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Fixed Point Results in $C^{\star}$-Algebra-Valued Partial b-Metric Spaces with Related Application

Gunaseelan Mani [1], Arul Joseph Gnanaprakasam [2], Ozgur Ege [3,*], Ahmad Aloqaily [4,5] and Nabil Mlaiki [4,*]

1. Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, Tamil Nadu, India
2. Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, SRM Nagar, Kattankulathur, Kanchipuram 603203, Tamil Nadu, India
3. Department of Mathematics, Ege University, Bornova, Izmir 35100, Türkiye
4. Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia
5. School of Computer, Data and Mathematical Sciences, Western Sydney University, Sydney, NSW 2150, Australia

* Correspondence: ozgur.ege@ege.edu.tr (O.E.); nmlaiki2012@gmail.com or nmlaiki@psu.edu.sa (N.M.)

Abstract: In this manuscript, we prove some fixed point theorems on $C^{\star}$-algebra-valued partial b-metric spaces by using generalized contraction. We give support and suitable examples of our main results. Moreover, we present a generative application of the main results.

Keywords: fixed point; $C^{\star}$-algebra-valued partial b-metric; $C^{\star}$-algebra-valued b-metric; $C^{\star}$-algebra

MSC: 47H09; 47H10; 46L05; 54H25

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1. Introduction

Bakhtin [1] defined b-metric space, and Czerwik [2] established the fixed point results using the Banach contraction principle (see, for instance, [3–5] and references therein). Abdou et al. [6] illustrated a new concept of locally α-ψ-contractive mapping, generalized α-ψ- rational contraction and established fixed point theorems for such mappings in the context of extended b-metric spaces. Gholidahneh et al. [7] demonstrated the concept of modular p-metric space and established some fixed point results for α-$\bar{v}$-Meir–Keeler contractions in this space. Furthermore, they established a relationship between the fuzzy concept of Meir–Keeler and extended p-metrics with modular p-metrics and obtained fixed point results in triangular p-metric spaces with fuzzy concepts.

In 2014, Ma et al. [8] proved some fixed point theorems for self-maps with contractive or expansive conditions on $C^{\star}$-algebra-valued metric spaces. Chandok et al. [9] presented the concept of $C^{\star}$-algebra-valued partial metric space ($C^{\star}$-AVPMS) and some fixed point results on such spaces using C-class functions in 2019. Mlaiki et al. [10] expanded the class of $C^{\star}$-AV_bMS ($C^{\star}$-algebra-valued b-metric space) and $C^{\star}$-AVPMS by introducing $C^{\star}$-AVP_bMS ($C^{\star}$-algebra-valued partial b-metric space) and used it to prove fixed point results in 2021.

In this paper, we prove fixed point theorems for generalized contraction in $C^{\star}$-AVP_bMS.

This paper consists of five sections, wherein Section 1 begins with an introduction. In Section 2 we first recall some definitions, lemma and theorem related to $C^{\star}$-AVP_bMS and discuss their related properties. In Section 3 we prove fixed point results as well as giving an example to support our main result. In Section 4 we apply our main result to examine the existence and uniqueness of a solution for the system of the Fredholm integral equation, and in the last section we present our conclusions.

2. Preliminaries

This section covers the basic definitions and properties of $C^\star$-algebras [11,12] with the following important consequences Suppose that $\mathbb{B}$ is a unital algebra with unit $\mathfrak{I}$. An involution on $\mathbb{B}$ is a conjugate-linear map $\vartheta \mapsto \vartheta^\star$ on $\mathbb{B}$ such that $\vartheta^{\star\star} = \vartheta$ and $(\vartheta\varrho)^\star = \varrho^\star\vartheta^\star \; \forall\, \vartheta, \varrho \in \mathbb{B}$. The pair $(\mathbb{B}, \star)$ is known as a $\star$-algebra . A Banach $\star$-algebra is a $\star$-algebra $\mathbb{B}$ together with a complete sub-multiplicative norm such that $\|\vartheta^\star\| = \|\vartheta\|$ for all $\vartheta \in \mathbb{B}$. A $C^\star$-algebra is a Banach $\star$-algebra with the property that $\|\vartheta^\star\vartheta\| = \|\vartheta\|^2$ for all $\vartheta \in \mathbb{B}$. In this paper, we prove some fixed point theorems on $C^\star$-algebra-valued partial b-metric spaces by using generalized contraction.

Throughout this paper, we denote a $C^\star$-algebra with unit $\mathfrak{I}$ by $\mathbb{B}$. Set $\mathbb{B}_{\mathfrak{h}} = \{\varrho \in \mathbb{B} : \varrho = \varrho^\star\}$. Consider a positive element $\varrho \in \mathbb{B}$, i.e., $\varrho \geq \theta$ if $\varrho \in \mathbb{B}_{\mathfrak{h}}$ and $\varsigma(\varrho) \subset [0, \infty)$, where $\varsigma(\varrho)$ is the spectrum of ϱ. We define a partial ordering on $\mathbb{B}_{\mathfrak{h}}$ as follows: $\varrho \preceq \xi$ iff $\xi - \varrho \succeq \theta$. Now, we denote the set $\{\varrho \in \mathbb{B} : \varrho \succeq \theta\}$ and $|\varrho| = (\varrho^\star\varrho)^{\frac{1}{2}}$ by $\mathbb{B}_+$.

