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Inverse Sturm–Liouville Problem with Spectral Parameter in the Boundary Conditions

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Abstract: In this paper, for the first time, we study the inverse Sturm–Liouville problem with polynomials of the spectral parameter in the first boundary condition and with entire analytic functions in the second one. For the investigation of this new inverse problem, we develop an approach based on the construction of a special vector functional sequence in a suitable Hilbert space. The uniqueness of recovering the potential and the polynomials of the boundary condition from a part of the spectrum is proved. Furthermore, our main results are applied to the Hochstadt–Lieberman-type problems with polynomial dependence on the spectral parameter not only in the boundary conditions but also in discontinuity (transmission) conditions inside the interval. We prove novel uniqueness theorems, which generalize and improve the previous results in this direction. Note that all the spectral problems in this paper are investigated in the general non-self-adjoint form, and our method does not require the simplicity of the spectrum. Moreover, our method is constructive and can be developed in the future for numerical solution and for the study of solvability and stability of inverse spectral problems.

Keywords: inverse spectral problems; Sturm–Liouville operator; polynomials in the boundary conditions; entire functions in the boundary conditions; uniqueness theorems; half-inverse problems; discontinuity inside the interval

MSC: 34A55; 34B07; 34B09; 34B24; 34L40

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1. Introduction

In this paper, we consider the following boundary value problem $L = L(q, p_1, p_2, f_1, f_2)$:

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (0, \pi), \tag{1}$$

$$p_1(\lambda)y'(0) + p_2(\lambda)y(0) = 0, \quad f_1(\lambda)y'(\pi) + f_2(\lambda)y(\pi) = 0, \tag{2}$$

where (1) is the Sturm–Liouville equation with the complex-valued potential $q \in L_2(0, \pi)$, λ is the spectral parameter, the boundary condition (BC) (2) at $x = 0$ contains relatively prime polynomials $p_j(\lambda)$, $j = 1, 2$, and the BC at $x = \pi$, arbitrary functions $f_j(\lambda)$, $j = 1, 2$, which are analytical in the whole λ-plane.

This paper aims to study the inverse spectral problem that consists in the recovery of the potential $q(x)$ and the polynomials $p_1(\lambda)$ and $p_2(\lambda)$ from some part of the problem L's spectrum. Inverse spectral theory for the Sturm–Liouville operators with *constant* coefficients in the boundary conditions has been developed fairly completely (see the monographs [1–4]). There is also a number of studies concerning eigenvalue problems with *polynomial* dependence on the spectral parameter in the BCs. Such problems arise in various physical applications, e.g., in mechanical engineering [5], in flow dust acoustics [6], in heat conduction, diffusion, and electric circuit problems (see [7,8] and references therein). The theory of *direct* spectral problems for general classes of differential operators depending nonlinearly on the spectral parameter can be found in [9–11].

Inverse Sturm–Liouville problems with polynomials in the BCs have been studied in [12–25], and other papers. We mention that there is a large number of research works on the Sturm–Liouville problems with linear or quadratic dependence on the spectral parameter (see, e.g., [26–28]). However, in this paper, we mostly focus on the bibliography concerning the inverse Sturm–Liouville problems with polynomials of arbitrary degrees in the BCs. The majority of the studies in this direction deal with self-adjoint problems containing rational Herglotz–Nevanlinna functions of the spectral parameter in the BCs (see, e.g., [13,14,21,23–25,29]). It is easy to check that the BCs of that type can be reduced to the form with polynomial dependence on the spectral parameter. A constructive solution of the inverse Sturm–Liouville problem on a finite interval with the polynomial BCs in the general non-self-adjoint form has been obtained by Freiling and Yurko [17] by using the method of spectral mappings. The case of the half-line was considered in [18]. In recent years, significant progress for the self-adjoint Sturm–Liouville inverse problems with Herglotz–Nevanlinna functions of λ in the BCs has been achieved by Guliyev (see [23–25,29]), who obtained the spectral data characterization for regular potentials of class $L_2(0, \pi)$, as well as for singular potentials of class $W_2^{-1}(0, \pi)$.

Recently, a new class of the inverse Sturm–Liouville problems with entire analytic functions in one of the BCs has started to be investigated (see [30–34]). Such problems cause interest in connection with the so-called *partial* inverse problems, which consist in the recovery of the differential expression coefficients (e.g., of the Sturm–Liouville potentials) on a part of an interval or a geometrical graph from the spectral data, while the coefficients on the remaining part are known a priori. Naturally, partial inverse problems require less spectral data than the complete ones. In particular, Hochstadt and Lieberman [35] have proved that, if the potential $q(x)$ is known on the half of the interval $(0, \pi)$, then $q(x)$ on the other half is uniquely specified by one spectrum. In general, due to the classical result by Borg [36], two spectra are required for the unique reconstruction of the potential. The Hochstadt–Lieberman-type problems for the Sturm–Liouville operators with polynomial BCs also attract the attention of scholars. For some special cases, such problems were considered in [19,22,24].

In the mentioned papers [30–34], a unified approach has been developed for a variety of partial inverse problems. That approach consists in the reduction in a partial inverse problem to the Sturm–Liouville inverse problem with entire functions in the BC. The idea of that method appeared from the investigation of partial inverse problems on metric graphs [37] and of the inverse transmission eigenvalue problem [38]. Later on, that approach was transferred to the discrete Jacobi systems (see [39]). We also mention that the Sturm–Liouville inverse problems with entire functions in the BC considered in [30–32] are closely related to the problem of the recovery of the potential from the values of the Weyl function at a countable set of points (see [40,41]).

This paper is concerned with the development of the inverse spectral theory for the Sturm–Liouville problem L of form (1)-(2), with polynomial dependence on λ in one of the BCs, and with analytical dependence in the other one. To the best of the authors' knowledge, inverse problems for L have not been considered before. For the investigation of this new inverse problem, we develop an approach based on the construction of a special vector functional sequence $\{v_n\}_{n=1}^{\infty}$ in a suitable Hilbert space. We prove that the completeness of $\{v_n\}_{n=1}^{\infty}$ is sufficient for the uniqueness of the inverse problem solution. Our approach relies on the ideas of [30] and on some results of [16,17] for the inverse problems with only polynomial BCs. Note that we consider the problem L in the general non-self-adjoint form, and our method does not require the simplicity of the spectrum. Moreover, our method is constructive and can be developed in the future for numerical solution and for the study of solvability and stability of inverse spectral problems.

Furthermore, we apply our main results to the Hochstadt–Lieberman-type problems, with polynomial dependence on the spectral parameter not only in BCs but also in discontinuity (transmission) conditions inside the interval. The developed approach allows us to investigate various cases in the same way. We prove the uniqueness theorems which generalize and improve the results of [19,22,24] for the case of polynomials contained only in BCs. In particular, we show that, in some cases, a part of the eigenvalues can be excluded, and the remaining subspectrum is still sufficient for the uniqueness. For the case of polynomials in the discontinuity conditions, our problem statement is novel, and the obtained results are the first ones in this direction.

It is worth mentioning that eigenvalue problems with discontinuity conditions depending on the spectral parameter have attracted the interest of mathematicians in recent years. Bartels et al. [42,43] obtained the Hilbert space formulation and the eigenvalue asymptotics for the Sturm–Liouville problems with Herglotz–Nevanlinna functions of λ in the discontinuity conditions arising in microelectronics. Some issues of inverse spectral theory for differential operators with linear dependence on the spectral parameter in the discontinuity conditions were considered in [44,45]. Polynomials of higher degree in the discontinuity conditions appear in the study of the inverse Sturm–Liouville problems on time scales (see [46,47]). However, there are only fragmentary results for boundary value problems with polynomials of λ in the discontinuity conditions, and the general inverse spectral theory of such problems has not been created yet. The methods of this paper may be useful for future research in this direction. In addition, we point out that spectral problems with differential expression coefficients depending on the eigenparameter also arise in applications. In particular, a problem of this kind appeared in the recent study [48] of the full-waveform inversion with frequency-dependent offset-preconditioning, having applications in exploration geophysics. From the inverse spectral theory viewpoint, boundary value problems' eigenparameter dependence in equation coefficients are different from the ones considered in this paper and so require a separate investigation.

The paper is organized as follows. In Section 2, the inverse problem statements and the main results are formulated. In Section 3, we prove the uniqueness theorem and provide a constructive algorithm for solving the inverse problem for L. In Section 4, we obtain the sufficient conditions of uniqueness, which are convenient for applications. In Section 5, the main results are applied to the Hochstadt–Lieberman-type problems.

2. Main Results

Consider the boundary value problem $L = L(q, p_1, p_2, f_1, f_2)$ of form (1)-(2). The spectrum of the problem L consists of the eigenvalues being the zeros of some analytic entire function which depends on $f_1(\lambda)$ and $f_2(\lambda)$. Therefore, we cannot say anything specific about the behavior of the spectrum. However, we can consider the reconstruction of the potential from some countable subset of the spectrum $\{\lambda_n\}_{n=1}^{\infty}$ and obtain sufficient conditions on the subspectrum $\{\lambda_n\}_{n=1}^{\infty}$ for the unique solvability of the inverse problem.

The polynomials $p_1(\lambda)$ and $p_2(\lambda)$ can be represented in the form

$$p_1(\lambda) = \sum_{n=0}^{N_1} a_n \lambda^n, \quad p_2(\lambda) = \sum_{n=0}^{N_2} b_n \lambda^n, \quad a_{N_1} \neq 0, \quad b_{N_2} \neq 0, \quad N_1, N_2 \geq 0. \tag{3}$$

Here, we exclude the case of the Dirichlet BC $y(0) = 0$, that is, $p_1(\lambda) \equiv 0$, $p_2(\lambda) \equiv 1$, since this case has been studied in [30]. Without loss of generality, we assume that $a_{N_1} = 1$ if $N_1 \geq N_2$ and $b_{N_2} = 1$ if $N_2 > N_1$. Introduce the notations

$$\omega = \frac{1}{2}\int_0^{\pi} q(t)\,dt, \quad \varpi = \begin{cases} \omega - b_{N_1}, & N_1 = N_2 \\ \omega + a_{N_1}, & N_1 = N_2 - 1 \\ \omega, & \textit{otherwise}. \end{cases} \tag{4}$$

In this paper, we consider the following inverse problem.

