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Stress–Strength Inference on the Multicomponent Model Based on Generalized Exponential Distributions under Type-I Hybrid Censoring

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Abstract: The stress–strength analysis is investigated for a multicomponent system, where all strength variables of components follow a generalized exponential distribution and are subject to the generalized exponential distributed stress. The estimation methods of the maximum likelihood and Bayesian are utilized to infer the system reliability. For the Bayesian estimation method, informative and non-informative priors combined with three loss functions are considered. Because the computational difficulty on working posteriors, the Markov chain Monte Carlo method is adopted to obtain the approximation of the reliability estimator posterior. In addition, the bootstrap method and highest probability density interval are used to obtain the reliability confidence intervals. The simulation study shows that the Bayes estimator with informative prior is superior to other competitors. Finally, two real examples are given to illustrate the proposed estimation methods.

Keywords: multicomponent stress–strength model; generalized exponential distribution; Bayesian method; Markov chain Monte Carlo method; highest probability density interval

MSC: 62F10; 62N05; 62P30

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1. Introduction

Let strength and stress variables in a single-component system be presented by X and Y, respectively. The single-component system is survived if the strength is over its stress borne. Hence, the performance of the single-component system can be evaluated by the reliability, $R = P(X > Y)$, which was connected to Mann–Whitney statistics by Birnbaum [1]. The aforementioned model was called the stress–strength model by Church and Harris [2] and has widely been used in industrial engineering, economics, psychology and medical research since then. Currently, many applications and inferences of the stress–strength model have appeared in the literature. For example, Kundu and Gupta [3] deduced the maximum likelihood, consistent minimum variance unbiased, and Bayes estimators of R based on the generalized exponential distributed stress and strength variables with different shape parameters. Given identical scale parameter for two Burr-type X distributions, Raqab and Kundu [4] investigated the stress–strength model and compared the maximum likelihood, uniformly minimum variance unbiased, and approximate Bayes estimators using simulation methods. Al-Mutairi et al. [5] proposed the uniformly minimum variance unbiased and maximum likelihood estimators of R under Lindley distributions with different shape parameters. Ali [6] derived the mathematical properties of the Lindley distribution and Bayesian method for the stress and strength analysis with different loss functions. Singh et al. [7] discussed the estimation problem of R via using the classical and Bayesian paradigms under generalized Lindley distribution. The Markov chain Monte Carlo (MCMC) technique was used to perform Bayesian calculation.

Sadek et al. [8] studied the R estimation problem utilizing stress and strength samples taken from the quasi Lindley distributions, respectively, and provided the maximum likelihood and Bayes estimators, where the MCMC technique via the Metropolis–Hastings algorithm was used to implement Bayesian estimation.

In many cases, the system can be composed of multicomponents. Bhattacharyya and Johnson [9] mentioned that the performance of a multicomponent system depends on the specific minimum number of components that must operate simultaneously. Thus, they proposed a multicomponent stress–strength (MSS) model with k independent components of identically distributed random strength to model the MSS system. Components in a MSS model are subject to independent stress variables. A MSS system survives if at least $s(\leq k)$ components work normally. They studied the uniformly minimum variance unbiased estimator and other relevant estimators under different exponential distributions. Pandey and Uddin [10] derived the maximum likelihood and Bayes estimators for the MSS parameters. Rao and Kantam [11] studied the performance of the reliability estimation in a MSS model by using different parameter estimation methods for the log-logistic distribution. Rao [12] used the maximum likelihood estimation method to obtain the asymptotic distribution and confidence interval of reliability in a MSS model for the generalized exponential distribution. Sharma and Sanku [13] studied the reliability estimation of the MSS model for the inverted exponentiated Rayleigh distribution.

All the aforementioned references for the MSS reliability inference methods were developed based on complete random lifetime observations. Due to technological advances, the component quality has been improved, and the component lifetime has been prolonged. Collecting complete lifetimes from all test components will no longer be easy. Producers suffer from cost restrictions or other reasons to utilize a complete lifetime sample for the reliability inference. The effects of incomplete data have been widely researched in recent years, and solutions have been proposed to deal with reliability problems. Gunasekera [14] established the maximum likelihood and Bayes estimators of the MSS reliability via using progressively type-II right-censored data from exponentiated inverted exponential distributions. Kohansal and Shoaee [15] considered the point and interval estimations of the MSS reliability based on the adaptive hybrid progressively censored data from Weibull distributions. Azhad et al. [16] proposed statistical methods to infer the MSS reliability by using upper recorded strength and stress samples that follow independent Pareto distributions with different shape parameters and common scale parameter.

Recently, a type-I hybrid censoring scheme was proposed to be used commonly; see, for example, Kundu and Prahan [17]. Let the component strength and corresponding stress observations be respectively collected from the life test with n and m items. The test is terminated as long as the predetermined number, r, of failures is obtained or the test time, T, is reached. Based on our best knowledge, the stress–strength inference of a muticomponent system based on type-I hybrid censored samples from generalized exponential distributions has not been studied in the literature. We are motivated to investigate the performance of the maximum likelihood and Bayes estimators for the MSS reliability when lifetime samples are collected using a type-I hybrid censoring scheme. The informative and non-informative priors with three loss functions are utilized to develop the Bayesian methods in this study.

The MSS reliability based on generalized exponential distributions is introduced in Section 2. Section 3 presents the maximum likelihood and two Bayesian estimation methods. Section 4 conducts a Monte Carlo simulation study to compare the performance of all proposed estimators. Next, two examples are presented in Section 5 for illustration. Conclusively, some remarks are addressed in Section 6.

2. Multicomponent Stress–Strength Model

Let $X \sim GE(\alpha, \lambda)$, where $GE(\alpha, \lambda)$ denotes the generalized exponential distribution. The probability density function (pdf) and cumulative distribution function (cdf) of $GE(\alpha, \lambda)$ can be respectively expressed by

$$f_X(x) \equiv f(x; \alpha, \lambda) = \alpha\lambda e^{-x\lambda}\left(1 - e^{-x\lambda}\right)^{\alpha-1},\ x > 0, \tag{1}$$

and

$$F_X(x) \equiv F(x; \alpha, \lambda) = \left(1 - e^{-x\lambda}\right)^{\alpha},\ x > 0, \tag{2}$$

where $\alpha(> 0)$ and $\lambda(> 0)$ are the shape and rate parameters. The $100p$th percentile and survival function of GE(α, λ) can be presented by

$$q_p \equiv q_p(\alpha, \lambda) = -\frac{1}{\lambda}\log\left(1 - p^{\frac{1}{\alpha}}\right),\ 0 < p < 1, \tag{3}$$

and

$$S(x; \alpha, \lambda) = 1 - F(x; \alpha, \lambda) = 1 - \left(1 - e^{-x\lambda}\right)^{\alpha}, \tag{4}$$

respectively. Let k independent strength components in a MSS system follow $GE(\alpha, \lambda_1)$ and denoted by $X_1,\ X_2, \ldots,\ X_k \sim GE(\alpha, \lambda_1)$ and the common random stress, $Y \sim GE(\beta, \lambda_2)$. According to Rao [12], the MSS reliability can be addressed by

$$\begin{aligned} R_{s,k} &\equiv P(\text{at least } s \text{ of the } X_1, X_2, \ldots, X_k \text{ exceed } Y) \\ &= \sum_{i=s}^{k}\binom{k}{i}\int_{-\infty}^{\infty}[1 - F_X(y)]^i[F_X(y)]^{k-i}dF_Y(y) \\ &= \sum_{i=s}^{k}\binom{k}{i}\int_{0}^{\infty}\left[1 - \left(1 - e^{-y\lambda_1}\right)^{\alpha}\right]^i\left[\left(1 - e^{-y\lambda_1}\right)^{\alpha}\right]^{k-i}\times \\ &\quad \beta\lambda_2 e^{-y\lambda_2}\left(1 - e^{-y\lambda_2}\right)^{\beta-1}\mathrm{dy}, \end{aligned} \tag{5}$$

where $0 < s \leq k$.

3. Type-I Hybrid Censoring and MSS Parameter Estimation Methods

Let n denote the number of items used for life testing, $n - r$ denote the censored number in a type-II censoring scheme, and T denote the termination time of a type-I censoring scheme. A type-I hybrid censoring scheme is a hybrid scheme of the type-I censoring and type-II censoring schemes. Let the resulting failure times for strength variable be denoted by $x_{1:n} \leq x_{2:n} \leq \ldots \leq x_{d_1:n}$, which are the realizations of $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{d_1:n}$ with d_1 failures observed at T. Thus, d_1 can also be expressed as

$$d_1 = \begin{cases} r, & x_{r:n} \leq T, \\ m_1, & x_{r:n} > T, x_{m_1:n} \leq T \text{ and } x_{m_1+1:n} > T. \end{cases} \tag{6}$$

Moreover, let c_1 denote the recorded lifetimes for all survived components, then c_1 can also be expressed as

$$c_1 = \begin{cases} x_{r:n}, & x_{r:n} \leq T, \\ T, & x_{r:n} > T. \end{cases} \tag{7}$$

Similarly, let the resulting failure times for stress variables be denoted by $y_{1:m} \leq y_{2:m} \leq \ldots \leq y_{d_2:m}$, which are the realizations of $Y_{1:m} \leq Y_{2:m} \leq \ldots \leq Y_{d_2:m}$ with d_2 failures observed at T. Thus, d_2 can also be expressed as

$$d_2 = \begin{cases} r, & y_{r:m} \leq T, \\ m_2, & y_{r:m} > T, y_{m_2:m} \leq T \text{ and } y_{m_2+1:m} > T. \end{cases} \tag{8}$$

Additionally, let c_2 denote the recorded stress of the stress variables at T, and then c_2 can also be expressed as

$$c_2 = \begin{cases} y_{r:m}, & y_{r:m} \leq T, \\ T, & y_{r:m} > T. \end{cases} \tag{9}$$

Therefore, we can arrange the observations of the type-I hybrid censored sample into $\mathbf{D} = \{X_{1:n}, \ldots, X_{d_1:n}, c_1, Y_1, \ldots, Y_{d_2:m}, c_2\}$.

