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Dynamics of Classical Solutions of a Two-Stage Structured Population Model with Nonlocal Dispersal

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Abstract: We study the dynamics of classical solutions of a two-stage structured population model with nonlocal dispersal in a spatially heterogeneous environment and address the question of the persistence of the species. In particular, we show that the species' persistence is completely determined by the sign of the principal spectrum point, λ_p , of the linearized system at the trivial solution: the species goes extinct if $\lambda_p \leq 0$, while it persists uniformly in space if $\lambda_p > 0$. Sufficient conditions are provided to guarantee the existence, uniqueness, and stability of a positive steady state when the parameters are spatially heterogeneous. Furthermore, when the model parameters are spatially homogeneous, we show that the unique positive equilibrium is globally stable with respect to positive perturbations.

Keywords: nonlocal-dispersal; stage-structured model; persistence; steady state; stability

MSC: 92D40; 92D50; 35P15; 35K57



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1. Introduction

In this work, we study the dynamics of solutions of the nonlocal system of partial differential equations (PDE)

$$\begin{cases} \partial_t u_1 = \mu_1 \int_{\Omega} \kappa(x, y)(u_1(t, y) - u_1(t, x)) dy + ru_2 - su_1 - (a + bu_1 + \tau cu_2)u_1 & x \in \Omega, t > 0, \\ \partial_t u_2 = \mu_2 \int_{\Omega} \kappa(x, y)(u_2(t, y) - u_2(t, x)) dy + su_1 - (e + fu_2 + \tau gu_1)u_2 & x \in \Omega, t > 0, \end{cases} \quad (1)$$

where $\mathbf{u}(t, x) = (u_1(t, x), u_2(t, x))$ represents the density function of a single species structured into two stages (juveniles and adults) and living in a bounded habitat $\Omega \subset \mathbb{R}^n$: $u_1(t, x)$ denotes the density function of the juveniles and $u_2(t, x)$ that of the adults who have reached reproductive maturity age. The adults have local reproductive rate $r(x)$ and the juveniles attain reproductive maturity at the rate $s(x)$. The juveniles have local death rate $a(x)$ and self-limitation rate $b(x)$ due to their size in space. The adults have a local death rate $e(x)$ and local self-limitation rate $f(x)$ induced by their size. Due to the interaction between the adults and the juveniles, there may exist an interspecific competition for local resources. Then, we denote by $\tau c(x)$ and $\tau g(x)$ the interspecific local competition rates of the adults and juveniles, respectively. Here, τ is a non-negative number and measures the strength of the interspecific competition. See Figure 1 below for a schematic illustration of the model. Throughout this work, we shall suppose that the kernel function κ is uniformly continuous, positive, and symmetric, i.e., $\kappa(x, y) = \kappa(y, x) > 0$ for every $x, y \in \overline{\Omega}$. Furthermore, we suppose that the parameters of system (1) satisfy the following standing hypothesis.

Hypothesis 1. The functions a, b, c, e, f, g, r , and s are Hölder continuous on $\overline{\Omega}$ and non-negative, r and s are non-identically zero, and b and f are strictly positive.

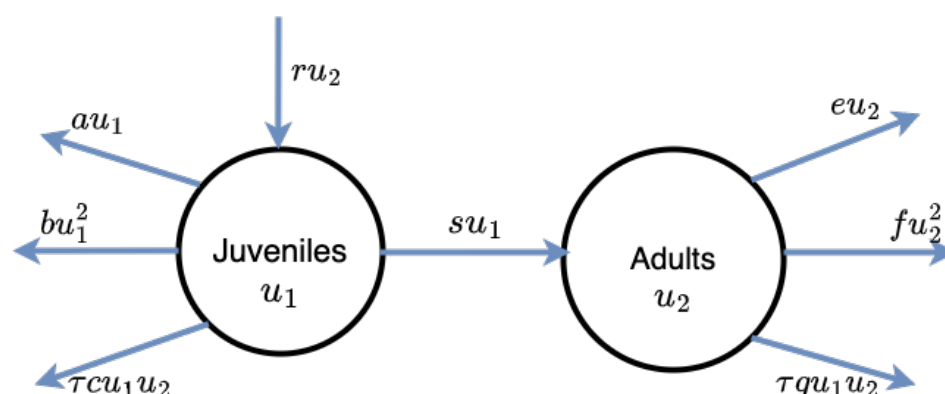


Figure 1. Illustrative diagram of the dynamics of a population structured in two stages, where u_1 is the density of the juveniles and u_2 is the density of the adults.

System (1) is the nonlocal dispersal counterpart of the following population system with random (local) dispersal and Neumann boundary conditions

$$\begin{cases} \partial_t u_1 = \mu_1 \Delta u_1 + ru_2 - su_1 - (a + bu_1 + \tau cu_2)u_1 & x \in \Omega, t > 0, \\ \partial_t u_2 = \mu_2 \Delta u_2 + su_1 - (e + fu_2 + \tau gu_1)u_2 & x \in \Omega, t > 0, \\ \nabla u_1 \cdot \nu = \nabla u_2 \cdot \nu = 0 \text{ on } \partial\Omega, t > 0, \end{cases} \quad (2)$$

where ν is the outward unit normal to $\partial\Omega$. The parameters of system (2) have the same meaning as those of system (1). The models (1) and (2) describe the population dynamics of a dispersing species structured into two-stage subpopulations. The authors in [1,2] studied the model (2) as a problem in optimal control. Most studies on population models typically investigate different species competing for shared resources under the assumption that each species has similar characteristics among its kind. However, it is natural for species to involve individuals at different maturity stages who exhibit varying characteristics which significantly influence their survival or extinction. Consequently, several authors have studied the population dynamics of stage-structured species with a random dispersal mechanism (see for example [1–7] and the references therein). Typically, their analysis on the dynamics of the system depends on the sign of the principal eigenvalue of the linearization of the system at the zero solution. In particular, it was shown in [3] that when $b = \tau c$ and $f = \tau g$ are positive constants, system (2) has a positive steady state if and only if the principal eigenvalue of its linearization at the zero solution is positive.

For unstructured population models for which species dispersal is modeled by pure diffusion, it is known that slow diffusion is advantageous for species' survival (see [8–11] and the references therein). Recently, in [4], the authors examined the influence of diffusion rates on the persistence and extinction of the stage-structured model with random dispersal (2). Their theoretical results indicated that the persistence and extinction of the species depend on the spatial distribution of favorable habitats for the different subpopulations. Essentially, if the regions where adults can reproduce are the same as those where juveniles can mature, typically slow diffusion guarantees persistence, but if those regions are separated, then faster diffusion may be advantageous for the species' survival.

It is typical to assume that a species' dispersal is governed by random walking between adjacent spatial locations. Such an assumption give rise to local dispersal models such as (2). However, some organisms can move between non-adjacent spatial locations, exhibiting long-range dispersal. Nonlocal dispersal equations of the form (1) have been employed to model the population dynamics of such species (see [12–17]). For the nonlocal dispersal operator $\int_{\Omega} \kappa(x, y)(u_i(t, y) - u_i(t, x))dy$ in (1), $\kappa(x, y)$ is a probability density function that gives the probability of jumping from location y to x . Thus, since $u_i(t, x)$ gives the population density at time t and location x , $\int_{\Omega} \kappa(x, y)u_i(t, y)dy$ gives the rate at which

individuals from other places arrive at location x and $-\int_{\Omega} \kappa(x, y) u_i(t, x) dy$ gives the rate at which they leave location x .

Typical examples of such kernel function include the probability density function of the normal distribution

$$\kappa(x, y) = \kappa(x - y) = \frac{1}{\sqrt{(2\pi)^n}} e^{-\frac{|x-y|^2}{2}} \quad x, y \in \mathbb{R}^n.$$

In [16], the authors established the relationship between such random dispersal equations such as (2) with Neumann boundary conditions and the nonlocal dispersal Equation (1). Hence, Equation (1) can be said to have a Neumann-type boundary condition. The techniques developed in the current work hold for general kernel functions having singularities at $x = y$. Depending on the strength of the singularity in the later types of kernels at $x = y$, they can be used to approximate solutions of fractional Laplacian operators (see [16,18–21] and the references therein).

Our goal in this study is to examine to what extent the results obtained for model (2) extend to (1); hence, we study the dynamics of the model (1). Our analysis is based on the spectral bound of the linearization of the model (1) at the zero solution. Our results show that the principal spectrum point (see Definition 2 below) completely determines the persistence or extinction of the species modeled by (1) in the sense that the species die out if the principal spectrum point is nonpositive, but persist uniformly in time when it is positive (see Theorem 1). When persistence occurs, it is appropriate to examine the existence of a strictly positive steady state for the model and its stability with respect to positive perturbations. Understandably, such a steady state depends on the interaction magnitude τ . In the case of a spatially homogeneous environment, our studies show that for any $\tau \geq 0$, the system (1) has a unique positive steady state which attracts every positive solution (see Theorem 2). It is pertinent to mention that due to the lack of compactness induced by the nonlocal operators, the analysis of nonlocal dispersal systems pose extra challenge than their local dispersal counterparts. Nevertheless, for the spatially heterogeneous environment, we obtained sufficient conditions for the existence, uniqueness and local stability of a positive stationary solution of (1) when $\tau > 0$ (see Theorems 3 and 4). Given that the persistence of the species modeled by (1) depends on the sign of the principal spectrum point, we established some criteria on the model parameters for the positivity or otherwise of the principal spectrum point. Thus, Theorem 5 provides sufficient conditions on the parameters for the persistence or extinction of the species.

The rest of the paper is organized as follows. Section 2 contains notations, definitions, and the main results. In Section 3, we provide a brief discussion of our theoretical results on model (1), compare them to some related results on model (2), and highlight some future directions. The proof of our main results are presented in Appendices A–C.

2. Notations, Definitions and Main Results

2.1. Notations and Definitions

Before stating our main results on the classical solutions of (1), we first introduce a few notations and definitions. Let $X := C(\bar{\Omega})$ denote the Banach space of uniformly continuous and bounded functions on $\bar{\Omega}$ endowed with the sup-norm; $\|u\|_{\infty} = \sup_{x \in \bar{\Omega}} |u(x)|$ for every $u \in X$. Since the density functions are non-negative, we will be interested in the following subsets of X :

$$X^+ = \{u \in X \mid u(x) \geq 0, \quad x \in \bar{\Omega}\},$$

and

$$X^{++} = \{u \in X^+ \mid \inf_{x \in \bar{\Omega}} u(x) > 0\}.$$

We shall use bold-face letters to represent vectors. In particular, we use $\mathbf{u} = (u_1, u_2)$ to denote vectors in $\mathbb{R}^2$. For convenience, we also use $\mathbf{u} = (u_1, u_2)$ for elements of $X \times X$.

Hence, any vector in $\mathbb{R}^2$ can be understood as a constant vector valued function on $\overline{\Omega}$; this would not cause any confusion in the paper. We endow $X \times X$ with the norm

$$\|\mathbf{u}\| := \max\{\|u_1\|_\infty, \|u_2\|_\infty\} \quad \forall \mathbf{u} = (u_1, u_2) \in X \times X.$$

Hence, $X \times X$ is also a Banach space. Given a function $l \in X$, we set

$$l_{\min} := \min_{x \in \overline{\Omega}} l(x) \quad \text{and} \quad l_{\max} = \max_{x \in \overline{\Omega}} l(x).$$

We consider the following ordering in $X \times X$:

$$\mathbf{u} \leq_1 \mathbf{v} \text{ if } v_i - u_i \in X^+, i = 1, 2 \quad \text{and} \quad \mathbf{u} <_1 \mathbf{v} \text{ if } v_i - u_i \in X^{++}, i = 1, 2. \quad (3)$$

We adopt the following definition for classical solutions of (1).

Definition 1. Let $\tau \geq 0$, $\mathbf{u}_0 = (u_{1,0}, u_{2,0}) \in X^+ \times X^+$ and $T \in (0, \infty]$. We say that a function $\mathbf{u}(\cdot, \cdot; \tau, \mathbf{u}_0) \in C([0, T] \times \overline{\Omega})$ is a classical solution of (1) with initial data $\mathbf{u}_0$ if it is continuously differentiable with respect to t on $[0, T)$, satisfies (1) in the classical sense, $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0) \in X^+ \times X^+$ for every $t \in [0, T)$ and

$$\lim_{t \rightarrow 0^+} \|\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0) - \mathbf{u}_0\| = 0.$$

When $T = \infty$, we say that the classical solution $\mathbf{u}(\cdot, \cdot; \tau, \mathbf{u}_0)$ is globally defined in time.

Thanks to the type of nonlinearity in (1), the following result on the existence and uniqueness of classical solutions of system (1) follows from standard arguments from the literature on semigroup theory and positivity of semigroup generated by nonlocal operators [22,23].

Proposition 1. Let $\tau \geq 0$. Given any initial data $\mathbf{u}_0 \in X^+ \times X^+$, there is a unique corresponding global and bounded classical solution $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)$. Moreover, if $\mathbf{u}_0$ is not identically null, then $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0) \in X^{++} \times X^{++}$ for all $t > 0$.

It is clear that the null function $\mathbf{0} \equiv (0, 0)$, called the trivial solution, is an equilibrium solution of (1). To study the stability of the trivial solution, the sign of the "principal spectrum point" of the linearization of model (1) at $\mathbf{0}$ will be essential. For convenience, we first recall the following definition.

Definition 2 (Principal spectrum point). Let E be a Banach space and $B : \text{dom}(B) \rightarrow E$ be a linear map where $\text{dom}(B)$ is a linear subspace of E . Let $\sigma(B)$ denote the spectrum of B . The principal spectrum point (PSP) of B , denoted by $\lambda_p(B)$, is defined as

$$\lambda_p(B) := \begin{cases} \sup\{\text{Re}(\lambda) : \lambda \in \sigma(B)\} & \text{if } \sigma(B) \neq \emptyset, \\ -\infty & \text{if } \sigma(B) = \emptyset. \end{cases}$$

For our purpose in the current work, the Banach space in Definition 2 will be either $E = X$ or $E = X \times X$, while the linear operator B will always be a bounded linear map on E . Hence, in the current work, $\sigma(B)$ will always be nonempty and bounded, so that the PSP $\lambda_p(B)$ is a real number.