Problem 1. *Suppose that the degrees N_1, N_2 of the polynomials and functions $f_1(\lambda)$, $f_2(\lambda)$ are known a priori. Given a subspectrum $\{\lambda_n\}_{n=1}^{\infty}$ of the problem L and the number ϖ, find the potential $q(x)$ and the polynomials $p_1(\lambda)$, $p_2(\lambda)$.*

The subspectrum $\{\lambda_n\}_{n=1}^{\infty}$ can contain multiple eigenvalues of finite multiplicities. Note that, in the applications to the Hochstadt–Lieberman-type problems, the constant ϖ usually can be found from the eigenvalue asymptotics.

For investigating Problem 1, we construct the special sequence of vector functions $\{v_n\}_{n=1}^{\infty}$ in the Hilbert space

$$\mathcal{H}_K = L_2(0,\pi) \oplus L_2(0,\pi) \oplus \underbrace{\mathbb{C} \oplus \mathbb{C} \oplus \dots \oplus \mathbb{C}}_{K},$$

where $K = \max\{2N_1, 2N_2 - 1\}$. The construction of the sequence $\{v_n\}_{n=1}^{\infty}$ is different for $N_1 \ge N_2$ and $N_1 < N_2$ and, moreover, is technically complicated (see Formulas (22), (26), and (29)), so we do not provide it here. It is important to note that $\{v_n\}_{n=1}^{\infty}$ are constructed by using only the given data of Problem 1, that is, N_j, $f_j(\lambda)$, $j = 1, 2$, $\{\lambda_n\}_{n=1}^{\infty}$, and ϖ.

Along with $L = L(q, p_1, p_2, f_1, f_2)$, we consider the problem $\tilde{L} = L(\tilde{q}, \tilde{p}_1, \tilde{p}_2, \tilde{f}_1, \tilde{f}_2)$ of the same form (1)-(2) but with different coefficients. We agree that, if a symbol γ denotes an object related to L, then the symbol $\tilde{\gamma}$ with tilde denotes the analogous object related to $\tilde{L}$. One of the main results of this paper is the following uniqueness theorem for Problem 1.

Theorem 1. *Let $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\tilde{\lambda}_n\}_{n=1}^{\infty}$ be subspectra of the problems L and $\tilde{L}$, respectively. Suppose that the sequence $\{v_n\}_{n=1}^{\infty}$ constructed for the problem L and its subspectrum $\{\lambda_n\}_{n=1}^{\infty}$ by formulas (22), (26), and (29) is complete in $L_2(0,\pi)$, and let $N_j = \tilde{N}_j$, $f_j(\lambda) \equiv \tilde{f}_j(\lambda)$, $j = 1, 2$, $\lambda_n = \tilde{\lambda}_n$, $n \ge 1$, $\varpi = \tilde{\varpi}$. Then $q = \tilde{q}$ in $L_2(0,\pi)$ and $p_j(\lambda) \equiv \tilde{p}_j(\lambda)$, $j = 1, 2$.*

For the case when the sequence $\{v_n\}_{n=1}^{\infty}$ is a Riesz basis in $\mathcal{H}_K$, we provide a constructive algorithm for solving Problem 1 (see Algorithm 1).

Since the sequence $\{v_n\}_{n=1}^{\infty}$ has a complex structure, it is important to find such sufficient conditions of its completeness that are (i) easy for checking and (ii) natural for applications. Such conditions are provided in the next theorem. For clarity, here, we formulate the result for the case of simple eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$. For multiple eigenvalues, the analogous theorem is provided in Section 4.

Theorem 2. *Suppose that the eigenvalues of the subspectrum $\{\lambda_n\}_{n=1}^{\infty}$ are simple, $f_1(\lambda_n) \ne 0$ or $f_2(\lambda_n) \ne 0$ for every $n \ge 1$, and the system $\{\cos \sqrt{\lambda_n} t\}_{n=\max\{2N_1+1, 2N_2\}}^{\infty}$ is complete in $L_2(0, 2\pi)$. Then, the system $\{v_n\}_{n=1}^{\infty}$ is complete in $\mathcal{H}_K$.*

Our next goal is to study the uniqueness of solution for the Hochstadt–Lieberman-type problems with polynomials of λ in the BCs. Consider the following boundary value problem $\mathscr{L} = \mathscr{L}(q, p_1, p_2, r_1, r_2)$:

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (0, 2\pi), \tag{5}$$

$$p_1(\lambda)y'(0) + p_2(\lambda)y(0) = 0, \quad r_1(\lambda)y'(2\pi) + r_2(\lambda)y(2\pi) = 0, \tag{6}$$

where $q(x)$ is the complex-valued potential of class $L_2(0, 2\pi)$, the BC at $x = 0$ contains relatively prime polynomials $p_j(\lambda)$, $j = 1, 2$, and the BC at $x = 2\pi$ contains relatively prime polynomials $r_j(\lambda)$, $j = 1, 2$. The polynomials $p_1(\lambda)$, $p_2(\lambda)$ can be represented in the form (3) and the polynomials $r_1(\lambda)$, $r_2(\lambda)$, in the following analogous form:

$$r_1(\lambda) = \sum_{n=0}^{M_1} c_n \lambda^n, \quad r_2(\lambda) = \sum_{n=0}^{M_2} d_n \lambda^n, \quad c_{M_1} \neq 0, \quad d_{M_2} \neq 0, \quad M_1, M_2 \geq 0, \tag{7}$$

Without loss of generality, we assume that $a_{N_1} = 1$ if $N_1 \geq N_2$, $b_{N_2} = 1$ if $N_1 < N_2$, $c_{M_1} = 1$ if $M_1 \geq M_2$, $d_{M_2} = 1$ if $M_1 < M_2$.

The spectrum of $\mathscr{L}$ is a countable set of eigenvalues, which are asymptotically simple (see [17]), but a finite number of eigenvalues can be multiple. Let us denote the eigenvalues of $\mathscr{L}$ by $\{\mu_n\}_{n=1}^{\infty}$ (counting with multiplicities), and formulate the Hochstadt–Lieberman-type problem.

Problem 2. *Suppose that the degrees N_1, N_2 of the polynomials $p_j(\lambda)$, $j = 1, 2$, the polynomials $r_j(\lambda)$, $j = 1, 2$, and the potential $q(x)$ for $x \in (\pi, 2\pi)$ are known a priori. Given a subspectrum $\{\mu_n\}$ of the problem $\mathscr{L}$, find the potential $q(x)$ for $x \in (0, \pi)$ and the polynomials $p_1(\lambda)$, $p_2(\lambda)$.*

By reducing Problem 2 to Problem 1, we prove the following uniqueness theorem.

Theorem 3. *Let $\{\mu_n\}_{n=1}^{\infty}$ and $\{\tilde{\mu}_n\}_{n=1}^{\infty}$ be the spectra of the problems $\mathscr{L} = \mathscr{L}(q, p_1, p_2, r_1, r_2)$ and $\tilde{\mathscr{L}} = \mathscr{L}(\tilde{q}, \tilde{p}_1, \tilde{p}_2, \tilde{r}_1, \tilde{r}_2)$, respectively. Assume that $N_j = \tilde{N}_j$, $r_j(\lambda) \equiv \tilde{r}_j(\lambda)$, $j = 1, 2$, and $q(x) = \tilde{q}(x)$ a.e. on $(\pi, 2\pi)$. Additionally, impose the following assumptions.*

- *In the case $N_1 \geq N_2$, $M_1 \geq M_2$, suppose that $M_1 \geq N_1$ and $\mu_n = \tilde{\mu}_n$ for all $n \geq M_1 - N_1 + 1$.*
- *In the case $N_1 < N_2$, $M_1 \geq M_2$, suppose that $M_1 \geq N_2 - 1$ and $\mu_n = \tilde{\mu}_n$ for all $n \geq M_1 - N_2 + 2$.*
- *In the case $N_1 \geq N_2$, $M_1 < M_2$, suppose that $M_2 \geq N_1$ and $\mu_n = \tilde{\mu}_n$ for all $n \geq M_2 - N_1 + 1$.*
- *In the case $N_1 < N_2$, $M_1 < M_2$, suppose that $M_2 \geq N_2$ and $\mu_n = \tilde{\mu}_n$ for all $n \geq M_2 - N_2 + 1$.*

  Then, $q(x) = \tilde{q}(x)$ a.e. on $(0, \pi)$ and $p_j(\lambda) \equiv \tilde{p}_j(\lambda)$, $j = 1, 2$.

Theorem 3 provides sufficient conditions for the uniqueness of solution of Problem 2. For instance, in the first case $N_1 \geq N_2$, $M_1 \geq M_2$, the potential $q(x)$ on $(0, \pi)$ and the polynomials $p_1(\lambda)$, $p_2(\lambda)$ are uniquely specified by the subspectrum $\{\mu_n\}_{n \geq M_1 - N_1 + 1}$ if $M_1 \geq N_1$. The numbering of the eigenvalues $\{\mu_n\}_{n=1}^{\infty}$ is not uniquely fixed, so if $M_1 > N_1$, then any $(M_1 - N_1)$ eigenvalues can be excluded (taking the multiplicities into account).

In order to prove Theorem 3, we analyze the asymptotics of the eigenvalues $\{\mu_n\}$ and conclude that, for the chosen subspectrum in each case, the conditions of Theorem 2 are fulfilled. Applying Theorem 4 and then Theorem 1, we arrive at the assertion of Theorem 3.