3.1. Maximum Likelihood Estimation

According to Kundu and Pradhan [17], the likelihood function based on the type-I hybrid samples can be represented by

$$\begin{aligned} L(\Theta|\mathbf{D}) &\propto \left[\prod_{i=1}^{d_1} f(x_{i:n};\Theta_1)(S(c_1|\,\Theta_1))^{n-d_1}\right]\left[\prod_{j=1}^{d_2} f\left(y_{j:m};\Theta_2\right)(S(c_2|\,\Theta_2))^{m-d_2}\right] \\ &= (\alpha\lambda_1)^{d_1}(\beta\lambda_2)^{d_2} e^{-\lambda_1 d_1 \bar{x}_{d_1}} e^{-\lambda_2 d_2 \bar{y}_{d_2}} \\ &\times \left(1-\left(1-e^{-\lambda_1 c_1}\right)^{\alpha}\right)^{n-d_1}\left(1-\left(1-e^{-\lambda_2 c_2}\right)^{\beta}\right)^{m-d_2} \\ &\times \prod_{i=1}^{d_1}\left(1-e^{-x_{i:n}\lambda_1}\right)^{\alpha-1}\prod_{j=1}^{d_2}\left(1-e^{-y_{j:m}\lambda_2}\right)^{\beta-1}, \end{aligned} \tag{10}$$

where $\Theta = (\Theta_1,\ \Theta_2)$, $\Theta_1 = (\alpha, \lambda_1)$, $\Theta_2 = (\beta, \lambda_2)$,

$$\bar{x}_{d_1} = \frac{\sum_{i=1}^{d_1} x_{i:n}}{d_1}$$

and

$$\bar{y}_{d_2} = \frac{\sum_{j=1}^{d_2} y_{j:m}}{d_2}.$$

Taking logarithm transformation to Equation (10) and letting $\ell_1 = \log(L(\Theta|\mathbf{D}))$, we can obtain the following results:

$$\begin{aligned} \ell_1 &\propto d_1(\log(\alpha)+\log(\lambda_1)) + d_2(\log(\beta)+\log(\lambda_2)) - d_1\lambda_1\,\bar{x}_d - d_2\lambda_2\bar{y}_{d_2} \\ &+ (\alpha-1)\sum_{i=1}^{d_1}\log\left(1-e^{-\lambda_1 x_{i:n}}\right) + (\beta-1)\sum_{j=1}^{d_2}\log\left(1-e^{-\lambda_2 y_{j:m}}\right) \\ &+ (n-d_1)\log\left(1-\left(1-e^{-\lambda_1 c_1}\right)^{\alpha}\right) + (m-d_2)\log\left(1-\left(1-e^{-\lambda_2 c_2}\right)^{\beta}\right). \end{aligned} \tag{11}$$

To obtain the maximum likelihood estimate (MLE) of Θ, let the partial derivatives of the ℓ_1 with respect to parameters respectively equate to zero. Then, the likelihood equations are given as

$$\begin{aligned} \frac{\partial \ell_1}{\partial \alpha} &= \frac{d_1}{\alpha} + \sum_{i=1}^{d_1}\log\left(1-e^{-\lambda_1 x_{i:n}}\right) \\ &\quad -(n-d_1)\frac{\log\left(1-e^{-\lambda_1 c_1}\right)\left(1-e^{-\lambda_1 c_1}\right)^{\alpha}}{1-\left(1-e^{-\lambda_1 c_1}\right)^{\alpha}} = 0, \end{aligned} \tag{12}$$

$$
\begin{aligned}
\frac{\partial \ell_1}{\partial \beta} &= \frac{d_2}{\beta} + \sum_{j=1}^{d_2} \log\left(1 - e^{-\lambda_2 y_{j:m}}\right) \\
&\quad -(m - d_2)\frac{\log\left(1 - e^{-\lambda_2 c_2}\right)\left(1 - e^{-\lambda_2 c_2}\right)^{\beta}}{1 - \left(1 - e^{-\lambda_2 c_2}\right)^{\beta}} = 0,
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\frac{\partial \ell_1}{\partial \lambda_1} &= \frac{d_1}{\lambda_1} - d_1 \bar{x}_{d_1} + (\alpha - 1)\sum_{i=1}^{d_1} \frac{x_i e^{-\lambda_1 x_{i:n}}}{1 - e^{-\lambda_1 x_{i:n}}} \\
&\quad -(n - d_1)\frac{\alpha c_1 e^{-\lambda_1 c_1}\left(1 - e^{-\lambda_1 c_1}\right)^{\alpha - 1}}{1 - \left(1 - e^{-\lambda_1 c_1}\right)^{\alpha}} = 0
\end{aligned} \tag{14}
$$

and

$$
\begin{aligned}
\frac{\partial \ell_1}{\partial \lambda_2} &= \frac{d_2}{\lambda_2} - d_2 \bar{y}_{d_2} + (\beta - 1)\sum_{i=1}^{d_2} \frac{y_j e^{-\lambda_2 y_{j:m}}}{1 - e^{-\lambda_2 y_{j:m}}} \\
&\quad -(m - d_2)\frac{\beta c_2 e^{-\lambda_1 c_2}\left(1 - e^{-\lambda_2 c_2}\right)^{\beta - 1}}{1 - \left(1 - e^{-\lambda_2 c_2}\right)^{\beta}} = 0.
\end{aligned} \tag{15}
$$

Therefore, the MLE of Θ, denoted by $\widehat{\Theta} = (\widehat{\alpha}, \widehat{\lambda}_1, \widehat{\beta}, \widehat{\lambda}_2)$, can be obtained using Equations (12)–(15) and a numerical computation method. Replacing Θ with $\widehat{\Theta}$ in Equation (5), the MLE of the MSS reliability can be expressed by

$$
\begin{aligned}
\widehat{R}_{s,k} &= \sum_{i=s}^{k} \binom{k}{i} \int_0^{\infty} \left[1 - \left(1 - e^{-y\widehat{\lambda}_1}\right)^{\widehat{\alpha}}\right]^i \left[\left(1 - e^{-y\widehat{\lambda}_1}\right)^{\widehat{\alpha}}\right]^{k-i} \times \\
&\quad \widehat{\beta}\widehat{\lambda}_2 e^{-y\widehat{\lambda}_2}\left(1 - e^{-y\widehat{\lambda}_2}\right)^{\widehat{\beta} - 1} dy.
\end{aligned} \tag{16}
$$

Because the expectations of the second derivatives of ℓ_1 do not have an explicit expression and have different function forms when the values of parameters vary under the type-I hybrid censoring, the Fisher information matrix is not easily derived. Therefore, the asymptotic properties of the MLEs cannot be obtained via utilizing the Fisher information matrix. Bootstrap methods can be used in this study to obtain an approximate confidence interval of $R_{s,k}$ instead of using the delta method with Fisher information matrix. In bootstrapping, a resampling procedure is used from a sample to generate an empirical distribution of the target statistic by repeatedly taking samples with replacement from the working sample. Based on the following parametric bootstrap procedure, an approximate confidence interval of $R_{s,k}$ can be established as follows:

Step 1: Obtain the MLE, $\widehat{\Theta}$, of Θ by solving Equations (12)–(15) simultaneously based on the type-I hybrid censored sample $\mathbf{D} = \{x_{1:n}, \ldots, x_{d_1:n}, c_1, y_1, \ldots, y_{d_2:m}, c_2\}$. Then, using Equation (16) to obtain the MLE, $\widehat{R}_{s,k}$, of $R_{s,k}$.

Step 2: Generate a new type-I hybrid censored sample from two generalized exponential distributions but Θ is replaced by $\widehat{\Theta} = \left(\widehat{\alpha},\ \widehat{\lambda}_1,\ \widehat{\beta},\ \widehat{\lambda}_2\right)$. Denote the new generated type-I hybrid censored sample by $\mathbf{D}^* = \left\{x^*_{1:n}, \ldots, x^*_{d_1:n}, c^*_1, y^*_{1:m}, \ \ldots, y^*_{d_2:m}, c^*_2\right\}$

Step 3: Calculate the MLEs of Θ and $R_{s,k}$ based on $\mathbf{D}^*$. Denoted them by $\widehat{\Theta}^*$ and $\widehat{R}^*_{s,k}$, respectively.

Step 4: Repeat Step 2 and Step 3 B times, and a bootstrap sample of size B,

$$
\widehat{\mathbf{R}}^*_{s,k} = \{\widehat{R}^*_{s,k,1}, \widehat{R}^*_{s,k,2}, \cdots, \widehat{R}^*_{s,k,B}\},
$$

is collected. Let G^* be the empirical distribution generated by $\widehat{\mathbf{R}}^*_{s,k}$.

Step 5: The $100 \times (1-\gamma)\%$ confidence interval of $R_{s,k}$ can be presented as the following:

$$\left(\widehat{R}^*_{s,k,[\frac{\gamma}{2}B]}, \widehat{R}^*_{s,k,[(1-\frac{\gamma}{2})B]} \right), \tag{17}$$

where $0 < \gamma < 1$, $\widehat{R}^*_{s,k,[\frac{\gamma}{2}B]}$ and $\widehat{R}^*_{s,k,[(1-\frac{\gamma}{2})B]}$ are the $(\gamma/2)$th and $(1-\gamma/2)$th quantiles of $\widehat{G}^*$, respectively.

3.2. Bayesian Methods

In this section, we present how to obtain the Bayes estimators of Θ and $R_{s,k}$. In the Bayesian estimation procedures, the prior distribution and loss function need to be selected first. The non-informative prior distribution and informative prior distribution are considered here to obtain Bayes estimators of the model parameters.

