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# Relational Contractions Involving (c)-Comparison Functions with Applications to Boundary Value Problems

**Ebrahem Ateatullah Algehyne *, Musaad Sabih Aldhabani * and Faizan Ahmad Khan ***

Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia
* Correspondence: e.algehyne@ut.edu.sa (E.A.A.); maldhabani@ut.edu.sa (M.S.A.); fkhan@ut.edu.sa (F.A.K.)

**Abstract:** After the introduction of Alam–Imdad's relation-theoretic contraction principle, the field of metric fixed point theory has attracted much attention. A number of fixed point theorems in the context of relational metric space employing various types of contractions has been appeared during the last seven years. In this manuscript, one proved a metrical fixed point theorem for $\phi$-contraction involving (c)-comparison functions employing an amorphous relation. The result proved in this paper refines, modifies, unifies and sharpens several existing fixed point results. We also constructed an example in order to attest the credibility of our results. Finally, we applied our result to establish the existence and uniqueness of solution of certain periodic boundary value problem.

**Keywords:** fixed points; $\phi$-contractions; $\varrho$-self-closedness; $\mathfrak{S}$-continuous functions

**MSC:** 47H10; 06A75; 34B15; 54H25

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## 1. Introduction

Banach contraction principle (abbreviated as: BCP) and its applications are well-known. In subsequent years, various generalizations of this pivotal result were obtained by improving the underlying contraction condition. One of the remarkable generalized contractions is $\phi$-contraction, which is obtained from usual contraction by replacing the Lipschitzian constant $c \in [0,1)$ with an auxiliary function $\phi : [0,\infty) \to [0,\infty)$. The concept of $\phi$-contraction is essentially investigated by Browder [1] in 1968, wherein the author considered $\phi$ to be increasing and right continuous control function and utilized the same to extend the BCP. Afterward, many researchers generalized Browder fixed point theorem by modifying the properties of control function $\phi$ (e.g., Boyd–Wong contractions [2] and Matkowski contractions [3]). Indeed, Matkowski [3] employed a class of control functions, which are later termed as comparison functions. Matkowski contractions have been further studied in [4–13] besides several others.

In 2015, Alam and Imdad [14] established a novel variant of the BCP in a metric space equipped with an amorphous relation (see also [15,16]). In the recent years, various fixed point results have been proved under different types of contractivity conditions in relational metric spaces. The contraction conditions utilized in such results are indeed desired to hold for the elements that are comparative with respect to the underlying relation. These fixed point theorems can be applied to compute the unique solutions of certain matrix equations and boundary value problems (abbreviated as: BVP).

Relation-theoretic variants of Boyd–Wong fixed point result and Matkowski fixed point result are obtained recently by Alam and Imdad [17] and Arif et al. [18], respectively. To ensure the existence of a fixed point of a mapping satisfying such types of $\phi$-contractions, transitivity condition on underlying relation is additionally required. Due to restrictive nature of transitivity requirement, the authors [17,18] used an optimal condition of transitivity (locally $\mathcal{H}$-transitive relation). On the other hand, Ahmadullah et al. [19] proved a fixed point theorem in a metric space endowed with an amorphous relation satisfying generalized $\phi$-contractions using the concept of (c)-comparison functions. The contractivity condition utilized in [19] is stronger than Matkowski contraction. However, the authors succeed to prove the results for amorphous relation instead of a transitive relation. In the setting of a relational metric space, to ascertain the uniqueness of fixed point, usually the image of ambient space must be $\mathfrak{S}^s$-connected set. However, the results of Ahmadullah et al. [19] hold for the $\mathfrak{S}$-directed set, which is a particular class of $\mathfrak{S}^s$-connected set. On the other hand, the results of Alam and Imdad [17] and Arif et al. [18] are proved for $\mathfrak{S}^s$-connected sets.

Motivated by the above existing works, we established a fixed point theorem for a natural version of $\phi$-contractions employing (c)-comparison functions in relational metric space. In our results, the underlying relation is amorphous, while the uniqueness part requires the image of ambient space to be $\mathfrak{S}$-connected. This substantiates the utility of our main result over the results of Alam and Imdad [17], Arif et al. [18] and Ahmadullah et al. [19]. Some examples are constructed to substantiate the utility of our findings. To demonstrate the degree of applicability of our result, we studied the existence and uniqueness of solution of certain BVP of order one.

## 2. Preliminaries

Throughout this presentation, $\mathbb{N}$ will denote the set of natural numbers. Any subset of $\mathcal{M}^2$ is termed as a relation (or more precisely, a binary relation) on the set $\mathcal{M}$.