Theorem 3 generalizes the previously known results of [19,24] on the Hochstadt–Lieberman-type problems with polynomial BCs. Namely, in [19], the uniqueness theorem has been proved for the case $N_1 = N_2$, $M_1 = M_2$ and, in [24], for the case $N_2 = N_1 + 1$, $M_2 = M_1 + 1$ under an additional restriction of the self-adjointness. Moreover, the authors of [19,24] use the whole spectrum for the reconstruction, even if $M_1 > N_1$ and $M_2 > N_2$, respectively, while our Theorem 3 shows that a finite number of eigenvalues can be removed.

Furthermore, we show that our approach can be applied to the following boundary value problem $\mathcal{L} = \mathcal{L}(q, p_1, p_2, r_1, r_2, p_{ij})$, which contains polynomials of λ not only in the BCs but also in the discontinuity conditions inside the interval:

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (0, \pi) \cup (\pi, 2\pi), \tag{8}$$

$$p_1(\lambda)y'(0) + p_2(\lambda)y(0) = 0, \quad r_1(\lambda)y'(2\pi) + r_2(\lambda)y(2\pi) = 0, \tag{9}$$

$$p_{1j}(\lambda)y^{(j)}(\pi - 0) = p_{2j}(\lambda)y(\pi + 0) + p_{3j}(\lambda)y'(\pi + 0), \quad j = 0,1. \tag{10}$$

Obviously, the problem $\mathscr{L}(q, p_1, p_2, r_1, r_2)$ is the special case of the problem $\mathcal{L}$ with $p_{10}(\lambda) \equiv p_{20}(\lambda) \equiv p_{11}(\lambda) \equiv p_{31}(\lambda) \equiv 1$, $p_{30}(\lambda) \equiv p_{21}(\lambda) \equiv 0$. So, we similarly denote the eigenvalues of $\mathcal{L}$ by $\{\mu_n\}_{n=1}^{\infty}$ (counting with multiplicities) and study the following Hochstadt–Lieberman-type problem.

Problem 3. *Suppose that the degrees N_1, N_2 of the polynomials $p_j(\lambda)$, the polynomials $r_j(\lambda)$, $j = 1,2$, the polynomials $p_{ij}(\lambda)$, $i = \overline{1,3}$, $j = 0,1$, and the potential $q(x)$ for $x \in (\pi, 2\pi)$ are known a priori. Given a subspectrum $\{\mu_n\}$ of the problem $\mathcal{L}$, find the potential $q(x)$ for $x \in (0, \pi)$ and the polynomials $p_1(\lambda)$, $p_2(\lambda)$.*

In Section 5, we prove the uniqueness theorem (Theorem 5) for the solution of Problem 3.

Throughout the paper, we use the following **notations**:

1. $\lambda = \rho^2$, $\tau := \operatorname{Im} \rho$, $\rho_n = \sqrt{\lambda_n}$, $\arg \rho_n \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
2. Denote by B_a^+ the class of entire functions $F(\rho)$ satisfying the conditions $F(\rho) = O(\exp(|\tau|a))$ in $\mathbb{C}$, $F \in L_2(\mathbb{R})$, and $F(\rho) = F(-\rho)$. Thus, B_a^+ is the class of even Paley–Wiener functions, which can be represented as $F(\rho) = \int\limits_0^a f(t) \cos \rho t \, dt$, $f \in L_2(0, a)$.

3. Proof of the Main Theorem

The goal of this section is to prove Theorem 1 on the uniqueness of solution for Problem 1. We begin with some preliminaries.

Let us define the functions $S(x, \lambda)$ and $C(x, \lambda)$ as the solutions of equation (1) satisfying the initial conditions: $S(0, \lambda) = 0$, $S'(0, \lambda) = 1$, $C(0, \lambda) = 1$, $C'(0, \lambda) = 0$. It can be easily seen that the eigenvalues of the problem L coincide with the zeros of the entire characteristic function

$$\Delta(\lambda) = f_1(\lambda)\Delta_1(\lambda) + f_2(\lambda)\Delta_0(\lambda), \tag{11}$$

where

$$\Delta_j(\lambda) = p_1(\lambda)C^{(j)}(\pi, \lambda) - p_2(\lambda)S^{(j)}(\pi, \lambda), \quad j = 0,1. \tag{12}$$

It is worth noting that, for $j = 0,1$, the zeros of the function $\Delta_j(\lambda)$ coincide with the eigenvalues of the corresponding boundary value problem L_j for equation (1) with the BCs

$$p_1(\lambda)y'(0) + p_2(\lambda)y(0) = 0, \quad y^{(j)}(\pi) = 0.$$

In order to prove the main result, we need the following technical lemma.

Lemma 1. *The functions $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$ can be represented as follows.*

In ***the first case*** *$N_1 \geq N_2$:*

$$\Delta_1(\lambda) = -\rho^{2N_1+1} \sin \rho\pi + \varpi \rho^{2N_1} \cos \rho\pi + \rho^{2N_1} \int_0^{\pi} \mathcal{G}(t) \cos \rho t \, dt + \sum_{j=1}^{N_1} C_j \rho^{2j-2}, \tag{13}$$

$$\Delta_0(\lambda) = \rho^{2N_1} \cos \rho\pi + \varpi \rho^{2N_1-1} \sin \rho\pi + \rho^{2N_1-1} \int_0^{\pi} \mathcal{Q}(t) \sin \rho t \, dt + \sum_{j=1}^{N_1} D_j \rho^{2j-2}. \tag{14}$$

In ***the second case*** *$N_2 > N_1$:*

$$\Delta_1(\lambda) = -\rho^{2N_2} \cos \rho\pi - \varpi\rho^{2N_2-1} \sin \rho\pi + \rho^{2N_2-1} \int_0^{\pi} \mathcal{G}(t) \sin \rho t \, dt + \sum_{j=1}^{N_2} C_j \rho^{2j-2}, \tag{15}$$

$$\Delta_0(\lambda) = -\rho^{2N_2-1} \sin \rho\pi + \varpi\rho^{2N_2-2} \cos \rho\pi + +\rho^{2N_2-2} \int_0^{\pi} \mathcal{Q}(t) \cos \rho t \, dt + \sum_{j=1}^{N_2-1} D_j \rho^{2j-2}. \tag{16}$$

In both cases, $\mathcal{G}$ and $\mathcal{Q}$ are some functions of $L_2(0,\pi)$, and C_j, D_j are constants.

Proof. The solutions $S(x,\lambda)$ and $C(x,\lambda)$ admit the following representations in terms of transformation operators (see, e.g., [1]):

$$S(x,\lambda) = \frac{\sin \rho x}{\rho} + \int_0^x K(x,t) \frac{\sin \rho t}{\rho} \, dt,$$
$$C(x,\lambda) = \cos \rho x + \int_0^x P(x,t) \cos \rho t \, dt,$$

where $K(x,x) = P(x,x) = \frac{1}{2} \int_0^x q(\xi) \, d\xi$. Using these representations, we obtain the following standard relations for $S(\pi,\lambda)$, $S'(\pi,\lambda)$, $C(\pi,\lambda)$, and $C'(\pi,\lambda)$:

$$\begin{cases} S(\pi,\lambda) = \dfrac{\sin \rho\pi}{\rho} - \dfrac{\omega \cos \rho\pi}{\lambda} + \dfrac{1}{\lambda} \displaystyle\int_0^{\pi} \mathcal{K}(t) \cos \rho t \, dt, \\ S'(\pi,\lambda) = \cos \rho\pi + \dfrac{\omega \sin \rho\pi}{\rho} + \dfrac{1}{\rho} \displaystyle\int_0^{\pi} \mathcal{N}(t) \sin \rho t \, dt, \\ C(\pi,\lambda) = \cos \rho\pi + \dfrac{\omega \sin \rho\pi}{\rho} + \dfrac{1}{\rho} \displaystyle\int_0^{\pi} \mathcal{M}(t) \sin \rho t \, dt, \\ C'(\pi,\lambda) = -\rho \sin \rho\pi + \omega \cos \rho\pi + \displaystyle\int_0^{\pi} \mathcal{P}(t) \cos \rho t \, dt, \end{cases} \tag{17}$$

where $\mathcal{K}(t), \mathcal{N}(t), \mathcal{M}(t), \mathcal{P}(t) \in L_2(0,\pi)$.