3.2.1. Bayesian Method with Non-Informative Prior Distribution

The constant prior information is considered to be the non-informative prior distribution, that is,

$$g(\Theta) \propto \text{ const.} \tag{18}$$

Therefore, the form of the posterior distribution can be obtained as follows:

$$\pi(\Theta|\mathbf{D}) = g(\Theta)L(\Theta|\mathbf{D}) \propto \text{const} \times L(\Theta|\mathbf{D}). \tag{19}$$

Based on Equation (19), we can find that the obtained Bayes estimators are close to the corresponding MLEs due to the posterior distribution being proportional to the likelihood function. The conditional posterior distributions of α, β, λ_1 and λ_2 are respectively presented as follows:

$$\begin{aligned}\pi_{\lambda_1}(\lambda_1|\Theta_{-1}, \mathbf{D}) &= g(\lambda_1|\Theta_{-1}, \mathbf{D})L(\lambda_1|\Theta_{-1}, \mathbf{D}) \\ &\propto \lambda_1^{d_1} e^{-\lambda_1 d_1 \bar{x}_{d_1}} \left(1 - \left(1 - e^{-\lambda_1 c_1}\right)^{\alpha}\right)^{n-d_1} \prod_{i=1}^{d_1} \left(1 - e^{-x_{i:n}\lambda_1}\right)^{\alpha-1},\end{aligned} \tag{20}$$

$$\begin{aligned}\pi_{\alpha}(\alpha|\Theta_{-2}, \mathbf{D}) &= g(\alpha|\Theta_{-2}, \mathbf{D})L(\alpha|\Theta_{-2}, \mathbf{D}) \\ &\propto \alpha^{d_1} \left(1 - \left(1 - e^{-\lambda_1 c_1}\right)^{\alpha}\right)^{n-d_1} \prod_{i=1}^{d_1} \left(1 - e^{-x_{i:n}\lambda_1}\right)^{\alpha-1},\end{aligned} \tag{21}$$

$$\begin{aligned}\pi_{\lambda_2}(\lambda_2|\Theta_{-3}, \mathbf{D}) &= g(\lambda_2|\Theta_{-3}, \mathbf{D})L(\lambda_2|\Theta_{-3}, \mathbf{D}) \\ &\propto \lambda_2^{d_2} e^{-\lambda_2 d_2 \bar{y}_{d_2}} \left(1 - \left(1 - e^{-\lambda_2 c_2}\right)^{\beta}\right)^{m-d_2} \prod_{j=1}^{d_2} \left(1 - e^{-y_{j:m}\lambda_2}\right)^{\beta-1}\end{aligned} \tag{22}$$

and

$$\begin{aligned}\pi_{\beta}(\beta|\Theta_{-4}, \mathbf{D}) &= g(\beta|\Theta_{-4}, \mathbf{D})L(\beta|\Theta_{-4}, \mathbf{D}) \\ &\propto \beta^{d_2} \times \left(1 - \left(1 - e^{-\lambda_2 c_2}\right)^{\beta}\right)^{m-d_2} \prod_{j=1}^{d_2} \left(1 - e^{-y_{j:m}\lambda_2}\right)^{\beta-1},\end{aligned} \tag{23}$$

where $\Theta_{-1} = (\alpha, \beta, \lambda_2)$, $\Theta_{-2} = (\beta, \lambda_1, \lambda_2)$, $\Theta_{-3} = (\alpha, \beta, \lambda_1)$ and $\Theta_{-4} = (\alpha, \lambda_1, \lambda_2)$.

Because the conditional posterior distributions in Equations (20)–(23) do not have complete closed forms, the Bayes estimators cannot be obtained by utilizing the Gibbs sampling. Hence, the MCMC approach via using the Metropolis–Hastings algorithm described below is implemented.

Step 0: Propose $q_i, i = 1, 2, 3, 4$ as the transition probability distributions.

Step 1: Let the initial values, $(\lambda_1^{(0)}, \alpha^{(0)}, \lambda_2^{(0)}, \ \beta^{(0)})$, be randomly selected from their respective domains, and let $U(0, 1)$ be the uniform distribution over the $(0, 1)$ interval.

Step 2: Set $t = 1$.

Step 3: Update $\lambda_1^{(t)}$, $\alpha^{(t)}$, $\lambda_2^{(t)}$ and $\beta^{(t)}$ according to the following sub-steps:

(a) Generate $\lambda_1^{(*)} \sim \ q_1\left(\lambda_1^{(t)} | \lambda_1^{(t-1)}\right)$ and $u \sim U(0, 1)$. Then,

$$\lambda_1^{(t)} = \begin{cases} \lambda_1^{(*)}, & u \le \min\left[1 \ \frac{\pi\left(\lambda_1^{(*)} | \alpha^{(t-1)}, \lambda_2^{(t-1)}, \ \beta^{(t-1)}; \mathbf{D}\right) q\left(\lambda_1^{(t-1)} \middle| \lambda_1^{(*)}\right)}{\pi\left(\lambda_1^{(t-1)} | \alpha^{(t-1)}, \lambda_2^{(t-1)}, \ \beta^{(t-1)}; \mathbf{D}\right) q\left(\lambda_1^{(*)} \middle| \lambda_1^{(t-1)}\right)}\right], \\ \lambda_1^{(t-1)}, & \text{otherwise.} \end{cases}$$

(b) Generate $\alpha^{(*)} \sim \ q_2(\alpha^{(t)} | \alpha^{(t-1)})$ and $u \sim U(0, 1)$. Then,

$$\alpha^{(t)} = \begin{cases} \alpha^{(*)}, & u \le \min\left[1 \ \frac{\pi\left(\alpha^{(*)} | \lambda_1^{(t)}, \lambda_2^{(t-1)}, \ \beta^{(t-1)}; \mathbf{D}\right) q\left(\alpha^{(t-1)} \middle| \alpha^{(*)}\right)}{\pi\left(\alpha^{(t-1)} | \lambda_1^{(t)}, \lambda_2^{(t-1)}, \ \beta^{(t-1)}; \mathbf{D}\right) q\left(\alpha^{(*)} \middle| \alpha^{(t-1)}\right)}\right], \\ \alpha^{(t-1)}, & \text{otherwise.} \end{cases}$$

(c) Generate $\lambda_2^{(*)} \sim \ q_3\left(\lambda_2^{(t)} | \lambda_2^{(t-1)}\right)$ and $u \sim U(0, 1)$. Then,

$$\lambda_2^{(t)} = \begin{cases} \lambda_2^{(*)}, & u \le \min\left[1 \ \frac{\pi\left(\lambda_2^{(*)} | \alpha^{(t)}, \lambda_1^{(t)}, \ \beta^{(t-1)}; \mathbf{D}\right) q\left(\lambda_2^{(t-1)} \middle| \lambda_2^{(*)}\right)}{\pi\left(\lambda_2^{(t-1)} | \alpha^{(t)}, \lambda_1^{(t)}, \ \beta^{(t-1)}; \mathbf{D}\right) q\left(\lambda_2^{(*)} \middle| \lambda_2^{(t-1)}\right)}\right], \\ \lambda_2^{(t-1)}, & \text{otherwise.} \end{cases}$$

(d) Generate $\beta^{(*)} \ \sim \ q_4(\beta^{(t)} | \beta^{(t-1)})$ and $u \sim U(0, 1)$. Then,

$$\beta^{(t)} = \begin{cases} \beta^{(*)}, & u \le \min\left[1 \ \frac{\pi\left(\beta^{(*)} | \alpha^{(t)}, \lambda_1^{(t)}, \lambda_2^{(t)} \ ; \mathbf{D}\right) q\left(\beta^{(t-1)} \middle| \beta^{(*)}\right)}{\pi\left(\beta^{(t-1)} | \alpha^{(t)}, \lambda_1^{(t)}, \lambda_2^{(t)}; \mathbf{D}\right) q\left(\beta^{(*)} \middle| \beta^{(t-1)}\right)}\right], \\ \beta^{(t-1)}, & \text{otherwise.} \end{cases}$$

Step 4: Calculate $R_{s,k}^{(t)}$ according to $(\alpha^{(t)}, \lambda_1^{(t)}, \ \beta^{(t)}, \lambda_2^{(t)})$.

Step 5: If $t = N$, go to Step 6, else $t = t + 1$ and go to Step 3.

Step 6: Remove the first N_1 Markov chains for burn-in.

Step 7: Replacing Θ with $(\alpha^{(i)}, \lambda_1^{(i)}, \ \beta^{(i)}, \lambda_2^{(i)})$ in Equation (5), we can obtain $R_{s,k}^{(i)}$, $i = N_1 + 1, N_1 + 2, \ldots, N$.

Three Bayes estimators via using the general entropy, linear exponential, and squared error loss functions, labeled by GEF, LINEX, SEF, are applied, and the corresponding results are shown in Table 1.

Table 1. Three loss functions and corresponding Bayes estimator.

SE:	$\widehat{R}_{s,k_{\mathrm{Se}}} = E_{\Theta \mid x,y}\left(R_{s,k}\right) \cong \frac{1}{N-N_1} \sum_{i=N_1+1}^{N} R_{s,k}^{(i)}$
LINEX:	$\widehat{R}_{s,k_L} = -\frac{1}{a} \log \ E_{\Theta \mid x,y}\left[\exp\left(aR_{s,k}\right)\right] \cong -\frac{1}{a} \log\left[\frac{1}{N-N_1} \sum_{i=N_1+1}^{N} e^{-a \ R_{s,k}^{(i)}}\right]$
GEF:	$\widehat{R}_{s,k_{\mathrm{Ge}}} = \left[E_{\Theta \mid x,y}\left(R_{s,k}^{-k}\right)\right]^{-\frac{1}{b}} \cong \left[\frac{1}{N-N_1} \sum_{i=N_1+1}^{N} \left(R_{s,k}^{(i)}\right)^{-b}\right]^{-\frac{1}{b}}$

To obtain the credible interval of $R_{s,k}$ for the Bayesian method, the highest probability density interval (HPDI), proposed by Chen and Shao [18], is recommended, and three steps are provided as follows.

