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On Monochromatic Clean Condition on Certain Finite Rings

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Abstract: For a finite commutative ring R, let $a, b, c \in R$ be fixed elements. Consider the equation $ax + by = cz$ where x, y, and z are idempotents, units, and any element in the ring R, respectively. We say that R satisfies the r-monochromatic clean condition if, for any r-colouring χ of the elements of the ring R, there exist $x, y, z \in R$ with $\chi(x) = \chi(y) = \chi(z)$ such that the equation holds. We define $m_{(a,b,c)}(R)$ to be the least positive integer r such that R does not satisfy the r-monochromatic clean condition. This means that there exists $\chi(i) = \chi(j)$ for some $i, j \in \{x, y, z\}$ where $i \neq j$. In this paper, we prove some results on $m_{(a,b,c)}(R)$ and then formulate various conditions on the ring R for when $m_{(1,1,1)}(R) = 2$ or 3, among other results concerning the ring $\mathbb{Z}_n$ of integers modulo n.

Keywords: finite commutative rings; monochromatic solution; monochromatic clean condition; generalised Ramsey theory

MSC: 05C55; 13M99

1. Introduction

Ramsey theory is the study of the preservation of properties under set partitions and has undergone a great deal of research in the past century. In this paper, we attempt to extend the investigation of Ramsey-theoretic numbers to include algebraic structures, particularly commutative rings with identity. We begin with some definitions. For an arithmetic progression $A = \{a + ld : 0 \leq l \leq k - 1\}$, we say that A is a *k-term arithmetic progression with difference d*, or we write $\{a + ld\}_{0 \leq l \leq k-1}$. For a positive integer t, we denote the set $\{1, 2, 3, \ldots, t\}$ by $[1 \ldots t]$. An *r-colouring* χ of a set S is a surjective function $\chi : S \to [1 \ldots r]$. A *monochromatic k-term arithmetic progression* refers to a k-term arithmetic progression such that all of its elements are of the same color. Recently, a variety of topics related to Ramsey theory have been investigated, see [1–3]. Particularly, unimaginable numbers and their graph-theory representation linked to Goodstein's theorem are shown in [4]. Investigation into the coloring of these unimaginable numbers which are beyond any physical application will further develop the application of Ramsey theory in computational number theory and computer science.

Van der Waerden's theorem [5] states that there exists a least positive integer $w = w(k; r)$ such that for any $n \geq w$, every r-colouring of $[1 \ldots n]$ admits a monochromatic k-term arithmetic progression. It should be noted that the exact value of $w(k; r)$ is not known in general. We refer the readers to [6–11] for some recent van der Waerden type of results. By van der Waerden theorem, in particular, for the case $k = 3$, any r-colouring χ of $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ must admit a monochromatic 3-term arithmetic progression $\{a, a + d, a + 2d\}$ for some $a, d \geq 1$. Following this, by letting $x = a, y = a + 2d$ and $z = a + d$, then we say that χ has a monochromatic solution to the equation $x + y = 2z$, where $x, y, z \in \mathbb{Z}^+$ and $x \neq y$. This leads to the question of whether there exists a monochromatic solution for a given equation.

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Given an r-colouring χ of a set S and an equation f in the variables of elements of S, we say that f has a *monochromatic solution* under χ if there exist elements $x_1, \ldots, x_n \in S$ that satisfy f and also are monochromatic under χ, i.e., $\chi(x_1) = \cdots = \chi(x_n)$. By considering different sets S and different equations f, the question of whether there exists a monochromatic solution for a given equation has led to many different lines of investigation. Below, we present one such investigation, which is Schur's theorem, a well-known Ramsey-type theorem in the integers.

Theorem 1 ([5] Schur's Theorem)**.** *For any $r \geq 1$, there exists a least positive integer $s = s(r)$ such that for every r-colouring of $[1 \ldots s] \subseteq \mathbb{Z}$ there exists a monochromatic solution to $x + y = z$.*

The numbers $s(r)$ are called the Schur numbers. Schur numbers are significant for being one of the earliest results in Ramsey theory of integers. It is known that $s(2) = 5$. This means that for the interval $[1 \ldots 4]$, there exists a 2-coloring such that there is no monochromatic solution to $x + y = z$. As it turns out, the only Schur numbers that are currently known are $s(1) = 2, s(2) = 5, s(3) = 14$, and $s(4) = 45$, as seen in [5]. Some interesting results have been shown lately on the monochromatic solution on the integers, see [12,13]. Together with the concept of clean rings, we are motivated to investigate the monochromatic solutions to equations involving the coloring of rings.