Lemma 1 ([11,13])**.** *Let $\mathbb{B}$ be a unital $C^\star$-algebra with a unit $\mathfrak{I}$.*

1. *For each $\varrho \in \mathbb{B}_+$, we have $\varrho \preceq \mathfrak{I} \iff \|\varrho\| \leq 1$.*
2. *If $\vartheta \in \mathbb{B}_+$ with $\|\vartheta\| < \frac{1}{2}$, then $\mathfrak{I} - \vartheta$ is invertible and $\|\vartheta(\mathfrak{I} - \vartheta)^{-1}\| < 1$.*
3. *Assume that $\vartheta, \varrho \in \mathbb{B}$ with $\vartheta, \varrho \succeq \theta$ and $\vartheta\varrho = \varrho\vartheta$, then $\vartheta\varrho \succeq \theta$.*
4. *Define $\mathbb{B}' = \{\vartheta \in \mathbb{B} : \vartheta\varrho = \varrho\vartheta, \forall \varrho \in \mathbb{B}\}$. Let $\vartheta \in \mathbb{B}'$. If $\varrho, \mathfrak{c} \in \mathbb{B}$ with $\varrho \succeq \mathfrak{c} \succeq \theta$, and $\mathfrak{I} - \vartheta \in \mathbb{B}'_+$ is an invertible operator, then*

$$(\mathfrak{I} - \vartheta)^{-1}\varrho \geq (\mathfrak{I} - \vartheta)^{-1}\mathfrak{c}.$$

Ma et al. [14] presented in the sequel definition:

Definition 1. *Let $\mho \neq \emptyset$ and $\ell \in \mathbb{B}$ such that $\ell \succeq \mathfrak{I}$. A mapping $\Phi : \mho \times \mho \to \mathbb{B}$ satisfies the following condition:*

1. *$\theta \preceq \Phi(\varrho, \xi)$ for all $\Phi(\varrho, \xi) = \theta \iff \varrho = \xi$;*
2. *$\Phi(\varrho, \xi) = \Phi(\xi, \varrho)$;*
3. *$\Phi(\varrho, \xi) \preceq \ell[\Phi(\varrho, \varsigma) + \Phi(\varsigma, \xi)], \forall\, \varrho, \xi, \varsigma \in \mho$.*

Then Φ is called a $C^\star$-algebra-valued b-metric space ($C^\star$-AV_bM) on $\mho$ and $(\mho, \mathbb{B}, \Phi)$ is a $C^\star$-AV_bMS.

Now, we remember that the definition of C^*-AVP_bMS introduced by Mlaiki et al. [10].

Definition 2. *Let $\mho$ be a non-void set and $\ell \in \mathbb{B}$ such that $\ell \geq \mathfrak{I}$. A function $\Phi : \mho \times \mho \to \mathbb{B}$ satisfies the following property:*

1. *$\theta \preceq \Phi(\varrho, \xi), \forall\, \varrho, \xi \in \mho$ and $\Phi(\varrho, \varrho) = \Phi(\xi, \xi) = \Phi(\varrho, \xi)$ if $\varrho = \xi$;*
2. *$\Phi(\varrho, \varrho) \preceq \Phi(\varrho, \xi)$;*
3. *$\Phi(\varrho, \xi) = \Phi(\xi, \varrho)$;*
4. *$\Phi(\varrho, \xi) \preceq \ell(\Phi(\varrho, \varsigma) + \Phi(\varsigma, \xi)) - \Phi(\varsigma, \varsigma), \forall\, \varrho, \xi, \varsigma \in \mho$.*

Then Φ is said to be a $C^\star$-AVP_bM on $\mho$ and $(\mho, \mathbb{B}, \Phi)$ is said to be a $C^\star$-AVP_bMS.

Definition 3. *Let $(\mho, \mathbb{B}, \Phi)$ be a C^*-AVP_bMS. A sequence $\{\varrho_{\mathfrak{q}}\}$ in $(\mho, \mathbb{B}, \Phi)$ is said to be convergent (with respect to $\mathbb{B}$) to a point $\varrho \in \mho$ if $\epsilon > 0$, for each $\alpha \in \mathbb{N}$ satisfying $||\Phi(\varrho_{\mathfrak{q}}, \varrho) - \Phi(\varrho, \varrho)|| < \epsilon$ for all $\mathfrak{q} > \alpha$.*

Definition 4. *Let $(\mho, \mathbb{B}, \Phi)$ be a C^*-AVP_bMS. A sequence $\{\varrho_{\mathfrak{q}}\}$ in $(\mho, \mathbb{B}, \Phi)$ is said to be Cauchy (with respect to $\mathbb{B}$) if $\lim\limits_{\mathfrak{q} \to \infty} \Phi(\varrho_{\mathfrak{q}}, \varrho_\beta)$ exists and it is finite.*

Definition 5. *Let $(\mho, \mathbb{B}, \Phi)$ be a C^*-AVP_bMS. A triplet $(\mho, \mathbb{B}, \Phi)$ is said to be complete C^*-AVP_bMS if every Cauchy sequence is convergent to ϱ in $\mho$ such that*

$$\lim_{\mathfrak{q},\beta\to\infty} \Phi(\varrho_{\mathfrak{q}}, \varrho_\beta) = \lim_{\mathfrak{q}\to\infty} \Phi(\varrho_{\mathfrak{q}}, \varrho) = \Phi(\varrho, \varrho).$$

Theorem 1 ([10])**.** *Let $(\mho, \mathbb{B}, \Phi)$ be a complete $C^\star$-AVP_bMS and $\Lambda : \mho \to \mho$ is a C_b^*-contraction. Then Λ has a unique fixed point $\varrho \in \mho$ such that $\Phi(\varrho, \varrho) = 0_{\mathbb{B}}$.*

Inspired by Theorem 1, we prove fixed point theorems for generalized contractions in C^*-AVP_bMS with an application.