Step 1: Sort $R_{s,k}^{(i)}$, $i = N_1+1,\ldots,N$ in ascending order, $R_{s,k,(i)}$, $i = N_1+1,\ldots,N$, where $R_{s,k,(i)} \leq R_{s,k,(i+1)}, i = N_1+1,\ldots,N$.

Step 2: All candidate credible intervals of $R_{s,k}$ with the $1-\gamma$ confidence can be obtained by

$$\left(R_{s,k,(i)},\ \ R_{s,k,(i+[(1-\gamma)B_1])}\right),\ i = N_1+1,\ldots,N-[(1-\gamma)(N-N_1)],$$

where $[y]$ is the maximum integer less than or equal to y.

Step 3: The candidate credible interval with the shortest length in Step 2 is the HPDI.

3.2.2. Bayesian Method with Informative Prior Distribution

According to the suggestions of Kundu and Pradha [17], we assume that all random parameters, α, β, λ_1 and λ_2, are independent and follow the gamma prior distributions below:

$$g_1(\alpha) = \frac{b_1^{a_1}\alpha^{a_1-1}}{\Gamma(a_1)}\exp(-b_1\alpha),\ \alpha > 0, \tag{24}$$

$$g_2(\beta) = \frac{b_2^{a_2}\beta^{a_2-1}}{\Gamma(a_2)}\exp(-b_2\beta),\ \beta > 0, \tag{25}$$

$$g_3(\lambda_1) = \frac{b_3^{a_3}\lambda_1^{a_3-1}}{\Gamma(a_3)}\exp(-b_3\lambda_1),\ \lambda_1 > 0, \tag{26}$$

and

$$g_4(\lambda_2) = \frac{b_4^{a_4}\lambda_2^{a_4-1}}{\Gamma(a_4)}\exp(-b_4\lambda_2),\ \lambda_2 > 0, \tag{27}$$

where $a_i > 0, b_i > 0, i = 1,2,3,4$ are known hyper-parameters. Hence, the joint prior distribution of α, β, λ_1 and λ_2 can be addressed by

$$\begin{aligned} g(\alpha,\beta,\lambda_1,\lambda_2) \quad = \quad & \frac{b_1^{a_1}b_2^{a_2}b_3^{a_3}b_4^{a_4}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)} \\ & \times\alpha^{a_1-1}\beta^{a_2-1}{\lambda_1}^{a_3-1}{\lambda_2}^{a_4-1}e^{-(b_1\alpha+b_2\beta+b_3\lambda_1+b_4\lambda_2)},\ \alpha,\beta,\lambda_1,\lambda_2 > 0 \end{aligned} \tag{28}$$

and the joint posterior is given as

$$\begin{aligned} \pi(\alpha,\beta,\lambda_1,\lambda_2|x,y) \quad = \quad & \frac{g(\alpha,\beta,\lambda_1,\lambda_2)L(\alpha,\beta,\lambda_1,\lambda_2)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty g(\alpha,\beta,\lambda_1,\lambda_2)L(\alpha,\beta,\lambda_1,\lambda_2)d\alpha d\beta d\lambda_1 d\lambda_2} \\ \propto \quad & \alpha^{d_1+a_1-1}\beta^{d_2+a_2-1}\lambda_1^{d_1+a_3-1}\lambda_2^{d_2+a_4-1}e^{-\lambda_1\left(b_3+d_1\lambda_1\bar{x}_{d_1}\right)}e^{-\lambda_2\left(b_4+d_2\bar{y}_{d_2}\right)} \\ & \times e^{-\alpha\left(b_1-\sum_{i=1}^{d_1}\ln\left(1-e^{-\lambda_1 x_{i:n}}\right)\right)}e^{-\beta\left(b_2-\sum_{j=1}^{d_2}\ln\left(1-e^{-\lambda_2 y_{j:m}}\right)\right)} \\ & \times\prod_{i=1}^{d_1}\left(1-e^{-\lambda_1 x_{i:n}}\right)^{\alpha-1}\prod_{j=1}^{d_2}\left(1-e^{-\lambda_2 y_{j:m}}\right)^{\beta-1} \\ & \times\left(1-\left(1-e^{-\lambda_1 c_1}\right)^{\alpha}\right)^{n-d_1}\left(1-\left(1-e^{-\lambda_2 c_2}\right)^{\beta}\right)^{m-d_2}, \\ & \alpha,\beta,\lambda_1,\lambda_2 > 0. \end{aligned} \tag{29}$$

The conditional posterior of α, β, λ_1, and λ_2 is respectively derived as follows:

$$\pi_\alpha(\alpha|\lambda_1,\ x) \ \sim\ \text{Gamma}\left(d_1 + a_1\ b_1 - \sum_{i=1}^{d_1} \ln\left(1 - e^{-\lambda_1 x_{i:n}}\right)\right),\ \alpha, \lambda_1 > 0, \tag{30}$$

$$\pi_\beta(\beta|\ \lambda_2,\ y) \ \sim\ \text{Gamma}\left(d_2 + a_2\, b_2 - \sum_{j=1}^{d_2} \ln\left(1 - e^{-\lambda_2 y_{j:m}}\right)\right),\ \beta, \lambda_2 > 0 \tag{31}$$

$$\pi_{\lambda_1}(\lambda_1|\ x) \ \sim\ \text{Gamma}\left(d_1 + a_3\, b_3 + d_1 \overline{x}_{d_1}\ \right),\ \lambda_1 > 0 \tag{32}$$

and

$$\pi_{\lambda_2}(\lambda_2|\ y) \ \sim\ \text{Gamma}\left(d_2 + a_4\, b_4 + d_2 \overline{y}_{d_2}\ \right),\ \lambda_2 > 0. \tag{33}$$

Therefore, the Bayes estimator and HPDI of $R_{s,k}$ can be obtained through using the MCMC approach in Section 3.2.1, except replacing $\pi_{\lambda_1}(\lambda_1|\Theta_{-1}, \mathbf{D})$, $\pi_\alpha(\alpha|\Theta_{-2}, \mathbf{D})$, $\pi_{\lambda_2}(\lambda_2|\Theta_{-3}, \mathbf{D})$ and $\pi_\beta(\beta|\Theta_{-4}, \mathbf{D})$ by $\pi_{\lambda_1}(\lambda_1|\ x)$, $\pi_\alpha(\alpha|\lambda_1,\ x)$, $\pi_{\lambda_2}(\lambda_2|\ y)$ and $\pi_\beta(\beta|\ \lambda_2,\ y)$.

4. Monte Carlo Simulations

The performance of maximum likelihood and Bayesian estimation methods for estimating $R_{s,k}$ is compared in terms of relative Bias (rBias) and relative mean squared error (rMSE) via Monte Carlo simulation in this section. The rBias and rMSE for each estimator are evaluated using 10,000 iterative runs. In the study, $(\alpha, \lambda_1, \beta, \lambda_2) = (2, 2, 2, 3), (1.5, 1, 2, 1)$, $(s, k) = (1, 4), (2, 7)$, (n, m) = (30, 30), (35, 35), (40, 40), (45, 45), (50, 50), $T = 1, 5$, $N = 10,000$, $N_1 = 1,000$ and $\gamma = 0.05$ will be used along with the number of failures r as half amount of the sample size. We use the non-information prior (Prior-I) and the information prior (Prior-II) for the MCMC method to obtain the Bayes estimators. Then, we compare the performance of Bayes estimators under different loss functions. The hyper-parameter setting values of the Gamma prior distributions are given as those in Case 1 and Case 2.

Case 1: $a_1 = 2, b_1 = 1, a_2 = 2, b_2 = 1, a_3 = 2, b_3 = 1, a_4 = 3$ and $b_4 = 1$,

Case 2: $a_1 = 1.5, b_1 = 1, a_2 = 2, b_2 = 1, a_3 = 1, b_3 = 1, a_4 = 1$ and $b_4 = 1$.

In addition, the parameters of LINEX and GE loss functions are respectively set as $a = b = -0.5, 1$. The simulation results for point estimators are presented in Tables 2–5. In view of Tables 2–5, the following results are given:

- As the sample size increases, rBias and rMSE decrease.
- Both MLE and Bayesian methods underestimate the nominal $R_{s,k}$.
- The performance of Prior-II outperforms MLE and Prior-I for almost all simulation settings.
- When $a = b = -0.5$, the Bayes estimator obtained by using the LINEX performs the best and is followed by using the SEF.

The results of 95% confidence interval of $R_{s,k}$ are presented in Tables 6–9, where "LCB" and "UCB" refer to upper and lower bounds of confidence interval, respectively, "AL" refers to the average length of confidence interval, and "Cover" refers to the proportion of 10,000 95% confidence intervals that cover the parameter, $R_{s,k}$. In viewing Tables 6–9, the following results are listed:

- As the sample size increases, the AL decreases.
- For AL, Prior I has the shortest interval length, followed by that of Prior II and MLE.
- For Cover, the performance of Prior II is the best, followed by that of Prior I and MLE.

Based on the above findings, the MLE does not work well for estimating $R_{s,k}$. In addition, although the confidence interval of Prior I is shorter than that of Prior II, the coverage of Prior II for the parameters is closer to the nominal value. Therefore, the performance of Prior II is also competitive with Prior I.

Table 2. The rBias and rMSE (in parentheses) for Case 1, $(\alpha, \lambda_1, \beta, \lambda_2) = (2,\ 2,\ 2,\ 3)$ and $T = 1$.