**Definition 1.** *Assume that $\mathcal{M}$ is a set, $\varrho$ is a metric on $\mathcal{M}$, $\mathfrak{S}$ is a relation on $\mathcal{M}$ and $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ is a function. One says that:*

- *Ref. [14] A pair $x, y \in \mathcal{M}$ is $\mathfrak{S}$-comparative, denoted by $[x, y] \in \mathfrak{S}$, if*
$$(x, y) \in \mathfrak{S} \quad \text{or} \quad (y, x) \in \mathfrak{S}.$$
- *Ref. [20] The relation $\mathfrak{S}^{-1} := \{(x, y) \in \mathcal{M}^2 : (y, x) \in \mathfrak{S}\}$ is transpose of $\mathfrak{S}$.*
- *Ref. [20] The relation $\mathfrak{S}^s := \mathfrak{S} \cup \mathfrak{S}^{-1}$ is symmetric closure of $\mathfrak{S}$.*
- *Ref. [21] A relation on a subset $\mathcal{S} \subseteq \mathcal{M}$ defined by*
$$\mathfrak{S}|_{\mathcal{S}} := \mathfrak{S} \cap \mathcal{S}^2$$
*is the restriction of $\mathfrak{S}$ on $\mathcal{S}$.*
- *Ref. [14] $\mathfrak{S}$ is $\mathcal{H}$-closed if it satisfies*
$$(\mathcal{H}x, \mathcal{H}y) \in \mathfrak{S},$$
*$\forall\, x, y \in \mathcal{M}$ verifying $(x, y) \in \mathfrak{S}$.*
- *Ref. [14] A sequence $\{r_n\} \subset \mathcal{M}$ verifying $(r_n, r_{n+1}) \in \mathfrak{S}$, $\forall\, n \in \mathbb{N}$, is $\mathfrak{S}$-preserving.*
- *Ref. [15] $(\mathcal{M}, \varrho)$ is $\mathfrak{S}$-complete if each $\mathfrak{S}$-preserving Cauchy sequence in $\mathcal{M}$ is convergent.*
- *Ref. [15] $\mathcal{H}$ is $\mathfrak{S}$-continuous at $p \in \mathcal{M}$ if for every $\mathfrak{S}$-preserving sequence $\{r_n\} \subset \mathcal{M}$ satisfying $r_n \xrightarrow{\varrho} p$, one has*
$$\mathcal{H}(r_n) \xrightarrow{\varrho} \mathcal{H}(p)$$
- *Ref. [15] $\mathcal{H}$ is $\mathfrak{S}$-continuous if it is $\mathfrak{S}$-continuous at all points of $\mathcal{M}$.*
- *Ref. [14] $\mathfrak{S}$ is $\varrho$-self-closed if every $\mathfrak{S}$-preserving sequence $\{r_n\} \subset \mathcal{M}$ verifying $r_n \xrightarrow{\varrho} p \in \mathcal{M}$ has a subsequence $\{r_{n_k}\}$ satisfying $[r_{n_k}, p] \in \mathfrak{S}$.*
- *Ref. [21] Given $x, y \in \mathcal{M}$, a subset $\{s_0, s_1, \ldots, s_l\} \subset \mathcal{M}$ is a path of length $l \in \mathbb{N}$ in $\mathfrak{S}$ from $x$ to $y$ if $s_0 = x$, $s_l = y$ and $(s_j, s_{j+1}) \in \mathfrak{S}, \quad 0 \leq j \leq l - 1$.*
- *Ref. [17] A subset $\mathcal{S} \subseteq \mathcal{M}$ is $\mathfrak{S}$-connected if each pair of elements of $\mathcal{S}$ admits path in $\mathfrak{S}$.*
- *Ref. [8] A subset $\mathcal{S} \subseteq \mathcal{M}$ is $\mathfrak{S}$-directed if for each pair $x, y \in \mathcal{S}$, $\exists\, z \in \mathcal{M}$ satisfying $(x, z) \in \mathfrak{S}$ and $(y, z) \in \mathfrak{S}$.*

**Proposition 1** ([14])**.** $(x, y) \in \mathfrak{S}^s \iff [x, y] \in \mathfrak{S}$.

**Proposition 2** ([17]). *$\mathfrak{S}$ is $\mathcal{H}^n$-closed provided $\mathfrak{S}$ is $\mathcal{H}$-closed.*

**Remark 1** (see [17]). *If a subset $\mathcal{S}$ of $\mathcal{M}$ is $\mathfrak{S}$-directed, then each pair of elements of $\mathcal{S}$ admits a path of length 2. Consequently, $\mathcal{S}$ is $\mathfrak{S}$-connected. Therefore, every $\mathfrak{S}$-directed subset of $\mathcal{M}$ is also $\mathfrak{S}$-connected.*

The following notations will be utilized in future.

- $F(\mathcal{H})$ := the set of fixed points of $\mathcal{H}$,
- $\mathcal{M}(\mathcal{H}, \mathfrak{S}) := \{r \in \mathcal{M} : (r, \mathcal{H}r) \in \mathfrak{S}\}$.

The following result proved by Alam and Imdad [14] is popular as relation-theoretic contraction principle.

**Theorem 1** ([14,16]). *Assume that $(\mathcal{M}, \varrho)$ is metric space equipped with a relation $\mathfrak{S}$ while $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ is a function. Additionally,*

(i) *$(\mathcal{M}, \varrho)$ is $\mathfrak{S}$-complete,*
(ii) *$\mathcal{M}(\mathcal{H}, \mathfrak{S}) \neq \emptyset$,*
(iii) *$\mathfrak{S}$ is $\mathcal{H}$-closed,*
(iv) *$\mathcal{H}$ is $\mathfrak{S}$-continuous or $\mathfrak{S}$ is $\varrho$-self-closed,*
(v) *$\exists\, c \in [0, 1)$ verifying*

$$\varrho(\mathcal{H}x, \mathcal{H}y) \leq c\varrho(x, y),\ \forall\, x, y \in \mathcal{M} \text{ with } (x, y) \in \mathfrak{S}.$$

*Then, $\mathcal{H}$ admits a fixed point. Additionally, if $\mathcal{H}(\mathcal{M})$ is $\mathfrak{S}^s$-connected, then $\mathcal{H}$ admits a unique fixed point.*

## 3. (c)-Comparison Functions

Let us recall two families of auxiliary functions of existing literature.