Given $h \in X$, consider the bounded linear operators $\mathcal{K}, \mathcal{I}, h\mathcal{I} : X \rightarrow X$

$$\mathcal{K}u(x) = \int_{\Omega} \kappa(x, y)(u(y) - u(x))dy \quad \forall u \in X, x \in \overline{\Omega}, \quad (4)$$

$$\mathcal{I}u(x) = u(x) \quad \forall u \in X, x \in \overline{\Omega}, \quad (5)$$

and

$$(h\mathcal{I})u(x) = h(x)u(x) \quad \forall u \in X, x \in \overline{\Omega}. \quad (6)$$

Next, given a bounded linear operator $\mathcal{B}$ on X and a real vector $\mathbf{z} = (z_1, z_2) \in \mathbb{R}^2$, we denote by $\mathbf{z} \circ \mathcal{B}$ the bounded linear operator on $X \times X$ given by

$$\mathbf{z} \circ \mathcal{B}\mathbf{u} = \begin{pmatrix} z_1 \mathcal{B}u_1 \\ z_2 \mathcal{B}u_2 \end{pmatrix} \quad \forall \mathbf{u} \in X \times X. \quad (7)$$

For convenience, set

$$A(x) = \begin{pmatrix} -(a(x) + s(x)) & r(x) \\ s(x) & -e(x) \end{pmatrix} \quad \forall x \in \overline{\Omega}, \quad (8)$$

and define the bounded linear operator $\mathcal{A} : X \times X \rightarrow X \times X$,

$$(\mathcal{A}\mathbf{u})(x) = A(x)\mathbf{u}(x) = \begin{pmatrix} r(x)u_2(x) - (a(x) + s(x))u_1(x) \\ s(x)u_1(x) - e(x)u_2(x) \end{pmatrix} \quad \forall \mathbf{u} \in X \times X, x \in \overline{\Omega}. \quad (9)$$

Note that by the Perron–Frobenius theorem, for every $x \in \overline{\Omega}$, the cooperative matrix $A(x)$ has a maximal eigenvalue, which we shall denote by $\Lambda(x)$.

2.2. Main-Results

We state our main results in the current subsection. Our first result is on the global stability of the trivial solution $\mathbf{0}$.

Theorem 1. Let $\tau \geq 0$ and $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A})$ be the PSP of the bounded linear map $\mu \circ \mathcal{K} + \mathcal{A}$. Let $\mathbf{u}(\cdot, \cdot; \tau, \mathbf{u}_0)$ be a classical solution of (1) with a positive initial data $\mathbf{u}_0 \in X^+ \times X^+ \setminus \{\mathbf{0}\}$.

(i) If $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) < 0$, then for every $\xi \in (0, 1)$, there is $t_\xi > 0$, independent of initial data and τ such that

$$\|\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)\| \leq e^{t\xi\lambda_p(\mu \circ \mathcal{K} + \mathcal{A})}\|\mathbf{u}_0\|, \quad \forall t > t_\xi. \quad (10)$$

(ii) If $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) = 0$, then

$$\limsup_{t \rightarrow \infty} \sup_{\tau \geq 0} \|\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)\| = 0. \quad (11)$$

(iii) If $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) > 0$, then there is $\eta_* > 0$, independent of initial data, such that

$$\limsup_{t \rightarrow \infty} \|\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)\| \geq \eta_*. \quad (12)$$

Furthermore if $r \in X^{++}$ and $s \in X^{++}$, then there is $\eta^* > 0$ which depends on the initial data $\mathbf{u}_0$ and τ , such that

$$u_i(t, x; \tau, \mathbf{u}_0) \geq \eta^*, \quad t \geq 1, x \in \Omega, i = 1, 2. \quad (13)$$

It follows from Theorem 1 that the sign of $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A})$ determines completely the persistence of the species. Indeed, if $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) \leq 0$, then by Theorem 1 (i)–(ii), every classical solution eventually goes extinct in the long run uniformly in space. However, if $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) > 0$, then the species persists uniformly in the sense of (13). It is important to note that $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A})$ is independent of the non-negative parameter $\tau \geq 0$. Hence, the persistence or extinction of the species is independent of the magnitude of τ .

A natural quest is to understand the long-time behavior of classical solutions of (1) when $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) > 0$. In particular, we are interested in whether (1) has a stable steady-state solution which attracts all positive solutions whenever $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) > 0$. In this direction, when all the parameters are independent of x , our next result shows that (1) has a

unique positive equilibrium point which attracts every classical solution. More specifically, the following result holds.

Theorem 2. Suppose that $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) > 0$ and a, b, c, e, f, r , and s are spatially homogeneous. Then for every $\tau \geq 0$, (1) has a unique constant positive steady-state solution $\mathbf{u}^\tau$. Moreover, given any positive initial data $\mathbf{u}_0$, $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)$ converges to $\mathbf{u}^\tau$ as t tends to infinity in the sense that

$$\lim_{t \rightarrow \infty} \|\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0) - \mathbf{u}^\tau\| = 0. \quad (14)$$

Thanks to Theorems 1 and 2, the global dynamics of classical solutions of system (1) are well understood when the parameters in the model are spatially homogeneous. However, when these parameters depend on x and $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) > 0$, the existence and uniqueness of a positive steady state remain unclear. This is due to the lack of compactness induced by the nonlocal operators used to describe the dispersal modes of the species and the fact that the flow of classical solutions induced by (1) is not monotone when $\tau > 0$.

When the parameters depend on x and $\tau = 0$, system (1) is cooperative, and some standard methods, based on the theory of monotone dynamical systems from the literature, can be employed to study the global dynamics of the classical solutions (see [24]). Our next result provides some sufficient conditions, which guarantees the existence, uniqueness, and local stability of a positive steady state of system (1) when $\tau > 0$. In the next two results, we suppose that the parameter functions depend on the variable x .

Theorem 3. (i) Let $\mu > 0$ such that $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) > 0$. Then there is $\tau^* \in (0, \infty]$ such that for every $\tau \in [0, \tau^*)$, (1) has a unique positive steady state solution $\mathbf{u}^\tau \in X^{++} \times X^{++}$. Moreover, for every $\tau \in [0, \tau^*)$, $\mathbf{u}^\tau$ is linearly stable.
(ii) If $\Lambda_{\min} > 0$, then for every $\tau > 0$, there is $\mu^\tau > 0$ such that for every diffusion rate $\mu = (\mu_1, \mu_2)$ satisfying $0 < \mu_i < \mu^\tau$, (1) has a unique positive steady state solution $\mathbf{u}^{\mu, \tau} \in X^{++} \times X^{++}$. Furthermore, $\mathbf{u}^{\mu, \tau}$ is linearly stable.

For each μ for which $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) > 0$, Theorem 3 (i) shows that system (1) has a unique and linearly stable positive steady state for every $\tau \in [0, \tau^*)$. Note from Theorem 2 that $\tau^* = \infty$ when the parameters are spatially homogeneous. In general, when the parameters depend on the spatial variable, it is unclear whether $\tau^* = \infty$. Our next result is concerned with the global stability of the positive steady state given by Theorem 3.

Theorem 4. Suppose that $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) > 0$. The following conclusions hold.

- (i) If $\tau = 0$, then the unique strictly positive steady state $\mathbf{u}^0$ of (1) is globally stable with respect to positive perturbations in the sense of (14).
- (ii) If $r_{\min} > 0$ and $s_{\min} > 0$, then there is $\tau^{**} \in (0, \tau^*]$, where τ^* is given by Theorem 3 (i) such that for every $\tau \in [0, \tau^{**})$ the unique positive steady state $\mathbf{u}^\tau$ of (1) is globally stable with respect to positive perturbations in the sense of (14).

Next, we examine some sufficient conditions on the spatial heterogeneity of model (1) to determine the sign of $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A})$ for any diffusion rate $\mu = (\mu_1, \mu_2)$. First, note that $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A})$ is independent of the diffusion rate μ when all the parameters are spatially homogeneous. In this case, $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A})$ equals the maximal eigenvalue of the cooperative matrix A . In general, when the parameters depend on the space variable x , $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A})$ may not be an eigenvalue (see [25,26] for examples). Moreover, in such a case, there is no explicit variational characterization formula for $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A})$. From the application viewpoint, it is of particular importance to find sufficient conditions to determine the sign of $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A})$. Inspired by the results in [4], we have the following result.

Theorem 5. Suppose that $(a + s)_{\min} > 0$ and $e_{\min} > 0$.

- (i) If either $\left(\frac{s}{e}\right)_{\max} \left(\frac{r}{s+a}\right)_{\max} < 1$ or $\left(\frac{4(r+s)^2}{e(a+s)}\right)_{\max} < 1$, then $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) < 0$ for any diffusion rate $\mu = (\mu_1, \mu_2)$.
- (ii) If either $\left(\frac{s}{e}\right)_{\min} \left(\frac{r}{s+a}\right)_{\min} > 1$ or $\int_{\Omega} \sqrt{rs} > \frac{1}{2} \int_{\Omega} (a + s + e)$, then $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) > 0$ for any diffusion rate $\mu = (\mu_1, \mu_2)$.

Theorem 5 provides sufficient hypotheses on the parameters which guarantee that the sign of $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A})$ is independent of the diffusion rate μ of the species. Note that, when the parameters are spatially homogeneous, $\Lambda := \Lambda(x)$ is independent of μ and $\Lambda > 0$ if $sr > (a + s)e$, $\Lambda = 0$ if $rs = (a + s)e$, and $\Lambda < 0$ if $rs < (a + s)e$. Hence, the first assumption in Theorem 5 (i) and (ii) are consistent with that of the ODE model. Next, observe that $\frac{s}{e}$ (resp. $\frac{r}{s+a}$) is the ratio of the growth rate to the self-limitation rate of the adults (resp. juveniles). Hence, on one hand, Theorem 5 (i) indicates that if the product of the maximum of these ratios is smaller than one, then species cannot survive and will go extinct in the long run irrespective of the diffusion rate. On the other hand, if the product of the minimum of these ratios is greater than one, Theorem 5 (ii) indicates that the species will always persist uniformly in space for any diffusion rate.

3. Conclusions and Future Directions

In this study, we examined the dynamics of classical solutions of a population adopting a long-range dispersal migration and structured in two stages, juveniles and adults. In particular, we addressed the question of the persistence of the species. In this direction, Theorem 1 indicates that the sign of the principal spectrum point λ_p of the linearization of system (1) at the trivial solution completely determines the species' persistence or extinction. More precisely, if $\lambda_p \leq 0$, the species eventually goes extinct uniformly in space while it persists uniformly in space if $\lambda_p > 0$. This result agrees with the dynamics of classical solutions of model (2) where a random local dispersal is used to describe the evolution of the species. Indeed, Cantrell, Cosner, and Martinez (Lemma 2 in [4]) showed that if the principal eigenvalue, $\lambda_{\text{eig},p}$, of the linearization of system (2) at the trivial solution is positive, then the species persists uniformly in space while it goes extinct otherwise. Note that the analyses of system (2) rely largely on the compactness of its solution operator; however, the semiflow generated by classical solutions of (1) is not compact. Consequently, λ_p might not be an eigenvalue. This lack of compactness of the semiflow generated by classical solutions of system (1) induces some difficulties, and several techniques developed for system (2) cannot be applied to system (1).

When $\lambda_p > 0$ and the environment is homogeneous, Theorem 2 shows that the species eventually stabilize in the long run at the unique stationary equilibrium solution. Our proof of this result relies on constructing a delicate Lyapunov function (see Lemma A3). This Lyapunov function is concave and can also be used to obtain the global stability of classical solutions of system (2) and the corresponding kinetic model. It is important to note that Bouguima and Mehlija (Theorem 2.1 in [27]) used the Poincaré–Bendixon theory to conclude the uniqueness and global stability of the unique stationary equilibrium for the kinetic model. However, this method is only suitable for the ODE model.

When the environment is heterogeneous, and $\lambda_p > 0$ under suitable smallest assumption on the interspecific competitions terms between the juveniles and the adults, Theorems 3 and 4 guaranteed the existence and stability of a positive steady state solution of system (1). It would be of great interest to know whether (1) always has a unique and globally stable positive steady state solution whenever $\lambda_p > 0$. This question also remains open for system (2). In this direction, for system (2) and when $\lambda_{\text{eig},p} > 0$, it is important to note that Brown and Zhang (Theorem 3.2 in [3]) established the existence of at least one positive steady state, Bouguima and Mehlija (Proposition 6.1 in [27]) obtained a sufficient condition of the initial data which guarantee the global stability of the unique positive steady state solution, and Cantrell, Cosner, and Martinez (Lemma 3 in [4]) conclude the global stability of the positive steady solution if τ is sufficiently small.

There are several important prospects to consider in future studies on the dynamics of solutions of (1).

- (i) **Effect of dispersal rates on λ_p .** The necessary and sufficient condition for persistence is the positivity of λ_p . Unlike the case of unstructured single species and local reaction diffusion equation, there is no explicit formula for λ_p . This makes the study of λ_p with respect to the diffusion rates difficult. However, in several instances, juveniles do not move or move very slowly whereas adults have a high movement rate. This is the case for some species of birds. Hence, it is of great biological interest to study the asymptotic behavior of λ_p when one of the diffusion rates is small while the other is large.
- (ii) **Global stability of positive steady states.** Understanding the long-time behavior of classical solutions of model (1) is essential since it helps to provide some accurate prediction on the future of the species. In the current work, we completely settled this question in the case of a homogeneous environment or the smallest assumption on τ . Further efforts are needed for the case of arbitrary $\tau > 0$.
- (iii) **Asymptotic profile of positive steady states with respect to diffusion rates.** When species persist and eventually stabilize, from an ecological viewpoint, it is important to know the spatial distribution of the species. This would be determined by the influence of the diffusion rates on the steady states. In this direction, we hope that future works would explore the dependence of positive steady-state solutions of system (1) on the diffusion rates.
- (iv) **Effect of temporal heterogeneity on the dynamics of (1).** An important fact not considered in our study is the effect of temporal heterogeneity on the dynamics of solutions of model (1). It would be of important biological interest to study the dynamics of solutions of (1) in time-periodic environments.

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Appendix A. Proof of Theorem 1

For convenience, we first present some preliminary results to be used in the proofs of our main results in the subsequent sections. Linearizing (1) at the trivial solution gives

$$\begin{cases} \partial_t U_1 = \mu_1 \mathcal{K}(U_1) + rU_2 - (a + s)U_1 & x \in \Omega, \\ \partial_t U_2 = \mu_2 \mathcal{K}(U_2) + sU_1 - eU_2 & x \in \Omega. \end{cases} \quad (\text{A1})$$

Note that (A1) can be rewritten as

$$\frac{d\mathbf{U}}{dt} = \mu \circ \mathcal{K}(\mathbf{U}) + \mathcal{A}\mathbf{U} \quad t > 0. \quad (\text{A2})$$

Let $\{\mathbf{U}^\mu(t)\}_{t \geq 0}$ denote the uniformly continuous semigroup generated by the bounded linear operator $\mu \circ \mathcal{K} + \mathcal{A}$ on $X \times X$. Hence, for every $\mathbf{U}_0 \in X \times X$, $\mathbf{U}(t, \mathbf{U}_0) := \mathbf{U}^\mu(t)\mathbf{U}_0$ is the unique solution of (A2), equivalently (A1), with initial data $\mathbf{U}_0$. Since $\mathcal{A}(x)$ is cooperative for each $x \in \overline{\Omega}$, the uniformly continuous semigroup $\{\mathbf{U}^\mu(t)\}_{t \geq 0}$ is strongly positive, in the sense that

$$\mathbf{U}^\mu(t)(X^+ \times X^+) \subset X^+ \times X^+ \quad \text{and} \quad \mathbf{U}^\mu(t)((X^+ \times X^+) \setminus \{\mathbf{0}\}) \subset X^{++} \times X^{++} \quad \forall t > 0. \quad (\text{A3})$$

Define the nonlinear continuous function $\mathcal{F} : [0, \infty) \times X \times X \rightarrow X \times X$,

$$\mathcal{F}(\tau, \mathbf{u}) = \begin{pmatrix} -bu_1 - \tau cu_2 \\ -fu_2 - \tau gu_1 \end{pmatrix} \circ \mathbf{u} \quad \forall \mathbf{u} \in X \times X, \tau \geq 0. \quad (\text{A4})$$

Hence, the nonlocal PDE (1) can be rewritten in the compact form

$$\partial_t \mathbf{u}(t, \cdot) = (\mu \circ \mathcal{K} + \mathcal{A})\mathbf{u}(t, \cdot) + \mathcal{F}(\tau, \mathbf{u}(t, \cdot)) \quad \forall t > 0. \quad (\text{A5})$$

Therefore, by the variation of constants formula, for a given initial data $\mathbf{u}_0$, if $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)$ exists on a time interval $[0, T)$ for some $T > 0$, then it can be rewritten as

$$\mathbf{u}(t, \cdot) = \mathbf{U}^\mu(t)\mathbf{u}_0 + \int_0^t \mathbf{U}^\mu(t-\eta)\mathcal{F}(\tau, \mathbf{u}(\eta, \cdot; \tau, \mathbf{u}_0))d\eta \quad \forall t \in [0, T). \quad (\text{A6})$$

The following result shows that the principal spectrum point $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A})$ equals the exponential growth bound of the uniformly continuous semigroup $\{\mathbf{U}^\mu(t)\}_{t \geq 0}$. This result turns out to be of particular importance in the arguments used to prove some of our main results.