(s, k)	n, m		MCMC (Prior-I)						MCMC (Prior-II)				
		$\widehat{R}_{s,k_{\mathrm{MLE}}}$	$\widehat{R}_{s,k_{\mathrm{Se}}}$	$\widehat{R}_{s,k_{\mathrm{L}}}$ $a=b=-0.5$	$\widehat{R}_{s,k_{\mathrm{Ge}}}$ $a=b=-0.5$	$\widehat{R}_{s,k_{\mathrm{L}}}$ $a=b=1$	$\widehat{R}_{s,k_{\mathrm{Ge}}}$ $a=b=1$	$\widehat{R}_{s,k_{\mathrm{Se}}}$	$\widehat{R}_{s,k_{\mathrm{L}}}$ $a=b=-0.5$	$\widehat{R}_{s,k_{\mathrm{Ge}}}$ $a=b=-0.5$	$\widehat{R}_{s,k_{L}}$ $a=b=1$	$\widehat{R}_{s,k_{\mathrm{Ge}}}$ $a=b=1$
(1, 4)	30,30	−0.1631	−0.0342	−0.0270	−0.0493	−0.0485	−0.1000	−0.0313	−0.0233	−0.0485	−0.0472	−0.1073
		(0.1908)	(0.0504)	(0.0501)	(0.0530)	(0.0514)	(0.0650)	(0.0493)	(0.0488)	(0.0524)	(0.0505)	(0.0682)
	35,35	−0.1577	−0.0338	−0.0274	−0.0464	−0.0906	−0.0470	−0.0271	−0.0202	−0.0410	−0.0912	−0.0418
		(0.1620)	(0.0456)	(0.0453)	(0.0477)	(0.0464)	(0.0573)	(0.0443)	(0.0440)	(0.0468)	(0.0453)	(0.0586)
	40,40	−0.1581	−0.0330	−0.0272	−0.0451	−0.0445	−0.0845	−0.0248	−0.0185	−0.0381	−0.0374	−0.0823
		(0.1498)	(0.0413)	(0.0409)	(0.0431)	(0.0421)	(0.0514)	(0.0401)	(0.0397)	(0.0422)	(0.0410)	(0.0520)
	45,45	−0.1599	−0.0370	−0.0316	−0.0481	−0.0477	−0.0845	−0.0276	−0.0218	−0.0398	−0.0392	−0.0803
		(0.1423)	(0.0397)	(0.0393)	(0.0415)	(0.0406)	(0.0493)	(0.0384)	(0.0380)	(0.0404)	(0.0393)	(0.0494)
	50,50	−0.1615	−0.0410	−0.0361	−0.0513	−0.0508	−0.0844	−0.0295	−0.0242	−0.0405	−0.0400	−0.0766
		(0.1282)	(0.0369)	(0.0365)	(0.0386)	(0.0378)	(0.0455)	(0.0354)	(0.0350)	(0.0371)	(0.0362)	(0.0447)
(2, 7)	30,30	−0.1751	−0.0529	−0.0450	−0.0708	−0.0684	−0.1333	−0.0489	−0.0401	−0.0695	−0.0664	−0.1439
		(0.2099)	(0.0560)	(0.0554)	(0.0599)	(0.0575)	(0.0790)	(0.0554)	(0.0547)	(0.0603)	(0.0573)	(0.0861)
	35,35	−0.1712	−0.0519	−0.0450	−0.0676	−0.0657	−0.1208	−0.0444	−0.0367	−0.0620	−0.0597	−0.1240
		(0.1803)	(0.0496)	(0.0491)	(0.0528)	(0.0510)	(0.0676)	(0.0488)	(0.0482)	(0.0526)	(0.0504)	(0.0717)
	40,40	−0.1707	−0.0512	−0.0449	−0.0655	−0.0638	−0.1137	−0.0416	−0.0347	−0.0575	−0.0555	−0.1128
		(0.1705)	(0.0468)	(0.0463)	(0.0497)	(0.0481)	(0.0627)	(0.0459)	(0.0453)	(0.0492)	(0.0473)	(0.0654)
	45,45	−0.1712	−0.0530	−0.0472	−0.0662	−0.0646	−0.1105	−0.0421	−0.0357	−0.0567	−0.0549	−0.1069
		(0.1556)	(0.0435)	(0.0430)	(0.0462)	(0.0448)	(0.0581)	(0.0424)	(0.0418)	(0.0454)	(0.0438)	(0.0597)
	50,50	−0.1709	−0.0525	−0.0472	−0.0645	−0.0632	−0.1043	−0.0397	−0.0339	−0.0527	−0.0512	−0.0969
		(0.1419)	(0.0409)	(0.0404)	(0.0433)	(0.0421)	(0.0536)	(0.0392)	(0.0387)	(0.0417)	(0.0404)	(0.0534)

Table 3. The rBias and rMSE (in parentheses) for Case 1, $(\alpha, \lambda_1, \beta, \lambda_2) = (1.5,\ 1,\ 2,\ 1)$ and $T = 1$.

(s, k)	n, m		MCMC (Prior-I)						MCMC (Prior-II)				
		$\widehat{R}_{s,k_{\mathrm{MLE}}}$	$\widehat{R}_{s,k_{\mathrm{Se}}}$	$\widehat{R}_{s,k_{\mathrm{L}}}$ $a=b=-0.5$	$\widehat{R}_{s,k_{\mathrm{Ge}}}$ $a=b=-0.5$	$\widehat{R}_{s,k_{\mathrm{L}}}$ $a=b=1$	$\widehat{R}_{s,k_{\mathrm{Ge}}}$ $a=b=1$	$\widehat{R}_{s,k_{\mathrm{Se}}}$	$\widehat{R}_{s,k_{\mathrm{L}}}$ $a=b=-0.5$	$\widehat{R}_{s,k_{\mathrm{Ge}}}$ $a=b=-0.5$	$\widehat{R}_{s,k_{L}}$ $a=b=1$	$\widehat{R}_{s,k_{\mathrm{Ge}}}$ $a=b=1$
(1, 4)	30,30	0.0045	−0.0987	−0.0934	−0.1065	−0.1096	−0.1345	−0.1053	−0.0995	−0.1140	−0.1174	−0.1455
		(0.0613)	(0.0255)	(0.0244)	(0.0276)	(0.0280)	(0.0368)	(0.0249)	(0.0236)	(0.0274)	(0.0278)	(0.0382)
	35,35	−0.0024	−0.0838	−0.0793	−0.0902	−0.0931	−0.1124	−0.0904	−0.0856	−0.0973	−0.1004	−0.1216
		(0.0552)	(0.0209)	(0.0201)	(0.0224)	(0.0227)	(0.0285)	(0.0202)	(0.0192)	(0.0218)	(0.0222)	(0.0289)
	40,40	−0.0003	−0.0768	−0.0728	−0.0824	−0.0850	−0.1013	−0.0828	−0.0786	−0.0887	−0.0915	−0.1089
		(0.0489)	(0.0186)	(0.0180)	(0.0198)	(0.0201)	(0.0246)	(0.0177)	(0.0169)	(0.0190)	(0.0193)	(0.0243)
	45,45	−0.0036	−0.0706	−0.0671	−0.0755	−0.0779	−0.0917	−0.0762	−0.0725	−0.0812	−0.0837	−0.0982
		(0.0433)	(0.0169)	(0.0163)	(0.0178)	(0.0181)	(0.0215)	(0.0158)	(0.0152)	(0.0168)	(0.0171)	(0.0209)
	50,50	−0.0085	−0.0616	−0.0585	−0.0659	−0.0681	−0.0799	−0.0674	−0.0642	−0.0718	−0.0740	−0.0862
		(0.0409)	(0.0142)	(0.0138)	(0.0149)	(0.0152)	(0.0177)	(0.0133)	(0.0128)	(0.0141)	(0.0143)	(0.0171)
(2, 7)	30,30	0.0096	−0.1121	−0.1055	−0.1223	−0.1256	−0.1614	−0.1162	−0.1089	−0.1277	−0.1311	−0.1720
		(0.0788)	(0.0317)	(0.0303)	(0.0347)	(0.0348)	(0.0493)	(0.0301)	(0.0285)	(0.0336)	(0.0338)	(0.0509)
	35,35	0.0071	−0.0946	−0.0889	−0.1031	−0.1061	−0.1340	−0.0999	−0.0938	−0.1092	−0.1124	−0.1433
		(0.0684)	(0.0260)	(0.0250)	(0.0281)	(0.0284)	(0.0380)	(0.0247)	(0.0235)	(0.0271)	(0.0274)	(0.0386)
	40,40	0.0041	−0.0851	−0.0801	−0.0924	−0.0953	−0.1182	−0.0900	−0.0848	−0.0979	−0.1008	−0.1258
		(0.0627)	(0.0235)	(0.0226)	(0.0251)	(0.0253)	(0.0322)	(0.0217)	(0.0208)	(0.0235)	(0.0238)	(0.0318)
	45,45	−0.0010	−0.0784	−0.0740	−0.0849	−0.0876	−0.1071	−0.0829	−0.0782	−0.0896	−0.0924	−0.1130
		(0.0547)	(0.0212)	(0.0205)	(0.0225)	(0.0227)	(0.0281)	(0.0194)	(0.0187)	(0.0209)	(0.0211)	(0.0270)
	50,50	−0.0078	−0.0716	−0.0676	−0.0773	−0.0798	−0.0962	−0.0769	−0.0728	−0.0827	−0.0853	−0.1024
		(0.0520)	(0.0192)	(0.0187)	(0.0203)	(0.0204)	(0.0245)	(0.0177)	(0.0171)	(0.0188)	(0.0190)	(0.0234)

Table 4. The rBias and rMSE (in parentheses) for Case 2, $(\alpha, \lambda_1, \beta, \lambda_2) = (2,\ 2,\ 2,\ 3)$ and $T = 5$.

