the function

$$v(t, \varepsilon) = \sum_{k=-\infty}^{\infty} \xi_k \exp(\lambda_k(\varepsilon)t), \tag{19}$$

where ξ_k are arbitrary complex constants, also satisfies Equation (6). Taking into account the asymptotic formulas for $\lambda_k(\varepsilon)$ presented in Lemma 5, the expression (19) can be written as

$$v(t, \varepsilon) = E(t, \varepsilon)\xi(\tau, x).$$

Here, $E(t, \varepsilon) = \exp\left[\left(i(\omega_0 \varepsilon^{-1} + \theta - \Omega_0) + \varepsilon R_1(\theta - \Omega_0)\right)t\right]$,

$$\xi(\tau, x) = \sum_{k=-\infty}^{\infty} \xi_k(\tau) \exp(2\pi i k x),\ \tau = \varepsilon^2 t,\ x = (1 + i\varepsilon R_1)t,$$

$\xi_k(\tau) = \xi_k \exp\left((\lambda_{k2} + O(\varepsilon))\tau\right)$. Recall that, according to (18), the value of iR_1 is real.

In the case under consideration, we seek solutions $u(t, x, \varepsilon)$ of the nonlinear Equation (3) in the form of a formal series

$$u(t, x, \varepsilon) = \varepsilon^{1/2}\left(\xi(\tau, x)E(t, \varepsilon) + \bar{\xi}(\tau, x)\bar{E}(t, \varepsilon)\right) + \varepsilon^{3/2} u_3(t, x) + \ldots, \tag{20}$$

where $\xi(\tau, x)$ is an unknown complex function to be determined, which is 1-periodic in the space variable x:

$$\xi(\tau, x + 1) \equiv \xi(\tau, x), \tag{21}$$

and the dependence of u_3 on t is periodic.

Let us introduce the boundary value problem

$$\frac{\partial \xi}{\partial \tau} = -R_2 \frac{\partial^2 \xi}{\partial x^2} + R_4 \frac{\partial \xi}{\partial x} + R_5 \xi - \xi^2 \frac{\partial \bar{\xi}}{\partial x} - 2|\xi|^2 \frac{\partial \xi}{\partial x}, \tag{22}$$

where $R_4 = -i(2R_2(\theta - \Omega_0) + R_3)$, $R_5 = R_2(\theta - \Omega_0)^2 + R_3(\theta - \Omega_0) + \gamma_1 \exp(-i\Omega_0)$ with periodic conditions (21).

We introduce some notation. We denote by $\varepsilon_n = \varepsilon_n(\theta_0)$ a sequence $\varepsilon_n \to 0$ such that $\theta(\varepsilon_n) = \theta_0$. We have the following.

Theorem 2. *Let (7) and (12) hold. Let the boundary value problem (21) and (22) have a solution $\xi(\tau)$ bounded at $\tau \to \infty$ for an arbitrarily fixed value $\theta_0 \in [0, 2\pi)$. Then, for the sequence $\varepsilon_n = \varepsilon_n(\theta_0)$, the function*

$$\begin{aligned} u(t, x, \varepsilon_n) =& \varepsilon_n^{1/2} \big[\xi(\tau, x) E(t, \varepsilon_n) + \bar{\xi}(\tau, x) \bar{E}(t, \varepsilon_n) \big] + \\ &+ \varepsilon_n^{3/2} \Big[-9\omega_0^2 + 1 + 3i\omega_0 - \exp(-3i\Omega_0) \Big]^{-1} i\omega_0 \xi^3 E^3(t, \varepsilon_n) + \\ &+ \varepsilon_n^{3/2} \Big[-9\omega_0^2 + 1 - 3i\omega_0 - \exp(3i\Omega_0) \Big]^{-1} i(-\omega_0) \bar{\xi}^3 \bar{E}^3(t, \varepsilon_n) \end{aligned}$$

where $\tau = \varepsilon_n^2 t$, $x = (1 - \varepsilon_n a)t$ satisfies (3) up to $o(\varepsilon_n^{3/2})$.

Justification. Let us substitute (20) into (3) and equate the coefficients at the same powers of ε. We take into account that

$$E(t-1, \varepsilon) = E(t, \varepsilon) \cdot \exp(-i\Omega_0) \Big[1 - \varepsilon i R_1 (\theta - \Omega_0) - \frac{1}{2} \varepsilon^2 R_1^2 (\theta - \Omega_0)^2 + O(\varepsilon^3) \Big]$$

and, when t is replaced by $t - 1$, function $\xi(\tau, x)$ becomes

$$\xi(\tau - \varepsilon^2, x - 1 - \varepsilon i R_1) = \xi(\tau, x) - \varepsilon^2 \frac{\partial \xi}{\partial \tau} - i\varepsilon R_1 \frac{\partial \xi}{\partial x} - \frac{1}{2} \varepsilon^2 R_1^2 \frac{\partial^2 \xi}{\partial x^2}.$$