We are interested in a monochromatic solution in commutative rings with identity. Given a commutative ring with identity R, an element $z \in R$ is called an *idempotent* if $z^2 = z$ and an element $w \in R$ is called a *unit* if there exists $a \in R$ such that $wa = aw = e$ where e is the identity in R. Below we define clean rings, a concept introduced by Nicholson [14] in 1977.

Definition 1. *A ring is a clean ring if every element can be written as the sum of an idempotent and a unit.*

Let $M_n(R)$ be the set of $n \times n$ matrices over the ring R with the usual operations of matrix addition and matrix multiplication. Then $M_n(R)$ is a ring. We note here that when $R = \mathbb{C}$, the set of complex numbers, then the ring $M_n(R)$ is a clean ring [14]. Furthermore, the ring $\mathbb{Z}_{p^n}$ where p is a prime and $n \geq 1$ is also a clean ring.

Let $\text{Idemp}(R)$ and $\text{Unit}(R)$ be the sets of all the idempotents and units in R, respectively. Motivated by the definition of clean rings, it will be interesting to consider monochromatic solutions to the equation $x + y = z$ over R such that $x \in \text{Idemp}(R)$, $y \in \text{Unit}(R)$ and $z \in R$. Generally, let $a, b, c \in R$ be fixed and consider the equation

$$ax + by = cz, \tag{1}$$

where $x \in \text{Idemp}(R)$, $y \in \text{Unit}(R)$ and $z \in R$. We say that R satisfies the *r-monochromatic clean condition* if for any r-colouring χ of the elements of the ring R, there exist $x \in \text{Idemp}(R)$, $y \in \text{Unit}(R)$ and $z \in R$ with $\chi(x) = \chi(y) = \chi(z)$ such that Equation (1) holds. Note that there is no restriction on the coloring of the elements a, b, c. Let $m_{(a,b,c)}(R)$ be the least positive integer r such that R does not satisfy the r-monochromatic clean condition. Clearly, $m_{(a,b,c)}(R) \geq 1$. Furthermore, if Equation (1) has no solution, then $m_{(a,b,c)}(R) = 1$. Here, we shall denote the ring of integers modulo n by $\mathbb{Z}_n$.

Example 1. *Consider the equation $2y = pz$ over the ring $\mathbb{Z}_{p^2}$ where p is an odd prime. This equation is obtained by setting $(a, b, c) = (0, 2, p)$ in Equation (1). Since y is a unit in $\mathbb{Z}_{p^2}$, we must have $\gcd(y, p) = 1$. If a solution exists, i.e., there exist $y \in \text{Unit}(\mathbb{Z}_{p^2})$ and $z \in \mathbb{Z}_{p^2}$ such that $2y = pz$ in $\mathbb{Z}_{p^2}$, then $2y \equiv 0 \mod p$, but this is not possible as $\gcd(y, p) = 1$. Hence, $m_{(0,2,p)}(\mathbb{Z}_{p^2}) = 1$.*

Clearly, if Equation (1) has a solution, then $m_{(a,b,c)}(R) > 1$. For convenience, if no such least positive integer r exists, then we write $m_{(a,b,c)}(R) = \infty$. It is obvious that $m_{(0,0,0)}(R) = \infty$.

Example 2. *Consider the equation $x + y = 2z$ over the ring $\mathbb{Z}_p$ where p is an odd prime. This equation is obtained by setting $(a, b, c) = (1, 1, 2)$ in Equation (1). Note that for any r-colouring χ of the elements of the ring $\mathbb{Z}_p$, $(x, y, z) = (1, 1, 1)$ is a monochromatic solution, i.e., $\chi(x) = \chi(y) = \chi(z) = \chi(1)$ and $x + y = 2z$. Hence, $m_{(1,1,2)}(\mathbb{Z}_p) = \infty$.*

The *order* of a ring R is the number of elements in the ring and is denoted as $|R|$. Given that we are investigating units in a ring, it is obvious that we can only consider rings with identities. Furthermore, we shall limit ourselves to commutative rings. For the remainder of this paper, R refers to commutative rings with identity unless otherwise stated.