(s, k)	n, m		MCMC (Prior-I)						MCMC (Prior-II)				
		$\widehat{R}_{s,k_{\mathrm{MLE}}}$	$\widehat{R}_{s,k_{\mathrm{Se}}}$	$\widehat{R}_{s,k_{\mathrm{L}}}$ $a=b=-0.5$	$\widehat{R}_{s,k_{\mathrm{Ge}}}$ $a=b=-0.5$	$\widehat{R}_{s,k_{\mathrm{L}}}$ $a=b=1$	$\widehat{R}_{s,k_{\mathrm{Ge}}}$ $a=b=1$	$\widehat{R}_{s,k_{\mathrm{Se}}}$	$\widehat{R}_{s,k_{\mathrm{L}}}$ $a=b=-0.5$	$\widehat{R}_{s,k_{\mathrm{Ge}}}$ $a=b=-0.5$	$\widehat{R}_{s,k_{L}}$ $a=b=1$	$\widehat{R}_{s,k_{\mathrm{Ge}}}$ $a=b=1$
(1, 4)	30,30	−0.1598	−0.0325	−0.0253	−0.0476	−0.0468	−0.0982	−0.0304	−0.0225	−0.0476	−0.0464	−0.1064
		(0.1928)	(0.0509)	(0.0505)	(0.0534)	(0.0517)	(0.0651)	(0.0499)	(0.0494)	(0.0530)	(0.0511)	(0.0688)
	35,35	−0.1567	0.1592	0.1627	0.1534	0.1523	0.1357	−0.0341	−0.0277	−0.0473	−0.0467	−0.0910
		(0.1634)	(0.0458)	(0.0455)	(0.0479)	(0.0467)	(0.0575)	(0.0445)	(0.0441)	(0.0469)	(0.0455)	(0.0587)
	40,40	−0.1640	−0.0377	−0.0319	−0.0498	−0.0493	−0.0895	−0.0293	−0.0230	−0.0426	−0.0419	−0.0870
		(0.1500)	(0.0416)	(0.0412)	(0.0436)	(0.0426)	(0.0524)	(0.0403)	(0.0399)	(0.0425)	(0.0413)	(0.0528)
	45,45	−0.1652	−0.0393	−0.0340	−0.0505	−0.0500	−0.0869	−0.0295	−0.0237	−0.0417	−0.0411	−0.0821
		(0.1394)	(0.0385)	(0.0381)	(0.0403)	(0.0394)	(0.0482)	(0.0372)	(0.0368)	(0.0392)	(0.0382)	(0.0482)
	50,50	−0.1515	−0.0344	−0.0295	−0.0446	−0.0442	−0.0775	−0.0229	−0.0176	−0.0338	−0.0334	−0.0696
		(0.1254)	(0.0357)	(0.0353)	(0.0372)	(0.0364)	(0.0437)	(0.0342)	(0.0340)	(0.0358)	(0.0350)	(0.0428)

Table 4. *Cont.*

(s, k)	n, m		MCMC (Prior-I)					MCMC (Prior-II)				
		$\widehat{R}_{s,k_{\text{MLE}}}$	$\widehat{R}_{s,k_{\text{Se}}}$	$\widehat{R}_{s,k_{\text{L}}}$	$\widehat{R}_{s,k_{\text{Ge}}}$	$\widehat{R}_{s,k_{\text{L}}}$	$\widehat{R}_{s,k_{\text{Ge}}}$	$\widehat{R}_{s,k_{\text{Se}}}$	$\widehat{R}_{s,k_{\text{L}}}$	$\widehat{R}_{s,k_{\text{Ge}}}$	$\widehat{R}_{s,k_{L}}$	$\widehat{R}_{s,k_{\text{Ge}}}$
				$a=b=-0.5$		$a=b=1$			$a=b=-0.5$		$a=b=1$	
(2, 7)	30,30	−0.1803	−0.0541	−0.0463	−0.0719	−0.0695	−0.1338	−0.0509	−0.0422	−0.0715	−0.0684	−0.1457
		(0.2131)	(0.0556)	(0.0551)	(0.0594)	(0.0571)	(0.0781)	(0.0555)	(0.0547)	(0.0603)	(0.0574)	(0.0862)
	35,35	−0.1713	−0.0512	−0.0443	−0.0667	−0.0649	−0.1196	−0.0431	−0.0355	−0.0606	−0.0583	−0.1219
		(0.1849)	(0.0512)	(0.0507)	(0.0543)	(0.0525)	(0.0691)	(0.0503)	(0.0497)	(0.0540)	(0.0518)	(0.0729)
	40,40	−0.1739	−0.0538	−0.0475	−0.0681	−0.0664	−0.1163	−0.0441	−0.0372	−0.0600	−0.0580	−0.1153
		(0.1686)	(0.0465)	(0.0459)	(0.0494)	(0.0478)	(0.0626)	(0.0453)	(0.0447)	(0.0487)	(0.0468)	(0.0650)
	45,45	−0.1721	−0.0529	−0.0471	−0.0661	−0.0646	−0.1104	−0.0423	−0.0360	−0.0569	−0.0551	−0.1071
		(0.1623)	(0.0449)	(0.0444)	(0.0476)	(0.0462)	(0.0595)	(0.0439)	(0.0434)	(0.0470)	(0.0453)	(0.0613)
	50,50	−0.1728	−0.0518	−0.0465	−0.0637	−0.0625	−0.1032	−0.0390	−0.0333	−0.0520	−0.0505	−0.0958
		(0.1387)	(0.0394)	(0.0388)	(0.0417)	(0.0405)	(0.0516)	(0.0380)	(0.0375)	(0.0405)	(0.0392)	(0.0519)

Table 5. The rBias and rMSE (in parentheses) for Case 2, $(\alpha, \lambda_1, \beta, \lambda_2) = (1.5,\ 1,\ 2,\ 1)$ and $T = 5$.

(s, k)	n, m		MCMC (Prior-I)					MCMC (Prior-II)				
		$\widehat{R}_{s,k_{\text{MLE}}}$	$\widehat{R}_{s,k_{\text{Se}}}$	$\widehat{R}_{s,k_{\text{L}}}$	$\widehat{R}_{s,k_{\text{Ge}}}$	$\widehat{R}_{s,k_{\text{L}}}$	$\widehat{R}_{s,k_{\text{Ge}}}$	$\widehat{R}_{s,k_{\text{Se}}}$	$\widehat{R}_{s,k_{\text{L}}}$	$\widehat{R}_{s,k_{\text{Ge}}}$	$\widehat{R}_{s,k_{L}}$	$\widehat{R}_{s,k_{\text{Ge}}}$
				$a=b=-0.5$		$a=b=1$			$a=b=-0.5$		$a=b=1$	
(1, 4)	30,30	−0.0025	−0.0932	−0.0885	−0.0998	−0.1029	−0.1220	−0.0890	−0.0841	−0.0958	−0.0990	−0.1190
		(0.0450)	(0.0235)	(0.0227)	(0.0251)	(0.0255)	(0.0310)	(0.0198)	(0.0189)	(0.0212)	(0.0217)	(0.0271)
	35,35	0.0071	−0.0787	−0.0748	−0.0840	−0.0867	−0.1015	−0.0763	−0.0723	−0.0817	−0.0845	−0.0995
		(0.0377)	(0.0196)	(0.0190)	(0.0207)	(0.0210)	(0.0246)	(0.0166)	(0.0160)	(0.0176)	(0.0179)	(0.0214)
	40,40	0.0071	−0.0787	−0.0748	−0.0840	−0.0867	−0.1015	−0.0763	−0.0723	−0.0817	−0.0845	−0.0995
		(0.0377)	(0.0196)	(0.0190)	(0.0207)	(0.0210)	(0.0246)	(0.0166)	(0.0160)	(0.0176)	(0.0179)	(0.0214)
	45,45	0.0217	−0.0688	−0.0656	−0.0730	−0.0753	−0.0866	−0.0683	−0.0651	−0.0725	−0.0748	−0.0861
		(0.0311)	(0.0159)	(0.0155)	(0.0167)	(0.0169)	(0.0193)	(0.0137)	(0.0133)	(0.0144)	(0.0146)	(0.0169)
	50,50	0.0257	−0.0598	−0.0569	−0.0635	−0.0655	−0.0751	−0.0600	−0.0572	−0.0636	−0.0657	−0.0751
		(0.0271)	(0.0138)	(0.0135)	(0.0144)	(0.0146)	(0.0163)	(0.0119)	(0.0116)	(0.0125)	(0.0127)	(0.0143)
(2, 7)	30,30	−0.0288	−0.1109	−0.1050	−0.1196	−0.1229	−0.1502	−0.1022	−0.0959	−0.1113	−0.1148	−0.1439
		(0.0607)	(0.0310)	(0.0298)	(0.0332)	(0.0335)	(0.0424)	(0.0251)	(0.0240)	(0.0272)	(0.0276)	(0.0364)
	35,35	−0.0168	−0.0919	−0.0869	−0.0989	−0.1019	−0.1225	−0.0856	−0.0805	−0.0928	−0.0959	−0.1170
		(0.0483)	(0.0252)	(0.0244)	(0.0266)	(0.0269)	(0.0324)	(0.0206)	(0.0199)	(0.0219)	(0.0222)	(0.0274)
	40,40	−0.0104	−0.0845	−0.0800	0.0908	−0.0936	−0.1116	−0.0799	−0.0754	−0.0862	−0.0891	−0.1072
		(0.0429)	(0.0227)	(0.0221)	(0.0240)	(0.0242)	(0.0287)	(0.0187)	(0.0181)	(0.0198)	(0.0201)	(0.0243)
	45,45	−0.0052	−0.0789	−0.0748	−0.0845	−0.0871	−0.1029	−0.0757	−0.0716	−0.0813	−0.0839	−0.0997
		(0.0391)	(0.0204)	(0.0199)	(0.0214)	(0.0217)	(0.0253)	(0.0169)	(0.0164)	(0.0178)	(0.0181)	(0.0214)
	50,50	0.0023	−0.0686	−0.0650	−0.0735	−0.0759	−0.0891	−0.0667	−0.0632	−0.0715	−0.0739	−0.0869
		(0.0336)	(0.0180)	(0.0176)	(0.0188)	(0.0189)	(0.0216)	(0.0151)	(0.0147)	(0.0158)	(0.0160)	(0.0184)

Table 6. The 95% confidence intervals of $R_{s,k}$ for Case 1, $(\alpha, \lambda_1, \beta, \lambda_2) = (2,\ 2,\ 2,\ 3)$ and $T = 1$.