Then, collecting the coefficients at $\varepsilon^{3/2}$, we arrive at an equation for u_3. We represent this function as the sum of the first and third harmonic (with respect to the frequencies $i\omega_0 \varepsilon^1$):

$$u_3 = u_{31}(\tau, x) E(t, \varepsilon) + \bar{u}_{31}(\tau, x) \bar{E}(t, \varepsilon) + u_{32}(\tau, x) E^3(t, \varepsilon) + \bar{u}_{32}(\tau, x) \bar{E}^3(t, \varepsilon).$$

Equating in the resulting equation for u_3 the coefficients at the first harmonic (i.e., at $E(t, \varepsilon)$) and separately at the third harmonic (i.e., at $E^3(t, \varepsilon)$), we first arrive at (22) and we can set $u_{31} \equiv 0$. Then, we obtain

$$u_{32} = \Big[-9\omega_0^2 + 1 + 3i\omega_0 - \exp(-3i\Omega_0) \Big]^{-1} i\omega_0 \xi^3.$$

Thus, the boundary value problem (22), (21) is a quasinormal form for the Equation (3) in the considered critical case. Due to condition (13), this boundary value problem is parabolic. The structure of its solutions can be complex. This boundary value problem determines the principal terms of asymptotic approximations of solutions to the original Equation (3). It follows that the local dynamics (3) can also be complex. We also note that Equation (22) has a more complicated form than the classical Ginzburg–Landau equation.

4. Van der Pol Equation with a Large Delay and a Small Coefficient of Delayed Feedback

It was shown above that the critical case for Equation (3), for $a = 0$, is realized for $\gamma = 0$. Here, we assume that the coefficients a and γ are sufficiently small, i.e., for some a_1 and γ_1 we have

$$a = \varepsilon a_1,\ \gamma = \varepsilon\gamma_1 \quad (0 < \varepsilon \ll 1). \tag{23}$$

Let us first consider the Van der Pol equation with delayed feedback

$$\ddot{u} + \varepsilon a_1\dot{u} + u + \dot{u}u^2 = \varepsilon\gamma_1 u(t-h), \tag{24}$$

without assuming that the value of h is large.

For the equation linearized at zero, the characteristic quasi-polynomial has the form

$$\lambda^2 + \varepsilon a_1\lambda + 1 = \varepsilon\gamma_1\exp(-\lambda h).$$

In this section, we first study the solutions to (24) from a small neighborhood of the zero equilibrium state for fixed values of h, and then we consider the case when the delay is asymptotically large.

This equation has two roots, $\lambda(\varepsilon)$ and $\bar{\lambda}(\varepsilon)$, the real parts of which tend to zero as $\varepsilon \to 0$, while all other roots have negative real parts separated from zero as $\varepsilon \to 0$. Note that

$$\lambda(\varepsilon) = i + \varepsilon\lambda_1 + \ldots,$$

where

$$\lambda_1 = -\frac{1}{2}a_1 - \frac{i}{2}\gamma_1\exp(-ih).$$

In the vicinity of the zero state of Equation (24), for sufficiently small ε, there is a stable two-dimensional local invariant integral manifold [17,18], on which Equation (24) can be written up to order $O(\varepsilon)$ in normal form, namely as a scalar complex first-order equation:

$$\frac{\partial\xi}{\partial\tau} = \alpha\xi - \frac{1}{2}\xi|\xi|^2, \quad \tau = \varepsilon t, \quad \alpha = -\frac{1}{2}a_1 - \frac{i}{2}\gamma_1\exp(-ih). \tag{25}$$

Equations (24) and (25) are related via the relation

$$u = \varepsilon^{1/2}\big[\xi(\tau)\exp(it) + \bar{\xi}(\tau)\exp(-it)\big] + O(\varepsilon^{3/2}).$$

Equation (25) can be integrated explicitly, so it is easy to conclude about the existence and stability of its cycle. According to (25), the same conclusion about the existence and stability of the cycle is true for Equation (24).

Let us then consider the case when the delay in (24) is sufficiently large. Let

$$h = T_1\varepsilon^{-1}. \tag{26}$$

Then, infinitely many roots, $\lambda_k(\varepsilon)$, $k = 0, \pm1, \pm2, \ldots$, of the characteristic equation

$$\lambda^2 + \varepsilon a_1\lambda + 1 = \varepsilon\gamma_1\exp(T_1\varepsilon^{-1}\lambda)$$

tend to the imaginary axis as $\varepsilon \to 0$, and there is no root with a positive real part separated from zero as $\varepsilon \to 0$. Thus, in the problem of stability of the zero solution of (24), the critical case of infinite dimension is realized. For $\lambda_k(\varepsilon)$, we have the following asymptotic representations:

$$\lambda_k(\varepsilon) = i + \varepsilon\lambda_{k1} + \ldots,$$

where λ_{k1} is the root of the quasipolynomial

$$\lambda_{k1} + \frac{1}{2}a_1 = -\frac{i}{2}\gamma_1\exp(i\theta - T_1\lambda_{k1}). \tag{27}$$

The quantity $\theta = \theta(\varepsilon) \in [0, 2\pi)$ appearing in (27) complements the value $T\varepsilon^{-1}$ to an integer multiple of 2π. Note that the quasipolynomial (27) has infinitely many roots.