**Definition 2** ([22]). *A mapping $\phi : [0, \infty) \to [0, \infty)$ is termed as comparison function if it enjoys the following ones:*

(i) *$\phi$ is monotonic increasing,*
(ii) $\lim\limits_{n\to\infty} \phi^n(t) = 0, \quad \forall\, t > 0.$

**Definition 3** ([22]). *A mapping $\phi : [0, \infty) \to [0, \infty)$ is termed as (c)-comparison function if it enjoys the following ones:*

(i) *$\phi$ is monotonic increasing,*
(ii) $\sum\limits_{n=1}^{\infty} \phi^n(t) < \infty, \quad \forall\, t > 0.$

Clearly, every (c)-comparison function is a comparison function.

**Remark 2** ([22]). *Let $\phi$ be a (c)-comparison function. Then*

(i) *$\phi(0) = 0$,*
(ii) *$\phi(t) < t$, $\forall\, t > 0$,*
(iii) *$\phi$ is right continuous at 0.*

In 2019, Ahmadullah et al. [19] proved the following result:

**Theorem 2** (see Theorem 2.5 [19]). *Assume that $(\mathcal{M}, \varrho)$ is metric space equipped with a relation $\mathfrak{S}$ while $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ is a function. Additionally,*

(i) *$(\mathcal{M}, \varrho)$ is $\mathfrak{S}$-complete,*
(ii) *$\mathcal{M}(\mathcal{H}, \mathfrak{S}) \neq \emptyset$,*
(iii) *$\mathfrak{S}$ is $\mathcal{H}$-closed,*

(iv) *$\mathcal{H}$ is $\mathfrak{S}$-continuous or $\mathfrak{S}$ is $\varrho$-self-closed,*
(v) *$\exists$ a (c)-comparison function $\phi$ verifying*

$$\varrho(\mathcal{H}x,\mathcal{H}y) \leq \phi\left(\max\left\{\varrho(x,y),\frac{1}{2}[\varrho(x,\mathcal{H}x)+\varrho(y,\mathcal{H}y)],\frac{1}{2}[\varrho(x,\mathcal{H}y)+\varrho(y,\mathcal{H}x)]\right\}\right),$$
$$\forall\, x,y \in \mathcal{M} \text{ with } (x,y) \in \mathfrak{S}.$$

*Then, $\mathcal{H}$ admits a fixed point. Additionally, if $\mathcal{H}(\mathcal{M})$ is $\mathfrak{S}^s$-directed, then $\mathcal{H}$ admits a unique fixed point.*

The following consequence of Theorem 2 is immediate.

**Corollary 1.** *Assume that $(\mathcal{M},\varrho)$ is metric space equipped with a relation $\mathfrak{S}$ while $\mathcal{H}:\mathcal{M}\to\mathcal{M}$ is a function. Additionally,*

(i) *$(\mathcal{M},\varrho)$ is $\mathfrak{S}$-complete,*
(ii) *$\mathcal{M}(\mathcal{H},\mathfrak{S}) \neq \emptyset$,*
(iii) *$\mathfrak{S}$ is $\mathcal{H}$-closed,*
(iv) *$\mathcal{H}$ is $\mathfrak{S}$-continuous or $\mathfrak{S}$ is $\varrho$-self-closed,*
(v) *$\exists$ a (c)-comparison function $\phi$ verifying*

$$\varrho(\mathcal{H}x,\mathcal{H}y) \leq \phi(\varrho(x,y)),\ \forall\, x,y \in \mathcal{M} \text{ with } (x,y) \in \mathfrak{S}.$$

*Then, $\mathcal{H}$ admits a fixed point. Additionally, if $\mathcal{H}(\mathcal{M})$ is $\mathfrak{S}^s$-directed, then $\mathcal{H}$ admits a unique fixed point.*

Now, one proposes the following fact.

**Proposition 3.** *Assume that $(\mathcal{M},\varrho)$ is metric space equipped with a relation $\mathfrak{S}$ while $\mathcal{H}:\mathcal{M}\to\mathcal{M}$ is a function. If $\phi$ is a (c)-comparison function, then the following contractivity conditions are equivalent:*

(I) *$\varrho(\mathcal{H}x,\mathcal{H}y) \leq \phi(\varrho(x,y)),\ \forall\, x,y \in \mathcal{M}$ with $(x,y) \in \mathfrak{S}$,*
(II) *$\varrho(\mathcal{H}x,\mathcal{H}y) \leq \phi(\varrho(x,y)),\ \forall\, x,y \in \mathcal{M}$ with $[x,y] \in \mathfrak{S}$.*

**Proof.** The conclusion (II)$\Rightarrow$(I) holds trivially. Conversely, let (I) holds. Assume that $x,y \in \mathcal{M}$ with $[x,y] \in \mathfrak{S}$. Then, in case $(x,y) \in \mathfrak{S}$, (I) implies (II). Otherwise, in case $(y,x) \in \mathfrak{S}$, by symmetry of $\varrho$ and (I), one obtains

$$\varrho(\mathcal{H}x,\mathcal{H}y) = \varrho(\mathcal{H}y,\mathcal{H}x) \leq \phi(\varrho(y,x)) = \phi(\varrho(x,y)).$$