Let R be a commutative ring with identity 1. If $(a, b, c) = (3, 4, 5)$, then the equation being considered is equivalent to $3x + 4y = 5z$ in R. Similarly, if $(a, b, c) = (1, 1, 1)$, then the equation being considered is equivalent to $x + y = z$ in R. For convenience, if $(a, b, c) \in R \times R \times R$, we write $m_{(a,b,c)}(R)$.

In the next section, we find all the possible values of $m_{(a,b,c)}(R)$ for any ring R, we show how $m_{(a,b,c)}(R)$ is preserved through homomorphism and prove several results for the case where $a = b = c = 1$, and end this paper with some results on the ring of integers modulo n. In particular, we give a necessary and sufficient condition of when a finite commutative ring R with identity 1 satisfies $m_{(a,b,c)}(R) = \infty$ (Theorem 2). We also give necessary and sufficient conditions for which $m_{(a,b,c)}(\mathbb{Z}_{p^n}) = 1$ (Theorem 3). We classify the field R that satisfies $m_{(1,1,1)}(R) = 2$ (Theorem 4) and determine all integers n that satisfy $m_{(1,1,1)}(\mathbb{Z}_n) = 2$ (Theorem 5). By applying these results, we are able to give a complete classification of $m_{(1,1,1)}(\mathbb{Z}_{p^n})$ in Corollary 5.

2. Main Results

In this section, we investigate $m_{(a,b,c)}(R)$ where R is a commutative ring with identity and $a, b, c \in R$. We begin with the following lemma.

Lemma 1. *Let R and S be two commutative rings with identity and $\psi : R \to S$ be a ring homomorphism. If ψ is surjective, then*

$$m_{(a,b,c)}(R) \leq m_{(\psi(a),\psi(b),\psi(c))}(S).$$

Proof. If $m_{(\psi(a),\psi(b),\psi(c))}(S) = \infty$, then the lemma holds. So, we may assume that $m_{(\psi(a),\psi(b),\psi(c))}(S) = r$ for some positive integer r. This means S does not satisfy the r-monochromatic clean condition. Therefore, there exists an r-colouring χ of the elements of S such that the set

$$\begin{aligned} P = \{&(x', y', z') \in \mathrm{Idemp}(S) \times \mathrm{Unit}(S) \times S \ : \\ &\psi(a)x' + \psi(b)y' = \psi(c)z' \ \text{ and } \ \chi(x') = \chi(y') = \chi(z')\} \end{aligned}$$

is empty.

Now, we define an r-colouring $\overline{\chi}$ of the elements of R as follows: for each $u \in R$, $\overline{\chi}(u) = \chi(\psi(u))$. Suppose $m_{(a,b,c)}(R) > r$. Then, R satisfies the r-monochromatic clean condition. So, there exist $x \in \mathrm{Idemp}(R)$, $y \in \mathrm{Unit}(R)$ and $z \in R$ with $\overline{\chi}(x) = \overline{\chi}(y) = \overline{\chi}(z)$ such that $ax + by = cz$. This implies that

$$\psi(a)\psi(x) + \psi(b)\psi(y) = \psi(c)\psi(z),$$

and $\chi(\psi(x)) = \chi(\psi(y)) = \chi(\psi(z))$. Now, $x^2 = x$ implies that $(\psi(x))^2 = \psi(x)$. Thus, $\psi(x) \in \mathrm{Idemp}(S)$. Let e be the identity in R. Since ψ is surjective, $\psi(e)$ is the identity in S. Now, $y \in \mathrm{Unit}(R)$ implies that there is a $y_1 \in R$ such that $yy_1 = y_1y = e$. This implies that

$\psi(y)\psi(y_1) = \psi(e)$. Thus, $\psi(y) \in \text{Unit}(S)$ and $(\psi(x), \psi(y), \psi(z)) \in P$. This contradicts that $P = \varnothing$. Hence, $m_{(a,b,c)}(R) \leq m_{(\psi(a),\psi(b),\psi(c))}(S)$. □