Proposition A1. $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) \in \sigma(\mu \circ \mathcal{K} + \mathcal{A})$ and

$$\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) = \lim_{t \rightarrow \infty} \frac{\ln(\|\mathbf{U}^\mu(t)\|)}{t}. \quad (\text{A7})$$

Proof. Since $\{\mathbf{U}^\mu(t)\}_{t \geq 0}$ is a uniformly continuous semigroup, (A7) follows from (Chapter IV, Proposition 2.2 and Corollary 2.4 in [28]). Furthermore, since $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A})$ is a finite number and the uniformly continuous semigroup $\{\mathbf{U}^\mu(t)\}_{t \geq 1}$ is positive, then $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) \in \sigma(\mu \circ \mathcal{K} + \mathcal{A})$ by (Chapter 6, Theorem 2.10 in [28]). $\square$

Let $\{e^{t\mathcal{K}}\}_{t \geq 0}$ denote the uniformly continuous semigroup of bounded linear operators generated by the bounded linear operator $\mathcal{K}$ on X . This means that given any $u_0 \in X$, $u(t, x; u_0) := (e^{t\mathcal{K}}u_0)(x)$ is the unique solution of

$$\begin{cases} \partial_t u = \mathcal{K}u(t, x) & x \in \bar{\Omega}, \\ u(0, \cdot) = u_0(x) & x \in \bar{\Omega}. \end{cases} \quad (\text{A8})$$

By the comparison principle for linear nonlocal operators, $\{e^{t\mathcal{K}}\}_{t \geq 0}$ is strongly monotone, in the sense that

$$e^{t\mathcal{K}}u < e^{t\mathcal{K}}v \quad \text{for all } t > 0 \text{ whenever } u \leq, \neq v, u, v \in X. \quad (\text{A9})$$

Finally, we introduce the function

$$K(x) = \int_{\Omega} \kappa(x, y)dy \quad \forall x \in \bar{\Omega}.$$

Note that $K \in X^{++}$.

Now, observe that when $\tau = 0$ in (1), it reduces to the following cooperative system

$$\begin{cases} \partial_t u_1 = \mu_1 \mathcal{K}u_1 + ru_2 - su_1 - (a + bu_1)u_1 & x \in \Omega, t > 0, \\ \partial_t u_2 = \mu_2 \mathcal{K}u_2 + su_1 - (e + fu_2)u_2 & x \in \Omega, t > 0. \end{cases} \quad (\text{A10})$$

Choose a positive constant vector $\mathbf{M} = (M_1, M_2)$ such that

$$\mathcal{A}(\mathbf{M}) + \mathcal{F}(0, \mathbf{M})(x) = A(x)\mathbf{M} - \begin{pmatrix} b(x)M_1^2 \\ f(x)M_2^2 \end{pmatrix} <_1 \mathbf{0} \quad \forall x \in \overline{\Omega}. \quad (\text{A11})$$

Then, $\mathbf{u}_0(x) \equiv \mathbf{M}$ is a supper solution of (A10), which implies that the unique classical solution $\mathbf{u}(t, \cdot; 0, \mathbf{M})$ of system (A10), equivalently of (1) with $\tau = 0$, with initial data $\mathbf{M}$ decreases over time. Let

$$\mathbf{u}^*(x; 0, \mathbf{M}) = \lim_{t \rightarrow \infty} \mathbf{u}(t, x; 0, \mathbf{M}) \quad \forall x \in \overline{\Omega}. \quad (\text{A12})$$

The following result holds.

Lemma A1. Let $\mathbf{u}^*(x; 0, \mathbf{M})$ be defined by (A12). Then $\mathbf{u}^*(\cdot; 0, \mathbf{M}) \in X^+ \times X^+$ and is a steady state solution of (A10). Moreover, if $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) \leq 0$, then $\mathbf{u}^*(\cdot; 0, \mathbf{M}) \equiv 0$. However, if $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) > 0$, then $\mathbf{u}^*(\cdot; 0, \mathbf{M}) \in X^{++} \times X^{++}$ and is the unique positive steady state solution of (A10).

Proof. Step 1. We first show that $\mathbf{u}^* := \mathbf{u}^*(\cdot; 0, \mathbf{M})$ is a steady state solution of (A10). To do this, we observe that $\mathbf{u}(t, x) := \mathbf{u}(t, x; 0, \mathbf{M})$ is a solution to (A10) with initial $\mathbf{M}$; therefore,

$$\begin{cases} \partial_t u_1(t, x) = \mu_1 \mathcal{K} u_1 + r u_2(t, x) - (a + s + b u_1(t, x)) u_1(t, x) \\ \partial_t u_2(t, x) = \mu_2 \mathcal{K} u_2 + s u_1(t, x) - (e + f u_2(t, x)) u_2(t, x). \end{cases}$$

Observe that, for each $x \in \Omega$,

$$\begin{aligned} u_1(t+1, x) - u_1(t, x) &= \int_0^1 \partial_t u_1(t + \eta, x) d\eta \\ &= \int_0^1 (\mu_1 \mathcal{K} u_1(t, x) + r u_2(t, x) - s u_1(t, x) - (a + b u_1(t, x)) u_1(t, x)) d\eta \end{aligned}$$

Taking the limit of both sides, using the dominated convergence theorem yields

$$0 = \int_0^1 (\mu_1 \mathcal{K} u_1^*(x) + r u_2^*(x) - (a + s + b u_1^*(x)) u_1^*(x)) d\eta \quad \forall x \in \Omega,$$

which implies that

$$0 = \mu_1 \int_{\Omega} \kappa(x, y) (u_1^*(y) - u_1^*(x)) dy + r u_2^*(x) - (a + s + b u_1^*(x)) u_1^*(x) \quad \forall x \in \Omega.$$

Similar arguments can be applied to the second equation of (A10) to show that

$$0 = \mu_2 \int_{\Omega} \kappa(x, y) (u_2^*(y) - u_2^*(x)) dy + s u_1^*(x) - (e(x) + f(x) u_2^*(x)) u_2^*(x) \quad \forall x \in \Omega.$$

Hence, $\mathbf{u}^*$ is a steady state solution of (A12).

Step 2. We show that this limit function is continuous, i.e., $\mathbf{u}^* \in X^+ \times X^+$. First note that $\mathbf{u}^*$ satisfies

$$\begin{cases} 0 = r(x) u_2^*(x) - (a(x) + s(x) + \mu_1 K(x) + b(x) u_1^*(x)) u_1^*(x) + \mu_1 K_1^*(x) & x \in \Omega, \\ 0 = s(x) u_1^*(x) - (e(x) + \mu_2 K(x) + f(x) u_2^*(x)) u_2^*(x) + \mu_2 K_2^*(x) & x \in \Omega \end{cases} \quad (\text{A13})$$

where

$$K_i^*(x) = \int_{\Omega} \kappa(x, y) u_i^*(y) dy \quad \forall x \in \Omega, i = 1, 2.$$

Now, we distinguish two cases.

Case 1. In this case, we suppose that there is some $x_0 \in \overline{\Omega}$ and $i \in \{1, 2\}$ such that $u_i^*(x_0) = 0$. Without loss of generality, we suppose that $i = 1$. Then, from the first equation of (A13), we obtain that

$$r(x_0)u_2^*(x_0) = 0 \quad \text{and} \quad \mu_1 K_1^*(x_0) = 0, \quad (\text{A14})$$

since $ru_2^* \geq 0$ and $\mu_1 K_1^* \geq 0$ on $\overline{\Omega}$. It then follows from the formula for K_1^* , the positivity of the kernel κ , and the fact that $u_1^* \geq 0$ on $\overline{\Omega}$ and measurable, that u_1^* equals zero almost everywhere on $\overline{\Omega}$ and $K_1^* \equiv 0$. Consequently, we obtain from the first equation of (A13) that $ru_2^* = 0$ almost everywhere on Ω . Thus, by multiplying the second equation of (A13) by r and using the fact that $u_1^* = 0$ almost everywhere on Ω , we obtain that

$$r(x)K_2^*(x) = 0 \quad \text{almost everywhere on } \Omega,$$

from which we deduce that $K_2^*(x_1) = 0$ for some $x_1 \in \Omega$ because $r > 0$ on a set of positive measure. Whence, as above, we also obtain that $u_2^* = 0$ almost everywhere on Ω and $K_2^* \equiv 0$. Let

$$\mathcal{N}_i := \{x \in \overline{\Omega} : u_i^*(x) > 0\}.$$

Since $K_i^* \equiv 0$ for every $i = 1, 2$ and $K > 0$ on $\overline{\Omega}$, we then obtain from (A13) and the fact that $u_i^* = 0$ almost everywhere on Ω , that

$$\mathcal{N} := \mathcal{N}_1 = \mathcal{N}_2 \subset \{x \in \overline{\Omega} : s(x) > 0\} \cap \{x \in \overline{\Omega} : r(x) > 0\} \quad \text{and} \quad |\mathcal{N}| = 0.$$

Now, we claim that $\mathcal{N} = \emptyset$. To this end, we assume that $\mathcal{N} \neq \emptyset$. Let $\tilde{a}, \tilde{b}, \tilde{e}, \tilde{f}, \tilde{r}$, and $\tilde{s}$ be Hölder continuous functions which extend the functions a, b, c, e, f, r , and s on some open neighborhood, $\tilde{\Omega}$, of $\overline{\Omega}$, respectively. Consider the function $\tilde{\mathbf{F}} : \tilde{\Omega} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\tilde{\mathbf{F}}(x, \mathbf{u}) = (\tilde{F}_1(x, \mathbf{u}), \tilde{F}_2(x, \mathbf{u}))$$

where

$$\tilde{F}_1 = \tilde{r}(x)u_2 - (\tilde{a}(x) + \tilde{s}(x) + \mu_1 K(x) + \tilde{b}(x)u_1)u_1 + \mu_1 K_1^*(x)$$

and

$$\tilde{F}_2 = \tilde{s}(x)u_1 - (\tilde{e}(x) + \mu_2 K(x) + \tilde{f}(x)u_2)u_2 + \mu_2 K_2^*(x).$$

Clearly, $\tilde{\mathbf{F}}$ is Hölder continuous in $x \in \tilde{\Omega}$ locally uniformly for $\mathbf{u}$ and of class C^∞ in $\mathbf{u} \in \mathbb{R}^2$. Calculating the Jacobian of $\tilde{\mathbf{F}}$ at $\mathbf{u}$ yields

$$\frac{\partial \tilde{\mathbf{F}}}{\partial \mathbf{u}}(x, \mathbf{u}) = \begin{pmatrix} -(\tilde{a}(x) + \tilde{s}(x) + \mu_1 K(x) + 2\tilde{b}(x)u_1) & \tilde{r}(x) \\ \tilde{s}(x) & -(\tilde{e}(x) + \mu_2 K(x) + 2\tilde{f}(x)u_2) \end{pmatrix}$$

Evaluating the determinant of $\frac{\partial \tilde{\mathbf{F}}}{\partial \mathbf{u}}$ at $(x^*, \mathbf{u}^*(x^*))$ for $x^* \in \mathcal{N}$, we obtain

$$\det\left(\frac{\partial \tilde{\mathbf{F}}}{\partial \mathbf{u}}(x^*, \mathbf{u}^*(x^*))\right) = (a(x^*) + s(x^*) + \mu_1 K(x^*) + 2b(x^*)u_1^*(x^*)) \left(e(x^*) + \mu_2 K(x^*) + 2f(x^*)u_2^*(x^*) \right) - r(x^*)s(x^*).$$

In this case, we observe that (A13) yields

$$(a(x^*) + s(x^*) + \mu_1 K(x^*) + b(x^*)u_1^*(x^*)) = r(x^*) \frac{u_2^*(x^*)}{u_1^*(x^*)}$$

and

$$(e(x^*) + \mu_2 K(x^*) + f(x)u_2^*(x^*)) = s(x^*) \frac{u_1^*(x^*)}{u_2^*(x^*)}$$

Therefore, the above determinant becomes

$$\det\left(\frac{\partial \tilde{\mathbf{F}}}{\partial \mathbf{u}}(x^*, \mathbf{u}(x^*))\right) = \left(r(x^*) \frac{u_2^*(x^*)}{u_1^*(x^*)} + b(x^*)u_1^*(x^*)\right) \left(s(x^*) \frac{u_1^*(x^*)}{u_2^*(x^*)} + f(x^*)u_2^*(x^*)\right) - r(x^*)s(x^*) > 0.$$

Hence, by the implicit function theorem (Theorem 1.1 in [29]), there exist open neighborhoods $\mathcal{O} \subset \tilde{\Omega}$ of x^* and $\mathcal{U} \subset \mathbb{R}^2$ of $\mathbf{u}(x^*)$, and a continuous function $\tilde{\mathbf{u}} : \mathcal{O} \rightarrow \mathcal{U}$ such that for every $x \in \mathcal{O}$, the algebraic equation $\tilde{\mathbf{F}}(x, \mathbf{u}) = \mathbf{0}$ has a solution if and only if $\mathbf{u} = \tilde{\mathbf{u}}(x)$. Therefore, $\mathbf{u}^*(x) = \tilde{\mathbf{u}}(x)$ for all $x \in \bar{\Omega} \cap \mathcal{O}$. Moreover, by the continuity of $\tilde{\mathbf{u}}$ at x^* and the fact that $\min\{u_1^*(x^*), u_2^*(x^*)\} > 0$, there is an open neighborhood $\tilde{\mathcal{O}} \subset \subset \mathcal{O}$ of x^* such that $\eta^* := \min_{x \in \tilde{\mathcal{O}}} \{\tilde{u}_1(x), \tilde{u}_2(x)\} > 0$. In particular,

$$\min\{u_1^*(x), u_2^*(x)\} \geq \eta^* \quad \forall x \in \tilde{\mathcal{O}} \cap \bar{\Omega},$$

which implies that $\tilde{\mathcal{O}} \cap \bar{\Omega} \subset \mathcal{N}$. Hence, $0 < |\tilde{\mathcal{O}} \cap \bar{\Omega}| \leq |\mathcal{N}|$, which gives a contradiction. Therefore, $\mathcal{N} = \emptyset$. Therefore, we obtain here that $\mathbf{u}^* \equiv \mathbf{0} \in X^+ \times X^+$.

