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Some Functionals and Approximation Operators Associated with a Family of Discrete Probability Distributions

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Abstract: A certain discrete probability distribution was considered in ["A discrete probability distribution and some applications", Mediterr. J. Math., 2023]. Its basic properties were investigated and some applications were presented. We now embed this distribution into a family of discrete distributions depending on two parameters and investigate the properties of the new distributions.

Keywords: probability distribution; positive linear operators; convexity properties; stochastic convex ordering; quadrature formula

MSC: 60E05; 60E15; 26A51; 65D32

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1. Introduction

Let $n \in \mathbb{N}$, $j \in \{0, 1, \cdots, n\}$ and $\alpha, \beta > 0$. Define

$$a_{n,j,\alpha,\beta} := \binom{n}{j} \frac{B(j+\alpha, n-j+\beta)}{B(\alpha, \beta)}, \tag{1}$$

where $B(\lambda, \mu)$ is Euler's Beta function defined by

$$\begin{aligned} B(\lambda, \mu) &:= \int_0^1 t^{\lambda-1} (1-t)^{\mu-1} \, \mathrm{d}t \qquad \big(\min\{\Re(\lambda), \Re(\mu)\} > 1\big) \\ &= \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)} \end{aligned}$$

in terms of the familiar (Euler's) Gamma function.

The probability distribution $(a_{n,j,\alpha,\alpha})_{j=0,1,\cdots,n}$ was investigated in [1]. In this paper, we extend the results from [1] to the distribution $(a_{n,j,\alpha,\beta})_{j=0,1,\cdots,n}$. This enlarges the family of the investigated distributions and the area of applications involving these distributions. In our present investigation, we are motivated also by several related recent developments on approximation operators and probability distributions by (for example) Ong et al. [2].

Let $B_n : C[0,1] \to C[0,1]$ be the classical Bernstein operators, defined as

$$B_n f(x) := \sum_{j=0}^{n} \binom{n}{j} x^j (1-x)^{n-j} f\left(\frac{k}{n}\right) \ (f \in C[0,1],\ x \in [0,1]).$$

Consider the functional $A_{n,\alpha,\beta} : C[0,1] \to \mathbb{R}$,

$$A_{n,\alpha,\beta}(f) := \frac{1}{B(\alpha,\beta)} \int_0^1 t^{\alpha-1} \, (1-t)^{\beta-1} \, B_n \, f(t) \, \mathrm{d}t. \tag{2}$$

Let $e_j(t) = t^j$ $(j = 0, 1, \ldots; t \in [0,1])$.
It is well known that $B_n e_0 = e_0$ and $B_n e_1 = e_1$, so that (2) yields

$$A_{n,\alpha,\beta}(e_0) = 1, \tag{3}$$

$$A_{n,\alpha,\beta}(e_1) = \frac{\alpha}{\alpha+\beta}. \tag{4}$$

On the other hand, from (2) it is easy to infer that

$$A_{n,\alpha,\beta}(f) = \sum_{j=0}^{n} a_{n,j,\alpha,\beta} f\left(\frac{j}{n}\right), \ f \in C[0,1]. \tag{5}$$

Combined with (3) and (4), (5) leads to

$$\sum_{j=0}^{n} a_{n,j,\alpha,\beta} = 1, \tag{6}$$

$$\sum_{j=0}^{n} j a_{n,j,\alpha,\beta} = \frac{n\alpha}{\alpha+\beta}. \tag{7}$$

In particular, for each $n \in \mathbb{N}$, $\alpha, \beta > 0$, $(a_{n,j,\alpha,\beta})_{j=0,1,\cdots,n}$ can be considered as a discrete probability distribution, concentrated on a suitable set $\{x_0, x_1, \cdots, x_n\}$.

In Section 2, besides the functionals $A_{n,\alpha,\beta}$, we consider the functional $A_{\alpha,\beta}$ from (8). One of the main results is Theorem 1, which shows that for each $f \in C[0,1]$ the sequence $(A_{n,\alpha,\beta}(f))_{n\geq 1}$ converges to $A_{\alpha,\beta}(f)$. The rate of convergence is estimated for $f \in C^2[0,1]$ and the convergence for convex functions f is investigated. Equation (15) represents a quadrature formula for the following integral:

$$\int_0^1 x^{\alpha-1} \, (1-x)^{\beta-1} \, f(x) \, \mathrm{d}x.$$

The remainder is estimated for $f \in C^2[0,1]$.