The relations (13)–(16) are obtained by substitution of (3) and (17) into (12). For definiteness, let us derive the relation (13) for $\Delta_1(\lambda)$ in the case $N_1 \geq N_2$. Substituting (3) and (17) into (12) for $j = 1$, we obtain

$$\Delta_1(\lambda) = \sum_{n=0}^{N_1} a_n \lambda^n \Big(-\rho \sin \rho\pi + \omega \cos \rho\pi + \int_0^{\pi} \mathcal{P}(t) \cos \rho t \, dt \Big) -$$
$$- \sum_{n=0}^{N_2} b_n \lambda^n \Big(\cos \rho\pi + \frac{\omega \sin \rho\pi}{\rho} + \frac{1}{\rho} \int_0^{\pi} \mathcal{N}(t) \sin \rho t \, dt \Big).$$

This expression can be easily converted to the form

$$\Delta_1(\lambda) = -\rho^{2N_1+1} \sin \rho\pi + \varpi\rho^{2N_1} \cos \rho\pi + \rho^{2N_1} \Big(\int_0^{\pi} \mathcal{P}(t) \cos \rho t \, dt + F_1(\rho) \Big), \tag{18}$$

where

$$F_1(\rho) = \sum_{n=0}^{N_1-1} a_n \rho^{-2(N_1-n)} \Big(-\rho \sin \rho\pi + \omega \cos \rho\pi + \int_0^{\pi} \mathcal{P}(t) \cos \rho t \, dt \Big)$$
$$- \sum_{n=0}^{N_2} b_n \rho^{-2(N_1-n)} \Big(\cos \rho\pi + \frac{\omega \sin \rho\pi}{\rho} + \frac{1}{\rho} \int_0^{\pi} \mathcal{N}(t) \sin \rho t \, dt \Big) + (\omega - \varpi) \cos \rho\pi.$$

Obviously, the function $F_1(\rho)$ is even and fulfills the estimate

$$|F_1(\rho)| \leq \frac{C \exp(|\tau|\pi)}{|\rho|}, \quad |\rho| \geq \rho^*. \tag{19}$$

Furthermore, $F(\rho)$ has a pole of order of at most $2N_1$ at $\rho = 0$, so the Laurent series has the form

$$F_1(\rho) = \frac{C_1}{\rho^{2N_1}} + \frac{C_2}{\rho^{2N_1-2}} + \ldots + \frac{C_{N_1-1}}{\rho^4} + \frac{C_{N_1}}{\rho^2} + F_2(\rho), \tag{20}$$

where $F_2(\rho)$ is an even entire function. It follows from (19) and (20) that $F_2(\rho)$ satisfies the same estimate as (19). Hence, $F_2(\rho) \in L_2(-\infty, +\infty)$, so $F_2(\rho) \in B^+_\pi$ and can be represented in the form $F_2(\rho) = \int_0^\pi \mathcal{S}(t) \cos \rho t \, dt$, where $\mathcal{S}(t) \in L_2(0, \pi)$. Substituting this equality into (20) and (18), we arrive at the relation (13) with $\mathcal{G}(t) = \mathcal{S}(t) + \mathcal{P}(t)$. $\square$

Consider the Hilbert space

$$\mathcal{H}_K = L_2(0, \pi) \oplus L_2(0, \pi) \oplus \underbrace{\mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{K}$$

of elements

$$h = [H_1, H_2, h_1, ..., h_K], \quad H_1, H_2 \in L_2(0, \pi), \quad h_j \in \mathbb{C}, \; j = \overline{1, K}.$$

The scalar product and the norm in $\mathcal{H}_K$ are defined as follows:

$$(g, h) = \int_0^\pi (\overline{G_1(t)} H_1(t) + \overline{G_2(t)} H_2(t)) \, dt + \sum_{j=1}^K \overline{g_j} h_j, \quad \|h\| = \sqrt{(h, h)},$$

where

$$g = [G_1, G_2, g_1, ..., g_K], \quad h = [H_1, H_2, h_1, ..., h_K].$$

Consider some countable set of eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of the problem L. Suppose that the sequence $\{\lambda_n\}_{n=1}^\infty$ may contain multiple values of finite multiplicities. Introduce the set $I = \{n \geq 1 : \lambda_n \neq \lambda_k, k = \overline{1, n}\}$ and the number $m_k = \#\{l \geq 1 : \lambda_l = \lambda_k\}$. Thus, I is the index set of all the distinct numbers in the sequence $\{\lambda_n\}_{n=1}^\infty$, and m_k is the multiplicity of λ_k in this sequence. Due to these notations, λ_k is the zero of the characteristic function $\Delta(\lambda)$ of multiplicity at least m_k.

Our next goal is to define the sequence $\{v_n\}_{n=1}^\infty$ by using N_j, $f_j(\lambda)$, $j = 1, 2$, $\{\lambda_n\}_{n=1}^\infty$, and ϖ. Consider the two cases.

The first case: $N_1 \geq N_2$.

In this case, put $K = 2N_1$. Define the vector functions

$$u(t) = [\overline{\mathcal{G}(t)}, \overline{\mathcal{Q}(t)}, \overline{C}_{N_1}, \ldots, \overline{C}_1, \overline{D}_{N_1}, ..., \overline{D}_1], \tag{21}$$

$$v(t, \lambda) = [f_1(\lambda)\rho^{2N_1} \cos \rho t, f_2(\lambda)\rho^{2N_1-1} \sin \rho t, f_1(\lambda)\rho^{2N_1-2}, \ldots, f_1(\lambda), f_2(\lambda)\rho^{2N_1-2}, \ldots, f_2(\lambda)], \tag{22}$$

and find their scalar product in $\mathcal{H}_K$:

$$\begin{gathered}(u(t), v(t, \lambda)) = f_1(\lambda)\rho^{2N_1} \int_0^\pi \mathcal{G}(t) \cos \rho t \, dt + f_2(\lambda)\rho^{2N_1-1} \int_0^\pi \mathcal{Q}(t) \sin \rho t \, dt + \\ + C_{N_1} f_1(\lambda)\rho^{2N_1-2} + \cdots + C_1 f_1(\lambda) + D_{N_1} f_2(\lambda)\rho^{2N_1-2} + \cdots + D_1 f_2(\lambda).\end{gathered}$$

According to (11), (13), and (14), we can conclude that

$$(u(t), v(t, \lambda)) = \Delta(\lambda) + w(\lambda), \tag{23}$$

where

$$w(\lambda) = \frac{f_1(\lambda)\lambda^{N_1+1} \sin \rho\pi}{\rho} - f_1(\lambda)\lambda^{N_1} \varpi \cos \rho\pi - f_2(\lambda)\lambda^{N_1} \cos \rho\pi - \frac{f_2(\lambda)\lambda^{N_1} \varpi \sin \rho\pi}{\rho}. \tag{24}$$

The second case: $N_2 > N_1$.

In this case, put $K = 2N_2 - 1$. Define the vector functions

$$u(t) = [\overline{\mathcal{G}(t)}, \overline{\mathcal{Q}(t)}, \overline{C}_{N_2}, \dots, \overline{C}_1, \overline{D}_{N_2-1}, ..., \overline{D}_1], \tag{25}$$

$$v(t,\lambda) = [f_1(\lambda)\rho^{2N_2-1}\sin\rho t, f_2(\lambda)\rho^{2N_2-2}\cos\rho t, f_1(\lambda)\rho^{2N_2-2}, \dots, f_1(\lambda), f_2(\lambda)\rho^{2N_2-4}, \dots, f_2(\lambda)], \tag{26}$$

Finding their scalar product in $\mathcal{H}_K$ and using (11), (15), (16), we conclude that

$$(u(t), v(t,\lambda)) = \Delta(\lambda) + w(\lambda), \tag{27}$$

where

$$w(\lambda) = f_1(\lambda)\lambda^{N_2}\cos\rho\pi + \frac{f_1(\lambda)\lambda^{N_2}\varpi\sin\rho\pi}{\rho} + \frac{f_2(\lambda)\lambda^{N_2}\sin\rho\pi}{\rho} - f_2(\lambda)\lambda^{N_2-1}\varpi\cos\rho\pi. \tag{28}$$

Introduce the notation

$$f^{<n>}(\lambda) = \frac{d^n f}{d\lambda^n}, \quad n \ge 0.$$

Since λ_k is the zero of $\Delta(\lambda)$ of multiplicity at least m_k, we have

$$\Delta^{<n>}(\lambda_k) = 0, \quad k \in I, \quad n = \overline{0, m_k - 1}.$$

Consequently, it follows from (23) and (27) that

$$(u(t), v^{<n>}(t,\lambda_k))_{\mathcal{H}} = w^{<n>}(\lambda_k), \quad k \in I, \quad n = \overline{0, m_k - 1},$$

in the both cases.

Put

$$v_{k+n}(t) = v^{<n>}(t,\lambda_k), \quad w_{k+n}(t) = w^{<n>}(t,\lambda_k), \quad k \in I, \quad n = \overline{0, m_k - 1}. \tag{29}$$

Thus, we defined the sequence $\{v_n\}_{n=1}^{\infty}$ in $\mathcal{H}_K$ and the sequence of complex numbers $\{w_n\}_{n=1}^{\infty}$. Using (23) and (27), we arrive at the relation

$$(u, v_n) = w_n, \quad n \ge 1, \tag{30}$$

which plays a crucial role in the investigation of the inverse problem. Here, $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ are constructed by using the known data of Problem 1, while $u \in \mathcal{H}_K$ is related to the unknown potential $q(x)$ and the polynomials $p_1(\lambda)$, $p_2(\lambda)$.

In order to prove Theorem 1, we use the relation (30) to reduce Problem 1 to the problem studied in [16]. Define the Weyl function $M(\lambda) := \dfrac{\Delta_0(\lambda)}{\Delta_1(\lambda)}$ of the boundary value problem L_1 and consider the following auxiliary inverse problem.

Problem 4. *Given the Weyl function $M(\lambda)$, find $q(x)$, $p_1(\lambda)$, and $p_2(\lambda)$.*

The uniqueness of solution for Problem 4 has been proved by Chernozhukova and Freiling [16]. We formulate the uniqueness result in the following proposition.

Proposition 1. *If $M(\lambda) \equiv \tilde{M}(\lambda)$, then $q(x) \equiv \tilde{q}(x)$ a.e. on $(0,\pi)$ and $p_j(\lambda) \equiv \tilde{p}_j(\lambda)$, $j = 1, 2$.*

Now, we are ready to prove the uniqueness theorem for Problem 1.

Proof of Theorem 1. Suppose that two boundary value problems L and $\tilde{L}$ of form (1)-(2) and their subspectra $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\tilde{\lambda}_n\}_{n=1}^{\infty}$ fulfill the conditions of Theorem 1. By construction, we have $v_n = \tilde{v}_n$ in the Hilbert space $\mathcal{H}_K$ and $w_n = \tilde{w}_n$ for all $n \ge 1$. Then, for the problem $\tilde{L}$, we obtain $(\tilde{u}, v_n) = w_n$, $n \ge 1$. Therefore, $(u - \tilde{u}, v_n)_{\mathcal{H}} = 0$, $n \ge 1$.