Estimator	$(s,\ k)$	$R_{s,k}$	$(n,\ m)$	LCB	UCB	AL	Cover
MLE	$(1,\ 4)$	0.5394	(30, 30)	0.1260	07327	0.6067	0.6068
Prior-I				0.2823	0.7570	0.4747	0.9381
Prior-II				0.2690	0.7706	0.5015	0.9543
MLE			(35, 35)	0.1355	0.7153	0.5798	0.6053
Prior-I				0.2965	0.7434	0.4469	0.9340
Prior-II				0.2879	0.7569	0.4690	0.9483
MLE			(40, 40)	0.1414	0.6921	0.5507	0.5718
Prior-I				0.3063	0.7340	0.4277	0.9349
Prior-II				0.2997	0.7471	0.4474	0.9507
MLE			(45, 45)	0.1465	0.6823	0.5358	0.5743
Prior-I				0.3123	0.7236	0.4113	0.9315
Prior-II				0.3073	0.7366	0.4293	0.9478
MLE			(50, 50)	0.1517	0.6656	0.5139	0.5680
Prior-I				0.3184	0.7134	0.3950	0.9329
Prior-II				0.3164	0.7261	0.4097	0.9487

Table 6. *Cont.*

Estimator	$(s,\ k)$	$R_{s,k}$	(n, m)	LCB	UCB	AL	Cover
MLE	(2, 7)	0.5151	(30, 30)	0.1054	0.7254	0.6200	0.6260
Prior-I				0.2446	0.7273	0.4827	0.9365
Prior-II				0.2300	0.7430	0.5131	0.9538
MLE			(35, 35)	0.1173	0.7079	0.5906	0.6155
Prior-I				0.2593	0.7145	0.4552	0.9387
Prior-II				0.2489	0.7295	0.4807	0.9535
MLE			(40, 40)	0.1221	0.6824	0.5604	0.5943
Prior-I				0.2690	0.7046	0.4355	0.9362
Prior-II				0.2615	0.7192	0.4577	0.9545
MLE			(45, 45)	0.1252	0.6845	0.5593	0.5805
Prior-I				0.2764	0.6953	0.4190	0.9386
Prior-II				0.2704	0.7096	0.4392	0.9539
MLE			(50, 50)	0.1369	0.6589	0.5220	0.5743
Prior-I				0.2857	0.6867	0.4010	0.9347
Prior-II				0.2827	0.7005	0.4179	0.9512

Table 7. The 95% confidence intervals of $R_{s,k}$ for Case 2, $(\alpha, \lambda_1, \beta, \lambda_2) = (1.5,\ 1,\ 2,\ 1)$, and $T = 1$.

Estimator	$(s,\ k)$	$R_{s,k}$	(n, m)	LCB	UCB	AL	Cover
MLE	(1, 4)	0.8654	(30, 30)	0.3211	1.1171	0.7960	0.8750
Prior-I				0.5098	1.0279	0.5181	0.9247
Prior-II				0.4900	1.0357	0.5457	0.9461
MLE			(35, 35)	0.3857	1.1137	0.7280	0.8760
Prior-I				0.5450	1.0226	0.4776	0.9296
Prior-II				0.5298	1.0265	0.4967	0.9490
MLE			(40, 40)	0.4154	1.1094	0.6940	0.8803
Prior-I				0.5658	1.0167	0.4509	0.9328
Prior-II				0.5546	1.0180	0.4634	0.9523
MLE			(45, 45)	0.4404	1.1075	0.6671	0.8878
Prior-I				0.5851	1.0104	0.4253	0.9331
Prior-II				0.5763	1.0100	0.4338	0.9490
MLE			(50, 50)	0.4808	1.0991	0.6184	0.8740
Prior-I				0.6052	1.0074	0.4022	0.9373
Prior-II				0.5980	1.0053	0.4072	0.9538
MLE	(2, 7)	0.8267	(30, 30)	0.2386	1.0857	0.8470	0.8493
Prior-I				0.4423	1.0082	0.5659	0.9171
Prior-II				0.4226	1.0204	0.5978	0.9438
MLE			(35, 35)	0.2991	1.0932	0.7941	0.8630
Prior-I				0.4785	1.0040	0.5255	0.9275
Prior-II				0.4633	1.0108	0.5475	0.9501
MLE			(40, 40)	0.3264	1.0864	0.7600	0.8643
Prior-I				0.5029	0.9978	0.4949	0.9217
Prior-II				0.4918	1.0018	0.5100	0.9472
MLE			(45, 45)	0.3473	1.0740	0.7262	0.8633
Prior-I				0.5221	0.9914	0.4693	0.9274
Prior-II				0.5142	0.9929	0.4786	0.9487
MLE			(50, 50)	0.3882	1.0664	0.6783	0.8613
Prior-I				0.5415	0.9851	0.4435	0.9272
Prior-II				0.5344	0.9839	0.4495	0.9479

Table 8. The 95% confidence intervals of $R_{s,k}$ for Case 1, $(\alpha, \lambda_1, \beta, \lambda_2) = (2,\ 2,\ 2,\ 3)$, and $T = 5$.

Estimator	$(s,\ k)$	$R_{s,k}$	(n, m)	LCB	UCB	AL	Cover
MLE	(1, 4)	0.5394	(30, 30)	0.1296	0.7321	0.6026	0.5820
Prior-I				0.2836	0.7574	0.4738	0.9358
Prior-II				0.2699	0.7708	0.5009	0.9539
MLE			(35, 35)	0.1353	0.7102	0.5748	0.5830
Prior-I				0.2967	0.7431	0.4464	0.9345
Prior-II				0.2886	0.7566	0.4680	0.9498
MLE			(40, 40)	0.1516	0.7211	0.5695	0.6010
Prior-I				0.3044	0.7311	0.4267	0.9363
Prior-II				0.2980	0.7443	0.4463	0.9522
MLE			(45, 45)	0.1443	0.6726	0.5283	0.5820
Prior-I				0.3117	0.7220	0.4103	0.9367
Prior-II				0.3070	0.7348	0.4278	0.9510
MLE			(50, 50)	0.1568	0.6649	0.5081	0.561
Prior-I				0.3223	0.7166	0.3942	0.9373
Prior-II				0.3204	0.7290	0.4086	0.9525
MLE	(2, 7)	0.5151	(30, 30)	0.1071	0.7531	0.6281	0.6430
Prior-I				0.2454	0.7256	0.4802	0.9362
Prior-II				0.2300	0.7412	0.5112	0.9540
MLE			(35, 35)	0.1416	0.7183	0.5766	0.6050
Prior-I				0.2607	0.7137	0.4530	0.9339
Prior-II				0.2510	0.7288	0.4778	0.9516
MLE			(40, 40)	0.1293	0.6917	0.5625	0.5900
Prior-I				0.2685	0.7026	0.4340	0.9340
Prior-II				0.2608	0.7173	0.4565	0.9516
MLE			(45, 45)	0.1299	0.6707	0.5407	0.5890
Prior-I				0.2768	0.6949	0.4181	0.9332
Prior-II				0.2710	0.7090	0.4380	0.9487
MLE			(50, 50)	0.1401	0.6606	0.5205	0.5620
Prior-I				0.2869	0.6861	0.3992	0.9389
Prior-II				0.2839	0.6998	0.4159	0.9531

Table 9. The 95% confidence intervals of $R_{s,k}$ for Case 2, $(\alpha, \lambda_1, \beta, \lambda_2) = (1.5,\ 1,\ 2,\ 1)$, and $T = 5$.

Estimator	$(s,\ k)$	$R_{s,k}$	(n, m)	LCB	UCB	AL	Cover
MLE	(1, 4)	0.8653	(30, 30)	0.3150	1.1020	0.7870	0.8890
Prior-I				0.5332	1.0235	0.4903	0.9189
Prior-II				0.5311	1.0340	0.5029	0.9525
MLE			(35, 35)	0.3659	1.1020	0.7361	0.9020
Prior-I				0.5677	1.0170	0.4493	0.9253
Prior-II				0.5670	1.0231	0.4561	0.9502
MLE			(40, 40)	0.4014	1.1037	0.7023	0.9130
Prior-I				0.5677	1.0170	0.4493	0.9253
Prior-II				0.5670	1.0231	0.4561	0.9502
MLE			(45, 45)	0.4180	1.0961	0.6781	0.9170
Prior-I				0.5989	1.0042	0.4053	0.9266
Prior-II				0.5995	1.0061	0.4066	0.9474
MLE			(50, 50)	0.4687	1.0931	0.6244	0.9310
Prior-I				0.6193	1.0009	0.3816	0.9265
Prior-II				0.6201	1.0012	0.3812	0.9479
MLE	(2, 7)	0.8267	(30, 30)	0.2148	1.0478	0.8330	0.8470
Prior-I				0.4622	0.9993	0.5371	0.9023
Prior-II				0.4616	1.0164	0.5548	0.9456
MLE			(35, 35)	0.2816	1.0589	0.7773	0.865
Prior-I				0.5013	0.9949	0.4935	0.9116
Prior-II				0.5032	1.0051	0.5019	0.9467
MLE			(40, 40)	0.2828	1.0416	0.7588	0.878
Prior-I				0.5194	0.9892	0.4698	0.9166
Prior-II				0.5219	0.9961	0.4742	0.9475
MLE			(45, 45)	0.3265	1.0484	0.7219	0.8860
Prior-I				0.5355	0.9828	0.4473	0.9188
Prior-II				0.5379	0.9870	0.4491	0.9483
MLE			(50, 50)	0.3544	1.0426	0.6882	0.8880
Prior-I				0.5578	0.9783	0.4205	0.9198
Prior-II				0.5604	0.9801	0.4197	0.9458

5. Real Data Analysis

Two practical data sets will be provided to illustrate the proposed three estimation methods.