Lemma 2. *Let R be a commutative ring with identity* 1. *Then*

$$\text{Idemp}(R) \cap \text{Unit}(R) = \{1\}.$$

Proof. Clearly, $1 \in \text{Idemp}(R) \cap \text{Unit}(R)$. Let $x \in \text{Idemp}(R) \cap \text{Unit}(R)$. Then $x^2 = x$, i.e., $x(x-1) = 0$. Since x is also a unit, we have $x - 1 = 0$, i.e., $x = 1$. Hence, the lemma follows. □

Theorem 2. *Let R be a finite commutative ring with identity* 1. *Then $m_{(a,b,c)}(R) = \infty$ if and only if $a + b = c$.*

Proof. Suppose $a + b = c$. Assume that R does not satisfy the r-monochromatic clean condition for some positive integer r. Then there is an r-colouring χ of the elements of R such that the set

$$P = \{(x, y, z) \in \text{Idemp}(R) \times \text{Unit}(R) \times R \ : \ ax + by = cz \text{ and } \chi(x) = \chi(y) = \chi(z)\}$$

is empty. By Lemma 2, $1 \in \text{Idemp}(R) \cap \text{Unit}(R)$. Let $(x_1, y_1, z_1) = (1, 1, 1)$. Then $ax_1 + by_1 = a + b = c = cz_1$. This implies that $(x_1, y_1, z_1) \in P$. Since $a(1) + b(1) = c(1)$, we have $(1, 1, 1) \in P$. But this contradicts that $P = \varnothing$. Hence, no such r exists, and this means $m_{(a,b,c)}(R) = \infty$.

Suppose $m_{(a,b,c)}(R) = \infty$. Let $|R| = n$ and χ be an n-colouring of the elements of R such that for all $\alpha, \beta \in R$ where $\alpha \neq \beta$, $\chi(\alpha) \neq \chi(\beta)$. Since $m_{(a,b,c)}(R) = \infty$, R satisfies the n-monochromatic clean condition. So, there exist $x \in \text{Idemp}(R)$, $y \in \text{Unit}(R)$ and $z \in R$ with $\chi(x) = \chi(y) = \chi(z)$ such that $ax + by = cz$. However, no two elements being monochromatic implies that $x = y = z$. It follows from Lemma 2 that $x = y = 1$. So, $z = 1$ and $a + b = c$.

This completes the proof of the theorem. □

Corollary 1. *Let R be a finite commutative ring with identity* 1. *If $m_{(a,b,c)}(R) \neq \infty$, then $1 \leq m_{(a,b,c)}(R) \leq 3$.*

Proof. It is sufficient to show that $m_{(a,b,c)}(R) \leq 3$. Note that $0, 1 \in \text{Idemp}(R)$. Define a 3-colouring χ such that $\chi(u) = 1$ for all $u \in R \setminus \text{Unit}(R)$, $\chi(1) = 2$ and $\chi(v) = 3$ for $v \in \text{Unit}(R) \setminus \{1\}$. Suppose R satisfies the 3-monochromatic clean condition. Then there exist $x \in \text{Idemp}(R)$, $y \in \text{Unit}(R)$ and $z \in R$ with $\chi(x) = \chi(y) = \chi(z)$ such that $ax + by = cz$.

Note that $\chi(y) \neq 1$ because all the units in R are colored with 2 or 3. This implies that x must be a unit. By Lemma 2, $x = 1$. This implies that $y = z = 1$. Thus, $a(1) + b(1) = c(1)$, i.e., $a + b = c$. By Theorem 2, $m_{(a,b,c)}(R) = \infty$, a contradiction. Hence, R does not satisfy the 3-monochromatic clean condition and $m_{(a,b,c)}(R) \leq 3$. □

The following corollary gives a sufficient condition for $m_{(a,b,c)}(R) \leq 2$.

Corollary 2. *Let R be a finite commutative ring with identity* 1. *Suppose* $\text{Idemp}(R) = \{0, 1\}$. *If $b \neq 0, c, -a, c - a$, then $m_{(a,b,c)}(R) \leq 2$.*

Proof. It is sufficient to show that R does not satisfy the 2-monochromatic clean condition. Let χ be the 2-colouring of R such that $\chi(0) = \chi(1) = 1$ and $\chi(u) = 2$ for $u \in R \setminus \{0, 1\}$.