Due to the completeness of the sequence $\{v_n\}_{n=1}^{\infty}$ in $\mathcal{H}_K$, this implies $u = \tilde{u}$ in $\mathcal{H}_K$. Hence, $\mathcal{G}(t) = \tilde{\mathcal{G}}(t)$, $\mathcal{Q}(t) = \tilde{\mathcal{Q}}(t)$ in $L_2(0, \pi)$, and

$$C_i = \tilde{C}_i, \quad i = \overline{1, \max\{N_1, N_2\}}, \qquad D_i = \tilde{D}_i, \quad i = \overline{1, \max\{N_1, N_2 - 1\}},$$

so it follows from (13)-(16) and $\varpi = \tilde{\varpi}$ that $\Delta_j(\lambda) \equiv \tilde{\Delta}_j(\lambda)$, $j = 0, 1$. Consequently, $M(\lambda) \equiv \tilde{M}(\lambda)$. According to Proposition 1, we conclude that $q = \tilde{q}$ in $L_2(0, \pi)$ and $p_j(\lambda) \equiv \tilde{p}_j(\lambda)$, $j = 1, 2$. □

If the sequence $\{v_n\}_{n=1}^{\infty}$ is a Riesz basis in $\mathcal{H}_K$, one can solve Problem 1 by Algorithm 1.

Algorithm 1: Solution of the inverse problem

Suppose that the integers N_1 and N_2, the entire functions $f_1(\lambda)$ and $f_2(\lambda)$, the subspectrum $\{\lambda_n\}_{n=1}^{\infty}$, and the number ϖ are given. We have to find $q(x)$, $p_1(\lambda)$, and $p_2(\lambda)$.

1. Put $K := \max\{2N_1, 2N_2 - 1\}$ and, depending on the case $N_1 \geq N_2$ or $N_1 < N_2$, construct the functions $v(t, \lambda)$ and $w(\lambda)$ by either (22), (24) or (26), (28).
2. Construct the sequences $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ by (29).
3. Find the biorthonormal sequence $\{v_n^*\}_{n=1}^{\infty}$ to $\{v_n\}_{n=1}^{\infty}$ in $\mathcal{H}_K$, that is,
$$(v_n, v_k^*) = \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases}$$
4. Find the element $u \in \mathcal{H}_K$ satisfying (30) by the formula
$$u = \sum_{n=1}^{\infty} \overline{w_n} v_n^*.$$
5. Using the entries of u (see (21) and (25)), find $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$ by the formulas of Lemma 1, and then find $M(\lambda) = \frac{\Delta_0(\lambda)}{\Delta_1(\lambda)}$.
6. Use the method of [17] to recover the potential $q(x)$ and the polynomials $p_1(\lambda)$, $p_2(\lambda)$ from the Weyl function $M(\lambda)$.

Algorithm 1 is theoretical. In this paper, we do not aim to elaborate in detail the algorithm's numerical implementation. This issue requires a separate work. Here, we only outline the main idea of the inverse problem solution.

4. Sufficient Conditions

In this section, we prove Theorem 2 and then generalize it to the case of multiple eigenvalues. First, we need the following proposition, which is analogous to Lemma 1 in [17].

Proposition 2. *If θ is a zero of $\Delta_1(\lambda)$, then $\Delta_0(\theta) \neq 0$.*

Proof of Theorem 2. Consider the problem $L = L(q, p_1, p_2, f_1, f_2)$ of form (1)-(2) and its simple subspectrum $\{\lambda_n\}_{n=1}^{\infty}$. This means $\lambda_n \neq \lambda_k$ for $n \neq k$.

In ***the first case*** $N_1 \geq N_2$, we have

$$v(t, \lambda) = [f_1(\lambda)\rho^{2N_1} \cos \rho t, f_2(\lambda)\rho^{2N_1 - 1} \sin \rho t, \rho^{2N_1 - 2} f_1(\lambda), \dots, f_1(\lambda), \rho^{2N_1 - 2} f_2(\lambda), \dots, f_2(\lambda)].$$

Consider an element

$$h = [H_1, H_2, h_1^1, \dots, h_1^{N_1}, h_2^1, \dots, h_2^{N_1}] \in \mathcal{H}_K \tag{31}$$

such that

$$(h, v_n) = 0, \quad n \geq 1, \tag{32}$$

where $v_n = v(t, \lambda_n)$, $n \geq 1$.

Let us find the scalar product

$$\begin{aligned} (h, v_n) = \lambda_n^{N_1} \textstyle\int_0^\pi \Big(H_1(t) f_1(\lambda_n) \cos \rho_n t + H_2(t) f_2(\lambda_n) \frac{\sin \rho_n t}{\rho_n} \Big)\, dt \\ + \sum_{m=1}^{N_1} \lambda^{N_1 - m} (h_1^m f_1(\lambda_n) + h_2^m f_2(\lambda_n)). \end{aligned} \tag{33}$$

From (11) and the relation $\Delta(\lambda_n) = 0$, we can obtain that

$$\begin{aligned} f_2(\lambda_n) &= -\frac{\Delta_1(\lambda_n)}{\Delta_0(\lambda_n)} f_1(\lambda_n), f_1(\lambda_n) \neq 0 \\ f_1(\lambda_n) &= -\frac{\Delta_0(\lambda_n)}{\Delta_1(\lambda_n)} f_2(\lambda_n), f_2(\lambda_n) \neq 0 \end{aligned} \tag{34}$$

In both expressions of system (34), the denominator is nonzero. Let us show this fact for the expression for $f_1(\lambda_n)$. Indeed, if $\Delta_1(\lambda_n) = 0$, then from (11), we obtain that $f_2(\lambda_n)\Delta_0(\lambda_n) = 0$. Using Proposition 2, we conclude that $f_2(\lambda_n) = 0$. However, $f_2(\lambda_n) \neq 0$ in this case. From this contradiction, we obtain that $\Delta_1(\lambda_n) \neq 0$.

Consider the case $f_1(\lambda_n) \neq 0$. The other case is similar. Using (32), (33), and (34), we obtain

$$\lambda_n^{N_1} \int_0^\pi \left(H_1(t) \cos \rho_n t - H_2(t) \frac{\Delta_1(\lambda_n)}{\Delta_0(\lambda_n)} \frac{\sin \rho_n t}{\rho_n} \right) dt + \sum_{m=1}^{N_1} \lambda^{N_1 - m} \left(h_1^m - \frac{\Delta_1(\lambda_n)}{\Delta_0(\lambda_n)} h_2^m \right) = 0, \quad n \geq 1. \tag{35}$$

Define the function

$$\begin{aligned} G(\lambda) := \lambda^{N_1} \int_0^\pi \left(H_1(t) \Delta_0(\lambda) \cos \rho t - H_2(t) \Delta_1(\lambda) \frac{\sin \rho t}{\rho} \right) dt \\ + \sum_{n=1}^{N_1} \lambda^{N_1 - n} (h_1^n \Delta_0(\lambda) - h_2^n \Delta_1(\lambda)). \end{aligned} \tag{36}$$

It follows from (35) and $\Delta_0(\lambda_n) \neq 0$ that $G(\lambda_n) = 0$, $n \geq 1$.

Using lemma 1, we can obtain the asymptotic formulas

$$\Delta_0(\lambda) = \rho^{2N_1} \cos \rho\pi + O(|\rho|^{2N_1 - 1} e^{\pi|\tau|}), \tag{37}$$

$$\Delta_1(\lambda) = -\rho^{2N_1 + 1} \sin \rho\pi + O(|\rho|^{2N_1} e^{\pi|\tau|}). \tag{38}$$

Substituting (37)–(38) into (36), we obtain

$$\begin{aligned} G(\lambda) &= \lambda^{2N_1} (G_1(\lambda) + O(|\rho|^{-1} e^{2\pi|\tau|})), \\ G_1(\lambda) &= \textstyle\int_0^\pi (H_1(t) \cos \rho t \cos \rho\pi + H_2(t) \sin \rho\pi \sin \rho t)\, dt. \end{aligned} \tag{39}$$

Clearly, $G_1(\rho^2) \in B_{2\pi}^+$, so

$$G(\lambda) = \lambda^{2N_1} \left(\int_0^{2\pi} g(t) \cos \rho t\, dt + O(|\rho|^{-1} e^{2\pi|\tau|}) \right), \quad g \in L_2(0, 2\pi). \tag{40}$$

Let us exclude the zeros $\{\lambda_n\}_{n=1}^{2N_1}$ of $G(\lambda)$ and define the function

$$R(\lambda) := \frac{G(\lambda)}{\prod\limits_{n=1}^{2N_1} (\lambda - \lambda_n)}. \tag{41}$$

It can be easily shown that $R(\rho^2) \in L_2(0, 2\pi)$, so $R(\lambda) = \int_0^{2\pi} r(t) \cos \rho t \, dt$, where $r(t) \in L_2(0, 2\pi)$. From (41), we conclude that

$$R(\lambda_n) = \int_0^{2\pi} r(t) \cos \rho_n t \, d = 0, \quad n \geq 2N_1 + 1.$$

Hence, if the system $\{\cos \rho_n t\}_{n=2N_1+1}^{\infty}$ is complete in $L_2(0, 2\pi)$, then $r(t) \equiv 0$, $R(\lambda) \equiv 0$, and so $G(\lambda) \equiv 0$.