Case 2. Suppose here that $u_j^*(x) > 0$ for every $x \in \bar{\Omega}$. We show that $\mathbf{u}^* \in X^+ \times X^+$. Consider again the function $\tilde{\mathbf{F}}$ defined above. This time, for every $x \in \bar{\Omega}$, we have

$$\begin{aligned} & \det\left(\frac{\partial \tilde{\mathbf{F}}}{\partial \mathbf{u}}(x, \mathbf{u}^*(x))\right) \\ &= \left(r(x) \frac{u_2^*(x)}{u_1^*(x)} + \mu_1 \frac{K_1^*(x)}{u_1^*(x)} + b(x^*)u_1^*(x)\right) \left(s(x) \frac{u_1^*(x)}{u_2^*(x)} + \mu_2 \frac{K_2^*(x)}{u_2^*(x)} + f(x)u_2^*(x)\right) - r(x)s(x) \\ &> 0. \end{aligned}$$

Hence, it follows from the implicit function theorem (Theorem 1.1 in [29]) that the continuous function $\tilde{\mathbf{u}}$ exists on $\bar{\Omega}$ as in the previous case and that $\mathbf{u}^* = \tilde{\mathbf{u}}$ on $\bar{\Omega}$. Therefore, $\mathbf{u}^*$ is continuous on $\bar{\Omega}$.

From cases 1 and 2 above, we have that $\mathbf{u}^* \in X^+ \times X^+$.

Step 3. Suppose that $\lambda_p(\mu \circ \mathcal{K}, A) \leq 0$; we show that $\mathbf{u}^* \equiv \mathbf{0}$. From the above, we have that either $\mathbf{u}^* \equiv \mathbf{0}$ or $\mathbf{0} <_1 \mathbf{u}^*$. Suppose, for contradiction, that $\mathbf{0} <_1 \mathbf{u}^*$. We observe that since $\mathbf{u}^*$ is a steady state solution of (A10) and (H) holds, taking $\lambda_0 := \min_{x \in \bar{D}} \{b(x)u_1^*(x), f(x)u_2^*(x)\} > 0$, then

$$\lambda_0 \mathbf{u}^* \leq_1 (\mu \circ \mathcal{K} + \mathcal{A}) \mathbf{u}^*.$$

This implies that

$$e^{t\lambda_0} \mathbf{u}^* \leq_1 \mathbf{U}^\mu(t) \mathbf{u}^* \quad \forall t > 0.$$

Hence, since $e^{t\lambda_0} \mathbf{u}^* \in X^{++} \times X^{++}$ for every $t > 0$, then

$$0 < e^{t\lambda_0} \|\mathbf{u}^*\| \leq \|\mathbf{U}^\mu(t) \mathbf{u}^*\| \leq \|\mathbf{U}^\mu(t)\| \|\mathbf{u}^*\| \quad \forall t > 0.$$

Thus, using (A7), we obtain

$$0 < \lambda_0 \leq \lim_{t \rightarrow \infty} \frac{\ln(\|\mathbf{U}^\mu(t)\|)}{t} = \lambda_p(\mu \circ \mathcal{K} + \mathcal{A}).$$

This contradicts the fact that $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) \leq 0$. Therefore, we must have $\mathbf{u}^* \equiv 0$.

Step 4. Suppose that $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) > 0$. We show that $\mathbf{u}^* \in X^{++} \times X^{++}$. Since $\mathbf{u}^* \in X^+ \times X^+$, by Dini's theorem, the limit in (A12) holds uniformly in $x \in \overline{\Omega}$. This implies that for every $\varepsilon > 0$, there is $t_\varepsilon > 0$ such that

$$\|\mathbf{u}(t, \cdot; 0, \mathbf{M})\| < \|\mathbf{u}^*\| + \varepsilon \quad \forall t > t_\varepsilon.$$

Hence, taking $\lambda_1 := \max\{\|b\|_\infty, \|f\|_\infty\}$, and observing that

$$(\mu \circ \mathcal{K} + \mathcal{A})\mathbf{u}(t + t_\varepsilon, \cdot; 0, \mathbf{M}) \leq_1 \lambda_1(\|\mathbf{u}^*\| + \varepsilon)\mathbf{u} + \partial_t \mathbf{u}(t + t_\varepsilon, \cdot; 0, \mathbf{M}) \quad \forall t > 0,$$

we obtain,

$$\mathbf{U}^\mu(t)\mathbf{u}(t_\varepsilon, \cdot; 0, \mathbf{M}) \leq_1 e^{\lambda_1(\|\mathbf{u}^*\| + \varepsilon)t}\mathbf{u}(t + t_\varepsilon, \cdot; 0, \mathbf{M}) \quad \forall t > 0.$$

Now, choose $l_1 \gg 1$ such that $l_1 u_i(t, \cdot; 0, \mathbf{M}) \geq 1$ for each $i = 1, 2$. It then follows from the above inequality that

$$\mathbf{U}^\mu(t)(l_1 \mathbf{u}(t_\varepsilon, \cdot; 0, \mathbf{M})) \leq_1 e^{\lambda_1(\|\mathbf{u}^*\| + \varepsilon)t}(l_1 \mathbf{u}(t + t_\varepsilon, \cdot; 0, \mathbf{M})) \quad \forall t > 0.$$

From which we deduce that

$$\|\mathbf{U}^\mu(t)(l_1 \mathbf{u}(t_\varepsilon, \cdot; 0, \mathbf{M}))\| \leq_1 e^{\lambda_1(\|\mathbf{u}^*\| + \varepsilon)t} \|l_1 \mathbf{u}(t + t_\varepsilon, \cdot; 0, \mathbf{M})\| \quad \forall t > 0. \quad (\text{A15})$$

Since $l_1 u_i^*(t_\varepsilon, \cdot; 0, \mathbf{M}) \geq 1$ for every $i = 1, 2$, we deduce that

$$\|\mathbf{U}^\mu(t)\| \leq \|\mathbf{U}^\mu(t)(l_1 \mathbf{u}(t_\varepsilon, \cdot; 0, \mathbf{M}))\| \quad \forall t > 0. \quad (\text{A16})$$

(A15) and (A16) yield

$$\lambda_1(\|\mathbf{u}^*\| + \varepsilon) + \frac{\ln(l_1) + \ln(\|\mathbf{u}(t + t_\varepsilon, \cdot; 0, \mathbf{M})\|)}{t} \geq \frac{\ln(\|\mathbf{U}^\mu(t)\|)}{t} \quad \forall t > 0. \quad (\text{A17})$$

Therefore, since (A7) holds and $\|\mathbf{u}(t, \cdot; 0, \mathbf{M})\| \leq \|\mathbf{M}\|$ for all $t \geq 0$, letting $t \rightarrow \infty$ in (A17) gives

$$\lambda_1(\|\mathbf{u}^*\| + \varepsilon) \geq \lambda_p(\mu \circ \mathcal{K} + \mathcal{A}).$$

We then conclude that $\|\mathbf{u}^*\| > 0$ since ε is arbitrary chosen and $\lambda_1 > 0$. It then follows from case 2 of step 1 that $\mathbf{u}^* \in X^{++} \times X^{++}$. Moreover, since $\tau = 0$, it is easily seen that (1) is strongly cooperative and subhomogeneous, therefore, $\mathbf{u}^*$ is the unique strictly positive steady-state of (1). $\square$

Remark A1. When $\tau = 0$ and $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) > 0$, Lemma A1 shows that (1) has a unique strictly positive steady state $\mathbf{u}^0$. This shows that the function $\mathbf{u}^*(\cdot; 0, \mathbf{M})$ in (A12) is independent of the positive constant vector $\mathbf{M}$. In what follows, we shall let $\mathbf{u}^0(\cdot) = \mathbf{u}^*(\cdot; 0, \mathbf{M})$.

Lemma A2. Suppose that $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) > 0$ and let $\tau > 0$ be fixed. Suppose also that $r_{\min} > 0$ and $s_{\min} > 0$. Let $\mathbf{u}^0$ be the unique positive steady state solution of (A10) given by Lemma A1. There is $\varepsilon_\tau > 0$ such that for every $0 < \varepsilon < \varepsilon_\tau$, $\varepsilon \mathbf{u}^0 \leq_1 \mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)$ for all $t \geq 0$ whenever $\varepsilon \mathbf{u}^0 \leq_1 \mathbf{u}_0$.

Proof. Let $\tilde{\varepsilon} := \min \left\{ \left(\frac{b(u_1^0)^2}{2u_2^0} \right)_{\min}, \left(\frac{f(u_2^0)^2}{2u_1^0} \right)_{\min}, r_{\min}, s_{\min} \right\}$ and

$$\varepsilon_\tau := \min \left\{ \frac{1}{2}, \frac{\tilde{\varepsilon}}{1 + \tau \|cu_1^0\|_\infty}, \frac{\tilde{\varepsilon}}{1 + \tau \|gu_2^0\|_\infty} \right\}.$$

Let $0 < \varepsilon < \varepsilon_\tau$ and $\mathbf{u}_0 \in X^{++} \times X^{++}$ satisfy $\varepsilon \mathbf{u}^0 \leq_1 \mathbf{u}_0$. Consider the mapping $\mathcal{M}$, defined by

$$\mathcal{M}_1(\phi_1) = \mu_1 \int_{\Omega} \kappa(x, y) [\phi_1(t, y) - \phi_1(t, x)] dy + (r - \tau c \phi_1) u_2(t, \cdot; \tau, \mathbf{u}_0) - (a + s + b \phi_1) \phi_1$$

and

$$\mathcal{M}_2(\phi_2) = \mu_2 \int_{\Omega} \kappa(x, y) [\phi_2(t, y) - \phi_2(t, x)] dy + (s - \tau g \phi_2) u_1(t, \cdot; \tau, \mathbf{u}_0) - (e + f \phi_2) \phi_2$$

for every $\phi_1, \phi_2 \in C([0, \infty) \times \overline{\Omega} : \mathbb{R}^+)$. Since $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)$ solves (1), then $\mathbf{u}(t, x; \tau, \mathbf{u}_0)$ satisfies

$$\partial_t \mathbf{u} = \mathcal{M}(\mathbf{u}) \quad x \in \Omega, t > 0. \quad (\text{A18})$$

Next, fix $\varepsilon \in (0, \varepsilon)$ and let

$$T_{\max, \varepsilon} := \sup\{T > 0 : \hat{\varepsilon} \mathbf{u}^0 <_1 \mathbf{u}(t, \cdot; \tau, \mathbf{u}_0) \forall 0 \leq t < T\}.$$

Since $\hat{\varepsilon} < \varepsilon$ and $\varepsilon \mathbf{u}^0 \leq_1 \mathbf{u}_0$, then $T_{\max, \varepsilon} \in (0, \infty]$. Now, we claim that $T_{\max, \varepsilon} = \infty$. If not, then there is some $x_0 \in \overline{\Omega}$ and $i \in \{1, 2\}$ such that

$$\hat{\varepsilon} u_i^0(x_0) = u_i(T_{\max, \varepsilon}, x_0; \tau, \mathbf{u}_0). \quad (\text{A19})$$

Without loss of generality, we may suppose that $i = 1$, since the case of $i = 2$ can be proved similarly. From (A18), $u_1(t, x; \tau, \mathbf{u}_0)$ is the unique classical solution of

$$\begin{cases} \partial_t u = \mathcal{M}_1(u) & x \in \Omega, t > 0, \\ u(0, \cdot) = u_1(0, \cdot; \tau, \mathbf{u}_0). \end{cases} \quad (\text{A20})$$

Now, since $\mathbf{u}^0$ is a positive steady state solution of (A10), then

$$\begin{aligned} \mathcal{M}_1(\hat{\varepsilon} u_1^0) &= \hat{\varepsilon} \mu_1 \int_{\Omega} \kappa(x, y) [u_1^0(y) - u_1^0(x)] dy + (r - \hat{\varepsilon} \tau c u_1^0) u_2(t, \cdot; \tau, \mathbf{u}_0) - (a + s + b \hat{\varepsilon} u_1^0) (\hat{\varepsilon} u_1^0) \\ &= (a + s + b u_1^0) (\hat{\varepsilon} u_1^0) - r (\hat{\varepsilon} u_2^0) + (r - \hat{\varepsilon} \tau c u_1^0) u_2 - (a + s + b (\hat{\varepsilon} u_1^0)) (\hat{\varepsilon} u_1^0) \\ &= \hat{\varepsilon} b (u_1^0)^2 - \hat{\varepsilon} \hat{\varepsilon} u_2^0 - (r - \hat{\varepsilon}) (\hat{\varepsilon} u_2^0) + ((r - \hat{\varepsilon}) + (\hat{\varepsilon} - \hat{\varepsilon} \tau c u_1^0)) u_2 - b (\hat{\varepsilon} u_1^0)^2 \\ &= \hat{\varepsilon} \left(\frac{b}{2} (u_1^0)^2 - \hat{\varepsilon} u_2^0 \right) + (r - \hat{\varepsilon}) (u_2 - \hat{\varepsilon} u_2^0) + (\hat{\varepsilon} - \hat{\varepsilon} \tau c u_1^0) u_2 + \left(\frac{1}{2} - \hat{\varepsilon} \right) \hat{\varepsilon} b (u_1^0)^2 \\ &\geq \nu \hat{\varepsilon} u_1^0 \end{aligned}$$

for all $t \in [0, T_{\max, \varepsilon}]$, where $\nu := (\frac{1}{2} - \hat{\varepsilon}) b_{\min} u_{1, \min}^0 > 0$. As a result, choosing $0 < \hat{\nu} \ll 1$ such that

$$(e^{\hat{\nu} t} - 1) \left(\|b\|_{\infty} \hat{\varepsilon} \|u_1^0\|_{\infty} + e^{-\hat{\nu} t} \frac{\|r\|_{\infty}}{\hat{\varepsilon} u_{1, \min}^0} \sup_{t \geq 0} \|\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)\| \right) + \hat{\nu} < \nu,$$

the function $\hat{u}(t, x) = e^{\hat{\nu} t} \hat{\varepsilon} \mathbf{u}^0(x)$ satisfies