Suppose R satisfies the 2-monochromatic clean condition. Then there exist $x \in \text{Idemp}(R)$, $y \in \text{Unit}(R)$ and $z \in R$ with $\chi(x) = \chi(y) = \chi(z)$ such that $ax + by = cz$. Note that either $x = 0$ or $x = 1$. If the former holds, then $y = 1$. If $z = 1$, then $b = c$, a contradiction. If $z = 0$, then $b = 0$, again, a contradiction. Suppose the latter holds, i.e.,

$x = 1$. Then $y = 1$. If $z = 1$, then $a + b = c$ and hence $b = c - a$, a contradiction. If $z = 0$, then $b = -a$, again, a contradiction. Hence, R does not satisfy the 2-monochromatic clean condition and $m_{(a,b,c)}(R) \leq 2$. □

In general, we know that $m_{(a,b,c)}(R) = 1$ if and only if $ax + by = cz$ has no solution $(x, y, z) \in \text{Idemp}(R) \times \text{Unit}(R) \times R$. For the special case $R = \mathbb{Z}_{p^n}$ where p is a prime and $n \geq 1$, we have Theorem 3. Firstly, we need the following lemma.

Lemma 3. *Let p be a prime and $n \geq 1$. Then* $\text{Idemp}(\mathbb{Z}_{p^n}) = \{0, 1\}$.

Proof. Clearly, $0, 1 \in \text{Idemp}(\mathbb{Z}_{p^n})$. Let $x \in \text{Idemp}(\mathbb{Z}_{p^n})$. Then $x(x-1) = 0$. If $\gcd(x, p) = 1$, then x is a unit. So, $x - 1 = 0$, i.e., $x = 1$. If p divides x, then $\gcd(x - 1, p) = 1$. Thus, $x - 1$ is a unit, and $x = 0$. Hence, the lemma holds. □

Theorem 3. *Let p be a prime and $n \geq 1$. Suppose $a, b, c \in \mathbb{Z}_{p^n}$ with $a + b \neq c$. Then $m_{(a,b,c)}(\mathbb{Z}_{p^n}) = 1$ if and only if*

(a) $\gcd(c, p^n)$ *does not divide b, and*
(b) $\gcd(a, p^n) \neq \gcd(b, p^n)$.

Proof. Since $a + b \neq c$, by Theorem 2, $m_{(a,b,c)}(\mathbb{Z}_{p^n}) \neq \infty$. It follows from Corollary 1 that $m_{(a,b,c)}(\mathbb{Z}_{p^n}) \leq 3$. We may assume that $a, b, c \in \{0, 1, 2 \ldots, p^n - 1\}$. Suppose $m_{(a,b,c)}(\mathbb{Z}_{p^n}) = 1$. This means the following equation

$$ax + by = cz \tag{2}$$

has no solution $(x, y, z) \in \text{Idemp}(\mathbb{Z}_{p^n}) \times \text{Unit}(\mathbb{Z}_{p^n}) \times \mathbb{Z}_{p^n}$.

By Lemma 3, $\text{Idemp}(\mathbb{Z}_{p^n}) = \{0, 1\}$. So, x may take values 0 or 1. Suppose $x = 0$. Note that if $\gcd(c, p^n)$ divides b, then given any $y \in \text{Unit}(\mathbb{Z}_{p^n})$, there exists an integer z satisfying $by \equiv cz \mod p^n$. So, $(0, y, z)$ is a solution to Equation (2), a contradiction. Hence, (a) holds.

Suppose $x = 1$. Since (a) holds, $\gcd(c, p^n) \neq 1$. So, we may assume that $\gcd(c, p^n) = p^i$ for some $1 \leq i \leq n$. Note that $a + by = cz$ in $\mathbb{Z}_{p^n}$ is equivalent to $a + by \equiv cz \mod p^n$. Suppose $\gcd(a, p^n) = \gcd(b, p^n) = p^j$ where $j \geq 0$. If $j \geq i$, then $\gcd(c, p^n) = p^i$ divides $a + by$ for any $y \in \text{Unit}(\mathbb{Z}_{p^n})$. Therefore, there exists an integer z satisfying $a + by \equiv cz \mod p^n$. So, we may assume that $0 \leq j < i$. Now, we consider the equation $a + by \equiv 0 \mod p^i$ with unknown y. Note that $i - j \geq 1$ and the equation

$$\left(\frac{a}{p^j}\right) + \left(\frac{b}{p^j}\right) y \equiv 0 \mod p^{i-j} \tag{3}$$