3. Main Results

Now, we prove fixed point theorems for generalized contractions in C^*-AVP_bMS.

Theorem 2. *Let $(\mho, \mathbb{B}, \Phi)$ be a complete $C^\star$-AVP_bMS. Suppose the mapping $\Lambda : \mho \to \mho$ satisfying the condition:*

$$\Phi(\Lambda\varrho, \Lambda\xi) \leq \gamma(\Phi(\Lambda\varrho, \xi) + \Phi(\Lambda\xi, \varrho)), \quad \forall \varrho, \xi \in \mho,$$

where $\gamma \in \mathbb{B}_+^{'}$ and $\|\gamma\ell\| < \frac{1}{2}$. Then Λ a unique fixed point $\varrho \in \mho$ such that $\Phi(\varrho, \varrho) = \theta$.

Proof. If we assume $\gamma = \theta$, then Λ maps $\mho$ into a single point. Thus, without loss of generality, we assume that $\gamma \neq \theta$. Notice that for $\gamma \in \mathbb{B}_+^{'}$, $\gamma(\Phi(\Lambda\varrho, \xi) + \Phi(\Lambda\xi, \varrho)) \geq \theta$. Choose $\varrho_0 \in \mho$ and set $\varrho_{\mathfrak{q}+1} = \Lambda\varrho_{\mathfrak{q}} = \Lambda^{\mathfrak{q}+1}\varrho_0$, $\mathfrak{q} = 1, 2, ...$, and $\Phi(\varrho_1, \varrho_0) = \gamma_0$. Then

$$\begin{aligned}
\Phi(\varrho_{\mathfrak{q}+1}, \varrho_{\mathfrak{q}}) &= \Phi(\Lambda\varrho_{\mathfrak{q}}, \Lambda\varrho_{\mathfrak{q}-1}) \\
&\preceq \gamma(\Phi(\Lambda\varrho_{\mathfrak{q}}, \varrho_{\mathfrak{q}-1}) + \Phi(\Lambda\varrho_{\mathfrak{q}-1}, \varrho_{\mathfrak{q}})) \\
&= \gamma(\Phi(\Lambda\varrho_{\mathfrak{q}}, \Lambda\varrho_{\mathfrak{q}-2}) + \Phi(\Lambda\varrho_{\mathfrak{q}-1}, \Lambda\varrho_{\mathfrak{q}-1})) \\
&\preceq \gamma\ell(\Phi(\Lambda\varrho_{\mathfrak{q}}, \Lambda\varrho_{\mathfrak{q}-1}) + \Phi(\Lambda\varrho_{\mathfrak{q}-1}, \Lambda\varrho_{\mathfrak{q}-2})) \\
&- \gamma\Phi(\Lambda\varrho_{\mathfrak{q}-1}, \Lambda\varrho_{\mathfrak{q}-1}) + \gamma\Phi(\Lambda\varrho_{\mathfrak{q}-1}, \Lambda\varrho_{\mathfrak{q}-1}) \\
&= \gamma\ell(\Phi(\Lambda\varrho_{\mathfrak{q}}, \Lambda\varrho_{\mathfrak{q}-1})) + \gamma\ell(\Phi(\Lambda\varrho_{\mathfrak{q}-1}, \Lambda\varrho_{\mathfrak{q}-2})) \\
&= \gamma\ell(\Phi(\varrho_{\mathfrak{q}+1}, \varrho_{\mathfrak{q}})) + \gamma\ell(\Phi(\varrho_{\mathfrak{q}}, \varrho_{\mathfrak{q}-1})).
\end{aligned}$$

By Lemma 1,

$$(\mathfrak{I} - \gamma\ell)\Phi(\varrho_{\mathfrak{q}+1}, \varrho_{\mathfrak{q}}) \ \leq \ \gamma\ell\Phi(\varrho_{\mathfrak{q}}, \varrho_{\mathfrak{q}-1}).$$

Since $\ell, \gamma \in \mathbb{B}_+^{'}$ with $\|\gamma\ell\| < \frac{1}{2}$ and $\ell \succeq \mathfrak{I}$, we have $\mathfrak{I} - \gamma\ell \preceq \mathfrak{I} - \gamma$ and furthermore $(\mathfrak{I} - \gamma\ell)^{-1} \in \mathbb{B}_+^{'}$ with $\|(\mathfrak{I} - \gamma\ell)^{-1}\gamma\ell\| < 1$ by Lemma 1. Therefore,

$$\Phi(\varrho_{\mathfrak{q}+1}, \varrho_{\mathfrak{q}}) \ \preceq (\mathfrak{I} - \gamma\ell)^{-1}\gamma\ell\Phi(\varrho_{\mathfrak{q}}, \varrho_{\mathfrak{q}-1}) = \chi\Phi(\varrho_{\mathfrak{q}}, \varrho_{\mathfrak{q}-1}),$$

where $\chi = (\mathfrak{I} - \gamma\ell)^{-1}\gamma\ell$.