$$\begin{aligned} \mathcal{M}_1(\hat{u}_1) &= \mu_1 \int_{\Omega} \kappa(x, y) [\hat{u}_1(t, y) - \hat{u}_1(t, x)] dy + (r - \tau c \hat{u}_1) u_2(t, \cdot; \tau, \mathbf{u}_0) - (a + s + b \hat{u}_1) (\hat{u}_1) \\ &= e^{\hat{\nu} t} \mathcal{M}_1(\hat{\varepsilon} u_1^0) + b \hat{\varepsilon} u_1^0 (1 - e^{\hat{\nu} t}) \hat{u}_1 + (1 - e^{\hat{\nu} t}) r u_2(t, \cdot; \tau, \mathbf{u}_0) \\ &\geq \nu \hat{u}_1 + b \hat{\varepsilon} u_1^0 (1 - e^{\hat{\nu} t}) \hat{u}_1 + (1 - e^{\hat{\nu} t}) r u_2(t, \cdot; \tau, \mathbf{u}_0) \\ &\geq \left(\nu - \|b\|_{\infty} \hat{\varepsilon} \|u_1^0\|_{\infty} (e^{\hat{\nu} t} - 1) - (1 - e^{-\hat{\nu} t}) \frac{\|r\|_{\infty} \|\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)\|}{\hat{\varepsilon} u_{1, \min}^0} \right) \hat{u}_1 \\ &= \left(\nu - (e^{\hat{\nu} t} - 1) \left(\|b\|_{\infty} \hat{\varepsilon} \|u_1^0\|_{\infty} + e^{-\hat{\nu} t} \frac{\|r\|_{\infty} \|\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)\|}{\hat{\varepsilon} u_{1, \min}^0} \right) \right) \hat{u}_1 \\ &> \hat{\nu} \hat{u}_1 = \partial_t \hat{u}_1 \end{aligned}$$

for all $t \in [0, T_{\max, \hat{\varepsilon}}]$. Therefore, by the comparison principle for single-species nonlocal equations we obtain that $\hat{u}_1(t, x) \leq u_1(t, x; \tau, \mathbf{u}_0)$ for all $t \in [0, T_{\max, \hat{\varepsilon}}]$. In particular,

$$\hat{u}_1^0(x_0) \leq e^{-\hat{\nu} T_{\max, \hat{\varepsilon}}} u_1(T_{\max, \hat{\varepsilon}}, x_0; \tau, \mathbf{u}_0) < u_1(T_{\max, \hat{\varepsilon}}, x_0; \tau, \mathbf{u}_0),$$

which contradicts (A19). Therefore, $T_{\max, \hat{\varepsilon}} = \infty$. This implies that $\hat{\varepsilon} \mathbf{u}^0 \leq_1 \mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)$ for any $t \geq 0$ and $0 < \hat{\varepsilon} < \varepsilon$. Letting $\hat{\varepsilon} \rightarrow \varepsilon$, yields the desired result. $\square$

Now, we present the proof of Theorem 1.

Proof of Theorem 1. (i) Let $\mathbf{u}(\cdot, \cdot; \tau, \mathbf{u}_0)$ be a classical solution of (1) with a non-negative initial data $\mathbf{u}_0$. We first suppose that $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) < 0$. Observe that $\mathbf{u}(t, x; \tau, \mathbf{u}_0)$ is a subsolution of the linear cooperative system (A1), that is

$$\begin{cases} \partial_t u_1 \leq \mu_1 \int_{\Omega} \kappa(x, y)(u_1(t, y) - u_1(t, x)) dy + ru_2 - (s + a)u_1 & x \in \Omega, t > 0, \\ \partial_t u_2 \leq \mu_2 \int_{\Omega} \kappa(x, y)(u_2(t, y) - u_2(t, x)) dy + su_1 - eu_2 & x \in \Omega, t > 0, \end{cases}$$

Hence $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0) \leq \mathbf{U}(t, \mathbf{u}_0)$ for all $t > 0$. If $\mathbf{u}_0 \equiv \mathbf{0}$, then $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0) \equiv \mathbf{0}$ for every $t > 0$, and the result follows. Now, suppose that $\mathbf{u}_0 \in X^+ \times X^+ \setminus \{\mathbf{0}\}$. Hence $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0) > \mathbf{0}$ for all $t > 0$. Since $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0) \leq \mathbf{U}(t, \mathbf{u}_0)$ for all $t > 0$, then

$$\frac{\ln(\|\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)\|)}{t} \leq \frac{\ln(\|\mathbf{U}(t, \cdot, \mathbf{u}_0)\|)}{t} \leq \frac{\ln(\|\mathbf{U}^\mu(t)\|)}{t} + \frac{\ln(\|\mathbf{u}_0\|)}{t} \quad \forall t > 0. \quad (\text{A21})$$

Now, let $\xi \in (0, 1)$. By (A7), there is $t_\xi \gg 1$, independent of $\mathbf{u}_0$ and τ , such that

$$\frac{\ln(\|\mathbf{U}^\mu(t)\|)}{t} \leq \xi \lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) \quad \forall t \geq t_\xi,$$

which together with (A21) yields

$$\|\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)\| \leq \|\mathbf{u}_0\| e^{t_\xi \xi \lambda_p(\mu \circ \mathcal{K} + \mathcal{A})} \quad \forall t \geq t_\xi.$$

Hence, (10) holds.

(ii) Choose $\mathbf{M}$ such that $\mathbf{u}_0 \leq_1 \mathbf{M}$. Since $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)$ is a subsolution of (A10), it follows that

$$\sup_{\tau > 0} \mathbf{u}(t, \cdot; \tau, \mathbf{u}_0) \leq \mathbf{u}(t, \cdot; 0, \mathbf{M})$$

Hence, from Lemma A1, we have that if $\lambda(\mu \circ \mathcal{K} + \mathcal{A}) = 0$, then

$$\limsup_{t \rightarrow \infty} \sup_{\tau > 0} \|\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)\| \leq \lim_{t \rightarrow \infty} \|\mathbf{u}(t, \cdot; 0, \mathbf{M})\| = 0.$$

(iii) Set $L := \limsup_{t \rightarrow \infty} \|\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)\|$. For all $\varepsilon > 0$ there exists t_ε such that

$$\|\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)\| \leq L + \varepsilon \quad \forall t \geq t_\varepsilon.$$

Taking $\lambda_1 = \max\{\|b\|_\infty, \|f\|_\infty\}$, $\lambda_2 = \max\{\|c\|_\infty, \|g\|_\infty\}$, and $\lambda_\tau = \lambda_1 + \tau \lambda_2$ we have

$$(\mu \circ \mathcal{K} + \mathcal{A})\mathbf{u}(t + t_\varepsilon, \cdot; \tau, \mathbf{u}_0) \leq_1 \lambda_\tau(L + \varepsilon)\mathbf{u} + \partial_t \mathbf{u}(t + t_\varepsilon, \cdot; \tau, \mathbf{u}_0) \quad \forall t > 0.$$

Whence,

$$\mathbf{U}^\mu(t)\mathbf{u}(t_\varepsilon, \cdot; \tau, \mathbf{u}_0) \leq_1 e^{\lambda_1(L+\varepsilon)t}\mathbf{u}(t + t_\varepsilon, \cdot; \tau, \mathbf{u}_0) \quad \forall t > 0.$$

Following a similar argument as in Step 4 of Lemma A1 we obtain that

$$\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) \leq \lambda_\tau(L + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\limsup_{t \rightarrow \infty} \|u(t, \cdot; \tau, \mathbf{u}_0)\| \geq \eta^*$$

where $\eta^* = \frac{\lambda_p(\mu \circ \mathcal{K} + \mathcal{A})}{\lambda_\tau}$ depends only on τ but is independent of the initial $\mathbf{u}_0$. This proves (12). Finally, since $\mathbf{u}(1, \cdot; \tau, \mathbf{u}_0) \in X^{++} \times X^{++}$, there is $0 < \varepsilon < \varepsilon_\tau$ (where ε_τ is given by Lemma A2) such that $\varepsilon \mathbf{u}^0 \leq \mathbf{u}(1, \cdot; \tau, \mathbf{u}_0)$. It then follows from Lemma A2 that $\varepsilon \mathbf{u}^0 \leq_1 \mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)$ for every $t \geq 1$, which yields (13). $\square$

Remark A2. Suppose that $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) > 0$ and let $\tau \geq 0$. Let $\mathbf{u}^0$ be the unique positive steady state solution of (A10) and $\varepsilon_\tau > 0$ be given by Lemma A2. Then, for every $0 < \varepsilon < \varepsilon_\tau$, the closed interval $[\varepsilon \mathbf{u}^0, \mathbf{u}^0]$ is an invariant set for the flow generated by the solution of (1).

Appendix B. Proof of Theorem 2

We shall suppose that all the parameters a, b, c, e, f, r , and s are constant. In this case, we have that $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A})$ is independent of the diffusion rate μ and $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) = \Lambda_{\max}$. Throughout this section, we suppose that $\Lambda_{\max} > 0$. By Remark A2, the kinetic model of (1), i.e., $\mu_1 = \mu_2 = 0$, is weakly uniformly persistent. Therefore, since solutions of (1) are eventually bounded, we can apply the persistence theory of dynamical systems [30,31] to conclude that for every $\tau > 0$, there is a positive equilibrium solution $\mathbf{u}^\tau$ of

$$\begin{cases} 0 = ru_2^\tau - (a + s + bu_1^\tau + \tau cu_2^\tau)u_1^\tau \\ 0 = su_1^\tau - (e + fu_2^\tau + \tau gu_1^\tau)u_2^\tau. \end{cases} \quad (\text{A22})$$

Note that $\mathbf{u}^\tau$ is a positive spatially homogeneous equilibrium solution of (1). As we shall see from Lemma A3 below, $\mathbf{u}^\tau$ is the unique positive solution of system (A22). For convenience, let

$$h_{1,2} := \frac{r}{u_1^\tau} - \tau c, \quad h_{2,1} := \frac{s}{u_2^\tau} - \tau g, \quad \text{and} \quad \Delta_1 := \frac{rs - (a + s)e - (a + s)fu_2^\tau - beu_1^\tau}{u_1^\tau u_2^\tau}.$$

Observe from (A22) that

$$h_{1,2} = \frac{a + s + bu_1^\tau}{u_2^\tau} > 0 \quad \text{and} \quad h_{2,1} = \frac{e + fu_2^\tau}{u_1^\tau} > 0.$$

Observe also from (A22) that

$$\begin{aligned} rs &= (a + s + bu_1^\tau + \tau cu_2^\tau)(e + fu_2^\tau + \tau gu_1^\tau) \\ &= (a + s)e + (a + s)fu_2^\tau + \tau(a + s)gu_1^\tau + beu_1^\tau + (fu_2^\tau + \tau gu_1^\tau)bu_1^\tau \\ &\quad + \tau(e + fu_2^\tau + \tau gu_1^\tau)cu_2^\tau. \end{aligned}$$

Hence,

$$\Delta_1 = \frac{\tau(a + s)gu_1^\tau + (fu_2^\tau + \tau gu_1^\tau)bu_1^\tau + \tau(e + fu_2^\tau + \tau gu_1^\tau)cu_2^\tau}{u_1^\tau u_2^\tau} > 0.$$

Next, let

$$\Delta_{1,1} := \min \left\{ \frac{\Delta_1}{b}, f \right\}, \quad \Delta_{2,2} := \min \left\{ \frac{\Delta_1}{f}, b \right\}, \quad \text{and} \quad h := \frac{h_{1,2}}{h_{2,1}}. \quad (\text{A23})$$

It follows from the above computations that $\Delta_{1,1} > 0$, $\Delta_{2,2} > 0$ and $h > 0$. Finally, we introduce the function

$$\tilde{\mathcal{L}}(\eta) = \eta - 1 - \ln(\eta) \quad \forall \eta > 0. \quad (\text{A24})$$

Note that $\tilde{\mathcal{L}}(\eta) \geq 0$ for all $\eta > 0$ with equality if and only if $\eta = 1$. Note also that the function $\tilde{\mathcal{L}}$ is strictly convex. The function $\tilde{\mathcal{L}}$ will be used to construct a Lyapunov function in the proof of Theorem 4.

Lemma A3. Let $\tau > 0$ be fixed, and $\mathbf{u}^\tau$ be a positive solution of (A22). Let $\Delta_{1,1}$, $\Delta_{2,2}$ and h be defined by (A23). Let $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)$ be a classical solution of (1) with a positive initial data $\mathbf{u}_0 \in X^{++} \times X^{++}$. Define

$$\mathcal{L}_i(t) = \int_{\Omega} u_i^\tau \tilde{\mathcal{L}}\left(\frac{u_i(t, x; \tau, \mathbf{u}_0)}{u_i^\tau}\right) dx \quad \forall t \geq 0, i = 1, 2,$$

where $\tilde{\mathcal{L}}$ is defined by (A24) and set

$$\mathcal{L}(t) = \mathcal{L}_1(t) + h\mathcal{L}_2(t) \quad \forall t > 0.$$

Then, for all $t > 0$,

$$\frac{d\mathcal{L}(t)}{dt} \leq - \sum_{i=1}^2 \frac{\mu_i u_i^\tau h^i}{2h} \int_{\Omega} \int_{\Omega} \kappa(x, y) \frac{(u_i(t, y) - u_i(t, x))^2}{u_i(t, x) u_i(t, y)} dy dx - \sum_{i=1}^2 \frac{h^2 \Delta_{i,i}}{2h^i} \int_{\Omega} (u_i - u_i^\tau)^2. \quad (\text{A25})$$

Proof. Note also that $h_{1,2}$, $h_{2,1}$ and h are constants. For convenience, we set

$$h_{1,1} := b + \frac{ru_2}{u_1^\tau u_1} \quad \text{and} \quad h_{2,2} = f + \frac{su_1}{u_2^\tau u_2}.$$

By computations, thanks to (A22) and the fact that κ is symmetric, we obtain

$$\begin{aligned} \frac{d\mathcal{L}_1(t)}{dt} &= \int_{\Omega} \partial_t u_1(t, x) \left(1 - \frac{u_1^\tau}{u_1(t, x)}\right) dx \\ &= \int_{\Omega} \left(r \frac{u_2}{u_1} - (a + s + bu_1 + \tau cu_2)\right) (u_1 - u_1^\tau) dx \\ &\quad + \mu_1 \int_{\Omega} \int_{\Omega} \kappa(x, y) [u_1(t, y) - u_1(t, x)] \left(1 - \frac{u_1^\tau}{u_1}\right) dy dx \\ &= \int_{\Omega} \left(r \frac{u_2}{u_1} - (a + s + bu_1^\tau + \tau cu_2^\tau) - (b(u_1 - u_1^\tau) + \tau c(u_2 - u_2^\tau))\right) (u_1 - u_1^\tau) dx \\ &\quad - \mu_1 u_1^\tau \int_{\Omega} \int_{\Omega} \kappa(x, y) \frac{[u_1(t, y) - u_1(t, x)]}{u_1(t, x)} dy dx \\ &= \int_{\Omega} \left(r \left(\frac{u_2}{u_1} - \frac{u_2^\tau}{u_1^\tau}\right) - (b(u_1 - u_1^\tau) + \tau c(u_2 - u_2^\tau))\right) (u_1 - u_1^\tau) dx \\ &\quad - \mu_1 u_1^\tau \int_{\Omega} \int_{\Omega} \kappa(x, y) \frac{[u_1(t, y) - u_1(t, x)]}{u_1(t, x)} dy dx \\ &= \int_{\Omega} \left(r \left(\frac{u_2}{u_1} - \frac{u_2}{u_1^\tau} + \frac{u_2}{u_1^\tau} - \frac{u_2^\tau}{u_1^\tau}\right) - (b(u_1 - u_1^\tau) + \tau c(u_2 - u_2^\tau))\right) (u_1 - u_1^\tau) dx \\ &\quad - \mu_1 u_1^\tau \int_{\Omega} \int_{\Omega} \kappa(x, y) \frac{[u_1(t, y) - u_1(t, x)]}{u_1(t, x)} dy dx \\ &= \int_{\Omega} \left(r \left(\frac{u_2}{u_1^\tau u_1} (u_1^\tau - u_1) + \frac{1}{u_1^\tau} (u_2 - u_2^\tau)\right) - (b(u_1 - u_1^\tau) + \tau c(u_2 - u_2^\tau))\right) (u_1 - u_1^\tau) dx \\ &\quad - \mu_1 u_1^\tau \int_{\Omega} \int_{\Omega} \kappa(x, y) \frac{[u_1(t, y) - u_1(t, x)]}{u_1(t, x)} dy dx \\ &= \int_{\Omega} \left(h_{1,2} (u_2 - u_2^\tau) (u_1 - u_1^\tau) - h_{1,1} (u_1 - u_1^\tau)^2\right) dx \\ &\quad - \mu_1 u_1^\tau \int_{\Omega} \int_{\Omega} \kappa(x, y) \frac{[u_1(t, y) - u_1(t, x)]}{u_1(t, x)} dy dx. \end{aligned} \quad (\text{A26})$$