has an integer solution $y = y_0$ because $\gcd\left(\frac{b}{p^j}, p^{n-j}\right) = 1$ and obviously, 1 divides $-\frac{a}{p^j}$. Furthermore, $\gcd(y_0, p) = 1$. In fact, if $\gcd(y_0, p) = p$, then from Equation (3), p divides $\frac{a}{p^j}$. This is not possible as $\gcd\left(\frac{a}{p^j}, p^{n-j}\right) = 1$. Thus, there is an integer y_0 with $\gcd(y_0, p) = 1$ satisfying $a + by_0 \equiv 0 \mod p^i$. Again, since $\gcd(c, p^n) = p^i$ divides $a + by_0$, there exists an integer z_0 satisfying $a + by_0 \equiv cz_0 \mod p^n$. This implies that $(1, y_0, z_0)$ is a solution to Equation (2), a contradiction. Hence, (b) holds.

Suppose (a) and (b) hold. Assume that $m_{(a,b,c)}(\mathbb{Z}_{p^n}) > 1$. This means there exist $(x, y, z) \in \text{Idemp}(\mathbb{Z}_{p^n}) \times \text{Unit}(\mathbb{Z}_{p^n}) \times \mathbb{Z}_{p^n}$, such that $ax + by = cz$ in $\mathbb{Z}_{p^n}$, i.e., $ax + by \equiv cz \mod p^n$. By Lemma 3, $x = 0$ or 1. Suppose $x = 0$. Then $by \equiv cz \mod p^n$. By (a), $\gcd(c, p^n)$ does not divide b. Thus, $\gcd(c, p^n) \neq 1$. Suppose $\gcd(c, p^n) = p^{i'}$ for some positive integer i'. Since y is a unit, $\gcd(y, p) = 1$. Therefore, $\gcd(c, p^n)$ divides b, a contradiction. Hence, $x \neq 0$.

Suppose $x = 1$. Then $a + by \equiv cz \mod p^n$. Again, by (a), $\gcd(c, p^n) \neq 1$. Let $\gcd(c, p^n) = p^i$ for some positive integer i. Then $a + by \equiv 0 \mod p^i$. Let $\gcd(a, p^n) = p^{j_1}$

and $\gcd(b, p^n) = p^{j_2}$ where $0 \leq j_1, j_2 \leq n$. By (b), $j_1 \neq j_2$. By (a), $j_2 < i$. It follows from $a + by \equiv 0 \mod p^i$ that p^{j_2} divides a. Therefore, $j_1 > j_2$. Since $i \geq j_2 + 1$ and $j_1 \geq j_2 + 1$, we have p^{j_2+1} divides by. This implies that p divides y. On the other hand, y is a unit that implies that $\gcd(y, p) = 1$. This is not possible. Hence, $x \neq 1$ and $m_{(a,b,c)}(\mathbb{Z}_{p^n}) = 1$. □

Now, we consider the case $(a, b, c) = (1, 1, 1)$.

Lemma 4. *Let R be a finite commutative ring with identity 1. Then $m_{(1,1,1)}(R) = 2$ or 3.*

Proof. $(a, b, c) = (1, 1, 1)$ implies that $2 = a + b \neq c = 1$. It follows from Theorem 2 that $m_{(1,1,1)}(R) \neq \infty$. Next, given any $x \in \text{Idemp}(R)$ and $y \in \text{Unit}(R)$, we set $z = x + y$. So, a solution (x, y, z) with $x + y = z$ exists. Thus, $m_{(1,1,1)}(R) > 1$. It follows from Corollary 1 that $m_{(1,1,1)}(R) = 2$ or 3. □

Lemma 5. *Let R be a finite commutative ring with identity 1. Let χ be a 2-coloring of R such that R does not satisfy the 2-monochromatic clean condition subject to the equation $x + y = z$. Then $\chi(0) \neq \chi(y)$ for all $y \in \text{Unit}(R)$.*

Proof. Suppose $\chi(0) = \chi(y_0)$ for some $y_0 \in \text{Unit}(R)$. Then $(x, y, z) = (0, y_0, y_0)$ is a solution to $x + y = z$ with $\chi(0) = \chi(y_0)$. This contradicts that R does not satisfy the 2-monochromatic clean condition subject to the equation $x + y = z$. Hence, $\chi(0) \neq \chi(y)$ for all $y \in \text{Unit}(R)$. □

We note here that $\mathbb{Z}_n$ is a field if and only if n is a prime. The next theorem states that the only field R with $m_{(1,1,1)}(R) = 2$ is $R = \mathbb{Z}_2$.