For any $\beta \geq 1$ and $\sigma \geq 1$, we have

$$\begin{aligned}
\Phi(\varrho_{\beta+\sigma}, \varrho_\beta) &\preceq \ell[\Phi(\varrho_{\beta+\sigma}, \varrho_{\beta+\sigma-1}) + \Phi(\varrho_{\beta+\sigma-1}, \varrho_\beta)] - \Phi(\varrho_{\beta+\sigma-1}, \varrho_{\beta+\sigma-1}) \\
&\preceq \ell\Phi(\varrho_{\beta+\sigma}, \varrho_{\beta+\sigma-1}) + \ell\Phi(\varrho_{\beta+\sigma-1}, \varrho_\beta) \\
&\preceq \ell\Phi(\varrho_{\beta+\sigma}, \varrho_{\beta+\sigma-1}) + \ell^2[\Phi(\varrho_{\beta+\sigma-1}, \varrho_{\beta+\sigma-2}) + \Phi(\varrho_{\beta+\sigma-2}, \varrho_\beta)] \\
&- \ell\Phi(\varrho_{\beta+\sigma-2}, \varrho_{\beta+\sigma-2}) \\
&\preceq \ell\Phi(\varrho_{\beta+\sigma}, \varrho_{\beta+\sigma-1}) + \ell^2\Phi(\varrho_{\beta+\sigma-1}, \varrho_{\beta+\sigma-2}) + \ell^2\Phi(\varrho_{\beta+\sigma-2}, \varrho_\beta) \\
&\preceq \ell\Phi(\varrho_{\beta+\sigma}, \varrho_{\beta+\sigma-1}) + \ell^2\Phi(\varrho_{\beta+\sigma-1}, \varrho_{\beta+\sigma-2}) + \cdots \\
&+ \ell^{\sigma-1}\Phi(\varrho_{\beta+2}, \varrho_{\beta+1}) + \ell^{\sigma-1}\Phi(\varrho_{\beta+1}, \varrho_\beta) \\
&\preceq \ell(\chi)^{\beta+\sigma-1}\gamma_0 + \ell^2(\chi)^{\beta+\sigma-2}\gamma_0 + \ell^3(\chi)^{\beta+\sigma-3}\gamma_0 + \dots \\
&+ \ell^{\sigma-1}(\chi)^{\beta+1}\gamma_0 + \ell^{\sigma-1}(\chi)^\beta\gamma_0 \\
&= \sum_{\alpha=1}^{\sigma-1} \ell^\alpha(\chi)^{\beta+\sigma-\alpha}\gamma_0 + \ell^{\sigma-1}(\chi)^\beta\gamma_0 \\
&= \sum_{\alpha=1}^{\sigma-1} |\gamma_0^{\frac{1}{2}}\chi^{\frac{\beta+\sigma-\alpha}{2}}\ell^{\frac{\alpha}{2}}|^2 + |\gamma_0^{\frac{1}{2}}\ell^{\frac{\sigma-1}{2}}\chi^{\frac{\beta}{2}}|^2 \\
&\preceq \|\gamma_0| \sum_{\alpha=1}^{\sigma-1} \|\chi\|^{\beta+\sigma-\alpha}\|\ell\|^\alpha\mathfrak{I} + \|\ell\|^{\sigma-1}\|\chi\|^\beta\|\gamma_0\|\mathfrak{I} \\
&\preceq \frac{\|\gamma_0\|\|\ell\|^\sigma\|\chi\|^{\beta+1}}{\|\ell\| - \|\chi\|}\mathfrak{I} + \|\ell\|^{\sigma-1}\|\chi\|^\beta\|\gamma_0\|\mathfrak{I} \\
&\to \theta \quad (\beta \to \infty).
\end{aligned}$$

This implies that $\{\varrho_{\mathfrak{q}}\}$ is a Cauchy sequence in $\mathbb{B}$. By the completeness of $(\mho, \mathbb{B}, \Phi)$, we can find $\varrho \in \mho$ satisfying $\lim_{\mathfrak{q}\to\infty} \varrho_{\mathfrak{q}} = \varrho$ and

$$\lim_{\mathfrak{q},\beta\to\infty} \Phi(\varrho_{\mathfrak{q}}, \varrho_\beta) = \lim_{\mathfrak{q}\to\infty} \Phi(\varrho_{\mathfrak{q}}, \varrho_{\mathfrak{q}}) = \lim_{\mathfrak{q}\to\infty} \Phi(\varrho_{\mathfrak{q}}, \varrho) = \Phi(\varrho, \varrho) = \theta.$$

So,

$$\begin{aligned}
\Phi(\mathbb{T}\varrho, \varrho) &\preceq \ell[\Phi(\Lambda\varrho, \Lambda\varrho_{\mathfrak{q}}) + \Phi(\Lambda\varrho_{\mathfrak{q}}, \varrho)] - \Phi(\Lambda\varrho_{\mathfrak{q}}, \varrho_{\mathfrak{q}}) \\
&\preceq \ell[\Phi(\Lambda\varrho, \Lambda\varrho_{\mathfrak{q}}) + \Phi(\Lambda\varrho_{\mathfrak{q}}, \varrho)] \\
&\preceq \ell[\gamma(\Phi(\Lambda\varrho, \varrho_{\mathfrak{q}}) + \Phi(\Lambda\varrho_{\mathfrak{q}}, \varrho)) + \Phi(\varrho_{\mathfrak{q}+1}, \varrho)] \\
&\preceq \ell\gamma\ell(\Phi(\Lambda\varrho, \varrho_{\mathfrak{q}}) + \Phi(\Lambda\varrho_{\mathfrak{q}}, \varrho)) + \ell\gamma\Phi(\varrho_{\mathfrak{q}+1}, \varrho) + \ell\Phi(\varrho_{\mathfrak{q}+1}, \varrho).
\end{aligned}$$