Similarly,

$$\begin{aligned}\frac{d\mathcal{L}_2(t)}{dt} &= \int_{\Omega} \left(h_{2,1}(u_1 - u_1^\tau)(u_2 - u_2^\tau) - h_{2,2}(u_2 - u_2^\tau)^2 \right) dx \\ &\quad - \mu_2 u_2^\tau \int_{\Omega} \int_{\Omega} \kappa(x, y) \frac{[u_2(t, y) - u_2(t, x)]}{u_2(t, x)} dy dx.\end{aligned}\quad (\text{A27})$$

It follows from (A26) and (A27) that

$$\begin{aligned}\frac{d\mathcal{L}(t)}{dt} &= - \int_{\Omega} \left(h_{1,1}(u_1 - u_1^\tau)^2 - (h_{1,2} + hh_{2,1})(u_1 - u_1^\tau)(u_2 - u_2^\tau) + hh_{2,2}(u_2 - u_2^\tau)^2 \right) \\ &\quad - \sum_{i=1}^2 \mu_i u_i^\tau h^{i-1} \int_{\Omega} \int_{\Omega} \kappa(x, y) \frac{[u_i(t, y) - u_i(t, x)]}{u_i(t, x)} dy dx.\end{aligned}\quad (\text{A28})$$

Next, since κ is symmetric, for every $w \in X^{++}$,

$$\begin{aligned}\int_{\Omega} \int_{\Omega} \kappa(x, y) \frac{(w(y) - w(x))}{w(x)} dy dx &= \frac{1}{2} \int_{\Omega} \int_{\Omega} \kappa(x, y) \left[\frac{(w(y) - w(x))}{w(x)} + \frac{(w(x) - w(y))}{w(y)} \right] dy dx \\ &= \frac{1}{2} \int_{\Omega} \int_{\Omega} \kappa(x, y) (w(y) - w(x)) \left(\frac{1}{w(x)} - \frac{1}{w(y)} \right) dy dx \\ &= \frac{1}{2} \int_{\Omega} \int_{\Omega} \kappa(x, y) \frac{(w(y) - w(x))^2}{w(y)w(x)} dy dx.\end{aligned}\quad (\text{A29})$$

Then, by (A28) and (A29),

$$\begin{aligned}\frac{d\mathcal{L}(t)}{dt} &= - \int_{\Omega} \left(h_{1,1}(u_1 - u_1^\tau)^2 - (h_{1,2} + hh_{2,1})(u_1 - u_1^\tau)(u_2 - u_2^\tau) + hh_{2,2}(u_2 - u_2^\tau)^2 \right) \\ &\quad - \sum_{i=1}^2 \frac{\mu_i u_i^\tau h^{i-1}}{2} \int_{\Omega} \int_{\Omega} \kappa(x, y) \frac{(u_i(t, y) - u_i(t, x))^2}{u_i(t, x)u_i(t, y)} dy dx\end{aligned}\quad (\text{A30})$$

Now, note that

$$\Delta := 4hh_{1,1}h_{2,2} - (h_{1,2} + hh_{2,1})^2 = 4h \left(bf + \frac{rs}{u_1^\tau u_2^\tau} + \frac{sbu_1}{u_2^\tau u_2} + \frac{fru_2}{u_1^\tau u_1} \right) - (h_{1,2} + hh_{2,1})^2. \quad (\text{A31})$$

Observe from (A23) that

$$h_{1,2}h_{2,1}u_1^\tau u_2^\tau = (a + s + bu_1^\tau)(e + fu_2^\tau) = (a + s)e + (a + s)fu_2^\tau + beu_1^\tau + bfu_1^\tau u_2^\tau,$$

from which we obtain

$$bf = h_{1,2}h_{2,1} - \frac{(a + s)e}{u_1^\tau u_2^\tau} - \frac{(a + s)f}{u_1^\tau} - \frac{be}{u_2^\tau} \quad (\text{A32})$$

Combining (A32) and (A31) and the fact that $h_{1,2} = hh_{2,1}$ yield

$$\begin{aligned}\Delta &= 4h \left(\frac{rs}{u_1^\tau u_2^\tau} + \frac{sbu_1}{u_2^\tau u_2} + \frac{fru_2}{u_1^\tau u_1} - \frac{(a + s)e}{u_1^\tau u_2^\tau} - \frac{(a + s)f}{u_1^\tau} - \frac{be}{u_2^\tau} \right) - (h_{1,2} - hh_{2,1})^2 \\ &= \frac{4h}{u_1^\tau u_2^\tau} \left(rs - (a + s)e - (a + s)fu_2^\tau - beu_1^\tau \right) + 4h \left(\frac{sbu_1}{u_2^\tau u_2} + \frac{fru_2}{u_1^\tau u_1} \right) \\ &= 4h(\Delta_1 + \Delta_2)\end{aligned}\quad (\text{A33})$$

where $\Delta_2 := \frac{sbu_1}{u_2^\tau u_2} + \frac{fru_2}{u_1^\tau u_1}$. It is clear that Δ_1 is a positive constant and that $\Delta_2 > 0$ for every $t > 0$. Therefore, by (A30), we have

$$\begin{aligned} \frac{d\mathcal{L}(t)}{dt} = & - \int_{\Omega} \left(h_{1,1} \left((u_1 - u_1^\tau) - \frac{(h_{1,2} + hh_{2,1})}{2h_{1,1}} (u_2 - u_2^\tau) \right)^2 + \frac{h}{h_{1,1}} (\Delta_1 + \Delta_2) (u_2 - u_2^\tau)^2 \right) dx \\ & - \sum_{i=1}^2 \frac{\mu_i u_i^\tau h^{i-1}}{2} \int_{\Omega} \int_{\Omega} \kappa(x, y) \frac{(u_i(t, y) - u_i(t, x))^2}{u_i(t, x) u_i(t, y)} dx dy. \end{aligned} \quad (\text{A34})$$

Using the inequality

$$\frac{m_1 + s}{m_2 + s} \geq \min\{1, \frac{m_1}{m_2}\} \quad \forall m_1 > 0, m_2 > 0, s > 0,$$

we obtain

$$\frac{\Delta_1 + \Delta_2}{h_{1,1}} = \frac{\Delta_1 + \frac{sbu_1}{u_2^\tau u_2} + \frac{fru_2}{u_1^\tau u_1}}{b + \frac{ru_2}{u_1^\tau u_1}} \geq \min\{\frac{\Delta_1}{b}, f\} = \Delta_{1,1} > 0 \quad (\text{A35})$$

and

$$\frac{\Delta_1 + \Delta_2}{h_{2,2}} = \frac{\Delta_1 + \frac{sbu_1}{u_2^\tau u_2} + \frac{fru_2}{u_1^\tau u_1}}{f + \frac{su_1}{u_2^\tau u_2}} \geq \min\{\frac{\Delta_1}{f}, b\} = \Delta_{2,2} > 0. \quad (\text{A36})$$

By (A34) and (A35),

$$\begin{aligned} \frac{d\mathcal{L}(t)}{dt} \leq & - \int_{\Omega} \left(h_{1,1} \left((u_1 - u_1^\tau) - \frac{(h_{1,2} + hh_{2,1})}{2h_{1,1}} (u_2 - u_2^\tau) \right)^2 + h\Delta_{1,1} (u_2 - u_2^\tau)^2 \right) dx \\ & - \sum_{i=1}^2 \frac{\mu_i u_i^\tau h^{i-1}}{2} \int_{\Omega} \int_{\Omega} \kappa(x, y) \frac{(u_i(t, y) - u_i(t, x))^2}{u_i(t, x) u_i(t, y)} dx dy. \end{aligned} \quad (\text{A37})$$

By rewriting (A28) as

$$\begin{aligned} \frac{d\mathcal{L}(t)}{dt} = & - \int_{\Omega} \left(hh_{2,2} \left((u_2 - u_2^\tau) - \frac{(h_{1,2} + hh_{2,1})}{2hh_{2,2}} (u_1 - u_1^\tau) \right)^2 + \frac{1}{h_{2,2}} (\Delta_1 + \Delta_2) (u_1 - u_1^\tau)^2 \right) dx \\ & - \sum_{i=1}^2 \frac{\mu_i u_i^\tau h^{i-1}}{2} \int_{\Omega} \int_{\Omega} \kappa(x, y) \frac{(u_i(t, y) - u_i(t, x))^2}{u_i(t, x) u_i(t, y)} dx dy, \end{aligned}$$

where we have used the fact

$$\frac{4hh_{1,1}h_{2,2} - (h_{1,2} + hh_{2,1})^2}{4hh_{2,2}} = \frac{(\Delta_1 + \Delta_2)}{h_{2,2}},$$

it follows from (A36) that

$$\begin{aligned} \frac{d\mathcal{L}(t)}{dt} \leq & - \int_{\Omega} \left(h_{2,2} \left((u_2 - u_2^\tau) - \frac{(h_{1,2} + hh_{2,1})}{2h_{2,2}} (u_1 - u_1^\tau) \right)^2 + \Delta_{2,2} (u_1 - u_1^\tau)^2 \right) dx \\ & - \sum_{i=1}^2 \frac{\mu_i u_i^\tau h^{i-1}}{2} \int_{\Omega} \int_{\Omega} \kappa(x, y) \frac{(u_i(t, y) - u_i(t, x))^2}{u_i(t, x) u_i(t, y)} dx dy. \end{aligned} \quad (\text{A38})$$

Finally, we deduce from (A37) and (A38) that

$$\frac{d\mathcal{L}(t)}{dt} \leq - \sum_{i=1}^2 \frac{\mu_i u_i^\tau h^{i-1}}{2} \int_{\Omega} \int_{\Omega} \kappa(x, y) \frac{(u_i(t, y) - u_i(t, x))^2}{u_i(t, x) u_i(t, y)} dy dx - \sum_{i=1}^2 \frac{h^{2-i} \Delta_{i,i}}{2} \int_{\Omega} (u_i - u_i^\tau)^2,$$

which shows that (A25) holds. $\square$

Observe from Lemma A3, mainly inequality (A25), that $\mathbf{u}^\tau$ is the unique positive steady state solution of (1). In particular, system (A22) has a unique positive solution. As a

first application of Lemma A3, we have the convergence of $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)$ to $\mathbf{u}^\tau$ as $t \rightarrow \infty$ in the $L^p(\Omega)$ -space for any finite $p \geq 1$.

Lemma A4. Let $\tau > 0$ be fixed, and $\mathbf{u}^\tau$ be a positive solution of (A22). Let $\Delta_{1,1}$, $\Delta_{2,2}$ and h be defined by (A23). Let $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)$ be a classical solution of (1) with a positive initial data $\mathbf{u}_0 \in X^{++} \times X^{++}$. Then,

$$\lim_{t \rightarrow \infty} \sum_{i=1}^2 \|u_i(t, \cdot; \tau, \mathbf{u}_0) - u_i^\tau\|_{L^p(\Omega)} = 0 \quad \forall p \geq 1.$$

Proof. Since $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0) \leq \mathbf{u}(t, \cdot; 0, \mathbf{u}_0)$ for every $\tau \geq 0$ and $\mathbf{u}(t, \cdot; 0, \mathbf{u}_0)$ is eventually bounded, to complete the proof of the lemma, it is enough to show the result for $p = 2$. It is easy to see from (1) and the uniform boundedness of $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)$ that the function $t \mapsto \sum_{i=1}^2 \|u_i(t, \cdot; \tau, \mathbf{u}_0) - u_i^\tau\|_{L^2(\Omega)}^2$ is Lipschitz continuous on $[0, \infty)$. On the other hand, by Lemma A3, there is a positive constant $m_* > 0$ such that

$$\int_0^t \sum_{i=1}^2 \|u_i(\eta, \cdot; \tau, \mathbf{u}_0) - u_i^\tau\|_{L^2(\Omega)}^2 d\eta \leq m_* (\mathcal{L}(0) - \mathcal{L}(t)) \quad \forall t \geq 0.$$

However, since $\tilde{\mathcal{L}}(\eta) \geq 0$ for all $\eta > 0$, then $\mathcal{L}(t) \geq 0$ for all $t \geq 0$, which in view of the above inequality yield

$$\int_0^\infty \sum_{i=1}^2 \|u_i(\eta, \cdot; \tau, \mathbf{u}_0) - u_i^\tau\|_{L^2(\Omega)}^2 d\eta \leq m_* \mathcal{L}(0).$$

Hence, $\sum_{i=1}^2 \|u_i(t, \cdot; \tau, \mathbf{u}_0) - u_i^\tau\|_{L^2(\Omega)}^2 \rightarrow 0$ as $t \rightarrow \infty$ since the mapping $[0, \infty) \ni t \mapsto \sum_{i=1}^2 \|u_i(t, \cdot; \tau, \mathbf{u}_0) - u_i^\tau\|_{L^2(\Omega)}^2$ is Lipschitz. $\square$

Next, we show that $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)$ is uniformly bounded away from zero.

Lemma A5. Let $\tau > 0$ be fixed. Let $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)$ be a classical solution of (1) with a positive initial data $\mathbf{u}_0 \in X^{++} \times X^{++}$. Then there is $\eta^* > 0$ such that

$$\liminf_{t \rightarrow \infty} \min_{x \in \bar{\Omega}} \min_{i=1,2} u_i(t, x; \tau, \mathbf{u}_0) \geq \eta^*.$$

Proof. It follows from Theorem 1(iii). $\square$

Thanks to Lemmas A4 and A5, we can now complete the proof of Theorem 2.