Section 3 is devoted to a sequence of random variables $(Z_{n,\alpha,\beta})_{n\geq 1}$. We describe the sequence of characteristic functions and its limit. Consequently, the sequence $(Z_{n,\alpha,\beta})_{n\geq 1}$ converges in law to a Beta-type random variable. This offers a new proof for the convergence to zero of the remainder in the quadrature formula.

In Section 4, we consider two classical sequences of positive linear operators investigated by Lupaş and Lupaş [3] (see also [4,5]). We estimate the difference of these two sequences by using results from Section 2 and from the paper [6] and the references therein.

In Section 5, using the numbers $a_{n,j,\alpha,\beta}$, we construct a polynomial logarithmically convex Heun function.

Section 6 is devoted to inequalities between random variables from the preceding sections in the sense of the convex stochastic order. In the particular case $\alpha = 1$ and $\beta = 1/2$ formula (33) was proved in [1].

In summary, our distribution $(a_{n,j,\alpha,\beta})_{j=0,1,\cdots,n}$ has connections with several mathematical objects, including a sequence of positive linear functionals, a quadrature formula and its remainder, a sequence of random variables and their characteristic functions, two sequences of positive linear operators and the differences between them, polynomial logarithmically convex Heun functions, inequalities between random variables in the sense of the stochastic convex order.

In Section 7, we present conclusions and suggestions for further work.

2. A Quadrature Formula

Besides the functionals $A_{n,\alpha,\beta}$, consider also the functional $A_{\alpha,\beta} : C[0,1] \to \mathbb{R}$,

$$A_{\alpha,\beta}(f) := \frac{1}{B(\alpha,\beta)} \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} f(t)\, \mathrm{d}t. \tag{8}$$

Theorem 1. *(1) If $f \in C[0,1]$, then*

$$\lim_{n\to\infty} A_{n,\alpha,\beta}(f) = A_{\alpha,\beta}(f). \tag{9}$$

(2) If $f \in C[0,1]$ is a convex function, then

$$A_{n,\alpha,\beta}(f) \geq A_{n+1,\alpha,\beta}(f) \geq A_{\alpha,\beta}(f),\ n \geq 1. \tag{10}$$

(3) If $f \in C^2[0,1]$ and $2m \leq f''(x) \leq 2M$, $x \in [0,1]$, then

$$\frac{m\alpha\beta}{n(\alpha+\beta)(\alpha+\beta+1)} \leq A_{n,\alpha,\beta}(f) - A_{\alpha,\beta}(f) \leq \frac{M\alpha\beta}{n(\alpha+\beta)(\alpha+\beta+1)}. \tag{11}$$

In particular, if $f \in C^2[0,1]$, then

$$|A_{n,\alpha,\beta}(f) - A_{\alpha,\beta}(f)| \leq \frac{\alpha\beta}{n(\alpha+\beta)(\alpha+\beta+1)} \|f''\|_\infty. \tag{12}$$

Proof. 1. It is well-known that

$$\lim_{n\to\infty} B_n f(x) = f(x),\ f \in C[0,1], \tag{13}$$

uniformly with respect to $x \in [0,1]$.

Using (2), (8), (13), and the Lebesgue-dominated convergence theorem, we obtain (9).

2. It is also well-known that

$$B_n f \geq B_{n+1} f \geq f \tag{14}$$

for each convex function $f \in C[0,1]$.

To prove (10), we combine (2), (8) and (14).

3. If $f \in C^2[0,1]$ and $2m \leq f''(x) \leq 2M$, $x \in [0,1]$, then the functions $f - me_2$ and $Me_2 - f$ are convex. According to Item 2, we have

$$\begin{aligned} m\big(A_{n,\alpha,\beta}(e_2) - A_{\alpha,\beta}(e_2)\big) &\leq A_{n,\alpha,\beta}(f) - A_{\alpha,\beta}(f) \\ &\leq M\big(A_{n,\alpha,\beta}(e_2) - A_{\alpha,\beta}(e_2)\big). \end{aligned}$$