This is equivalent to

$$(\mathfrak{I} - \ell^2\gamma)\Phi(\Lambda\varrho, \varrho) \preceq \ell^2\gamma\Phi(\varrho, \varrho_{\mathfrak{q}}) + (\ell\gamma + \ell)\Phi(\varrho_{\mathfrak{q}+1}, \varrho).$$

Thus,

$$\begin{aligned}
\|\Phi(\Lambda\varrho, \varrho)\| &\leq \|(\mathfrak{I} - \ell^2\gamma)^{-1}\ell^2\gamma\|\|\Phi(\varrho, \varrho_{\mathfrak{q}})\| + \|(\mathfrak{I} - \ell^2\gamma)^{-1}(\ell\gamma + \ell)\|\|\Phi(\varrho_{\mathfrak{q}+1}, \varrho)\| \\
&\to \theta \quad (\mathfrak{q} \to \infty).
\end{aligned}$$

Therefore, $\Lambda\varrho = \varrho$.
Now if $\xi(\neq \varrho)$ is another fixed point of Λ, then

$$\begin{aligned}
\theta \preceq \Phi(\varrho, \xi) &= \Phi(\Lambda\varrho, \Lambda\xi) \\
&\preceq \gamma(\Phi(\Lambda\varrho, \xi) + \Phi(\Lambda\xi, \varrho)) \\
&= \gamma(\Phi(\varrho, \xi) + \Phi(\xi, \varrho)).
\end{aligned}$$

That is,

$$\Phi(\varrho,\xi) \preceq (\mathfrak{I}-\gamma)^{-1}\gamma\Phi(\Lambda\varrho,\Lambda\xi).$$

Since $\|\gamma(\mathfrak{I}-\gamma)^{-1}\| < 1$,

$$\begin{aligned}0 \leq \|\Phi(\varrho,\xi)\| &= \|\Phi(\Lambda\varrho,\Lambda\xi)\| \\ &\leq \|(\mathfrak{I}-\gamma)^{-1}\gamma\Phi(\varrho,\xi)\| \\ &\leq \|(\mathfrak{I}-\gamma)^{-1}\gamma\|\|\Phi(\varrho,\xi)\| \\ &< \|\Phi(\varrho,\xi)\|.\end{aligned}$$

This means that

$$\Phi(\varrho,\xi) = \theta \quad \Longleftrightarrow \quad \varrho = \xi.$$

□

Theorem 3. *Let $(\mho,\mathbb{B},\Phi)$ be a complete $C^\star$-AVP_bMS. Suppose the mapping $\Lambda:\mho\to\mho$ satisfying the following condition:*

$$\Phi(\Lambda\varrho,\Lambda\xi) \preceq \gamma(\Phi(\Lambda\varrho,\varrho)+\Phi(\Lambda\xi,\xi)), \quad \forall\varrho,\xi\in\mho,$$

where $\gamma \in \mathbb{B}_+^{'}$ and $\|\gamma\| < \frac{1}{2}$. Then Λ has a unique fixed point in $\mho$.

Proof. We assume that $\gamma \neq \theta$, without loss of generality. Notice that for $\gamma \in \mathbb{B}_+^{'}$, $\gamma(\Phi(\Lambda\varrho,\varrho)+\Phi(\Lambda\xi,\xi)) \geq \theta$. Choose $\varrho_0 \in \mho$ and set $\varrho_{\mathfrak{q}+1} = \Lambda\varrho_{\mathfrak{q}} = \Lambda^{\mathfrak{q}+1}\varrho_0, \mathfrak{q} = 1,2,....$ and $\Phi(\varrho_1,\varrho_0) = \gamma_0$. Then

$$\begin{aligned}\Phi(\varrho_{\mathfrak{q}+1},\varrho_{\mathfrak{q}}) &= \Phi(\Lambda\varrho_{\mathfrak{q}},\Lambda\varrho_{\mathfrak{q}-1}) \\ &\preceq \gamma(\Phi(\Lambda\varrho_{\mathfrak{q}},\varrho_{\mathfrak{q}})+\Phi(\Lambda\varrho_{\mathfrak{q}-1},\varrho_{\mathfrak{q}-1})) \\ &= \gamma(\Phi(\varrho_{\mathfrak{q}+1},\varrho_{\mathfrak{q}})+\Phi(\varrho_{\mathfrak{q}},\varrho_{\mathfrak{q}-1})).\end{aligned}$$

Thus,

$$\Phi(\varrho_{\mathfrak{q}+1},\varrho_{\mathfrak{q}}) \preceq (\mathfrak{I}-\gamma)^{-1}\gamma\Phi(\varrho_{\mathfrak{q}},\varrho_{\mathfrak{q}-1}) = \chi\Phi(\varrho_{\mathfrak{q}},\varrho_{\mathfrak{q}-1}),$$

where $\chi = (\mathfrak{I}-\gamma)^{-1}\gamma$.