Proof of Theorem 2. We proceed by contradiction. Suppose that there exist $\{(t_n, x_n)\}_{n \geq 1}$ with $t_n \rightarrow \infty$ such that

$$\liminf_{n \rightarrow \infty} \max_{i=1,2} |u_i(t_n, x_n; \tau, \mathbf{u}_0) - u_i^\tau| > 0. \quad (\text{A39})$$

Consider the sequence $\mathbf{v}^n(t) = \mathbf{u}(t_n + t, x_n; \tau, \mathbf{u}_0)$. Observe that for $t > -t_n$, $\mathbf{v}^n$ solves

$$\begin{cases} \frac{dv_1^n}{dt} = \mu_1 \int_{\Omega} \kappa(x_n, y) (u_1(t_n + t, y) - u_1^\tau) dy - \mu_1 (v_1^n - u_1^\tau) K(x_n) + r v_2^n - (a + s + b v_1^n + \tau c v_2^n) v_1^n \\ \frac{dv_2^n}{dt} = \mu_2 \int_{\Omega} \kappa(x_n, y) (u_2(t_n + t, y) - u_2^\tau) dy - \mu_2 (v_2^n - u_2^\tau) K(x_n) + s v_1^n - (e + f v_2^n + \tau g v_1^n) v_2^n. \end{cases}$$

Since $\bar{\Omega}$ is compact, without loss of generality (if possible after passing to a subsequence), we may suppose there is $x_\infty \in \bar{\Omega}$ such that $x_n \rightarrow x_\infty$ as $n \rightarrow \infty$. By Lemma A4 and Holder's inequality, we have

$$\begin{aligned} \left| \int_{\Omega} \kappa(x_n, y)(u_i(t+t_n, y) - u_i^{\tau}) dy \right| &\leq \|\kappa(x_n, \cdot)\|_{L^2(\Omega)} \|u_i(t+t_n, \cdot) - u_i^{\tau}\|_{L^2(\Omega)} \\ &\leq \|\kappa\|_{\infty} |\Omega|^{\frac{1}{2}} \|u_i(t+t_n, \cdot) - u_i^{\tau}\|_{L^2(\Omega)} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

uniformly for $t \in [T, \infty)$ for every $T \in \mathbb{R}$. Furthermore, since $\sup_{t \geq 0} \|\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)\| < \infty$, then

$$M_1 := \sup_{n \geq 1} \sup_{t \geq -t_n} \|\mathbf{v}^n(t)\| < \infty \quad \text{and} \quad \sup_{n \geq 1} \sup_{t \geq -t_n} \max_{i=1,2} \left| \frac{dv_i^n(t)}{dt} \right| < \infty.$$

Therefore, it follows from Arzela–Ascoli’s theorem that there is a non-negative function $\mathbf{v}^{\infty} \in C(\mathbb{R} : \mathbb{R}^2)$ such that, up to a subsequence, $\mathbf{v}^n \rightarrow \mathbf{v}^{\infty}$ as $n \rightarrow \infty$, locally uniformly on $\mathbb{R}$. Moreover, $\mathbf{v}^{\infty}$ satisfies

$$\begin{cases} \frac{dv_1^{\infty}}{dt} = -\mu_1(v_1^{\infty} - u_1^{\tau})K(x_{\infty}) + rv_2^{\infty} - (a + s + bv_1^{\infty} + \tau cv_2^{\infty})v_1^{\infty} & t \in \mathbb{R}, \\ \frac{dv_2^{\infty}}{dt} = -\mu_2(v_2^{\infty} - u_2^{\tau})K(x_{\infty}) + sv_1^{\infty} - (e + fv_2^{\infty} + \tau gv_1^{\infty})v_2^{\infty} & t \in \mathbb{R}. \end{cases} \quad (\text{A40})$$

Furthermore, using Lemma A5 and uniform boundedness of $\mathbf{u}(t, \cdot; \tau, \mathbf{u}_0)$, there is $M_* > 0$ such that

$$\frac{1}{M_*} \leq v_i^{\infty}(t) \leq M_* \quad \forall t \in \mathbb{R}, i = 1, 2. \quad (\text{A41})$$

Moreover, it follows from (A39) that

$$\|\mathbf{v}^{\infty}(0) - \mathbf{u}^{\tau}\| > 0. \quad (\text{A42})$$

Next, define the Lyapunov function

$$\hat{\mathcal{L}}(t) = \tilde{\mathcal{L}}\left(\frac{v_1^{\infty}(t)}{u_1^{\tau}}\right) + h\tilde{\mathcal{L}}\left(\frac{v_2^{\infty}(t)}{u_2^{\tau}}\right) \quad \forall t \in \mathbb{R},$$

where h is the positive number in (A23) and $\tilde{\mathcal{L}}$ is given by (A24). Using computations similar to those leading to (A25), we obtain

$$\frac{d\hat{\mathcal{L}}(t)}{dt} \leq -\sum_{i=1}^2 \frac{\mu_i h^{i-1} K(x_{\infty})}{v_i^{\infty}(t) u_i^{\tau}} (v_i^{\infty} - u_i^{\tau})^2 - \sum_{i=1}^2 \frac{\Delta_{i,i} h^{2-i}}{2} (v_i^{\infty} - u_i^{\tau})^2, \quad (\text{A43})$$

where $\Delta_{1,1}$ and $\Delta_{2,2}$ are the constants in (A23). Taking $\delta^* = \min\left\{\frac{\mu_1 K(x_{\infty})}{M_* u_1^{\tau}}, \frac{\mu_2 h K(x_{\infty})}{M_* u_2^{\tau}}, \frac{h\Delta_{1,1}}{2}, \frac{\Delta_{2,2}}{2}\right\}$, it follows from (A41) and (A43) that

$$\frac{d\hat{\mathcal{L}}(t)}{dt} \leq -\delta^* \left((v_1^{\infty} - u_1^{\tau})^2 + (v_2^{\infty} - u_2^{\tau})^2 \right) \quad \forall t \in \mathbb{R}. \quad (\text{A44})$$

Hence,

$$\int_{t_0}^t \sum_{i=1}^2 (v_i^{\infty}(\sigma) - u_i^{\tau})^2 d\sigma \leq \frac{1}{\delta^*} (\hat{\mathcal{L}}(t_0) - \hat{\mathcal{L}}(t)) \leq \frac{1}{\delta^*} \hat{\mathcal{L}}(t_0) \quad \forall t > t_0.$$

Notice from (A43) and (A41) that $\sup_{t \in \mathbb{R}} \hat{\mathcal{L}}(t) < \infty$, we then conclude from the last inequality that

$$\int_{-\infty}^{\infty} \sum_{i=1}^2 (v_i^{\infty}(\sigma) - u_i^{\tau})^2 d\sigma < \infty$$

Hence,

$$\lim_{t \rightarrow \pm\infty} \sum_{i=1}^2 (v_i^{\infty}(t) - u_i^{\tau})^2 = 0.$$

This, in turn, yields

$$\lim_{t \rightarrow \pm\infty} \hat{\mathcal{L}}(t) = 0.$$

Consequently, we deduce from (A44) that $\hat{\mathcal{L}}(t) \equiv 0$, which yield that $\mathbf{v}^\infty(t) = \mathbf{u}^\tau$ for all $t \in \mathbb{R}$. This clearly contradicts (A42). Therefore, the result follows. $\square$

Appendix C. Proof of Theorems 3 and 4

Notice that the function $\mathcal{F}$ in (A4) satisfies $\mathcal{F} \in C^\infty([0, \infty) \times X \times X)$ and

$$\partial_{\mathbf{u}}\mathcal{F}(\tau, \mathbf{u})(\mathbf{v}) = \begin{pmatrix} -(2bu_1 + \tau cu_2) & -\tau cu_1 \\ -\tau gu_2 & -(2fu_2 + \tau gu_1) \end{pmatrix} \mathbf{v} \quad \forall \mathbf{u}, \mathbf{v} \in X \times X, \tau > 0.$$

Thus, we have the following:

Lemma A6. Suppose that $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) > 0$ and let $\mathbf{u}^0$ be given by Lemma A1. Then there is $\tau^* > 0$ and a C^1 -map $\mathbf{u} : [0, \tau^*) \ni \tau \mapsto \mathbf{u}(\cdot; \tau) \in X^{++} \times X^{++}$ satisfying $\mathbf{u}(\cdot; 0) = \mathbf{u}^0$ such that

$$(\mu \circ \mathcal{K} + \mathcal{A})\mathbf{u}(\cdot; \tau) + \mathcal{F}(\tau, \mathbf{u}(\cdot; \tau)) = \mathbf{0} \quad \forall \tau \in [0, \tau^*). \quad (\text{A45})$$

Moreover, for every $\tau \in [0, \tau^*)$, $\mathbf{u}(\cdot; \tau)$ is the unique solution of (A45) in $X^{++} \times X^{++}$ and $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A} + \partial_{\mathbf{u}}\mathcal{F}(\tau, \mathbf{u}(\cdot; \tau))) < 0$.

Proof. Observe that the mapping on $X \times X$, defined by

$$\mathbf{v} \mapsto (\mu \circ \mathcal{K} + \mathcal{A})(\mathbf{v}) + \partial_{\mathbf{u}}\mathcal{F}(0, \mathbf{u}^0)(\mathbf{v}) = \mu \circ \mathcal{K}(\mathbf{v}) + \begin{pmatrix} -(a + s + 2bu_1^0) & r \\ s & -(e + 2fu_2^0) \end{pmatrix} \mathbf{v},$$

is cooperative. Furthermore, setting $\lambda^0 := \min_{x \in \bar{\Omega}} \{b(x)\mathbf{u}_1^0(x), f(x)\mathbf{u}_2^0(x)\} > 0$, since $\mathbf{u}^0$ is a steady state solution of (1), we have

$$\begin{aligned} (\mu \circ \mathcal{K} + \mathcal{A})(\mathbf{u}^0) + \partial_{\mathbf{u}}\mathcal{F}(0, \mathbf{u}^0)(\mathbf{u}^0) &= (\mu \circ \mathcal{K} + \mathcal{A})(\mathbf{u}^0) + \mathcal{F}(0, \mathbf{u}^0)(\mathbf{u}^0) - \begin{pmatrix} bu_1^0 & 0 \\ 0 & fu_2^0 \end{pmatrix} \mathbf{u}^0 \\ &= - \begin{pmatrix} bu_1^0 & 0 \\ 0 & fu_2^0 \end{pmatrix} \mathbf{u}^0 \\ &\leq -\lambda^0 \mathbf{u}^0. \end{aligned}$$

Hence, since $\mathbf{u}^0 \in X^{++} \times X^{++}$ and $\lambda^0 > 0$, $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A} + \partial_{\mathbf{u}}\mathcal{F}(0, \mathbf{u}^0)) \leq -\lambda^0 < 0$. This implies that the linear functional $\mu \circ \mathcal{K} + \mathcal{A} + \partial_{\mathbf{u}}\mathcal{F}(0, \mathbf{u}^0)$ is invertible. Hence, by the implicit function theorem, there is $\tau^* > 0$ and a C^1 map $\mathbf{u} : [0, \tau^*) \ni \tau \mapsto \mathbf{u}(\cdot; \tau) \in X^{++} \times X^{++}$ satisfying $\mathbf{u}(\cdot; 0) = \mathbf{u}^0(\cdot)$ such that $(\mu \circ \mathcal{K} + \mathcal{A})(\mathbf{u}(\cdot; \tau)) + \mathcal{F}(\tau, \mathbf{u}(\cdot; \tau)) = \mathbf{0}$ for all $\tau \in [0, \tau^*)$. Moreover, for every $\tau \in [0, \tau^*)$, $\mathbf{u}(\cdot; \tau)$ is the unique positive solution of $(\mu \circ \mathcal{K} + \mathcal{A})(\mathbf{u}) + \mathcal{F}(\tau, \mathbf{u}) = \mathbf{0}$ in $X^{++} \times X^{++}$. Furthermore, since $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A} + \partial_{\mathbf{u}}\mathcal{F}(0, \mathbf{u}^0)) \leq -\lambda^0 < 0$ and $\mathbf{u}(\cdot; \tau)$ is continuous in $\tau \in [0, \tau^*)$, if possible, by taking τ^* sufficiently small, we have that $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A} + \partial_{\mathbf{u}}\mathcal{F}(\tau, \mathbf{u}(\cdot; \tau))) < 0$ for every $\tau \in [0, \tau^*)$. $\square$

Proof of Theorem 3. (i) It follows from Lemma A6.

(ii) Suppose that $\Lambda_{\min} > 0$ and let $\tau > 0$. Then, by Theorem 2, for every $x \in \bar{\Omega}$, there is a unique positive equilibrium solution of the system

$$\begin{cases} 0 = r(x)u_2(x) - (a(x) + s(x) + b(x)u_1 + \tau c(x)u_2)u_1 \\ 0 = s(x)u_1 - (e(x) + f(x)u_2 + \tau g(x)u_1)u_2, \end{cases} \quad (\text{A46})$$

which we denote by $\mathbf{u}^{0,\tau}(x)$. Moreover, the function $\mathbf{u}^{0,\tau}(\cdot) \in X^{++} \times X^{++}$. Now, consider the function $\mathcal{G} : \mathbb{R}^2 \times X \times X \rightarrow X \times X$ defined by

$$\mathcal{G}(\mu, \mathbf{u}) = (\mu \circ \mathcal{K} + \mathcal{A})\mathbf{u} + \mathcal{F}(\tau, \mathbf{u})$$

where $\mathcal{F}$ is defined in (A4). Hence, the function $\mathcal{G}$ is of class C^∞ . Furthermore, for every $\mathbf{v} \in X \times X$,

$$\partial_{\mathbf{u}}\mathcal{G}(0, \mathbf{u}^{0,\tau})\mathbf{v} = \begin{pmatrix} -(a+s+2bu_1^{0,\tau}+\tau cu_2^{0,\tau}) & r-\tau cu_1^{0,\tau} \\ s-\tau gu_2^{0,\tau} & -(e+2fu_2^{0,\tau}+\tau gu_1^{0,\tau}) \end{pmatrix} \mathbf{v}.$$

Note from (A46) that

$$r(x) - \tau c(x)u_1^{0,\tau}(x) = \frac{(a(x)+s(x)+b(x)u_1^{0,\tau}(x))u_1^{0,\tau}(x)}{u_2^{0,\tau}(x)} > 0$$

and

$$s(x) - \tau g(x)u_2^{0,\tau}(x) = \frac{(e(x)+f(x)u_2^{0,\tau}(x))u_2^{0,\tau}(x)}{u_1^{0,\tau}(x)} > 0$$

for every $x \in \bar{\Omega}$. Hence, $\partial_{\mathbf{u}}\mathcal{G}(0, \mathbf{u}^{0,\tau})$ is cooperative. Furthermore,

$$\partial_{\mathbf{u}}\mathcal{G}(0, \mathbf{u}^{0,\tau})\mathbf{u}^{0,\tau} = - \begin{pmatrix} b(u_1^{0,\tau})^2 \\ f(u_2^{0,\tau})^2(\cdot, 0) \end{pmatrix} \leq -\tilde{\lambda}_0\mathbf{u}^{0,\tau},$$

where $\tilde{\lambda}_0 = \min\{(bu_1^{0,\tau})_{\min}, (gu_2^{0,\tau})_{\min}\} > 0$. This shows that $\lambda_p(\partial_{\mathbf{u}}\mathcal{G}(0, \mathbf{u}^{0,\tau})\mathbf{u}^{0,\tau}) \leq -\tilde{\lambda}_0 < 0$. Therefore, we can apply the implicit function theorem as in the proof of Lemma A6 to derive the desired result. $\square$

We give a proof of Theorem 4.