Since

$$A_{n,\alpha,\beta}(e_2) - A_{\alpha,\beta}(e_2) = \frac{\alpha\beta}{n(\alpha+\beta)(\alpha+\beta+1)},$$

we obtain (11) and also (12). □

We now consider the following quadrature formula:

$$\frac{1}{B(\alpha,\beta)}\int_0^1 x^{\alpha-1}\,(1-x)^{\beta-1}\,f(x)\,\mathrm{d}x=\sum_{j=0}^{n}a_{n,j,\alpha,\beta}f\left(\frac{j}{n}\right)+R_{n,\alpha,\beta}(f). \tag{15}$$

Theorem 2. *The remainder $R_{n,\alpha,\beta}$ satisfies*

$$\lim_{n\to\infty}R_{n,\alpha,\beta}(f)=0,\ \ f\in C[0,1]. \tag{16}$$

If $f\in C^2[0,1]$ and $2m\le f''(x)\le 2M$, $x\in[0,1]$, then

$$\frac{-M\alpha\beta}{n(\alpha+\beta)(\alpha+\beta+1)}\le R_{n,\alpha,\beta}(f)\le\frac{-m\alpha\beta}{n(\alpha+\beta)(\alpha+\beta+1)}. \tag{17}$$

In particular,

$$|R_{n,\alpha,\beta}(f)|\le\frac{\alpha\beta}{n(\alpha+\beta)(\alpha+\beta+1)}\|f''\|_\infty. \tag{18}$$

Proof. Using (8) and (5), (15) can be written as

$$A_{\alpha,\beta}(f)=A_{n,\alpha,\beta}(f)+R_{n,\alpha,\beta}(f).$$

Now (16), (17) and (18) are consequences of (9), (11) and (12). □

3. A Random Variable

Consider the random variable $Z_{n,\alpha,\beta}$ defined by $P\left(Z_{n,\alpha,\beta}=\frac{j}{n}\right)=a_{n,j,\alpha,\beta},\ j=0,1,\cdots,n.$ According to (7), $E(Z_{n,\alpha,\beta})=\frac{\alpha}{\alpha+\beta}$, where E stands for mathematical expectation. Let $\mathcal{B}_{\alpha,\beta}$ be the Beta-type random variable with density

$$\begin{cases}\dfrac{t^{\alpha-1}\,(1-t)^{\beta-1}}{B(\alpha,\beta)} & (0<t<1)\\[2mm] 0 & (\text{otherwise}).\end{cases}$$

Theorem 3. *Each of the following assertions holds true:*

1. *The characteristic function of $Z_{n,\alpha,\beta}$ is given by*

$$g_{n,\alpha,\beta}(s)=\frac{1}{B(\alpha,\beta)}\int_0^1 t^{\alpha-1}\,(1-t)^{\beta-1}\left(1+t(e^{is/n}-1)\right)^n\mathrm{d}t\quad(s\in\mathbb{R}). \tag{19}$$

2. *It is asserted that*

$$\lim_{n\to\infty}g_{n,\alpha,\beta}(s)=\int_0^1\frac{t^{\alpha-1}\,(1-t)^{\beta-1}}{B(\alpha,\beta)}\,e^{ist}\,\mathrm{d}t\quad(s\in\mathbb{R}). \tag{20}$$

3. *$(Z_{n,\alpha,\beta})_{n\in\mathbb{N}}$ converges in law to $\mathcal{B}_{\alpha,\beta}$, as $n\to\infty$.*

Proof. The characteristic function $Z_{n,\alpha,\beta}$ is by definition

$$g_{n,\alpha,\beta}(s):=E\left[e^{isZ_{n,\alpha,\beta}}\right]=\sum_{j=0}^{n}a_{n,j,\alpha,\beta}e^{isj/n}. \tag{21}$$

We have

$$\sum_{j=0}^{n} a_{n,j,\alpha,\beta} z^j = \frac{1}{B(\alpha,\beta)} \sum_{j=0}^{n} z^j \binom{n}{j} \int_0^1 t^{j+\alpha-1} (1-t)^{n-j+\beta-1} \, \mathrm{d}t$$
$$= \frac{1}{B(\alpha,\beta)} \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} (1+t(z-1))^n \, \mathrm{d}t. \quad (22)$$