For any $\beta \geq 1$ and $\sigma \geq 1$, we have

$$\begin{aligned}\Phi(\varrho_{\beta+\sigma},\varrho_\beta) &\preceq \ell[\Phi(\varrho_{\beta+\sigma},\varrho_{\beta+\sigma-1})+\Phi(\varrho_{\beta+\sigma-1},\varrho_\beta)]-\Phi(\varrho_{\beta+\sigma-1},\varrho_{\beta+\sigma-1}) \\ &\preceq \ell\Phi(\varrho_{\beta+\sigma},\varrho_{\beta+\sigma-1})+\ell\Phi(\varrho_{\beta+\sigma-1},\varrho_\beta) \\ &\preceq \ell\Phi(\varrho_{\beta+\sigma},\varrho_{\beta+\sigma-1})+\ell^2[\Phi(\varrho_{\beta+\sigma-1},\varrho_{\beta+\sigma-2})+\Phi(\varrho_{\beta+\sigma-2},\varrho_\beta)] \\ &-\ell\Phi(\varrho_{\beta+\sigma-2},\varrho_{\beta+\sigma-2}) \\ &\preceq \ell\Phi(\varrho_{\beta+\sigma},\varrho_{\beta+\sigma-1})+\ell^2\Phi(\varrho_{\beta+\sigma-1},\varrho_{\beta+\sigma-2})+\ell^2\Phi(\varrho_{\beta+\sigma-2},\varrho_\beta) \\ &\preceq \ell\Phi(\varrho_{\beta+\sigma},\varrho_{\beta+\sigma-1})+\ell^2\Phi(\varrho_{\beta+\sigma-1},\varrho_{\beta+\sigma-2})+\cdots \\ &+\ell^{\sigma-1}\Phi(\varrho_{\beta+2},\varrho_{\beta+1})+\ell^{\sigma-1}\Phi(\varrho_{\beta+1},\varrho_\beta)\end{aligned}$$

$$
\begin{aligned}
&\preceq \ell(\chi)^{\beta+\sigma-1}\gamma_0 + \ell^2(\chi)^{\beta+\sigma-2}\gamma_0 + \ell^3(\chi)^{\beta+\sigma-3}\gamma_0 + \dots \\
&+ \ell^{\sigma-1}(\chi)^{\beta+1}\gamma_0 + \ell^{\sigma-1}(\chi)^{\beta}\gamma_0 \\
&= \sum_{\alpha=1}^{\sigma-1} \ell^{\alpha}(\chi)^{\beta+\sigma-\alpha}\gamma_0 + \ell^{\sigma-1}(\chi)^{\beta}\gamma_0 \\
&= \sum_{\alpha=1}^{\sigma-1} |\gamma_0^{\frac{1}{2}}\chi^{\frac{\beta+\sigma-\alpha}{2}}\ell^{\frac{\alpha}{2}}|^2 + |\gamma_0^{\frac{1}{2}}\ell^{\frac{\sigma-1}{2}}\chi^{\frac{\beta}{2}}|^2 \\
&\preceq \|\gamma_0\| \sum_{\alpha=1}^{\sigma-1} \|\chi\|^{\beta+\sigma-\alpha}\|\ell\|^{\alpha}\Im + \|\ell\|^{\sigma-1}\|\chi\|^{\beta}\|\gamma_0\|\Im \\
&\preceq \frac{\|\gamma_0\|\|\ell\|^{\sigma}\|\chi\|^{\beta+1}}{\|\ell\|-\|\chi\|}\Im + \|\ell\|^{\sigma-1}\|\chi\|^{\beta}\|\gamma_0\|\Im \\
&\to \theta \quad (\beta \to \infty).
\end{aligned}
$$

This implies $\{\varrho_{\mathfrak{q}}\}$ is a Cauchy sequence in $\mathbb{B}$. By the completeness of $(\mho, \mathbb{B}, \Phi)$, we can find $\varrho \in \mho$ satisfying $\lim_{\mathfrak{q}\to\infty} \varrho_{\mathfrak{q}} = \varrho$ and

$$\lim_{\mathfrak{q},\beta\to\infty} \Phi(\varrho_{\mathfrak{q}}, \varrho_{\beta}) = \lim_{\mathfrak{q}\to\infty} \Phi(\varrho_{\mathfrak{q}}, \varrho_{\mathfrak{q}}) = \lim_{\mathfrak{q}\to\infty} \Phi(\varrho_{\mathfrak{q}}, \varrho) = \Phi(\varrho, \varrho) = \theta.$$

So,

$$
\begin{aligned}
\Phi(\mathbb{T}\varrho, \varrho) &\preceq \ell[\Phi(\Lambda\varrho, \Lambda\varrho_{\mathfrak{q}}) + \Phi(\Lambda\varrho_{\mathfrak{q}}, \varrho)] \\
&\preceq \ell[\gamma(\Phi(\Lambda\varrho, \varrho) + \Phi(\Lambda\varrho_{\mathfrak{q}}, \varrho_{\mathfrak{q}}) + \Phi(\Lambda\varrho_{\mathfrak{q}}, \varrho)] \\
&= \ell\gamma(\Phi(\Lambda\varrho, \varrho) + \Phi(\Lambda\varrho_{\mathfrak{q}}, \varrho_{\mathfrak{q}})) + \ell\Phi(\Lambda\varrho_{\mathfrak{q}}, \varrho).
\end{aligned}
$$

This is equivalent to

$$\Phi(\Lambda\varrho, \varrho) \preceq (\Im - \ell\gamma)^{-1}\ell\gamma\Phi(\Lambda\varrho_{\mathfrak{q}}, \Lambda\varrho_{\mathfrak{q}-1}) + (\Im - \ell\gamma)^{-1}\ell\Phi(\Lambda\varrho_{\mathfrak{q}}, \varrho).$$