Theorem 4. *Let R be a field. Then $m_{(1,1,1)}(R) = 2$ if and only if $R = \mathbb{Z}_2$.*

Proof. By Lemma 4, $m_{(1,1,1)}(R) = 2$ or 3.

Suppose $R = \mathbb{Z}_2$. Since $R = \{0, 1\}$, define a 2-colouring χ of the elements of R such that $\chi(0) = 1$ and $\chi(1) = 2$. Clearly, R does not satisfy the 2-monochromatic clean condition. Hence, $m_{(1,1,1)}(R) = 2$.

Suppose $m_{(1,1,1)}(R) = 2$. Then there exists a 2-coloring χ of the elements of R such that R does not satisfy the 2-monochromatic clean condition. By Lemma 5, $\chi(0) \neq \chi(y)$ for all $y \in \text{Unit}(R)$. Since R is a field, $\text{Unit}(R) = R \setminus \{0\}$. Without loss of generality, let $\chi(0) = 1$. Then $\chi(y) = 2$ for all $y \in R \setminus \{0\}$. In particular, $\chi(1) = 2$. If the characteristic of R is not 2, then 2 is a unit and $\chi(2) = 2$. Therefore, $(x, y, z) = (1, 1, 2)$ is a solution to $x + y = z$ and $\chi(1) = \chi(2)$, a contradiction. Hence, the characteristic of R must be 2. Suppose $R \neq \mathbb{Z}_2$. Then $|R| = 2^l$ for some integer $l \geq 2$. So, there is a $y_1 \in R \setminus \{0, 1, -1\}$. Since $1 + y_1 \neq 0$ is in R, it is also a unit. This means $\chi(1 + y_1) = \chi(1) = \chi(y_1)$. Now, $(x, y, z) = (1, y_1, 1 + y_1)$ is a solution to $x + y = z$ with $\chi(1 + y_1) = \chi(1) = \chi(y_1)$, a contradiction. Hence, $l = 1$ and $R = \mathbb{Z}_2$.

This completes the proof of the theorem. □

The following corollary is an immediate consequence of Lemma 4 and Theorem 4.

Corollary 3. *Let R be a field. Then $m_{(1,1,1)}(R) = 3$ if and only if $R \neq \mathbb{Z}_2$.*

Now, we consider the specific ring $\mathbb{Z}_n$.

Theorem 5. *Let $n > 1$. Then $m_{(1,1,1)}(\mathbb{Z}_n) = 2$ if and only if n is even.*

Proof. By Lemma 4, $m_{(1,1,1)}(\mathbb{Z}_n) = 2$ or 3.

Suppose n is even. We will show that $m_{(1,1,1)}(\mathbb{Z}_n) = 2$. This can be achieved by specifying a coloring χ of $\mathbb{Z}_n$ and showing that $\mathbb{Z}_n$ does not satisfy the 2-monochromatic clean condition. Now, let χ be a 2-colouring of $\mathbb{Z}_n$ such that $\chi(u) = 1$ for all odd integers

$u \in \mathbb{Z}_n$ and $\chi(v) = 2$ for all even integers $v \in \mathbb{Z}_n$. Note that if $u_1 \equiv u_2 \mod n$, then either both u_1, u_2 are odd or both u_1, u_2 are even, for n is even. Therefore, χ is well-defined. In particular, $\chi(0) = 2$ and $\chi(1) = 1$. Assume at the moment that $\mathbb{Z}_n$ satisfies the 2-monochromatic clean condition. Then there exist $x \in \text{Idemp}(\mathbb{Z}_n)$, $y \in \text{Unit}(\mathbb{Z}_n)$ and $z \in \mathbb{Z}_n$ with $\chi(x) = \chi(y) = \chi(z)$ such that $x + y = z$. Since y is a unit, $\gcd(y, n) = 1$. This implies that y is odd, for n is even. So, $\chi(y) = 1$. Since $\chi(x) = \chi(y) = 1$, x also must be odd. This implies that $z = x + y$ is even and $\chi(z) = 2$, a contradiction. Hence, $\mathbb{Z}_n$ does not satisfy the 2-monochromatic clean condition and $m_{(1,1,1)}(\mathbb{Z}_n) = 2$.