5.1. Strength Data of Carbon Fiber

Badar and Priest [19] provided a fiber strength testing example, where the strength under tension measured in GPA for single carbon fibers at the gauge lengths of 1, 10, 20, and 50 mm and impregnated 1000 carbon fiber tows. Raqab and Kundu [4] transformed the data sets using the 20 mm single fibers (with the size of $n = 69$) and 10 mm single fibers (with the size of $m = 63$) as the strength and stress samples, respectively. The two data sets are reported in Tables 10 and 11. The generalized exponential distribution modeling is applied for these two data sets. The MLEs of the model parameters for these two data sets are given as $\widehat{\alpha} = 8.8284, \widehat{\lambda}_1 = 1.8966, \widehat{\beta} = 3.1671, \widehat{\lambda}_2 = 1.5753$, and the Kolmogorov–Smirnov (K-S) test value and associated P-value are 0.8321 and 0.2122, respectively. The results indicate that the generalized exponential distribution is good for modeling these two data sets. In addition, $(s, k) = (1, 4)$ and $(2, 7)$ are considered in this study for MSS reliability evaluation. Based on the complete data sets, we have $\widehat{R}_{1,4} = 0.5515$, $\widehat{R}_{2,7} = 0.5165$, and the 95% confidence interval of $R_{1,4}$ and $R_{2,7}$ are $(0.4206, 0.7170)$ and $(0.4151, 0.7352)$, respectively.

In order to make the data meet the conditions of the type-I hybrid censoring scheme, the two data sets are respectively cut into type-I hybrid censored samples. Considering that the number of failures r is half of the sample size, and the T of the strength and stress data are selected as 0.7 and 0.8 quantiles, respectively. The results of Bayesian estimation are displaced in Table 12, which shows the interval length using Prior I is shorter than that using Prior II, and the estimated result of Prior II is closer to the MLE result under complete data.

Table 10. Breaking strength of a single carbon fiber with a gauge length of 20 mm.

0.312	0.314	0.479	0.552	0.7	0.803	0.861
0.865	0.944	0.958	0.966	0.997	1.006	1.021
1.027	1.055	1.063	1.098	1.14	1.179	1.224
1.24	1.253	1.27	1.272	1.274	1.301	1.301
1.359	1.382	1.382	1.426	1.434	1.435	1.478
1.49	1.511	1.514	1.535	1.554	1.566	1.57
1.586	1.629	1.633	1.642	1.648	1.684	1.697
1.726	1.77	1.773	1.8	1.809	1.818	1.821
1.848	1.88	1.954	2.012	2.067	2.084	2.09
2.096	2.128	2.233	2.433	2.585	2.585	

Table 11. Breaking strength of a single carbon fiber with a gauge length of 10 mm.

0.101	0.332	0.403	0.428	0.457	0.55	0.561
0.596	0.597	0.645	0.654	0.674	0.718	0.722
0.725	0.732	0.775	0.814	0.816	0.818	0.824
0.859	0.875	0.938	0.94	1.056	1.117	1.128
1.137	1.137	1.177	1.196	1.23	1.325	1.339
1.345	1.42	1.423	1.435	1.443	1.464	1.472
1.494	1.532	1.546	1.577	1.608	1.635	1.693
1.701	1.737	1.754	1.762	1.828	2.052	2.071
2.086	2.171	2.224	2.227	2.425	2.595	3.22

Table 12. The result of Bayesian estimation for strength data.

Estimator	(s,k)	$\widehat{R}_{s,k_{Se}}$	$\widehat{R}_{s,k_L}$ $a=b=-0.5$	$\widehat{R}_{s,k_{Ge}}$ $a=b=-0.5$	$\widehat{R}_{s,k_L}$ $a=b=1$	$\widehat{R}_{s,k_{Ge}}$ $a=b=1$	Confidence Interval
Prior-I	(1, 4)	0.6118	0.6130	0.6097	0.6093	0.6033	(0.4705, 0.7446)
Prior-II		0.5734	0.5746	0.5713	0.5710	0.5649	(0.4356, 0.7042)
Prior-I	(2, 7)	0.5818	0.5830	0.5797	0.5793	0.5731	(0.4457, 0.714)
Prior-II		0.5389	0.5400	0.5368	0.5367	0.5303	(0.4098, 0.6721)

5.2. Waiting Time before Customer Service of the Bank

The data sets that were studied by Al-Mutairi et al. [5], Ali [6], Singh et al. [7] and Sadek et al. [8] are used. The data sets are composed of the customers' waiting times (in minutes) before services in two banks, A and B, and are shown in Table 13 and Table 14, with sample sizes, $n = 100$ and $m = 60$, respectively. The MLEs of the generalized exponential distribution parameters, based on these two data sets, are respectively represented as $\widehat{\alpha} = 2.1834, \widehat{\lambda}_1 = 0.1592, \widehat{\beta} = 1.4071, \widehat{\lambda}_2 = 0.2368$. The K-S test and the corresponding P-values are, respectively, 0.9996 and 0.0733. These results show the two data sets are fitted with the generalized exponential distributions. In addition, $(s,k) = (1,4)$ and (2, 7) are considered, then $\widehat{R}_{1,4} = 0.4814$, $\widehat{R}_{2,7} = 0.4557$, and the 95% confidence intervals of $R_{1,4}$ and $R_{2,7}$ are $(0.3569, 0.6157)$ and $(0.2935, 0.6173)$, respectively.

In order to make the data sets meet the conditions of the type-I hybrid censoring scheme, the two data sets are respectively converted to type-I hybrid censored samples with r as half of the sample size, and T as 0.7 and 0.8 quantiles. The results of the Bayesian estimations are given in Table 15 that again show that the interval length with Prior I is shorter than that with Prior II.

Table 13. Bank A's customer waiting time.

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7
2.9	3.1	3.2	3.3	3.5	3.6	4	4.1	4.2	4.2
4.3	4.3	4.4	4.4	4.6	4.7	4.7	4.8	4.9	4.9
5	5.3	5.5	5.7	5.7	6.1	6.2	6.2	6.2	6.3
6.7	6.9	7.1	7.1	7.1	7.1	7.4	7.6	7.7	8
8.2	8.6	8.6	8.6	8.8	8.8	8.9	8.9	9.5	9.6
9.7	9.8	10.7	10.9	11	11	11.1	11.2	11.2	11.5
11.9	12.4	12.5	12.9	13	13.1	13.3	13.6	13.7	13.9
14.1	15.4	15.4	17.3	17.3	18.1	18.2	18.4	18.9	19
19.9	20.6	21.3	21.4	21.9	23	27	31.6	33.1	38.5

Table 14. Bank B's customer waiting time.

0.1	0.2	0.3	0.7	0.9	1.1	1.2	1.8	1.9	2
2.2	2.3	2.3	2.3	2.5	2.6	2.7	2.7	2.9	3.1
3.1	3.2	3.4	3.4	3.5	3.9	4	4.2	4.5	4.7
5.3	5.6	5.6	6.2	6.3	6.6	6.8	7.3	7.5	7.7
7.7	8	8	8.5	8.5	8.7	9.5	10.7	10.9	11
12.1	12.3	12.8	12.9	13.2	13.7	14.5	16	16.5	28

Table 15. The result of Bayesian estimation for waiting time data.

Estimator	(s,k)	$\widehat{R}_{s,k_{Se}}$	$\widehat{R}_{s,k_L}$ $a=b=-0.5$	$\widehat{R}_{s,k_{Ge}}$ $a=b=-0.5$	$\widehat{R}_{s,k_L}$ $a=b=1$	$\widehat{R}_{s,k_{Ge}}$ $a=b=1$	Confidence Interval
Prior-I	(1, 4)	0.4740	0.4752	0.4714	0.4715	0.4634	(0.3416, 0.6110)
Prior-II		0.4873	0.4886	0.4847	0.4848	0.4764	(0.3565, 0.6351)
Prior-I	(2, 7)	0.4478	0.4489	0.4452	0.4456	0.4374	(0.3201, 0.5795)
Prior-II		0.4570	0.4583	0.4542	0.4545	0.4454	(0.3218, 0.5971)

6. Concluding Remarks

The MSS reliability estimation problem based on type-I hybrid censored samples from generalized exponential distributions was investigated by the maximum likelihood estimation method and Bayesian approaches with informative or non-informative prior distribution. In the Bayesian estimation procedures, because the marginal posterior distributions do not have closed forms, it is impossible to use the Gibbs sampling algorithm to draw sample of posteriors to evaluate the Bayes estimators of the model parameters and the reliability $R_{s,k}$ in Equation (5). The MCMC approach with the Metropolis–Hastings algorithm is used to implement the sampling for Bayesian estimation. Meanwhile linear exponential, general entropy, and squared error loss functions are used to obtain Bayes estimators of the MSS reliability.

To overcome the complexity of asymptotic normality based on the Fisher information matrix with the maximum likelihood estimation method, the HPDI based on the Bayesian estimation method is used to obtain the confidence interval of the MSS reliability $R_{s,k}$. According to the simulation results, the Bayes estimator with information prior performs better than the Bayes estimator with non-informative prior and the MLE in terms of the performance metrics of rBias and rMSE. However, the performance of the Bayes estimators based on the informative and non-informative prior distributions are competitive.

We also find that the performance of the maximum likelihood estimation is the worst among the three compared methods. The findings mean that if the prior distribution can be well selected, the Bayes estimator with informative prior distributions is more reliable. Otherwise, the Bayes estimator with using non-informative prior distributions is recommended. Two practical examples are used to illustrate the proposed estimation methods.

The MSS reliability inferences for different distributions and under other censoring schemes by using pivotal estimation methods and the robustness of reliability estimators are interesting and merit future investigations.

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