From (21) and (22) with $z := e^{is/n}$, we get (19).
Now (20) is a consequence of (19) and Statement 3 follows from (20). □

Corollary 1. *For $f \in C[0,1]$, it is asserted that*

$$\lim_{n\to\infty} E(Z_{n,\alpha,\beta}) = E(\mathcal{B}_{\alpha,\beta}). \quad (23)$$

Proof. The relation (23) is a consequence of 3) from Theorem 3. □

Remark 1. *The relation (23) can be written as*

$$\lim_{n\to\infty} \sum_{j=0}^{n} a_{n,j,\alpha,\beta} f\left(\frac{j}{n}\right) = \frac{1}{B(\alpha,\beta)} \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} f(t) \, \mathrm{d}t,$$

i.e.,

$$\lim_{n\to\infty} A_{n,\alpha,\beta}(f) = A_{\alpha,\beta}(f).$$

So, we have another proof of (9).

4. Two Sequences of Operators

Let $\overline{\mathbb{B}}_n : C[0,1] \to C[0,1]$,

$$\overline{\mathbb{B}}_n f(x) := \begin{cases} f(0), \ x = 0, \\ \dfrac{1}{B(nx, n(1-x))} \int_0^1 t^{nx-1} (1-t)^{n(1-x)-1} f(t) \, \mathrm{d}t \ (0 < x < 1), \\ f(1) \ (x = 1). \end{cases}$$

This operator was introduced by Mühlbach [7,8] and Lupaş [3,9].
We use the notation $(a)_k := a(a+1)\cdots(a+k-1)$, $k \geq 1$, $(a)_0 = 1$.
The operators $L_n : C[0,1] \to C[0,1]$,

$$L_n f(x) := \sum_{j=0}^{n} \binom{n}{j} \frac{(nx)_j (n-nx)_{n-j}}{(n)_n} f\left(\frac{j}{n}\right),$$

were investigated in [3] (see also [4,5]). Clearly, $L_n f(0) = f(0)$, $L_n f(1) = f(1)$.

Theorem 4. *If $f \in C^2[0,1]$ and $2m \leq f''(x) \leq 2M$, $x \in [0,1]$, then*

$$m\frac{x(1-x)}{n+1} \leq L_n f(x) - \overline{\mathbb{B}}_n f(x) \leq M\frac{x(1-x)}{n+1}, \ x \in [0,1]. \quad (24)$$

Proof. Clearly, (24) is satisfied for $x = 0$ and $x = 1$. Let $0 < x < 1$. Then, according to (8),

$$\overline{\mathbb{B}}_n f(x) = A_{nx,n(1-x)}(f). \quad (25)$$

On the other hand, using (2) we obtain

$$\begin{aligned}
A_{n,nx,n(1-x)}(f) &= \frac{1}{B(nx,n(1-x))}\sum_{j=0}^{n}\binom{n}{j}B(j+nx,n-j+n(1-x))f\left(\frac{j}{n}\right) \\
&= \sum_{j=0}^{n}\binom{n}{j}\frac{\Gamma(j+nx)\Gamma(2n-j-nx)}{\Gamma(2n)}\cdot\frac{\Gamma(n)}{\Gamma(nx)\Gamma(n(1-x))}f\left(\frac{j}{n}\right) \\
&= \sum_{j=0}^{n}\binom{n}{j}\frac{(nx)_j(n-nx)_{n-j}}{(n)_n}f\left(\frac{j}{n}\right),
\end{aligned}$$

and so

$$L_n f(x) = A_{n,nx,n(1-x)}(f). \tag{26}$$

Now, (24) is a consequence of (25), (26) and (11). □

5. A Heun Function

Consider the function

$$h_{n,\alpha,\beta}(x) := \sum_{j=0}^{n} a_{n,j,\alpha,\beta}(2x-1)^{2j},\ x \in \mathbb{R}. \tag{27}$$

Theorem 5. *If $0 < \alpha \le \beta$, then $h_{n,\alpha,\beta}$ is a solution to the Heun differential equation*

$$\begin{aligned}
x(x-1)\left(x-\frac{1}{2}\right)u''(x) + \left(\frac{\alpha+\beta}{x} + \frac{\alpha+\beta}{x-1} + \frac{1-2n-2\beta}{x-1/2}\right)u'(x) \\
- \frac{4n\alpha x - 2n\alpha}{x(x-1)(x-1/2)}u(x) = 0.
\end{aligned} \tag{28}$$

Moreover, $h_{n,\alpha,\beta}$ is a logarithmically convex function.