It follows that (I)$\Rightarrow$(II). $\square$

## 4. Main Results

In view of Remark 1, $\mathfrak{S}^s$-connectedness is weaker than $\mathfrak{S}^s$-directedness. However, Ahmadullah et al. [19] could not succeed to extend Theorem 2 to $\mathfrak{S}^s$-connected sets. Our main result, which extends Corollary 1 to $\mathfrak{S}^s$-connected sets, runs as follows:

**Theorem 3.** *Assume that $(\mathcal{M},\varrho)$ is metric space equipped with a relation $\mathfrak{S}$ while $\mathcal{H}:\mathcal{M}\to\mathcal{M}$ is a function. Additionally,*

(i) *$(\mathcal{M},\varrho)$ is $\mathfrak{S}$-complete,*
(ii) *$\mathcal{M}(\mathcal{H},\mathfrak{S}) \neq \emptyset$,*
(iii) *$\mathfrak{S}$ is $\mathcal{H}$-closed,*
(iv) *$\mathcal{H}$ is $\mathfrak{S}$-continuous or $\mathfrak{S}$ is $\varrho$-self-closed,*
(v) *$\exists$ a (c)-comparison function $\phi$ verifying*

$$\varrho(\mathcal{H}x,\mathcal{H}y) \leq \phi(\varrho(x,y)),\ \forall\, x,y \in \mathcal{M} \text{ with } (x,y) \in \mathfrak{S}.$$

*Then, $\mathcal{H}$ admits a fixed point. Additionally, if $\mathcal{H}(\mathcal{M})$ is $\mathfrak{S}^s$-connected, then $\mathcal{H}$ admits a unique fixed point.*

**Proof.** Using assumption (ii), one can choose $r_0 \in \mathcal{M}(\mathcal{H}, \mathfrak{S})$. Define the sequence $\{r_n\} \subset \mathcal{M}$ verifying

$$r_n = \mathcal{H}^n(r_0) = \mathcal{H}(r_{n-1}), \quad \forall\, n \in \mathbb{N}. \tag{1}$$

Using the fact $(r_0, \mathcal{H}r_0) \in \mathfrak{S}$, $\mathcal{H}$-closedness of $\mathfrak{S}$ and Proposition 2, one obtains

$$(\mathcal{H}^n r_0, \mathcal{H}^{n+1} r_0) \in \mathfrak{S}$$

which, making use of (1), becomes

$$(r_n, r_{n+1}) \in \mathfrak{S}, \quad \forall\, n \in \mathbb{N}. \tag{2}$$

Thus, $\{r_n\}$ is an $\mathfrak{S}$-preserving sequence. Applying hypothesis (v) to (2), one finds

$$\varrho(r_n, r_{n+1}) \leq \phi(\varrho(r_{n-1}, r_n)),\ \forall\, n \in \mathbb{N}$$

which by induction and monotonicity of $\phi$ gives rise to

$$\varrho(r_n, r_{n+1}) \leq \phi^n \varrho(r_0, \mathcal{H}r_0), \quad \forall\, n \in \mathbb{N}. \tag{3}$$

$\forall\, m, n \in \mathbb{N}$ with $m < n$, making use of (3) and triangular inequality, one obtains

$$\begin{aligned}
\varrho(r_m, r_n) &\leq \varrho(r_m, r_{m+1}) + \varrho(r_{m+1}, r_{m+2}) + \cdots + \varrho(r_{n-1}, r_n) \\
&\leq \phi^m(\varrho(r_0, \mathcal{H}r_0)) + \phi^{m+1}(\varrho(r_0, \mathcal{H}r_0)) + \cdots + \phi^{n-1}(\varrho(r_0, \mathcal{H}r_0)) \\
&= \sum_{k=m}^{n-1} \phi^k(\varrho(r_0, \mathcal{H}r_0)) \\
&\leq \sum_{k \geq m} \phi^k(\varrho(r_0, \mathcal{H}r_0)) \\
&\to 0 \text{ as } m \text{ (and hence } n) \to \infty,
\end{aligned}$$

which yields that $\{r_n\}$ is Cauchy. Hence, $\{r_n\}$ is an $\mathfrak{S}$-preserving Cauchy sequence and hence by $\mathfrak{S}$-completeness of $\mathcal{M}$, $\exists\, p \in \mathcal{M}$ verifying $r_n \xrightarrow{\varrho} p$.

Now, one has to use condition (iv) to prove that $p$ is a fixed point of $\mathcal{H}$. Let $\mathcal{H}$ be $\mathfrak{S}$-continuous. Since $\{r_n\}$ is $\mathfrak{S}$-preserving verifying $r_n \xrightarrow{\varrho} p$, therefore by $\mathfrak{S}$-continuity of $\mathcal{H}$, one concludes that $r_{n+1} = \mathcal{H}(r_n) \xrightarrow{\varrho} \mathcal{H}(p)$. Uniqueness property of limit gives rise to $\mathcal{H}(p) = p$. Alternately, suppose that $\mathfrak{S}$ is $\varrho$-self-closed. Since $\{r_n\}$ is $\mathfrak{S}$-preserving and $r_n \xrightarrow{\varrho} p$, therefore $\{r_n\}$ has a subsequence $\{r_{n_k}\}$ satisfying $[r_{n_k}, p] \in \mathfrak{S}$, $\forall\, k \in \mathbb{N}$. By assumption (v), Proposition 3 and $[r_{n_k}, p] \in \mathfrak{S}$, one obtains

$$\varrho(r_{n_k+1}, \mathcal{H}p) = \varrho(\mathcal{H}r_{n_k}, \mathcal{H}p) \leq \phi(\varrho(r_{n_k}, p)).$$