Let $\{\theta_n\}_{n=1}^{\infty}$ be the zeros of $\Delta_1(\lambda)$, so $\{\theta_n\}_{n=1}^{\infty}$ are the eigenvalues of the boundary value problem L_1. Then, we obtain from (36) that

$$G(\theta_n) = \theta_n^{N_1} \int_0^{\pi} H_1(t) \Delta_0(\theta_n) \cos \sqrt{\theta_n} t dt + \sum_{n=1}^{N_1} \theta_n^{N_1 - n} h_1^n \Delta_0(\theta_n) = 0, \quad n \geq 1. \tag{42}$$

Consider the function

$$H(\lambda) = \lambda^{N_1} \int_0^{\pi} H_1(t) \cos \rho t \, dt + \sum_{n=1}^{N_1} \lambda^{N_1 - n} h_1^n.$$

The relation (42) implies that $H_1(\theta_n) = 0$, $n \geq 1$. Let us obtain the first N_1 values from $\{\theta_n\}_{n=1}^{\infty}$. Define function

$$F(\lambda) := \frac{H(\lambda)}{\prod\limits_{n=1}^{N_1} (\lambda - \theta_n)}.$$

Obviously, $F(\rho^2) \in B_{\pi}^{+}$, so it can be represented in the form $F(\lambda) = \int_0^{\pi} f(t) \cos \rho t \, dt$, where $f(t) \in L_2(0, \pi)$. Clearly, we have

$$\int_0^{\pi} f(t) \cos \sqrt{\theta_n} t \, dt = 0, \quad n \geq N_1 + 1. \tag{43}$$

Using the methods of [17], one can obtain the asymptotic formula

$$\sqrt{\theta_n} = n - N_1 - 1 + O(n^{-1}), \quad n \geq 1. \tag{44}$$

For simplicity, assume that the values $\{\theta_n\}_{n=N_1+1}$ are distinct. The opposite case requires minor changes. Then, it follows from (44) the the sequence $\{\cos \sqrt{\theta_n} t\}_{n=N_1+1}^{\infty}$ is complete in $L_2(0, \pi)$. Hence, (43) implies $f(t) = 0$ a.e. on $(0, \pi)$ and so and $H_1 = 0$ in $L_2(0, \pi)$, $h_1^j = 0$, $j = \overline{1, N_1}$. Taking (36) and $G(\lambda) \equiv 0$ into account, we conclude that $H_2 = 0$ in $L_2(0, \pi)$, $h_2^j = 0$, $j = \overline{1, N_1}$. Thus, we proved that, if $h \in \mathcal{H}_K$ fulfills (32), then $h = 0$. Consequently, the system $\{v_n\}_{n=1}^{\infty}$ is complete in $\mathcal{H}_K$.

<u>The second case</u> $N_2 > N_1$ is similar to the first one. In this case, it can be shown that the completeness of the system $\{\cos \rho_n t\}_{n=2N_2}^{\infty}$ in $L_2(0, 2\pi)$ is sufficient for the completeness of $\{v_n\}_{n=1}^{\infty}$ in $\mathcal{H}_K$.

We have

$$v(t, \lambda) = [f_1(\lambda) \rho^{2N_2 - 1} \sin \rho t, f_2(\lambda) \rho^{2N_2 - 2} \cos \rho t, \rho^{2N_2 - 2} f_1(\lambda), \ldots, f_1(\lambda), \rho^{2N_1 - 4} f_2(\lambda), \ldots, f_2(\lambda)].$$

Consider an element

$$h = [H_1, H_2, h_1^1, \ldots, h_1^{N_2}, h_2^1, \ldots, h_2^{N_2 - 1}] \in \mathcal{H}_K$$

that $(h, v_n) = 0$, $n \geq 1$, where $v_n = v(t, \lambda_n)$.

Analogously to the first case, we obtain that $G(\lambda_n) = 0$, $n \geq 0$, for the function

$$\begin{aligned} G(\lambda) &:= \lambda^{N_2-1} \int_0^{\pi} \Big(H_1(t)\Delta_0(\lambda)\rho \sin \rho t - H_2(t)\Delta_1(\lambda) \cos \rho t \Big)\, dt \\ &+ \lambda^{N_2-1} h_1^{N_2-1} \Delta_0(\lambda) + \sum_{n=2}^{N_2} \lambda^{N_2-n} (h_1^n \Delta_0(\lambda) - h_2^{n-1}\Delta_1(\lambda)), \end{aligned} \tag{45}$$

Then, using the asymptotics for $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$ and (45), we obtain

$$G(\lambda) = \lambda^{2N_2-1} \bigg(\int_0^{2\pi} g(t) \cos \rho t\, dt + O(|\rho|^{-1} e^{2\pi|\tau|}) \bigg), \quad g \in L_2(0, 2\pi). \tag{46}$$

Excluding the first $(2N_2 - 1)$ zeros of $G(\lambda)$, we obtain the function

$$R(\lambda) := \frac{G(\lambda)}{\prod\limits_{n=1}^{2N_2-1} (\lambda - \lambda_n)}. \tag{47}$$

We have $R(\rho^2) \in B_{2\pi}^+$, and so $R(\lambda) = \int_0^{2\pi} r(t) \cos \rho t\, dt$, where $r(t) \in L_2(0, 2\pi)$. From (41), we can conclude that

$$R(\lambda_n) = \int_0^{2\pi} r(t) \cos \rho_n t\, dt = 0, \quad n \geq 2N_2.$$

Therefore, if system $\{\cos \rho_n t\}_{n=2N_2}^{\infty}$ is complete in $L_2(0, 2\pi)$, then $r(t) \equiv 0$, $R(\lambda) \equiv 0$ and $G(\lambda) \equiv 0$. Consequently, one can show that $h = 0$ in $\mathcal{H}_K$, which concludes the proof. □

Now, we consider the general situation when the subspectrum $\{\lambda_n\}_{n=1}^{\infty}$ may contain multiple eigenvalues of finite multiplicities. Put $N := \max\{2N_1 + 1, 2N_2\}$. Denote by $\mathcal{A}$ any subset of indices $\mathbb{N}$ such that $|\mathcal{A}| = N$ and put $\mathcal{B} := \mathbb{N} \setminus \mathcal{A}$. Consider the subset $\{\lambda_n\}_{n \in \mathcal{B}}$. Thus, we have excluded arbitrary N values (counting with multiplicities) from the sequence $\{\lambda_n\}_{n=1}^{\infty}$. Denote by $\mathcal{I}$ the set of indices of distinct eigenvalues among $\{\lambda_n\}_{n \in \mathcal{B}}$ and by $\{\nu_k\}_{k \in \mathcal{I}}$ the multiplicities of the corresponding values $\{\lambda_k\}_{k \in \mathcal{I}}$:

$$\mathcal{I} := \{k \in \mathcal{B} \colon \lambda_k \neq \lambda_n,\, n \in \mathcal{B},\, n < k\}, \quad \nu_k := \#\{\lambda_n = \lambda_k \colon n \in \mathcal{B}\},\, k \in \mathcal{I}.$$

Define the functions

$$c(x, \lambda) := \cos \sqrt{\lambda} x, \quad c_{k,j}(x) = c^{<j>}(x, \lambda_k), \quad k \in \mathcal{I},\, j = \overline{0, \nu_k - 1}.$$

Then, by using the technique of [30], we obtain the following generalization of Theorem 2 to the case of multiple eigenvalues.

Theorem 4. *Suppose that $f_1(\lambda_n) \neq 0$ or $f_2(\lambda_n) \neq 0$ for every $n \geq 1$, and the system $\{c_{k,j}(x)\}_{k \in \mathcal{I},\, j=\overline{0,\nu_k-1}}$ is complete in $L_2(0, 2\pi)$. Then, the system $\{v_n\}_{n=1}^{\infty}$ is complete in $\mathcal{H}_K$.*

5. Hochstadt–Lieberman-Type Problems

In this section, we prove the uniqueness theorems for the Hochstadt–Lieberman-type inverse problems, namely, for Problems 2 and 3. The method of the proofs is based on the reduction in the Hochstadt–Lieberman-type problems to Problem 1, with entire functions in the right-hand side BC. Then, we successively apply Theorems 2 and 1.

Consider the boundary value problem $\mathscr{L} = \mathscr{L}(q, p_1, p_2, r_1, r_2)$ of form (5)-(6). Define the functions $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ as the solutions of equation (5) satisfying the initial conditions

$$\varphi(0, \lambda) = p_1(\lambda), \quad \varphi'(0, \lambda) = -p_2(\lambda), \quad \psi(2\pi, \lambda) = r_1(\lambda), \quad \psi'(2\pi, \lambda) = -r_2(\lambda). \tag{48}$$

It can be easily seen that the eigenvalues $\{\mu_n\}_{n=1}^{\infty}$ of the problem $\mathscr{L}$ coincide with the zeros of the characteristic function

$$\Delta(\lambda) = \varphi(\pi,\lambda)\psi'(\pi,\lambda) - \varphi'(\pi,\lambda)\psi(\pi,\lambda). \tag{49}$$

The function $\varphi(x,\lambda)$ can be represented in the form

$$\varphi(x,\lambda) = p_1(\lambda)C(x,\lambda) - p_2(\lambda)S(x,\lambda). \tag{50}$$

Substituting (50) into (49), we obtain

$$\Delta(\lambda) = \sum_{j=0}^{1}(-1)^j\psi^{(1-j)}(\pi,\lambda)(p_1(\lambda)C^{(j)}(\pi,\lambda) - p_2(\lambda)S^{(j)}(\pi,\lambda)). \tag{51}$$

Comparing (51) with (11), we conclude that the eigenvalues of the boundary value problem $\mathscr{L} = \mathscr{L}(q,p_1,p_2,r_1,r_2)$ coincide with the eigenvalues of the problem $L = L(q,p_1,p_2,f_1,f_2)$ with $f_1(\lambda) = -\psi(\pi,\lambda)$ and $f_2(\lambda) = \psi'(\pi,\lambda)$.