Proof. It was proved in [1] that if $0 < 2\theta \le \gamma$, then the function

$$H_{n,\gamma,\theta}(x) := \sum_{j=0}^{n}\binom{n}{j}\frac{(\theta)_j(\gamma-\theta)_{n-j}}{(\gamma)_n}(2x-1)^{2j}$$

is a logarithmically convex solution to the Heun differential equation

$$\begin{aligned}
x(x-1)\left(x-\frac{1}{2}\right)u''(x) + \left(\frac{\gamma}{x} + \frac{\gamma}{x-1} + \frac{2\theta+1-2n-2\gamma}{x-1/2}\right)u'(x) \\
- \frac{4n\theta x - 2n\theta}{x(x-1)(x-\frac{1}{2})}u(x) = 0.
\end{aligned} \tag{29}$$

Setting $\theta = \alpha$, $\gamma = \alpha+\beta$, we obtain

$$\binom{n}{j}\frac{(\theta)_j(\gamma-\theta)_{n-j}}{(\gamma)_n} = a_{n,j,\alpha,\beta},$$

So, (29) becomes (28), and $H_{n,\gamma,\theta}(x)$ becomes $h_{n,\alpha,\beta}(x)$. This concludes the proof. □

Remark 2. *From (27) and (6), we see that $h_{n,\alpha,\beta}(0) = 1$. Therefore $h_{n,\alpha,\beta}$ is a polynomial, logarithmically convex, Heun function.*

Remark 3. *The function $h_{n,\alpha,\beta}(x) = H_{n,\alpha+\beta,\alpha}(x)$ can be expressed also in terms of Appell polynomials. For further details, (see [1], Section 6).*

6. Stochastic Convex Orderings

Let X and Y be random variables on the same probability space. We say that X is dominated by Y (and write $X \leq_{cx} Y$) in the sense of the convex stochastic order if

$$Ef(X) \leq Ef(Y)$$

for all convex functions $f : \mathbb{R} \to \mathbb{R}$ such that the expectations exist (see [10,11]).

Theorem 6. *Let $0 < \beta \leq \alpha$ and $n \in \mathbb{N}$. Then, with respect to the convex stochastic order, we have*

$$\mathcal{B}_{\alpha,\alpha} \leq_{cx} \mathcal{B}_{\beta,\beta}, \tag{30}$$

and

$$Z_{n,\alpha,\alpha} \leq_{cx} Z_{n,\beta,\beta}. \tag{31}$$

Proof. It was proved that (see [1] (Theorem 10.1))

$$\frac{1}{B(\alpha,\alpha)} \int_0^1 x^{\alpha-1}\,(1-x)^{\alpha-1}\,f(x)\,\mathrm{d}x \leq \frac{1}{B(\beta,\beta)} \int_0^1 x^{\beta-1}\,(1-x)^{\beta-1}\,f(x)\,\mathrm{d}x, \tag{32}$$

for each convex function $f \in [0,1]$, provided that $0 < \beta \leq \alpha$.

This proves (30). Now, let $f \in C[0,1]$ be convex. Then, $B_n f$ is also convex, and (32) shows that

$$\frac{1}{B(\alpha,\alpha)} \int_0^1 x^{\alpha-1}\,(1-x)^{\alpha-1}\,B_n\,f(x)\,\mathrm{d}x \leq \frac{1}{B(\beta,\beta)} \int_0^1 x^{\beta-1}\,(1-x)^{\beta-1}\,B_n\,f(x)\,\mathrm{d}x.$$

Using (2) and (5), we obtain

$$A_{n,\alpha,\alpha}(f) \leq A_{n,\beta,\beta}(f),$$

i.e.,

$$\sum_{j=0}^{n} a_{n,j,\alpha,\alpha} f\left(\frac{j}{n}\right) \leq \sum_{j=0}^{n} a_{n,j,\beta,\beta} f\left(\frac{j}{n}\right). \tag{33}$$

This means that $Ef(Z_{n,\alpha,\alpha}) \leq Ef(Z_{n,\beta,\beta})$, therefore, (31) is proved. □