Thus,

$$
\begin{aligned}
&\|\Phi(\Lambda\varrho, \varrho)\| \leq \|(\Im - \ell\gamma)^{-1}\ell\gamma\|\|\Phi(\Lambda\varrho_{\mathfrak{q}}, \varrho_{\mathfrak{q}})\| + \|(\Im - \ell\gamma)^{-1}\ell\|\|\Phi(\Lambda\varrho_{\mathfrak{q}}, \varrho)\| \\
&\qquad \to 0 \quad (\mathfrak{q} \to \infty).
\end{aligned}
$$

It follows that $\Lambda\varrho = \varrho$. Hence, ϱ is a fixed point of Λ. Let $\xi(\neq \varrho)$ be a other fixed point of Λ, then

$$\theta \preceq \Phi(\varrho, \xi) = \Phi(\Lambda\varrho, \Lambda\xi) \preceq \gamma(\Phi(\Lambda\varrho, \varrho) + \Phi(\Lambda\xi, \xi)) = \theta.$$

Hence, $\varrho = \xi$. □

Example 1. *Let $\mho = [0, 1]$ and $\mathbb{B} = \mathcal{M}_2(\mathbb{C})$ and a mapping $\Phi : \mho \times \mho \to \mathbb{B}$ is defined by*

$$\Phi(\varrho, \xi) = \begin{bmatrix} |\varrho - \xi|^2 & 0 \\ 0 & \mathbb{W}|\varrho - \xi|^2 \end{bmatrix} + \begin{bmatrix} \max\{\varrho, \xi\}^2 & 0 \\ 0 & \mathbb{W}\max\{\varrho, \xi\}^2 \end{bmatrix},$$

where $\mathbb{W} \geq 0$ is a constant. For any $\mathcal{B} \in \mathbb{B}$, we denote its norm as, $||\mathbb{B}|| = \max_{1\leq \mathfrak{i} \leq 4}\{|\mathfrak{a}_{\mathfrak{i}}|\}$. Then, $(\mho, \mathbb{B}, \Phi)$ is a complete $C^{\star}$-AVP_bMS. Define a mapping $\Lambda : \mho \to \mho$ by $\Lambda(\varrho) = \frac{\varrho}{2}$ for all $\varrho \in \mho$. Observe that

$$\Phi(\Lambda\varrho, \Lambda\xi) \preceq \gamma(\Phi(\Lambda\varrho, \varrho) + \Phi(\Lambda\xi, \xi)), \quad \forall \varrho, \xi \in \mho,$$

which satisfies

$$\gamma = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

and $||\gamma|| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} < \frac{1}{2}$. *Therefore, all the postulates of Theorem 3 are fulfilled and* Λ *has the unique fixed point* $\varrho = 0$.

Example 2. *Let* $\mathbb{B} = \mathbb{R}^2$ *and* $\mho = [0, \infty)$. *Let* $\preceq$ *be the partial order on* $\mathbb{B}$ *given by*

$$(\mathfrak{a}_1, \mathfrak{b}_1) \preceq (\mathfrak{a}_2, \mathfrak{b}_2) \Leftrightarrow \mathfrak{a}_1 \leq \mathfrak{a}_2 \text{ and } \mathfrak{b}_1 \leq \mathfrak{b}_2$$

with the norm $||(\mathfrak{a}_1, \mathfrak{b}_1)|| = \max\{|\mathfrak{a}_1|, |\mathfrak{b}_1|\}$. *Define*

$$\Phi_{\mathfrak{b}} : \mho \times \mho \to \mathbb{B},$$

is defined by

$$\Phi_{\mathfrak{b}}(\varrho, \xi) = ((\varrho - \xi)^2, 0) + (\max\{\varrho, \xi\}^2, 0). \tag{1}$$

Then $(\mho, \mathbb{B}, \Phi)$ *is a complete* $C^\star$-AVP_bMS. *Define a mapping* $\Lambda : \mho \to \mho$ *by* $\Lambda(\varrho) = 1 - 2^{-\varrho}$ *for all* $\varrho \in \mho$. *Observe that*

$$\Phi(\Lambda\varrho, \Lambda\xi) \preceq \gamma(\Phi(\Lambda\varrho, \varrho) + \Phi(\Lambda\xi, \xi)), \quad \forall \varrho, \xi \in \mho,$$

which satisfies $\gamma = (\frac{1}{3}, 0)$ *and* $||\gamma|| < \frac{1}{2}$. *Therefore, all the postulates of Theorem 3 are fulfilled and* Λ *has the unique fixed point* $\varrho = 0$.

4. Application

We consider the Fredholm integral equation:

$$\varrho(\pounds) = \int_{\mathcal{B}} \mathcal{Q}(\pounds, \hbar, \varrho(\hbar)) d\hbar + \delta(\pounds), \ \pounds, \hbar \in \mathcal{B}, \tag{2}$$

where $\mathcal{B}$ is a measurable, $\mathcal{Q} : \mathcal{B} \times \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ and $\delta \in \mathcal{L}^\infty(\mathcal{B})$. Let $\mho = \mathcal{L}^\infty(\mathcal{B})$, $\mathcal{W} = \mathcal{L}^2(\mathcal{B})$ and $\mathcal{L}(\mathcal{W}) = \mathbb{B}$. Define a mapping $\rho : \mho \times \mho \to \mathbb{B}$ by

$$\rho(\delta, \mathfrak{w}) = \pi_{|\delta - \mathfrak{w}|^2} + \Im,$$

for all $\delta, \mathfrak{w}, \Im \in \mho$ with $||\lambda|| = \mathfrak{w} < 1$, where $\pi_\flat : \mathcal{W} \to \mathcal{W}$ is the multiplicative operator, defined by