Proof of Theorem 4. (i) Suppose that $\tau = 0$ and $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) > 0$. In this case, (1) is cooperative and subhomogeneous. By Lemma A1, we know that (1) has a unique strictly positive steady state, $\mathbf{u}^0$. Furthermore, since (1) is cooperative and subhomogeneous, it follows from standard arguments that $\mathbf{u}^0$ is globally stable with respect to positive perturbations.

(ii) Suppose that $r_{\min} > 0$ and $s_{\min} > 0$. Take

$$\tau^{**} = \min \left\{ \frac{r_{\min}}{(\|c\|_{\infty} + 2)\|u_2^0\|_{\infty}}, \frac{s_{\min}}{(\|g\|_{\infty} + 2)\|u_1^0\|_{\infty}}, \tau^* \right\}$$

where $\mathbf{u}^0 = (u_1^0, u_2^0)$ is the unique strictly positive steady state solution of (1) for $\tau = 0$, τ^* is given by Theorem 3. Next, consider the closed-bounded interval

$$[0, 2\mathbf{u}^0] := \{\mathbf{u} \in X^+ \times X^+ : \mathbf{u} \leq 1 \cdot 2\mathbf{u}^0\}.$$

Let $\tau \in (0, \tau^{**}]$ be fixed and $\mathbf{u}^{\tau}$ be the unique strictly positive steady state solution of (1) guaranteed by Theorem 3. Since every non-negative classical solution of (1) is subsolution of (A10), and $\mathbf{u}^0$ is globally stable for (A10), then every solution of (1) eventually lies in $[0, 2\mathbf{u}^0]$. Observe that, on $[0, 2\mathbf{u}^0]$, the semiflow generated by the classical solution of (1) is positive and subhomogeneous. Therefore, the unique strictly positive steady state, $\mathbf{u}^{\tau}$, is globally stable with respect to positive perturbations. $\square$

The rest of this section will be devoted to prove Theorem 5. Hence, throughout the rest of the paper, we suppose that $(a+s)_{\min} > 0$ and $e_{\min} > 0$ and prove Theorem 5.

Proof. (i) We divide the proof in two cases.

Case 1. Suppose that $\left(\frac{s}{e}\right)_{\max} \left(\frac{r}{s+a}\right)_{\max} < 1$. Chose $0 < \lambda_0 < \min\{\|e\|_{\infty}, \|a+s\|_{\infty}\}$ satisfying

$$\frac{\left(\frac{s}{e}\right)_{\max}}{1 - \frac{\lambda_0}{\|e\|_{\infty}}} < \frac{\left(1 - \frac{\lambda_0}{\|a+s\|_{\infty}}\right)}{\left(\frac{r}{a+s}\right)_{\max}}.$$

Now, choose two positive numbers $q_1 > 1$ and $q_2 > 1$ satisfying

$$\frac{\left(\frac{s}{e}\right)_{\max}}{1 - \frac{\lambda_0}{\|e\|_{\infty}}} < \frac{q_2}{q_1} < \frac{\left(1 - \frac{\lambda_0}{\|a+s\|_{\infty}}\right)}{\left(\frac{r}{a+s}\right)_{\max}}, \quad (\text{A47})$$

and consider the constant vector $\mathbf{q} = (q_1, q_2)$. It follows from (A47) that

$$-(a(x) + s(x))q_1 + r(x)q_2 \leq -\lambda_0 q_1 \quad \text{and} \quad s(x)q_1 - e(x)q_2 \leq -\lambda_0 q_2 \quad \forall x \in \overline{\Omega}.$$

Therefore,

$$(\mu \circ \mathcal{K} + \mathcal{A})\mathbf{q} = A(x)\mathbf{q} \leq_1 -\lambda_0 \mathbf{q}.$$

Hence, in view of the monotonicity of $\{\mathbf{U}^{\mu}(t)\}_{t \geq 0}$, we obtain

$$\mathbf{0} <_1 \mathbf{U}^{\mu}(t)\mathbf{q} \leq_1 e^{-\lambda_0 t} \mathbf{q} \quad \forall t > 0.$$

This along with the fact that $q_1, q_2 > 1$ implies that

$$\|\mathbf{U}^{\mu}(t)\| \leq \|\mathbf{U}^{\mu}(t)\mathbf{q}\| \leq e^{-\lambda_0 t} \|\mathbf{q}\| \quad \forall t > 0.$$

Whence, by Proposition A1

$$\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) = \lim_{t \rightarrow \infty} \frac{\ln(\|\mathbf{U}^{\mu}(t)\|)}{t} \leq -\lambda_0 < 0.$$

Case 2. Suppose that $\left(\frac{(r+s)^2}{4(a+s)e}\right)_{\max} < 1$. Let $\mu = (\mu_1, \mu_2)$ be fixed and $\varepsilon > 0$. Consider, the cooperative two-stage nonlocal system,

$$\begin{cases} \partial_t u_1 = \mu_1 \mathcal{K} u_1 + r u_2 - (a + s + \lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) - \varepsilon + b u_1) u_1 & x \in \Omega, t > 0, \\ \partial_t u_2 = \mu_2 \mathcal{K} u_2 + s u_1 - (e + \lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) - \varepsilon + b u_2) u_2 & x \in \Omega, t > 0. \end{cases} \quad (\text{A48})$$

Since $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A} - \lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) + \varepsilon) = \varepsilon > 0$, it follows from Lemma A1 that (A48) has a unique positive steady state solution $\mathbf{u}(\cdot; \varepsilon)$, that is $\mathbf{u}(\cdot; \varepsilon)$ solves

$$\begin{cases} 0 = \mu_1 \mathcal{K} u_1 + r u_2 - (a + s + \lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) - \varepsilon + b u_1) u_1 & x \in \Omega, \\ 0 = \mu_2 \mathcal{K} u_2 + s u_1 - (e + \lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) - \varepsilon + b u_2) u_2 & x \in \Omega. \end{cases} \quad (\text{A49})$$

Moreover, since the right hand side of (A48) is strictly increasing in ε , $\mathbf{u}(\cdot; \varepsilon)$ is strictly increasing in ε . Therefore, by the arguments in the proof of Lemma A1, there is $\mathbf{u}(\cdot; 0) \in X^+ \times X^+$ such that $\mathbf{u}(\cdot; \varepsilon) \rightarrow \mathbf{u}(\cdot; 0)$ as $\varepsilon \rightarrow 0$ uniformly in Ω . In addition, $\mathbf{u}(\cdot; 0)$ solves

$$\begin{cases} 0 = \mu_1 \mathcal{K} u_1 + r u_2 - (a + s + \lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) + b u_1) u_1 & x \in \Omega, \\ 0 = \mu_2 \mathcal{K} u_2 + s u_1 - (e + \lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) + b u_2) u_2 & x \in \Omega. \end{cases} \quad (\text{A50})$$

Hence, since $\lambda_p(\mu \circ \mathcal{K} + \mathcal{A} - \lambda_p(\mu \circ \mathcal{K} + \mathcal{A})) = 0$, we conclude from Lemma A1 and (A50) that $\mathbf{u}(\cdot; 0) \equiv \mathbf{0}$. Now, for every $\varepsilon > 0$, multiply the first equation of (A49) by $u_1(\cdot; \varepsilon)$ and integrate on Ω to obtain

$$\begin{aligned}
(\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) - \varepsilon) \int_{\Omega} u_1^2(\cdot; \varepsilon) + \int_{\Omega} bu_1^3(\cdot; \varepsilon) &= \mu_1 \int_{\Omega} \int_{\Omega} \kappa(x, y)(u_1(y; \varepsilon) - u_1(x; \varepsilon))u_1(x; \varepsilon) dx dy \\
&\quad + \int_{\Omega} ru_1u_2 - \int_{\Omega} (a + s)u_1^2 \\
&\leq \int_{\Omega} ru_1u_2 - \int_{\Omega} (a + s)u_1^2.
\end{aligned} \tag{A51}$$

Similarly, multiplying the second equation of (A49) by $u_2(\cdot; \varepsilon)$ yields

$$(\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) - \varepsilon) \int_{\Omega} u_2^2(\cdot; \varepsilon) + \int_{\Omega} fu_2^3(\cdot; \varepsilon) \leq \int_{\Omega} su_1u_2 - \int_{\Omega} eu_2^2. \tag{A52}$$

From (A51) and (A52), we obtain

$$(\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) - \varepsilon) \int_{\Omega} (u_1^2(\cdot; \varepsilon) + u_2^2(\cdot; \varepsilon)) \leq - \int_{\Omega} ((a + s)u_1^2 - (r + s)u_1u_2 + eu_2^2) \tag{A53}$$

Now, observe that

$$\begin{aligned}
(a + s)u_1^2 - (r + s)u_1u_2 + eu_2^2 &= (a + s)\left(u_1 - \frac{r + s}{2(a + s)}u_2\right)^2 + \frac{4(a + s)e - (r + s)^2}{4(a + s)}u_2^2 \\
&\geq e_{\min} \left(1 - \left(\frac{(r + s)^2}{4(a + s)e}\right)_{\max}\right) u_2^2
\end{aligned}$$

and

$$\begin{aligned}
(a + s)u_1^2 - (r + s)u_1u_2 + eu_2^2 &= e\left(u_2 - \frac{r + s}{2e}u_1\right)^2 + \frac{4(a + s)e - (r + s)^2}{4e}u_1^2 \\
&\geq (a + s)_{\min} \left(1 - \left(\frac{(r + s)^2}{4(a + s)e}\right)_{\max}\right) u_1^2
\end{aligned}$$

Hence, it follows from (A53) that

$$(\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) - \varepsilon + \frac{1}{2} \min\{e_{\min}, (a + s)_{\min}\} \left(1 - \left(\frac{(r + s)^2}{4(a + s)e}\right)_{\max}\right)) \int_{\Omega} (u_1^2(\cdot; \varepsilon) + u_2^2(\cdot; \varepsilon)) \leq 0,$$

from which we deduce that

$$\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) \leq \varepsilon - \frac{1}{2} \min\{e_{\min}, (a + s)_{\min}\} \left(1 - \left(\frac{(r + s)^2}{4(a + s)e}\right)_{\max}\right) \quad \forall \varepsilon > 0,$$

since $\int_{\Omega} (u_1^2(\cdot; \varepsilon) + u_2^2(\cdot; \varepsilon)) > 0$. Letting $\varepsilon \rightarrow 0$ in the last inequality yields

$$\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) \leq -\frac{1}{2} \min\{e_{\min}, (a + s)_{\min}\} \left(1 - \left(\frac{(r + s)^2}{4(a + s)e}\right)_{\max}\right) < 0.$$

(ii) We also distinguish two cases.

Case 1. Suppose that $\left(\frac{s}{e}\right)_{\min} \left(\frac{r}{s + a}\right)_{\min} > 1$ and choose $\lambda_0 > 0$ such that

$$\frac{\left(\frac{s}{e}\right)_{\min}}{1 + \frac{\lambda_0}{e_{\min}}} > \frac{\left(1 + \frac{\lambda_0}{(a + s)_{\min}}\right)}{\left(\frac{r}{a + s}\right)_{\min}}.$$

Now, choose two positive numbers q_1, q_2 such that $q_1 + q_2 = 1$ and

$$\frac{\left(\frac{s}{e}\right)_{\min}}{1 + \frac{\lambda_0}{e_{\min}}} > \frac{q_2}{q_1} > \frac{\left(1 + \frac{\lambda_0}{(a+s)_{\min}}\right)}{\left(\frac{r}{a+s}\right)_{\min}}, \quad (\text{A54})$$

and consider the constant vector $\mathbf{q} = (q_1, q_2)$. It follows from (A54) that

$$-(a(x) + s(x))q_1 + r(x)q_2 \geq \lambda_0 q_1 \quad \text{and} \quad s(x)q_1 - e(x)q_2 \geq \lambda_0 q_2 \quad \forall x \in \bar{\Omega}.$$

Therefore,

$$(\mu \circ \mathcal{K} + \mathcal{A})\mathbf{q} = A(x)\mathbf{q} \geq \lambda_0 \mathbf{q}$$

Hence, the monotonicity of $\{\mathbf{U}^\mu(t)\}_{t \geq 0}$ yields

$$\mathbf{0} <_1 e^{\lambda_0 t} \mathbf{q} \leq_1 \mathbf{U}^\mu(t) \mathbf{q} \quad \forall t > 0.$$

This along with the fact that $q_1 + q_2 = 1$ implies that

$$\|\mathbf{U}^\mu(t)\| \geq \|\mathbf{U}^\mu(t)\mathbf{q}\| \geq e^{\lambda_0 t} \|\mathbf{q}\| \quad \forall t > 0.$$

Whence, by Proposition A1

$$\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) = \lim_{t \rightarrow \infty} \frac{\ln(\|\mathbf{U}^\mu(t)\|)}{t} \geq \lambda_0 > 0.$$

Case 2. Next, we suppose that $\int_{\Omega} \sqrt{rs} > \frac{1}{2} \int_{\Omega} (a + s + e)$. Let $\mu = (\mu_1, \mu_2)$ be fixed and $\varepsilon > 0$. Let $\mathbf{u}(\cdot; \varepsilon)$ be the unique positive solution of (A49). Divide the first equation of (A49) by $u_1(\cdot; \varepsilon)$, integrate the resulting equation, and after using (A29), we obtain

$$|\Omega| \lambda_p(\mu \circ \mathcal{K} + \mathcal{A} - \varepsilon) + \int_{\Omega} b u_1 \geq \int_{\Omega} r \frac{u_2}{u_1} - \int_{\Omega} (a + s).$$

Similarly, we obtain

$$|\Omega| \lambda_p(\mu \circ \mathcal{K} + \mathcal{A} - \varepsilon) + \int_{\Omega} f u_2 \geq \int_{\Omega} s \frac{u_1}{u_2} - \int_{\Omega} e.$$

Adding up the last two inequality and using the inequality $r \frac{u_2}{u_1} + s \frac{u_1}{u_2} \geq 2\sqrt{rs}$, we obtain

$$2|\Omega| \lambda_p(\mu \circ \mathcal{K} + \mathcal{A} - \varepsilon) + (\|b\|_{\infty} + \|f\|_{\infty}) \int_{\Omega} (u_1(\cdot; \varepsilon) + u_2(\cdot; \varepsilon)) \geq 2 \int_{\Omega} \sqrt{rs} - \int_{\Omega} (a + s + e).$$

Therefore, since $\mathbf{u}(\cdot; \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ uniformly in Ω , we deduce from the last inequality, after letting $\varepsilon \rightarrow 0$,

$$\lambda_p(\mu \circ \mathcal{K} + \mathcal{A}) \geq \frac{1}{|\Omega|} \left(\int_{\Omega} \sqrt{rs} - \frac{1}{2} \int_{\Omega} (a + s + e) \right) > 0.$$

□

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