We seek solutions to the nonlinear equation in the form of formal series:

$$u = \varepsilon^{1/2}\big(\xi(\tau)\exp(it) + \bar{\xi}(\tau)\exp(-it)\big) + \varepsilon^{3/2}u_3(t,\tau) + \dots, \tag{28}$$

where the dependence on the argument t is 2π-periodic. We substitute (28) into (24). After standard calculations, we arrive at an equation for u_3. From the condition of its solvability in the class of 2π-periodic functions in t, we obtain an equation with a fixed delay for determining the unknown complex amplitude $\xi(\tau)$:

$$\frac{\partial\xi}{\partial\tau} = -\frac{1}{2}a_1\xi - \frac{i}{2}\gamma_1\exp(i\theta)\cdot\xi(\tau - T_1) - \frac{1}{2}\xi|\xi|^2. \tag{29}$$

Let us introduce some notation. We denote by $\varepsilon_n = \varepsilon_n(\theta_0)$ a sequence $\varepsilon_n \to 0$ such that $\theta(\varepsilon_n) = \theta_0$. We now formulate the final result; its justification is obtained according to the above standard scheme.

Theorem 3. *Let (23) and (26) hold. Let Equation (29) have a solution $\xi(\tau)$ bounded at $\tau \to \infty$ for an arbitrarily fixed value $\theta_0 \in [0, 2\pi)$. Then, for the sequence $\varepsilon_n = \varepsilon_n(\theta_0)$, the function*

$$u(t, \varepsilon_n) = \varepsilon_n^{1/2}\big(\xi(\tau)\exp(it) + \bar{\xi}(\tau)exp(-it)\big)$$

where $\tau = \varepsilon_n t$ satisfies (24) up to $O(\varepsilon_n^{3/2})$.

It follows from this theorem that the distributed Equation (29) is a quasinormal form in the case under consideration. For Equation (29), it is easy to investigate the existence and stability of the simplest cycles of the form $\rho\exp(i\sigma\tau)$. We do not dwell on this in more detail.

5. Investigation of the Solutions in the Neighborhood of a Cycle of the Van der Pol Equation with a Small Perturbation with a Large Delay

Let us first consider the Van der Pol equation with a small perturbation and a fixed delay h

$$\ddot{u} + a\dot{u} + u + \dot{u}u^2 = \varepsilon\gamma_1 u(t-h), \quad 0 < \varepsilon \ll 1. \tag{30}$$

Assume that for $\varepsilon = 0$, the Van der Pol equation

$$\ddot{u} + a\dot{u} + u + \dot{u}u^2 = 0 \tag{31}$$

has an exponentially orbitally stable periodic solution $u_0(t)$ of period L. Equation (31) can be written as the system

$$\dot{u} = F(u), \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad F(u) = \begin{pmatrix} u_2 \\ -u_1 - au_2 - u_2u_1^2 \end{pmatrix}. \tag{32}$$

The system

$$\dot{V} = A(t)V, \quad A(t) = \begin{pmatrix} 0 & 1 \\ -1 - 2\dot{u}_0u_0 & -1 - u_0^2 \end{pmatrix} \tag{33}$$

linearized on

$$U_0(t) = \begin{pmatrix} u_0(t) \\ \dot{u}_0(t) \end{pmatrix}$$

has periodic solutions

$$V_0(t) = \begin{pmatrix} \dot{u}_0(t) \\ \ddot{u}_0(t) \end{pmatrix}.$$

By

$$W_0(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} \neq 0$$

we denote a periodic solution of the system conjugate to (33). It is convenient to assume that

$$< V_0(t), W_0(t) >= \frac{1}{L} \int_0^L (V_0(t), W_0(t)) dt = 1.$$

Equation (30) defines a 'perturbed' system with respect to (32)

$$\dot{u} = F(u) + \varepsilon\gamma K(u(t-h)), \text{ where } K(u) = \begin{pmatrix} 0 \\ u \end{pmatrix}. \tag{34}$$