7. Conclusions and Directions for Further Work

The probability distribution $(a_{n,j,\alpha})_{j=0,1,\cdots,n}$ was investigated from several points of view in [1]. In this paper, we generalize the corresponding results by considering the distribution $(a_{n,j,\alpha,\beta})_{j=0,1,\cdots,n}$ such that $(a_{n,j,\alpha,\alpha})_{j=0,1,\cdots,n}$ is $(a_{n,j,\alpha})_{j=0,1,\cdots,n}$ from [1]. A sequence of positive linear functionals is constructed in terms of the probability distribution. This sequence is convergent to another functional and this gives rise to a quadrature formula. The remainder of this formula is estimated for functions in $C^2[0,1]$ in terms of the uniform norm of f''. We intend to extend this result to functions in $C[0,1]$ by considering suitable moduli of continuity or K functionals. We also estimate the difference between two classical operators acting on functions in $C^2[0,1]$ and we study the same problem for functions in $C[0,1]$. A sequence of random variables is constructed using the probability distribution and is investigated from the point of view of the characteristic functions and their convergence. The probability distribution is useful for constructing a polynomial, logarithmically convex, Heun function. An inequality in the sense of the convex stochastic order is also established.

The numbers $a_{n,j,\alpha,\beta}$ satisfy the following recurrence relations:

$$a_{0,0,\alpha,\beta} := 1,$$
$$a_{n+1,0,\alpha,\beta} = \frac{n+\beta}{n+\alpha+\beta} a_{n,0,\alpha,\beta},\ n \geq 0,$$
$$a_{n+1,j+1,\alpha,\beta} = \frac{n+1}{n+\alpha+\beta} \cdot \frac{j+\alpha}{j+1} a_{n,j,\alpha,\beta},\ 0 \leq j \leq n.$$

For certain values of α, β, n, j, we present, in Table 1, the numerical values of the numbers $a_{n,j,\alpha,\beta}$.

Table 1. Values of $a_{n,j,\alpha,\beta}$, $\alpha = 1/2$, $\beta = 2$ and $n, j = 0, 1, \cdots 9$.

$n \setminus j$	0	1	2	3	4	5	6	7	8	9
0	1									
1	$\frac{4}{5}$	$\frac{1}{5}$								
2	$\frac{24}{35}$	$\frac{8}{35}$	$\frac{3}{35}$							
3	$\frac{64}{105}$	$\frac{8}{35}$	$\frac{4}{35}$	$\frac{1}{21}$						
4	$\frac{128}{231}$	$\frac{256}{155}$	$\frac{48}{385}$	$\frac{16}{231}$	$\frac{1}{33}$					
5	$\frac{512}{1001}$	$\frac{640}{3003}$	$\frac{128}{1001}$	$\frac{80}{1001}$	$\frac{20}{429}$	$\frac{3}{143}$				
6	$\frac{1024}{2145}$	$\frac{1024}{5005}$	$\frac{128}{1001}$	$\frac{256}{3003}$	$\frac{8}{143}$	$\frac{24}{715}$	$\frac{1}{65}$			
7	$\frac{16384}{36465}$	$\frac{7168}{36465}$	$\frac{1536}{12155}$	$\frac{640}{7293}$	$\frac{448}{7293}$	$\frac{504}{12155}$	$\frac{28}{1105}$	$\frac{1}{85}$		
8	$\frac{98304}{230945}$	$\frac{131072}{692835}$	$\frac{28672}{230945}$	$\frac{4096}{46189}$	$\frac{8960}{138567}$	$\frac{10752}{230945}$	$\frac{672}{20995}$	$\frac{32}{1615}$	$\frac{3}{323}$	
9	$\frac{131072}{323323}$	$\frac{294912}{1616615}$	$\frac{196608}{1616615}$	$\frac{4096}{46189}$	$\frac{3072}{46189}$	$\frac{2304}{46189}$	$\frac{768}{20995}$	$\frac{288}{11305}$	$\frac{36}{2261}$	$\frac{1}{133}$

We will study the possibility of extending the definitions of the numbers $a_{n,j,\alpha,\beta}$, when α and β tend individually or simultaneously to 0 or to ∞. This would increase the family of results and examples related to the probability distribution which we have considered herein.

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