Using items (i) and (ii) of Remark 2 (whether $\varrho(r_{n_k}, p)$ is non-zero or zero) and $r_{n_k} \xrightarrow{\varrho} p$, we obtain

$$\varrho(r_{n_k+1}, \mathcal{H}p) \leq \varrho(r_{n_k}, p) \to 0, \text{ as } k \to \infty$$

so that $r_{n_k+1} \xrightarrow{\varrho} \mathcal{H}(p)$. Uniqueness property of limit gives rise to $\mathcal{H}(p) = p$. Hence in both the cases, $p$ is a fixed point of $\mathcal{H}$.

To prove uniqueness, take $p, q \in F(\mathcal{H})$, one obtains

$$\mathcal{H}^n(p) = p \text{ and } \mathcal{H}^n(q) = q. \tag{4}$$

Since $p, q \in \mathcal{H}(\mathcal{M})$ and $\mathcal{H}(\mathcal{M})$ is $\mathfrak{S}^s$-connected, therefore $\exists$ a path $\{s_0, s_1, s_2, \ldots, s_l\}$ in $\mathfrak{S}^s$ from $p$ to $q$ so that

$$s_0 = p,\ s_l = q \text{ and } [s_j, s_{j+1}] \in \mathfrak{S}, \quad \forall\, j \in \{0, 1, \ldots, l-1\}. \tag{5}$$

Using $\mathcal{H}$-closedness of $\mathfrak{S}$ and Proposition 2, one has

$$[\mathcal{H}^n s_j, \mathcal{H}^n s_{j+1}] \in \mathfrak{S}, \quad \forall\, j \in \{0, 1, \ldots, l-1\} \text{ and } \forall\ n \in \mathbb{N}. \tag{6}$$

By (6), hypothesis (v) and Proposition 3, one has

$$t^j_{n+1} := \varrho(\mathcal{H}^{n+1} s_j, \mathcal{H}^{n+1} s_{j+1}) \le \phi(\varrho(\mathcal{H}^n s_j, \mathcal{H}^n s_{j+1})) = \phi(t^j_n), \quad \forall\, j \in \{0, 1, \ldots, l-1\}. \tag{7}$$

One will prove that

$$\lim_{n\to\infty} t^j_n = 0, \quad \forall\, j \in \{0, 1, \ldots, l-1\}. \tag{8}$$

For each fixed $j$ $(0 \le j \le l-1)$, two cases arise. Firstly, suppose that $t^j_{n_0} = 0$ for some $n_0 \in \mathbb{N}$, i.e., $\mathcal{H}^{n_0}(s_j) = \mathcal{H}^{n_0}(s_{j+1})$ implying thereby $\mathcal{H}^{n_0+1}(s_j) = \mathcal{H}^{n_0+1}(s_{j+1})$. Consequently, one finds $t^j_{n_0+1} = 0$ and thus by induction, we get $t^j_n = 0,\ \forall\ n \ge n_0$ so that $\lim\limits_{n\to\infty} t^j_n = 0$. Secondly, suppose that $t^j_n > 0,\ \forall\ n \in \mathbb{N}$. Using induction on $n$ and monotonicity of $\phi$ in (7), one has

$$t^j_{n+1} \le \phi(t^j_n) \le \phi^2(t^j_{n-1}) \le \cdots \le \phi^n(t^j_1)$$

so that

$$t^j_{n+1} \le \phi^n(t^j_1), \quad \forall\, j\ (0 \le j \le l-1). \tag{9}$$

Letting $n \to \infty$ in (9) and owing to the definition of (c)-comparison functions, one has

$$\lim_{n\to\infty} t^j_{n+1} \le \lim_{n\to\infty} \phi^n(t^j_1) = 0.$$

Therefore, in each case, (8) is verified. In lieu of (4), (5), (8), and triangular inequality, one has

$$\varrho(p, q) = \varrho(\mathcal{H}^n s_0, \mathcal{H}^n s_l) \le t^0_n + t^1_n + \cdots + t^{l-1}_n \to 0, \ \ as \ \ n \to \infty$$

so that $p = q$. Hence $\mathcal{H}$ admits a unique fixed point. $\square$

## 5. Illustrative Examples

To attest the credibility of Theorem 3, let us adopt the following examples.