It is known that, for small ε, system (34) has an exponentially orbitally stable L-periodic solution $V_0(t, \varepsilon)$, and

$$V(t, \varepsilon) = V_0(\tau) + \varepsilon V_1(\tau) + \dots, \tag{35}$$

where $\tau = \tau(t)$ is related to t via

$$\frac{d\tau}{dt} = 1 + \varepsilon\alpha_1 + \dots. \tag{36}$$

Substituting (35) and (36) into (34), and collecting the coefficients of the various powers of ε, one can determine the coefficients of the formal series (35) and (36). For example, by collecting the coefficients of ε, we obtain an equation for $V_1(\tau)$, and from the condition of its solvability in the class of L-periodic functions in τ we arrive at

$$\alpha = \gamma_1 < u_0(t) \cdot w_2(t) > .$$

In this section, we study the structure of solutions to system (34) in a neighborhood of the cycle $V_0(t)$ under the condition that the parameter ε is sufficiently small and the delay h is sufficiently large:

$$h = T_1 \varepsilon^{-1}.$$

Let us consider formal series more general than (35) and (36), namely

$$\dot{V}(t, \varepsilon) = V_0(\tau) + \varepsilon V_1(\tau, s) + \dots, \tag{37}$$

$$\frac{d\tau}{dt} = 1 + u(s) + \dots, \quad s = \varepsilon t, \tag{38}$$

where functions $V_j(t, s)$ are L-periodic in τ, and $u(s)$ is a scalar almost periodic function. Substitute (37), (38) into (34). Collecting the coefficients at the first power of ε, we arrive at the system

$$\varphi(s)\dot{V}_0 + \frac{dV_1}{d\tau} = A(\tau)V_1 + \gamma_1 K\big(u_0(\tau(t - T_1\varepsilon^{-1}))\big). \tag{39}$$

Let us introduce some notation. Let $\theta = \theta(\varepsilon) \in [0, L)$ denote the value that completes the expression $T_1\varepsilon^{-1}$ to an integer multiple of L. Thus, for $\varepsilon \to 0$, the value of θ changes infinitely many times from 0 to L. Then, consider the L-periodic function $g(z)$ defined by

$$g(z) = -\gamma_1 < u_0(t - z) \cdot w_2(t) > .$$

From (38), we obtain that

$$\tau(t) = t - t_0 + \varepsilon \int_{t_0}^{t} (\varphi(\varepsilon s) + \dots) ds,$$

and therefore,

$$\tau(t - T_1\varepsilon^{-1}) = \tau(t) - T_1\varepsilon^{-1} - \int_{s-T_1}^{s} (\varphi(\tau) + \dots) d\tau.$$

We return to system (39). The condition for the existence of a solution L-periodic in τ is that the terms of function $W_0(t)$ which do not contain V_1 are orthogonal on average. From here, we arrive at

$$\varphi(s) = g\Big(\theta + \int_{-T_1}^{0} \varphi(s+\tau) d\tau\Big). \tag{40}$$

After the solution (40) is found, the algorithm for determining successively the elements of the series (37) and (38) can be continued indefinitely.

We examine the equilibrium state $\varphi(s) \equiv \varphi$ of the Equation (40). In order to find them, we obtain the equation

$$\varphi = g(\theta + T_1\varphi). \tag{41}$$

The number of solutions to this equation depends on function $g(z)$. Let (41) have a solution φ_0, for some $\theta = \theta_0$, i.e., for $\theta = \theta_0$; Equation (40) has the equilibrium state $\varphi(s) \equiv \varphi_0$. We study the stability of this solution. To do this, consider the equation linearized on φ_0

$$\psi(s) = g'(\theta_0 + T_1\varphi_0) \int_{-T_1}^{0} \psi(s+\tau) d\tau. \tag{42}$$

We have the following.

Lemma 6. *When*

$$T_1 g'(\theta_0 + T_1\varphi_0) < 1,$$

all roots of the characteristic quasi-polynomial for the Equation (42) have negative real parts, and when

$$T_1 g'(\theta_0 + T_1\varphi_0) > 1$$

there is a positive root of this quasi-polynomial.

The following proposition reveals the main result.

Theorem 4. *Let Equation (41) have an equilibrium φ_0, for some $\theta = \theta_0$, and let*

$$T_1 g'(\theta_0 + T_1\varphi_0) \neq 1.$$

Then, there exists a sequence $\varepsilon_n \to 0$, determined by the condition $\theta(\varepsilon) = \theta_0$, such that for $\varepsilon = \varepsilon_n$ and for sufficiently large n, the function

$$V(t,\varepsilon) = V_0(\tau) + \varepsilon V_1(\tau), \text{ where } \tau = (1 + \varepsilon\varphi_0 + o(\varepsilon^2))t \tag{43}$$

satisfies (34) up to $O(\varepsilon^2)$.

Remark 2. *The results in [27] indicate that there is a stronger statement than Theorem 4 about the existence, stability, and asymptotic behavior of a periodic solution (34) close to (43).*

Note that Equation (40) can have any number of equilibrium states, which means that system (34) can have any number of stable periodic solutions. In addition, (34) is characterized by an unbounded process of forward and backward bifurcations as $\varepsilon \to 0$. This is due to the fact that the number and stability of equilibrium states in (40) depend on the quantity $\theta = \theta(\varepsilon)$, which changes infinitely many times from 0 to L as $\varepsilon \to 0$.