Proof of Theorem 3. Consider the case $N_1 \geq N_2, M_1 \geq M_2$. The other cases can be treated similarly. Introduce the notations

$$\Omega = \frac{1}{2}\int_0^{2\pi} q(t)\,dt, \quad \Theta = \begin{cases} \Omega + d_{M_2} - b_{N_2}, & N_1 = N_2, M_1 = M_2 \\ \Omega - b_{N_2}, & N_1 = N_2, M_1 \neq M_2 \\ \Omega + d_{M_2}, & N_1 > N_2, M_1 = M_2 \\ \Omega, & \textit{otherwise} \end{cases} \tag{52}$$

Instead of (51), it is more convenient to use another representation of the characteristic function:

$$\Delta(\lambda) = \sum_{j=0}^{1}(-1)^j r_{2-j}(\lambda)(p_1(\lambda)C^{(j)}(2\pi,\lambda) - p_2(\lambda)S^{(j)}(2\pi,\lambda)). \tag{53}$$

Using (53), we obtain the following asymptotics for $\Delta(\lambda)$:

$$\Delta(\lambda) = \rho^{2(N_1+M_1)+1}\sin 2\rho\pi + O(|\rho|^{2(N_1+M_1)}e^{2\pi|\tau|})$$

and for the eigenvalues

$$\sqrt{\mu_n} = \frac{n-1}{2} - (N_1+M_1) + \frac{\Theta}{\pi n} + \frac{\chi_n}{n}, \quad n \geq 1, \quad \{\chi_n\} \in l_2. \tag{54}$$

For simplicity, assume that the eigenvalues $\{\mu_n\}_{n=N_1+M_1+1}^{\infty}$ are simple. The general case requires technical changes. Then, the asymptotics (54) imply that the system $\{\cos\sqrt{\mu_n}t\}_{n=N_1+M_1+1}^{\infty}$ is complete in $L_2(0,2\pi)$.

Let us pass from the problem $\mathscr{L}(q,p_1,p_2,r_1,r_2)$ to the corresponding problem $L(q,p_1,p_2,f_1,f_2)$ with $f_1(\lambda) = -\psi(\pi,\lambda)$ and $f_2(\lambda) = \psi'(\pi,\lambda)$. It follows from Proposition 2 that $f_1(\lambda)$ and $f_2(\lambda)$ do not have common zeros. Suppose that $M_1 \geq N_1$ and consider the subspectrum $\{\lambda_n\}_{n=1}^{\infty} := \{\mu_n\}_{n=M_1-N_1+1}^{\infty}$ of the problem L. Thus, in the case $N_1 = M_1$, we consider the whole spectrum of $\mathscr{L}$ and, in the case $N_1 < M_1$, we exclude $(M_1 - N_1)$ eigenvalues. The excluded eigenvalues can be chosen arbitrarily. According to the above arguments, we have that the system $\{\cos\sqrt{\lambda_n}t\}_{n=2N_1+1}^{\infty}$ is complete in $L_2(0,2\pi)$. Hence, the conditions of Theorem 2 are fulfilled for the sequence $\{v_n\}_{n=1}^{\infty}$ constructed by L and $\{\lambda_n\}_{n=1}^{\infty}$.

Now, consider two boundary value problems $\mathscr{L} = \mathscr{L}(q,p_1,p_2,r_1,r_2)$ and $\tilde{\mathscr{L}} = \mathscr{L}(\tilde{q},\tilde{p}_1,\tilde{p}_2,\tilde{r}_1,\tilde{r}_2)$ satisfying the conditions of the theorem, that is, $N_1 = \tilde{N}_1$, $N_2 = \tilde{N}_2$,

$r_j(\lambda) = \tilde{r}_j(\lambda)$, $j = 1, 2$, $q(x) = \tilde{q}(x)$ a.e. on $(\pi, 2\pi)$, and $\mu_n = \tilde{\mu}_n$ for $n \geq M_1 - N_1 + 1$. Then, it follows from (54) that $\Theta = \tilde{\Theta}$. Observe that

$$\varpi = \Theta - d_{M_2} - \int_{\pi}^{2\pi} q(t)\, dt,$$

where ϖ and Θ are defined by (4) and (52), respectively. Moreover, since $q(t) = \tilde{q}(t)$ a.e. on $(\pi, 2\pi)$, then $\int_{\pi}^{2\pi} q(t)\, dt = \int_{\pi}^{2\pi} \tilde{q}(t)\, dt$. Hence, $\varpi = \tilde{\varpi}$. Furthermore, the solution $\psi(x, \lambda)$ on $[\pi, 2\pi]$ is uniquely specified by the polynomials $r_j(\lambda)$, $j = 1, 2$, and the potential $q(x)$ on $(\pi, 2\pi)$. Consequently, $\psi(x, \lambda) \equiv \tilde{\psi}(x, \lambda)$, $x \in [\pi, 2\pi]$. Consider the equivalent problems $L = L(q, p_1, p_2, f_1, f_2)$ and $\tilde{L} = L(\tilde{q}, \tilde{p}_1, \tilde{p}_2, \tilde{f}_1, \tilde{f}_2)$ for the problems $\mathscr{L}$ and $\tilde{\mathscr{L}}$, respectively. By the above arguments, we have $f_j(\lambda) \equiv \tilde{f}_j(\lambda)$, $j = 1, 2$. Consider the subspectra $\{\lambda_n\}_{n=1}^{\infty} := \{\mu_n\}_{n=M_1-N_1+1}^{\infty}$ and $\{\tilde{\lambda}_n\}_{n=1}^{\infty} := \{\tilde{\mu}_n\}_{n=M_1-N_1+1}^{\infty}$ of the problems L and $\tilde{L}$, respectively. By virtue of Theorem 2, the sequence $\{v_n\}_{n=1}^{\infty}$ constructed by L and $\{\lambda_n\}_{n=1}^{\infty}$ is complete in $\mathcal{H}_K$. Thus, the conditions of Theorem 1 hold. Applying Theorem 1, we conclude that $q(x) = \tilde{q}(x)$ a.e. on $(0, \pi)$ and $p_j(\lambda) \equiv \tilde{p}_j(\lambda)$, $j = 1, 2$. $\square$

Proceed to the second boundary value problem $\mathcal{L} = \mathcal{L}(q, p_1, p_2, r_1, r_2, p_{ij})$ for the Sturm–Liouville equation (8) with the complex-valued potential $q \in L_2(0, 2\pi)$, the BC at $x = 0$ containing the relatively prime polynomials $p_j(\lambda)$, $j = 1, 2$, the BC at $x = 2\pi$, the relatively prime polynomials $r_j(\lambda)$, $j = 1, 2$, and the discontinuity conditions (10), the polynomials $p_{ij}(\lambda)$, $i = \overline{1, 3}$, $j = \overline{0, 1}$. Suppose that the polynomials $p_1(\lambda)$, $p_2(\lambda)$ have the form (3), the polynomials $r_1(\lambda)$, $r_2(\lambda)$, the form (7), and the polynomials $p_{ij}(\lambda)$, $i = \overline{1, 3}$, $j = 0, 1$, the form

$$p_{ij}(\lambda) = \sum_{n=0}^{K_{ij}} g_n^{ij} \lambda^n, \quad g_{K_{ij}}^{ij} \neq 0, \quad K_{ij} \geq 0, \quad i = \overline{1, 3}, \quad j = 0, 1. \tag{55}$$

For definiteness, we confine ourselves to the case $N_1 > N_2$, $M_1 > M_2$, $K_{20} > K_{30}$, $K_{21} > K_{31}$, $K_{10} + K_{21} > K_{11} + K_{20}$. Without loss of generality, we assume that $a_{N_1} = 1$, $c_{M_1} = 1$, $g_{K_{10}}^{10} = g_{K_{11}}^{11} = 1$.

Consider the solutions $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ of equation (5) satisfying the initial conditions (48) on the segments $[0, \pi]$ and $[\pi, 2\pi]$, respectively. It can be easily seen that the eigenvalues of the problem $\mathcal{L}$ coincide with the zeros of the characteristic function

$$\Delta(\lambda) = \sum_{j=0}^{1} (-1)^{1-j} \varphi^{(j)}(\pi, \lambda) p_{1,j}(\lambda) (p_{2,1-j}(\lambda) \psi(\pi, \lambda) + p_{3,1-j}(\lambda) \psi'(\pi, \lambda)). \tag{56}$$

The function $\varphi(x, \lambda)$ can be represented in the form (50). So, substituting (50) into (56), we obtain

$$\begin{aligned} \Delta(\lambda) = \sum_{j=0}^{1} (-1)^{1-j} &(p_1(\lambda) C^{(j)}(\pi, \lambda) - p_2(\lambda) S^{(j)}(\pi, \lambda)) \\ &\times p_{1,j}(\lambda)(p_{2,1-j}(\lambda) \psi(\pi, \lambda) + p_{3,1-j}(\lambda) \psi'(\pi, \lambda)). \end{aligned} \tag{57}$$

Comparing (57) with (11), we conclude that the eigenvalues of the boundary value problem $\mathcal{L} = \mathcal{L}(q, p_1, p_2, r_1, r_2, p_{ij})$ coincide with the eigenvalues of the problem $L = L(q, p_1, p_2, f_1, f_2)$ with

$$f_1(\lambda) = p_{11}(\lambda)(p_{20}(\lambda) \psi(\pi, \lambda) + p_{30}(\lambda) \psi'(\pi, \lambda)), \tag{58}$$

$$f_2(\lambda) = -p_{10}(\lambda)(p_{21}(\lambda) \psi(\pi, \lambda) + p_{31}(\lambda) \psi'(\pi, \lambda)). \tag{59}$$

The following theorem implies the uniqueness of solution for Problem 3.