**Example 1.** *Consider $\mathcal{M} = (0, 1]$ equipped with standard metric $\varrho$ and a relation $\mathfrak{S} = \{(x, y) : \frac{1}{4} \le x \le y \le \frac{1}{3}$ or $\frac{1}{2} \le x \le y \le 1\}$. Define a function $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ by*

$$\mathcal{H}(x) = \begin{cases} \dfrac{1}{4}, & \text{if } 0 \le x < \dfrac{1}{2} \\ 1, & \text{if } \dfrac{1}{2} \le x \le 1. \end{cases}$$

*Then $(\mathcal{M}, \varrho)$ is $\mathfrak{S}$-complete metric space, $\mathcal{H}$ is $\mathfrak{S}$-continuous and $\mathfrak{S}$ is $\mathcal{H}$-closed. One can easily check that assumption* (v) *of Theorem 3 holds for an arbitrary (c)-comparison function $\phi$. Therefore, the assumptions* (i)–(v) *of Theorem 3 hold. Consequently, $\mathcal{H}$ admits a fixed point. Since there in no path between $\dfrac{1}{4}$ and $1$, therefore $\mathcal{H}(\mathcal{M})$ is not $\mathfrak{S}^s$-connected. Note that $p = \dfrac{1}{4}$ and $q = 1$ are two fixed points of $\mathcal{H}$.*

**Example 2.** *Consider $\mathcal{M} = [0,2]$ equipped with standard metric $\varrho$ and a relation $\mathfrak{S} = \{(0,0),(0,1),(1,0),(1,1),(0,2)\}$ Define a function $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ by*

$$\mathcal{H}(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 < x \leq 2 \end{cases}$$

*Then $(\mathcal{M}, \varrho)$ is $\mathfrak{S}$-complete metric space while $\mathfrak{S}$ is $\mathcal{H}$-closed. Let $\{r_n\} \subset \mathcal{M}$ be an $\mathfrak{S}$-preserving sequence verifying $r_n \xrightarrow{\varrho} l$ so that $(r_n, r_{n+1}) \in \mathfrak{S}$, $\forall\, n \in \mathbb{N}$. Notice that $(r_n, r_{n+1}) \notin \{(0,2)\}$ yielding thereby $(r_n, r_{n+1}) \in \{(0,0),(0,1),(1,0),(1,1)\}$, $\forall\, n \in \mathbb{N}$ so that $\{r_n\} \subset \{0,1\}$. Since $\{0,1\}$ is closed, therefore one has $[r_n, l] \in \mathfrak{S}$. Thus, $\mathfrak{S}$ is $\varrho$-self-closed. One can verify contractivity condition* (v) *of Theorem 3 with $\phi(t) = \dfrac{t}{2}$. Rest of the hypotheses of Theorem 3 also hold. Consequently, $\mathcal{H}$ has a unique fixed point $p = 0$.*

## 6. Application to Boundary Value Problem

Consider the following BVP:

$$\begin{cases} \theta'(s) = \Phi(s, \theta(s)), & s \in [0,a] \\ \theta(0) = \theta(a) \end{cases} \tag{10}$$

where $a > 0$ and $\Phi : [0,a] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

As usual, $\mathcal{C}[0,a]$ will denote the family of all real valued continuous functions on $[0,a]$. Let us revisit to the following notions:

**Definition 4** ([23])**.** *One says that $\mu \in \mathcal{C}^1[0,a]$ forms a lower solution of* (10) *if*

$$\begin{cases} \mu'(s) \leq \Phi(s, \mu(s)), & s \in [0,a] \\ \mu(0) \leq \mu(a). \end{cases}$$

**Definition 5** ([23])**.** *One says that $\mu \in \mathcal{C}^1[0,a]$ forms an upper solution of* (10) *if*

$$\begin{cases} \mu'(s) \geq \Phi(s, \mu(s)), & s \in [0,a] \\ \mu(0) \geq \mu(a). \end{cases}$$

One presents the existence and uniqueness theorem to determine a solution of Problem (10) as follows.

**Theorem 4.** *In addition to Problem* (10)*, if $\exists\, k > 0$ and $\exists$ a (c)-comparison function $\phi$ verifying $\forall\, r, t \in \mathbb{R}$ with $r \leq t$ that*

$$0 \leq [\Phi(s,t) + kt] - [\Phi(s,r) + kr] \leq k\phi(t - r). \tag{11}$$

*Further, if* (10) *has a lower solution, then $\exists$ a unique solution of Problem* (10)*.*

**Proof.** Rewrite Problem (10) as

$$\begin{cases} \theta'(s) + k\theta(s) = \Phi(s, \theta(s)) + k\theta(s), & \forall\, s \in [0,a] \\ \theta(0) = \theta(a). \end{cases} \tag{12}$$

Equation (12) is equivalent to the integral equation

$$\theta(s) = \int_0^a G(s,\xi)[\Phi(\xi, \theta(\xi)) + k\theta(\xi)]d\xi \tag{13}$$

where $G(s,\xi)$ is Green function defined by

$$G(s,\xi)=\begin{cases}\frac{e^{k(a+\xi-s)}}{e^{ka}-1}, & 0\le\xi<s\le a\\ \frac{e^{k(\xi-s)}}{e^{ka}-1}, & 0\le s<\xi\le a.\end{cases}$$

Define a mapping $\mathcal{H}:\mathcal{C}[0,a]\to\mathcal{C}[0,a]$ by

$$(\mathcal{H}\theta)(s)=\int_0^a G(s,\xi)[\Phi(\xi,\theta(\xi))+k\theta(\xi)]d\xi,\quad \forall\, s\in[0,a]. \tag{14}$$

Thus, $\theta\in\mathcal{C}[0,a]$ is a fixed point of $\mathcal{H}$ if, and only if, $\theta\in\mathcal{C}^1[0,a]$ forms a solution of (13) and, hence, of (10).