6. Van der Pol Equations with a Large Delay Control Coefficient

The Van der Pol equation with delay control reads

$$\ddot{u} + a\dot{u} + u + \dot{u}u^2 = \gamma\big([\dot{u}(t-T) - \dot{u}] + \alpha[u(t-T) - u]\big). \tag{44}$$

Here, the coefficients γ and α are positive. Equations of this form arise in many applied problems (see, for example, [27–31]). The basic assumption that paves the way for the application of asymptotic methods is that the parameter γ is large enough:

$$\gamma \gg 1 \quad (\varepsilon = \gamma^{-1} \ll 1).$$

After dividing by γ both sides of (44), we obtain

$$\varepsilon\ddot{u} + \varepsilon a\dot{u} + \varepsilon u + \varepsilon\dot{u}u^2 = \dot{u}(t-T) - \dot{u} + \alpha\big(u(t-T) - u\big). \tag{45}$$

Note that the simpler equation

$$\varepsilon\ddot{u} + \varepsilon a\dot{u} + \varepsilon u + \varepsilon\dot{u}u^2 = \alpha\big(u(t-T) - u\big),$$

in a region of the phase space $C_{[-T,0]} \times C_{[-T,0]}$ bounded at $\varepsilon \to 0$ cannot have stable solutions to Equation (45) for sufficiently large γ. This follows from the results of Section 3.

Equation (45) can be written equivalently as the system

$$\dot{u} = v - \alpha u, \tag{46}$$

$$\varepsilon\dot{v} = \varepsilon\big[(\alpha - a)v - (1 - \alpha a + \alpha^2)u - (v - \alpha u)u^2\big] + v(t-T) - v. \tag{47}$$

We employ the results obtained in [21] for solutions to

$$\varepsilon\dot{v} = \varepsilon^{\delta}\beta(\tau)v + v(t-T) - v + \varepsilon p v^3, \tag{48}$$

where the parameter δ takes one of the values $\delta = 1$ or $\delta = 2$, and $\beta(\tau)$ is some bounded smooth function dependent on 'slow' time as $\tau \to \infty$:

$$\tau = \begin{cases} \varepsilon t & \text{at } \delta = 1, \\ \varepsilon^{1/2} t & \text{at } \delta = 2. \end{cases}$$

For each of the two values of δ, the case of infinite dimension, which is critical in the stability of the zero solution, is realized. Therefore, the structure of solutions is determined by special quasi-normal forms. The quasi-normal form in the case $\delta = 2$ is a parabolic boundary value problem for an unknown function $\xi(\tau, x)$, where

$$\tau = \varepsilon^{1/2} t, \quad x = (1 + O(\varepsilon))t. \tag{49}$$

The leading term of the asymptotic representation of the solution, $v(t, \varepsilon)$, to Equation (48) is defined by

$$v(t, \varepsilon) = \varepsilon^{1/2}\xi(\tau, x) + O(\varepsilon^{3/2}).$$

In the case when $\delta = 1$, the situation is much more complicated, and it is studied in [21,32]. Here, the quasi-normal form is a family of parabolic-type boundary value problems for $\xi(\tau, x)$, depending on the continuum parameters $z \in (0, \infty)$ and $\theta \in [0, 1)$. Unlike (49), here, the spatial variable x and variable τ are related to t via

$$\tau = \varepsilon t, \quad x = (z\varepsilon^{-1/2} + \theta + O(\varepsilon))t, \tag{50}$$

whereas for the solutions to (48), $v(t, \varepsilon)$, we have the asymptotic relation

$$v(t, \varepsilon) = \xi(\tau, x) + o(1). \tag{51}$$

We return to system (46) and (47). Suppose that in (47) for some function $\varphi(\tau)$, we have the asymptotic representation

$$u = \varphi(\tau) + o(1),$$

where $\tau = \varepsilon t$. Then, this equation belongs to the class of Equations (48), where $\delta = 1$, and hence the asymptotic behavior in the principal function $u(t, \varepsilon)$ is determined by an expression of the form (51), and $\xi(\tau, x)$ changes slowly with respect to its first argument ($\tau = \varepsilon t$) and oscillates rapidly with respect to its second argument ($x = (z\varepsilon^{-1/2} + \theta + o(1))t$). With this in mind, consider Equation (46). The right side of it can be averaged [33] over the rapidly oscillating variable x. Then, we arrive at

$$\dot{u} = M_x(\xi(\tau, x)) - \alpha u, \quad M_x(\psi(x)) = \frac{1}{T}\int\limits_0^T \psi(x)dx.$$