Theorem 5. *Let $\{\mu_n\}_{n=1}^{\infty}$ and $\{\tilde{\mu}_n\}_{n=1}^{\infty}$ be the spectra of the problems $\mathcal{L} = \mathcal{L}(q, p_1, p_2, r_1, r_2, p_{ij})$ and $\tilde{\mathcal{L}} = \mathcal{L}(\tilde{q}, \tilde{p}_1, \tilde{p}_2, \tilde{r}_1, \tilde{r}_2, \tilde{p}_{ij})$, respectively. Suppose that $N_j = \tilde{N}_j$, $r_j(\lambda) \equiv \tilde{r}_j(\lambda)$, $j = 1, 2$, $p_{ij}(\lambda) \equiv p_{ij}(\lambda)$, $i = \overline{1,3}$, $j = 0, 1$, $q(x) = \tilde{q}(x)$ a.e. on $(\pi, 2\pi)$, $N_1 \le M_1 + K_{10} + K_{21}$, and $\mu_n = \tilde{\mu}_n$ for all $n \ge n_1$, $n_1 := -N_1 + M_1 + K_{10} + K_{21} + 1$. In addition, assume that $f_1(\mu_n) \ne 0$ or $f_2(\mu_n) \ne 0$ for each $n \ge n_1$, where the functions $f_1(\lambda)$ and $f_2(\lambda)$ are defined by (58)–(59). Then, $q(x) = \tilde{q}(x)$ a.e. on $(0, \pi)$ and $p_j(\lambda) \equiv \tilde{p}_j(\lambda)$, $j = 1, 2$.*

Proof. The idea of the proof is based on the reduction in Problem 3 to Problem 1.

Using (57), we obtain the following asymptotics for the characteristic function

$$\Delta(\lambda) = \rho^{2n_0}\left(g_{K_{21}}^{21}\cos^2\rho\pi + \frac{g_{K_{21}}^{21}\Omega}{2\rho}\cos\rho\pi\sin\rho\pi + o(\rho^{-1}e^{2|\tau|\pi})\right), \tag{60}$$

and for the eigenvalues

$$\sqrt{\mu_n} = n - \frac{1}{2} - n_0 + O(n^{-\frac{1}{2}}), \quad n \ge 1, \tag{61}$$

where $\Omega = \frac{1}{2}\int_0^{2\pi} q(t)\,dt$ and $n_0 := N_1 + M_1 + K_{10} + K_{21}$.

Define the function

$$\Delta^1(\lambda) = \frac{\Delta(\lambda)}{\prod\limits_{n=1}^{n_0}(\lambda - \mu_n)}. \tag{62}$$

It has only zeros $\{\mu_n\}_{n=n_0+1}^{\infty}$. Suppose that the eigenvalues $\{\mu_n\}_{n>n_0}$ are simple. The general case requires minor technical changes. Let us prove that the sequence $\{\cos\sqrt{\mu_n}t\}_{n=n_0+1}^{\infty}$ is complete in $L_2(0, 2\pi)$. Let $h \in L_2(0, 2\pi)$ be such a function that

$$\int_0^{2\pi} h(t)\cos\sqrt{\mu_n}t\,dt = 0, \quad n > n_0.$$

We have to show that $h \equiv 0$. Consider the function $H(\lambda) := \int\limits_0^{2\pi} h(t)\cos\rho t\,dt$. Clearly, $\frac{H(\lambda)}{\Delta^1(\lambda)}$ is an entire function and $H(\lambda) = o(e^{2|\tau|\pi})$, $|\lambda| \to \infty$. It can be shown that $|\Delta^1(\lambda)| \ge C_\delta e^{2|\tau|\pi}$ in the region

$$G_\delta = \{\rho \in \mathbb{C}\colon |\rho - (n - \frac{1}{2})| \ge \delta, n \in \mathbb{Z}\}, \quad |\rho| \ge \rho^*,$$

for some positive constants δ, ρ^*, and C_δ. So, we can conclude that $\dfrac{H(\lambda)}{\Delta^1(\lambda)} \to 0$ as $|\lambda| \to \infty$, $\lambda = \rho^2$, $\rho \in G_\delta$. By Liouville's theorem, we conclude that $\dfrac{H(\lambda)}{\Delta^1(\lambda)} \equiv 0$, then $H(\lambda) \equiv 0$ and $h(t) = 0$ a.e. on $(0, 2\pi)$. Hence, the system $\{\cos\sqrt{\mu_n}t\}_{n\ge n_0+1}$ is complete in $L_2(0, 2\pi)$.

Let us pass from the problem $\mathcal{L}(q, p_1, p_2, r_1, r_2, p_{ij})$ to the corresponding problem $L(q, p_1, p_2, f_1, f_2)$, with the functions $f_1(\lambda)$ and $f_2(\lambda)$ defined by (58) and (59), respectively. Suppose that $N_1 \le M_1 + K_{10} + K_{21}$ and consider the subspectrum $\{\lambda_n\}_{n=1}^{\infty} := \{\mu_n\}_{n=n_1}^{\infty}$ of the problem L. Thus, in the case $N_1 = M_1 + K_{10} + K_{21}$, we consider the whole spectrum of $\mathcal{L}$ and, in the case $N_1 < M_1 + K_{10} + K_{21}$, we exclude $(M_1 + K_{10} + K_{21} - N_1)$ eigenvalues. The excluded eigenvalues can be chosen arbitrarily. According to the above arguments, we have that the system $\{\cos\sqrt{\lambda_n}t\}_{n=2N_1+1}^{\infty}$ is complete in $L_2(0, 2\pi)$. Hence, the conditions of Theorem 2 are fulfilled for the sequence $\{v_n\}_{n=1}^{\infty}$ constructed by L and $\{\lambda_n\}_{n=1}^{\infty}$.

Let us show that the value ϖ is uniquely specified by the subspectrum $\{\mu_n\}_{n=n_0+1}^{\infty}$ and the potential $q(x)$ on $(\pi, 2\pi)$. By using Hadamard's factorization theorem, one can reconstruct the function $\Delta^1(\lambda)$ from its zeros $\{\mu_n\}_{n=n_0+1}^{\infty}$ uniquely up to a multiplicative constant:

$$P(\lambda) := \prod_{n=n_0+1}^{\infty} \left(1 - \frac{\lambda}{\mu_n}\right), \quad P(\lambda) = c_1 \Delta^1(\lambda), \quad c_1 \neq 0.$$

From (60) and (62), we obtain

$$P(\lambda) = c_1 \big(g_1 \cos^2 \rho\pi + g_2 \rho^{-1} \sin \rho\pi \cos \rho\pi + o(\rho^{-1} e^{2|\tau|\pi})\big), \tag{63}$$

where $g_1 = g_{K_{21}}^{21}$, $g_2 = g_{K_{21}}^{21} \frac{\Omega}{2}$.

Taking $\rho = i\tau$ in (63), we derive

$$c_1 g_1 = 2 \lim_{\tau \to +\infty} P(-\tau^2) e^{-2\tau\pi} =: \kappa_1,$$

$$c_1 g_2 = 2 \lim_{\tau \to +\infty} \tau (2P(-\tau^2) e^{-2\tau\pi} - \kappa_1) =: \kappa_2.$$

Then, we can find $\Omega = \dfrac{2\kappa_2}{\kappa_1}$. Observe that

$$\varpi = \Omega - \int_{\pi}^{2\pi} q(t)\, dt.$$

Now, consider two boundary value problems $\mathcal{L} = \mathcal{L}(q, p_1, p_2, r_1, r_2, p_{ij})$ and $\tilde{\mathcal{L}} = \mathcal{L}(\tilde{q}, \tilde{p}_1, \tilde{p}_2, \tilde{r}_1, \tilde{r}_2, \tilde{p}_{ij})$ and their subspectra $\{\mu_n\}_{n \ge n_1}$ and $\{\tilde{\mu}_n\}_{n \ge n_1}$ satisfying the conditions of the theorem. Since $\mu_n = \tilde{\mu}_n$, $n \ge n_1$, then $\Delta^1(\lambda) = \tilde{\Delta}^1(\lambda)$. Hence, $\kappa_j = \tilde{\kappa}_j$, $j = 1, 2$, and, consequently, $\Omega = \tilde{\Omega}$. Moreover, since $q(t) = \tilde{q}(t)$ a.e. on $(\pi, 2\pi)$, then $\int_{\pi}^{2\pi} q(t)\, dt = \int_{\pi}^{2\pi} \tilde{q}(t)\, dt$. Hence, $\varpi = \tilde{\varpi}$. Furthermore, the solution $\psi(x, \lambda)$ on $(\pi, 2\pi)$ is uniquely specified by the polynomials $r_j(\lambda)$, $j = 1, 2$, and the potential $q(x)$ on $(\pi, 2\pi)$. Consequently, $\psi(x, \lambda) \equiv \tilde{\psi}(x, \lambda)$, $x \in (\pi, 2\pi)$. Consider the equivalent problems $L = L(q, p_1, p_2, f_1, f_2)$ and $\tilde{L} = L(\tilde{q}, \tilde{p}_1, \tilde{p}_2, \tilde{f}_1, \tilde{f}_2)$ for the problems $\mathcal{L}$ and $\tilde{\mathcal{L}}$, respectively. By the above arguments, we have $f_j(\lambda) \equiv \tilde{f}_j(\lambda)$, $j = 1, 2$. Consider the subspectra $\{\lambda_n\}_{n=1}^{\infty} := \{\mu_n\}_{n=n_1}^{\infty}$ and $\{\tilde{\lambda}_n\}_{n=1}^{\infty} := \{\tilde{\mu}_n\}_{n=n_1}^{\infty}$ of the problems L and $\tilde{L}$, respectively. By virtue of Theorem 2, the sequence $\{v_n\}_{n=1}^{\infty}$ constructed by L and $\{\lambda_n\}_{n=1}^{\infty}$ is complete in $\mathcal{H}_K$. Thus, the conditions of Theorem 1 hold. Applying Theorem 1, we conclude that $q(x) = \tilde{q}(x)$ a.e. on $(0, \pi)$ and $p_j(\lambda) \equiv \tilde{p}_j(\lambda)$, $j = 1, 2$. $\square$

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