Define a metric $\varrho$ on $\mathcal{C}[0,a]$ by

$$\varrho(\theta,\vartheta)=\sup_{s\in[0,a]}|\theta(s)-\vartheta(s)|,\quad \forall\,\theta,\vartheta\in\mathcal{C}[0,a]. \tag{15}$$

Undertake a relation $\mathfrak{S}$ on $\mathcal{C}[0,a]$ defined by

$$\mathfrak{S}=\{(\theta,\vartheta)\in\mathcal{C}[0,a]\times\mathcal{C}[0,a]:\theta(s)\le\vartheta(s),\ \forall\, s\in[0,a]\}. \tag{16}$$

Now, one will verify all the hypotheses of Theorem 3.

(i) Obviously, $(\mathcal{C}[0,a],\varrho)$ is $\mathfrak{S}$-complete metric space.

(ii) Let $\mu\in\mathcal{C}^1[0,a]$ be a lower solution of (10), then one has

$$\mu'(s)+k\mu(s)\le\Phi(s,\mu(s))+k\mu(s),\quad \forall\, s\in[0,a].$$

Multiplying on both the sides by $e^{ks}$, one obtains

$$(\mu(s)e^{ks})'\le[\Phi(s,\mu(s))+k\mu(s)]e^{ks},\quad \forall\, s\in[0,a]$$

yielding thereby

$$\mu(s)e^{ks}\le\mu(0)+\int_0^s[\Phi(\xi,\mu(\xi))+k\mu(\xi)]e^{k\xi}d\xi,\quad \forall\, s\in[0,a]. \tag{17}$$

Owing to $\mu(0)\le\mu(a)$, one obtains

$$\mu(0)e^{ka}\le\mu(a)e^{ka}\le\mu(0)+\int_0^a[\Phi(\xi,\mu(\xi))+k\mu(\xi)]e^{k\xi}d\xi$$

so that

$$\mu(0)\le\int_0^a\frac{e^{k\xi}}{e^{ka}-1}[\Phi(\xi,\mu(\xi))+k\mu(\xi)]d\xi. \tag{18}$$

By (17) and (18), one finds

$$\begin{aligned}\mu(s)e^{ks}&\le\int_0^a\frac{e^{k\xi}}{e^{ka}-1}[\Phi(\xi,\mu(\xi))+k\mu(\xi)]d\xi+\int_0^s e^{k\xi}[\Phi(\xi,\mu(\xi))+k\mu(\xi)]d\xi\\ &=\int_0^s\frac{e^{k(a+\xi)}}{e^{ka}-1}[\Phi(\xi,\mu(\xi))+k\mu(\xi)]d\xi+\int_s^a\frac{e^{k\xi}}{e^{ka}-1}[\Phi(\xi,\mu(\xi))+k\mu(\xi)]d\xi\end{aligned}$$

so that

$$\begin{aligned}\mu(s) &\leq \int_0^s \frac{e^{k(a+\xi-s)}}{e^{ka}-1}[\Phi(\xi,\mu(\xi))+k\mu(\xi)]d\xi + \int_s^a \frac{e^{k(\xi-s)}}{e^{ka}-1}[\Phi(\xi,\mu(\xi))+k\mu(\xi)]d\xi \\ &= \int_0^a G(s,\xi)[\Phi(\xi,\mu(\xi))+k\mu(\xi)]d\xi \\ &= (\mathcal{H}\mu)(s), \quad \forall\, s\in[0,a]\end{aligned}$$

which yields that $(\mu,\mathcal{H}\mu)\in\mathfrak{S}$ so that $\mathcal{C}[0,a](\mathcal{H},\mathfrak{S})$ is non-empty.

(iii) Take $\theta,\vartheta\in\mathcal{C}[0,a]$ verifying $(\theta,\vartheta)\in\mathfrak{S}$. Using (11), one obtains

$$\Phi(s,\theta(s))+k\theta(s)\leq\Phi(s,\vartheta(s))+k\vartheta(s), \quad \forall\, s\in[0,a]. \tag{19}$$

By (14), (19), and due to $G(s,\xi)>0$, $\forall\, s,\xi\in[0,a]$, one obtains

$$\begin{aligned}(\mathcal{H}\theta)(s) &= \int_0^a G(s,\xi)[\Phi(\xi,\theta(\xi))+k\theta(\xi)]d\xi \\ &\leq \int_0^a G(s,\xi)[\Phi(\xi,\vartheta(\xi))+k\vartheta(\xi)]d\xi \\ &= (\mathcal{H}\vartheta)(s), \quad \forall\, s\in[0,a],\end{aligned}$$

which by using (16) yields that $(\mathcal{H}\theta,\mathcal{H}\vartheta)\in\mathfrak{S}$ and hence $\mathfrak{S}$ is $\mathcal{H}$-closed.

(iv) Let $\{\theta_n\}\subset\mathcal{C}[0,a]$ be an $\mathfrak{S}$-preserving sequence converging to $\theta\in\mathcal{C}[0,a]$. Then for each $s\in[0,a]$, $\{\theta_n(s)\}$ is monotone increasing sequence in $\mathbb{R}$ converging to $\theta(s)$. Consequently, $\forall\, n\in\mathbb{N}$ and $\forall\, s\in[0,a]$, one has $\theta_n(s)\leq\theta(s)$. Again due to (16), it follows that $(\theta_n,\theta)\in\mathfrak{S}$, $\forall\, n\in\mathbb{N}$ and hence $\mathfrak{S}$ is $\varrho$-self-closed.