Since the expression $M_x(\xi(\tau, x))$ changes slowly with respect to the time variable, we can use the well-known results (see, for example, [34]) on the applicability of the frozen coefficients method. Hence, we obtain that

$$u(t, \varepsilon) = \alpha^{-1} M_x(\xi(\tau, x)) + o(1). \tag{52}$$

Using (50) and (52), and the asymptotic representation

$$v(t, \varepsilon) = \xi(\tau, x) + O(\varepsilon),$$

for the definition of function $\xi(\tau, x)$ from (47), we obtain the boundary value problem

$$\begin{aligned} \frac{\partial \xi}{\partial \tau} =& (2T)^{-1} z^2 \frac{\partial^2 \xi}{\partial x^2} - \theta_z T^{-1} \frac{\partial \xi}{\partial x} + (\alpha - a)\xi - \\ &- \alpha^{-1}\big(1 - \alpha(a - \alpha)\big) M_x(\xi) + \alpha^{-2}\big(\xi - M_x(\xi)\big)\big(M_x(\xi)\big)^2, \end{aligned} \tag{53}$$

with periodic boundary conditions

$$\xi(\tau, x + T) \equiv \xi(\tau, x). \tag{54}$$

We have the following.

Theorem 5. *We arbitrarily fix $z \neq 0$ and $\theta_z \in [0, 1)$, and let $\xi(\tau, x)$ be a bounded solution of the boundary value problem (53) and (54), for $\tau \to \infty, x \in [0, T]$. Then, for sufficiently small and ε such that $\theta(\varepsilon) = \theta_z$, the vector function*

$$\begin{pmatrix} u(t, \varepsilon) \\ v(t, \varepsilon) \end{pmatrix} = \begin{pmatrix} \alpha^{-1} M_x(\xi, \tau, x) + o(1), \\ \xi(\tau, x) + O(\varepsilon) \end{pmatrix}$$

for $\tau = \varepsilon t, \quad x = (z\varepsilon^{-1/2} + \theta + O(\varepsilon))t$, satisfies the system (46) and (47) up to $o(1)$.

The theorem states that the boundary value problem (53) and (54) plays the role of a normal form for the system of Equations (46) and (47). Note that, in the formulation of the theorem, the case $z = 0$ is excluded. This case is special. Solutions to the boundary value problem (46), (47) are no longer rapidly oscillating, so the averaging principle cannot be used. In this case, it is necessary to assume that the Van der Pol Equation (1) has a T-periodic solution $u_0(t)(\not\equiv 0)$. In (44), we set $u(t,\varepsilon) = \xi(\tau, x) + o(1)$, where $\tau = \varepsilon t, x = t$. Then, for $\xi(\tau, x)$, we arrive at the boundary value problem

$$\frac{\partial^2 \xi}{\partial x^2} = -\xi^2 \frac{\partial \xi}{\partial x} - T\frac{\partial^2 \xi}{\partial x \partial \tau} - \alpha T \frac{\partial \xi}{\partial \tau}, \tag{55}$$

$$\xi(\tau, x + T) \equiv \xi(\tau, x). \tag{56}$$

This boundary value problem also plays the role of a normal form for Equation (44) for sufficiently small ε. Its solutions determine the leading terms of the asymptotics of the solutions to Equation (44). Note that $u_0(x)$ is the equilibrium state of the boundary value problem (55) and (56).

From the Theorem 5 we can conclude the following. For large values of the delay control coefficients in (44), there is no stabilization to the homogeneous cycle of the Van der Pol equation. The structures that arise in such systems are complex, but on the other hand, they have a detailed analytical description.

7. Van der Pol Equation With Delayed Nonlinearity

We study the behavior of all solutions from a sufficiently small neighborhood of the zero equilibrium state for the Van der Pol equation with delayed nonlinearity

$$\ddot{u} + a\dot{u} + u + \dot{u}(t - T)u^2(t - T) = 0, \tag{57}$$

which differs from (1) only by the presence of a delay in the nonlinearity. The critical case in the zero stability problem is determined by the linear part, so we assume that for some fixed value a_1,

$$a = \varepsilon a_1,\ 0 < \varepsilon \ll 1.$$

In the absence of delay, i.e., for $T = 0$, the local behavior of solutions of (57) is determined primarily by the normal form

$$\frac{\partial \xi}{\partial \tau} = -\frac{1}{2}a_1\xi - \frac{1}{2}\xi|\xi|^2, \quad \tau = \varepsilon t. \tag{58}$$

Functions $u(t,\varepsilon)$, for $\tau = 0$, and $\xi(\tau)$ are related via

$$u(t,\varepsilon) = \varepsilon^{1/2}\big(\xi(\tau)\exp(it) + \bar{\xi}(\tau)exp(-it)\big) + O(\varepsilon^{3/2}).$$