$$\pi_\flat(\psi) = \flat \cdot \psi.$$

Theorem 4. *For all* $\varrho, \xi \in \mho$, *suppose that*

1. $\exists\, \kappa : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ *be a continuous function and* $\mathfrak{w} \in (0, 1)$ *such that*

$$|\mathcal{Q}(\pounds, \hbar, \varrho(\hbar)) - \mathcal{Q}(\pounds, \hbar, \xi(p))| \leq \mathfrak{w}|\kappa(\pounds, \hbar)|(|\int_{\mathcal{B}} \mathcal{Q}(\pounds, \hbar, \varrho(\hbar)) d\hbar + \delta(\pounds) - \xi(\hbar)|$$
$$+ |\int_{\mathcal{B}} \mathcal{Q}(\pounds, \hbar, \xi(\hbar)) d\hbar + \delta(\pounds) - \varrho(\hbar)| + \Im - \mathfrak{w}^{-1}\Im)$$

for all $\pounds, \hbar \in \mathcal{B}$;

2. $\sup_{\pounds \in \mathcal{B}} \int_{\mathcal{B}} |\kappa(\pounds, \hbar)| d\hbar \leq 1$.

Then the integral Equation (2) has a unique solution in $\mho$.

Proof. Define $\Lambda : \mho \to \mho$ by

$$\Lambda\varrho(\pounds) = \int_{\mathcal{B}} \mathcal{Q}(\pounds, \hbar, \varrho(\hbar)) d\hbar + \delta(\pounds), \ \forall \pounds, \hbar \in \mathcal{B}.$$

Set $\lambda = \mathfrak{w}\mathfrak{I}$. Then $\lambda \in \mathbb{B}$. For any $\varkappa \in \mathcal{W}$, we have

$$\begin{aligned}
||\Phi(\Lambda\varrho, \Lambda\xi)|| &= \sup_{||\varkappa||=1} (\pi_{|\Lambda\varrho - \Lambda\xi|^2 + \mathfrak{I}}\varkappa, \varkappa) \\
&= \sup_{||\varkappa||=1} \int_{\mathcal{B}} (|\Lambda\varrho - \Lambda\xi|^2 + \mathfrak{I})\varkappa(\pounds)\overline{\varkappa(\pounds)}d\pounds \\
&\leq \sup_{||z||=1} \int_{\mathcal{B}} \left[\int_{\mathcal{B}} |\mathcal{Q}(\pounds, \hbar, \varrho(\hbar)) - \mathcal{Q}(\pounds, \hbar, \xi(\hbar))| \right]^2 d\hbar |z(\pounds)|^2 d\pounds \\
&+ \sup_{||\varkappa||=1} \int_{\mathcal{B}} \int_{\mathcal{B}} d\hbar |\varkappa(\pounds)|^2 d\pounds \mathfrak{I} \\
&\leq \sup_{||\varkappa||=1} \int_{\mathcal{B}} \left[\int_{\mathcal{B}} \mathfrak{w} |\kappa(\pounds, \hbar)| (| \int_{\mathcal{B}} \mathcal{Q}(\pounds, \hbar, \varrho(\hbar))d\hbar + \delta(\pounds) - \xi(\hbar)| \right. \\
&\left. + | \int_{\mathcal{B}} \mathcal{Q}(\pounds, \hbar, \xi(\hbar))d\hbar + \delta(\pounds) - \varrho(\hbar)| + \mathfrak{I} - \mathfrak{w}^{-1}\mathfrak{I})d\hbar \right]^2 |z(\pounds)|^2 d\pounds + \mathfrak{I} \\
&\leq \mathfrak{w}^2 \sup_{||z||=1} \int_{\mathcal{B}} \left[\int_{\mathcal{B}} |\kappa(\pounds, \hbar)| d\hbar \right]^2 |z(\pounds)|^2 d\pounds (||\Lambda\varrho - \xi||_\infty^2 + ||\Lambda\xi - \varrho||_\infty^2) \\
&\leq \mathfrak{w} \sup_{\pounds \in \mathcal{B}} \int_{\mathcal{B}} |\kappa(\pounds, \hbar)| d\hbar \sup_{||z||=1} \int_{\mathcal{B}} |z(\pounds)|^2 d\pounds (||\Lambda\varrho - \xi||_\infty^2 + ||\Lambda\xi - \varrho||_\infty^2) \\
&\leq \mathfrak{w}[||\Lambda\varrho - \xi||_\infty^2 + ||\Lambda\xi - \varrho||_\infty^2] \\
&= ||\lambda||[||\Phi(\Lambda\varrho, \xi)|| + ||\Phi(\Lambda\xi, \varrho)||].
\end{aligned}$$

Hence all the hypotheses of Theorem 2 are fulfilled, and thus Equation (2) has a unique solution. □

5. Conclusions

In this paper, we presented fixed point theorems for generalized contractions on C^*-AVP_bMS. The examples and applications on C^*-AVP_bMS are presented to strengthen our main results. Samreen et al. [15] proved fixed point theorems on extended b-metric spaces. It is an interesting open problem to prove fixed theorems on $C^\star$-algebra-valued extended partial b-metric spaces. Arabnia Firozjah et al. [16] proved fixed point results on cone b-metric spaces over Banach algebras. Furthermore, it is an interesting open problem to prove fixed theorems on $C^\star$-algebra-valued cone b-metric spaces.

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