Suppose $m_{(1,1,1)}(\mathbb{Z}_n) = 2$. We will show that n is even. Then there exists a 2-coloring χ of $\mathbb{Z}_n$ such that $\mathbb{Z}_n$ does not satisfy the 2-monochromatic clean condition. By Lemma 5, $\chi(0) \neq \chi(y)$ for all $y \in \text{Unit}(\mathbb{Z}_n)$. Assume that n is odd. Without loss of generality, we may assume that $\chi(0) = 1$. Note that y is a unit in $\mathbb{Z}_n$ if and only if $\gcd(y, n) = 1$. So, $\chi(1) = 2$. Next, $\gcd(2, n) = 1$ implies that 2 is a unit. Therefore, $\chi(2) = 2$. Now, $(x, y, z) = (1, 1, 2)$ is also a solution to $x + y = z$ with $\chi(1) = \chi(2) = 2$, which is not possible as $\mathbb{Z}_n$ does not satisfy the 2-monochromatic clean condition. Hence, n must be even.

This completes the proof of the theorem. □

Corollary 4. *Let* $1 \leq i \leq l$, $n_i > 1$ *for all* i, *and*

$$R = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_l}.$$

Then

$$m_{(1,1,1)}(R) = \begin{cases} 2, & \text{if } n_i \text{ is even for some } i; \\ 3, & \text{otherwise.} \end{cases}$$

Proof. Suppose n_{i_0} is even for some i_0. Note that the projection mapping $\psi : R \to \mathbb{Z}_{n_{i_0}}$ is a ring homomorphism and ψ is surjective. Furthermore, $\psi(e) = 1$ where

$$e = (\overbrace{1, 1, \ldots, 1}^{l \text{ times}})$$

is the identity in R. By Lemma 1 and Theorem 5, $m_{(1,1,1)}(R) \leq m_{(1,1,1)}(\mathbb{Z}_{n_{i_0}}) = 2$. It follows from Lemma 4 that $m_{(1,1,1)}(R) = 2$.

Suppose n_i is odd for all i. Note that an element $(u_1, u_2, \ldots, u_l) \in R$ is a unit if and only if $\gcd(u_i, n_i) = 1$ for all i. Furthermore, the zero elements in R is

$$\mathbf{0} = (\overbrace{0, 0, \ldots, 0}^{l \text{ times}}),$$

and

$$e = (\overbrace{1, 1, \ldots, 1}^{l \text{ times}})$$

is the identity element in R. Since all n_i are odd, $2e \in \text{Unit}(R)$. By Lemma 4, $m_{(1,1,1)}(R) = 2$ or 3. Assume that $m_{(1,1,1)}(R) = 2$. Then there exists a 2-coloring χ of the elements of R such that R does not satisfy the 2-monochromatic clean condition. By Lemma 5, $\chi(0) \neq \chi(y)$ for all $y \in \text{Unit}(R)$. Let $\chi(\mathbf{0}) = 1$. Then $\chi(e) = \chi(2e) = 2$. Therefore, we have $e + e = 2e$, which is not possible as R does not satisfy the 2-monochromatic clean condition. Hence, $m_{(1,1,1)}(R) = 3$. □

The following corollary is an immediate consequence of Corollary 4,

Corollary 5. *Let p be a prime and $n \geq 1$. Then*

$$m_{(1,1,1)}(\mathbb{Z}_{p^n}) = \begin{cases} 2, & \text{if } p = 2; \\ 3, & \text{otherwise.} \end{cases}$$

3. Concluding Remark

In this paper, we have considered the monochromatic clean condition for finite commutative rings with identity, with a special focus on the value of $m_{(1,1,1)}(R)$. We have determined the values of $m_{(1,1,1)}(\mathbb{Z}_{p^n})$ where p is a prime and $n \geq 1$. The values of $m_{(1,1,1)}(R)$ where $R = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_l}$ are also determined.

There are a few future research directions. One of them is to investigate the monochromatic clean condition for other types of rings, including non-commutative rings. The other is to look at other monochromatic conditions on other types of equations like a quadratic or higher power. This is because the equation considered in this paper is linear and of the form $ax + by = cz$.

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