The same formula relates the solutions to (57) for $T \neq 0$ and the solutions to (58), and for the normal form, we obtain

$$\frac{\partial \xi}{\partial \tau} = -\frac{1}{2}a_1\xi - \frac{1}{2}b(T)\xi|\xi|^2, \tag{59}$$

where $\tau = \varepsilon t$, $b(T) = \exp(-iT)$. The stable non-zero periodic solution

$$\xi_0(\tau) = \xi_0 \exp(i\varphi\tau),\ \text{where}\ \xi_0 = (-a_1 \cos T)^{1/2},\ \varphi_0 = \frac{1}{2}\xi_0^2 \sin T,$$

exists under the condition

$$a_1 < 0, \quad \cos T > 0.$$

Thus, the existence and stability of the corresponding cycle in (57)

$$u_0(t,\varepsilon) = \varepsilon^{1/2}\big(\xi_0 \exp(i(1+\varphi_0\varepsilon)t) + \bar{\xi}_0 exp(-i(1+\varphi_0\varepsilon)t)\big) + O(\varepsilon^{3/2})$$

can be controlled by choosing the appropriate parameter T.

The situation when the parameter T in (57) is sufficiently large, namely

$$T = T_1 \varepsilon^{-1}.$$

for some $T_1 > 0$, is more interesting. In this case, instead of the ordinary differential equation (59), as a normal form, we obtain an equation with delay

$$\frac{\partial \xi}{\partial \tau} = -\frac{1}{2} a_1 \xi - \frac{1}{2} \exp(i\theta(\varepsilon)) \cdot \xi(\tau - T_1) |\xi(\tau - T_1)|^2, \tag{60}$$

where $\theta(\varepsilon) = -\left(T_1 \varepsilon^{-1}\right)|_{\text{ mod } 2\pi}$.

We fix arbitrarily $\theta_0 \in [0, 2\pi)$. Then, for $\varepsilon = \varepsilon_n = \varepsilon_n(\theta_0) = (T_1\theta_0)(2\pi n)^{-1}$ $(n = n_0, n_0 + 1, \ldots)$, we have $\exp\left(i\theta(\varepsilon_n)\right) = \exp\left(i\theta_0\right)$. By $\varphi_j = \varphi_j(\theta_0)$ we denote the roots of the equation numbered in ascending order

$$\tan\left(\theta_0 - \varphi T_1\right) = 2\varphi a_1^{-1}.$$

and set

$$\rho_j = \left(a_1 + 4\varphi_j^2\right)^{1/2}.$$

Unlike (58), (60) has infinitely many different periodic solutions $\xi_j(\tau) = \rho_j \exp(i\varphi_j \tau)$. This implies, in particular, that Equation (60), and hence Equation (57), is not dissipative.

The periodic solutions $\xi_j(\tau)$ allow us to determine the corresponding periodic solutions for Equation (57) for $\varepsilon = \varepsilon_n(\theta_0)$

$$u_j(t, \varepsilon) = \varepsilon^{1/2}\left(\xi_j(\tau)\exp(it) + \bar{\xi}_j(\tau) exp(-it)\right) + O(\varepsilon^{3/2}).$$

The stability of these solutions mainly depends on the location of the roots of the characteristic quasi-polynomial of the equation linearized on $u_j(t, \varepsilon)$

$$\lambda^2 - 2\lambda + (1 + u_j)\left(1 - \exp(-2\lambda)\right) = 0. \tag{61}$$

For stability, it is necessary that all roots of (61) have negative real parts. It can be shown that, for all sufficiently large numbers j, Equation (61) has a root with a positive real part.

The given results indicate that the dynamics of solutions to Equation (57) for large delays is much more complicated than in the case of the ordinary differential Equation (1).

8. Conclusions

The local structure of solutions to the Van der Pol equation with delayed feedback is studied under the assumption that the delay parameter T is sufficiently large. All possible critical cases in the problem of the stability of the zero equilibrium state are singled out. It is shown that they all have an infinite dimension in the sense that infinitely many roots of the characteristic equation of the linearized problem tend to the imaginary axis as the small parameter $\varepsilon = T^{-1}$ (or $\varepsilon = \gamma^{-1}$ in Section 6). A technique was developed for constructing analogs of normal forms, the so-called quasi-normal forms. Their nonlocal dynamics allows us to describe the asymptotics of solutions to the original equation in the vicinity of the equilibrium state.

Under the condition $a^2 > 2$ and $\gamma = \gamma_0$, the solutions to (3) for small ε are mainly formed on relatively small modes. They are described by quasi-normal forms, which are the Ginzburg–Landau Equations (14), with real coefficients and boundary conditions, (15) or (16). In the case $a^2 < 2$, the solutions are rapidly oscillating. The asymptotic behavior of the amplitudes of these solutions is determined by the quasi-normal form, namely a complex boundary value problem of the Ginzburg–Landau type (22) and (21). For the important case, when in (3) $a, \gamma \sim \varepsilon$, the quasi-normal form is a first-order nonlinear complex equation with delay (29). For the case when $a < 0$, the behavior of solutions to (3) in a neighborhood of a nonlocal cycle $u_0(t)$ of the Equation (1) is studied. In order to find other cycles in the neighborhood $u_0(t)$, an equation with delay (40) is constructed. The existence and stability of equilibrium states (40) is investigated. It is these equilibrium states that correspond to cycles from the neighborhood $u_0(t)$.