(v) Take $\theta,\vartheta\in\mathcal{C}[0,a]$ verifying $(\theta,\vartheta)\in\mathfrak{S}$. Then by (11), (14), and (15), one has

$$\begin{aligned}\varrho(\mathcal{H}\theta,\mathcal{H}\vartheta) &= \sup_{s\in[0,a]}|(\mathcal{H}\theta)(s)-(\mathcal{H}\vartheta)(s)| = \sup_{s\in[0,a]}\big((\mathcal{H}\vartheta)(s)-(\mathcal{H}\theta)(s)\big) \\ &\leq \sup_{s\in[0,a]}\int_0^a G(s,\xi)[\Phi(\xi,\vartheta(\xi))+k\vartheta(\xi)-\Phi(\xi,\theta(\xi))-k\theta(\xi)]d\xi \\ &\leq \sup_{s\in[0,a]}\int_0^a G(s,\xi)k\phi(\vartheta(\xi)-\theta(\xi))d\xi.\end{aligned} \tag{20}$$

Now, $0\leq\vartheta(\xi)-\theta(\xi)\leq\varrho(\theta,\vartheta)$. Using the monotonicity of $\phi$, one obtains $\phi(\vartheta(\xi)-\theta(\xi))\leq\phi(\varrho(\theta,\vartheta))$ and, hence, (20) becomes

$$\begin{aligned}\varrho(\mathcal{H}\theta,\mathcal{H}\vartheta) &\leq k\phi(\varrho(\theta,\vartheta))\sup_{s\in[0,a]}\int_0^a G(s,\xi)d\xi \\ &= k\phi(\varrho(\theta,\vartheta))\sup_{s\in[0,a]}\frac{1}{e^{ka}-1}\left[\frac{1}{k}e^{k(a+\xi-s)}\Big|_0^s+\frac{1}{k}e^{k(\xi-s)}\Big|_s^a\right] \\ &= k\phi(\varrho(\theta,\vartheta))\frac{1}{k(e^{ka}-1)}(e^{ka}-1) \\ &= \phi(\varrho(\theta,\vartheta))\end{aligned}$$

so that

$$\varrho(\mathcal{H}\theta,\mathcal{H}\vartheta)\leq\phi(\varrho(\theta,\vartheta)), \quad \forall\,\theta,\vartheta\in\mathcal{C}[0,a] \text{ such that } (\theta,\vartheta)\in\mathfrak{S}.$$

Let $\theta,\vartheta\in\mathcal{C}[0,a]$ be arbitrary. Then, one has $\varphi:=\max\{\mathcal{H}\theta,\mathcal{H}\vartheta\}\in\mathcal{C}[0,a]$. As $(\mathcal{H}\theta,\varphi)\in\mathfrak{S}$ and $(\mathcal{H}\vartheta,\varphi)\in\mathfrak{S}$, $\{\mathcal{H}\theta,\varphi,\mathcal{H}\vartheta\}$ is path in $\mathfrak{S}^s$ between $\mathcal{H}(\theta)$ and $\mathcal{H}(\vartheta)$. Thus, $\mathcal{H}(\mathcal{C}[0,a])$ is $\mathfrak{S}^s$-connected and hence by Theorem 3, $\mathcal{H}$ admits a unique fixed point, which forms the unique solution of Problem (10). □

Intending to illustrate Theorem 4, one considers the following example.

**Example 3.** *Let $\Phi(s,\theta(s)) = \cos s$ for $0 \leq s \leq \pi$, then $\Phi$ is a continuous functions. Define a function $\phi : [0,\infty) \to [0,\infty)$ by*

$$\phi(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq 2, \\ 2t-4, & \text{if } 2 < t \leq 3, \\ \frac{2}{3}t, & \text{if } t > 3. \end{cases}$$

*Then $\phi$ is a (c)-comparison function. Additionally, for any arbitrary pair $r,t \in \mathbb{R}$ with $r \leq t$, the inequality (11) holds. Moreover, $\theta = 0$ is a lower solution for $\frac{d\theta}{ds} = \cos s$. Therefore, Theorem 4 can be applied for the given problem and, hence, $\theta(s) = \sin s$ forms the unique solution.*

## 7. Conclusions

Alam and Imdad [17] and Arif et al. [18] established fixed point results under relation-theoretic Boyd–Wong contractions and Matkowski contractions employing a class of transitive relation, namely: locally $\mathcal{H}$-transitive relation. In the process, they observed that their results cannot be extended to an amorphous relation. The contraction condition utilized in our main result is restrictive as compared to Boyd–Wong contractions and Matkowski contractions, but we succeed to extend the fixed point theorem up to an amorphous binary relation. The existence part of our main result can be deduced from the result of Ahmadullah et al. [19] but the hypothesis of uniqueness part of our result is relatively more general, which concludes that our result is independent from that of Ahmadullah et al. [19]. Thus Theorem 3 is different from the results of Ahmadullah et al. [19], Alam and Imdad [17], and Arif et al. [18].

A relatively weaker contraction condition is utilized compared with those in the recent literature, as in this work, the contraction condition is desired to hold merely for comparative elements via underlying relation and not for the entire space. Owing to this restrictive nature, results employing relation-theoretic contractions are applicable in fields of non-linear matrix equations and BVP. In future, readers can extend our results to a pair of self-mappings by proving coincidence and common fixed point theorems, which are very influential and applicable areas by their own.

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