The dynamics of the Van der Pol equation with a large delay control coefficient, which is closely related to problems with a large delay, is studied. Here, quasi-normal forms are constructed, namely complex equations of the Ginzburg–Landau-type (53) for rapidly oscillating solutions of Equation (44), and the boundary value problem (55) and (56) for slowly oscillating solutions.

In the problem of the dynamics of the Van der Pol equation with a large delay in cubic nonlinearity (57), the role of the quasi-normal form is played by a complex equation with delay also in cubic nonlinearity (60). An explicit form of an infinite set of quasi-normal form cycles is given.

To conclude, we note that analytical statements about the behavior of solutions to the original problem in cases close to critical are given. Moreover, asymptotic formulas for solutions are presented, which is particularly important for applications. From these formulas, it follows that the use of the degenerate Equation (3) for $\varepsilon = 0$ does not make sense. Indeed, the local dynamics of the corresponding one-dimensional map $u(t) + f(u(t), 0) = \gamma u(t-1)$ is well known, even for an arbitrary smooth function $f(u, \dot{u})$ (see, for example, [35]) and has nothing to do with the behavior of solutions of Equation (3) described above for small ε. In addition, note that the numerical study of (3) for small ε is difficult since it was shown that the solutions can be asymptotically rapidly oscillating and their dynamics can be sensitive even to small changes in the problem parameters. We formulate some of the most important conclusions that follow from the analysis of the constructed quasi-normal forms:

1. In all the cases considered above, the local dynamics of the initial equations is determined by the nonlocal dynamics of distributed quasi-normal forms of nonlinear equations, either with partial derivatives or with delay. Thus, the considered problems are characterized by a complex structure of solutions. It is known that the Ginzburg–Landau equations can have irregular dynamics. Therefore, the same conclusion is also true for the original equation.
2. Separately, we note the effect of multistability. For example, it is demonstrated for Equation (44). The point is that these equations involve an arbitrary parameter $z \in (-\infty, \infty)$, and each dynamic mode of the corresponding equations for $\xi(\tau, x)$ for each fixed z corresponds to a solution of the original equation. Thus, pseudo-continuum families of solutions to (44) may arise.
3. After the leading asymptotic approximations for the solutions (44) are found, it is possible to obtain, using standard methods, asymptotic expansions that are more accurate with respect to the small parameter ε. This way, it becomes possible to justify the proximity of the constructed solutions to the exact solutions of the corresponding equations and conclude about the inheritance of stability properties. Note that there are significant technical difficulties here.
4. It should be noted that the dynamic properties of Equation (44) are highly "sensitive" to changes in the small parameter $\varepsilon = T^{-1}$ or $\varepsilon = \gamma^{-1}$. This is due to the fact that the construction involves the quantity $\theta_z(\varepsilon)$, which varies infinitely often from 0 to 1 for $\varepsilon \to 0$. It remains to be noted that for various θ, the dynamic properties can also be different [36]. Thus, at $\varepsilon \to \infty$, an unlimited process of forward and backward bifurcations occurs. Hence, it follows that the application of numerical methods in the problems under consideration requires serious efforts.

5. Note that solutions (3) with arbitrary initial conditions as $t \to \infty$ tend to a "critical" set of solutions, i.e., to solutions corresponding to the main function

$$v(t,\varepsilon) = \varepsilon \sum_{k=-\infty}^{\infty} \xi_k \exp(\lambda_k(\varepsilon)t) = \xi(\tau, x) \quad (x = (1-\varepsilon a)t).$$

Therefore, the choice of initial conditions is arbitrary. Of course, the process of establishing solutions will be faster if the initial conditions at $t \in [0,1]$ are taken as an arbitrary function close to $\xi(\tau, (1-\varepsilon a)t)$. The specificity of the Van der Pol equation was practically not used. The same study could be applied to an arbitrary second-order differential equation with sufficiently smooth nonlinearity. It is interesting to note that quasi-normal forms very similar to those given above arise in the analysis of critical cases for many classes of partial differential equations (see, for example, [17,18]) and for evolution equations describing chains of coupled oscillators with a large number of elements (see, for example, [8,31]). The dynamics of the Van der Pol equations, in which the feedback delay is not large enough, does not have such effects. The critical cases in them are finite dimensional, which means that they are described by much simpler normal forms, finite-dimensional ordinary differential equations.

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