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The General Fractional Integrals and Derivatives on a Finite Interval

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Abstract: The general fractional integrals and derivatives considered so far in the Fractional Calculus literature have been defined for the functions on the real positive semi-axis. The main contribution of this paper is in introducing the general fractional integrals and derivatives of the functions on a finite interval. As in the case of the Riemann–Liouville fractional integrals and derivatives on a finite interval, we define both the left- and the right-sided operators and investigate their interconnections. The main results presented in the paper are the 1st and the 2nd fundamental theorems of Fractional Calculus formulated for the general fractional integrals and derivatives of the functions on a finite interval as well as the formulas for integration by parts that involve the general fractional integrals and derivatives.

Keywords: Sonin kernels; Sonin condition; general fractional integral; general fractional derivative; fundamental theorems of fractional calculus

MSC: 26A33; 26A24; 45E10

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1. Introduction

The so-called general fractional integrals (GFIs) and the general fractional derivatives (GFDs) in their nowadays form appeared for the first time in the paper [1] by Sonin published in 1884. In this famous paper, the Sonin condition and the Sonin kernels that satisfy this condition were defined and an important class of such kernels in form of products of the power law functions and analytical functions was introduced. However, Sonin did not mention any connection of his operators to Fractional Calculus (FC). Moreover, his derivations were mostly formal and without providing exact conditions for their validity, including the spaces of functions.

The first publication devoted to the GFIs and the GFDs embedded in the framework of FC was the paper [2] by Kochubei published in 2011. In this paper, Kochubei first introduced a very important class of the Sonin kernels in terms of their Laplace transforms. He also established a connection of these kernels to the complete Bernstein functions and the Stieltjes functions and introduced a regularized form of the general fractional derivatives with these kernels, nowadays referred to as the Kochubei kernels. Moreover, Kochubei initiated a new line of FC research devoted to the ordinary and partial differential equations with the GFDs. In particular, he deduced some important results for the fractional relaxation equation and the Cauchy problem for the time-fractional diffusion equations with the GFDs with the Kochubei kernels; see the paper [3] for a survey of the recent results regarding these fractional differential equations.

The next publication devoted to the GFIs, the GFDs, and the fractional differential equations with the GFDs was the paper [4] by Luchko and Yamamoto published in 2018. In Ref. [4], some important estimates for the GFDs of the functions at their maximum points were first derived. Then these estimates were applied to prove a weak maximum principle for solutions to the initial-boundary-value problems for the general time-fractional diffusion equations with the GFDs. It is worth mentioning that in Ref. [4], the GFIs and the GFDs with the Sonin kernels from a different class of kernels compared to the Kochubei set were considered.

Very recently, a series of papers [5–11] devoted to the GFIs and the GFDs with the kernels from several different sets was published by Luchko. In Refs. [5,7,8], the general FC operators with the Sonin kernels that are continuous of the positive real semi-axis and can possess an integrable singularity of a power function type at the origin were defined and investigated. In Ref. [6], the Sonin condition was extended in the manner that allows introducing the GFIs and the GFDs of arbitrary order (please note that in the previous publications only the case of the order less than or equal to one has been treated). In Ref. [11], another extension of the Sonin condition to the case of three kernels has been suggested and the so-called 1st level GFDs with these kernels were defined for the first time. The 1st level GFDs contain the GFDs and the regularized GFDs introduced so far as their particular cases just in the same manner as the Hilfer fractional derivative covers both the Riemann–Liouville and the Caputo fractional derivatives. It is worth mentioning that the GFIs and the GFDs with the Luchko kernels have already been applied in FC literature both for mathematical and applied problems. In particular, in Ref. [9], they were employed for derivation of two different forms of a generalized convolution Taylor formula that provides a representation of a function as a convolution polynomial with a remainder in form of a composition of the n-fold GFIs and the n-fold sequential GFDs or the regularized GFDs. In Refs. [12–18], Tarasov used these operators for formulation of a general fractional dynamics, a general non-Markovian quantum dynamics, a general fractional vector calculus, a general non-local continuum mechanics, a non-local probability theory, a non-local statistical mechanics, and a non-local gravity theory, respectively.

The framework of the GFIs and the GFDs introduced by Sonin in Ref. [1] is very general. For developing a reasonable theory of these general FC operators, both the special sets of the kernels (such as, e.g., the Kochubei set, the Luchko set, etc.) and the suitable spaces of functions are needed. In this sense, there exists not only one but several different theories of the GFIs and the GFDs.

Another aspect of the GFIs and the GFDs that was not yet taken into consideration in the FC literature concerns their domains. It is well-known that the properties and even the definitions of the classical Riemann–Liouville fractional integrals and derivatives are very different in the case of the functions defined on a final interval, on the semi-axes, or on the real axes, respectively (see, e.g., Ref. [19]). The GFIs and the GFDs considered so far were introduced for the functions defined on the real positive semi-axes. In this paper, we suggest the definitions of the GFIs and the GFDs for the functions defined on a finite interval and study their basic properties for the first time in the FC literature.

It is worth mentioning that there exist some other concepts of the general FC operators defined in a completely different form compared to those mentioned above. In particular, we refer to Refs. [20,21], and Ref. [22] devoted to this topic. However, in this paper, we restrict ourselves to the general FC operators generated by the modified Sonin kernels and do not consider the approaches suggested in Refs. [20–22], and in other publications of this type.

The rest of this paper is organized as follows: In Section 2, we define the spaces of functions that we use in the further discussions, formulate a suitably modified Sonin condition, and introduce the GFIs with the kernels from a certain set in the case of the functions defined on a finite interval. In contrast to the GFIs defined for the functions on the real positive semi-axes, we define both the left- and the right-sided operators and then investigate their interconnections. Section 3 is devoted to the GFDs and their properties. The main results presented here are the 1st and the 2nd fundamental theorems of FC formulated for the GFIs and the GFDs for the functions defined on a finite interval as well as the formulas for integration by parts that involve the GFDs. In Section 4, some conclusions and directions for further research are formulated.

2. The General Fractional Integrals on a Finite Interval

We start this section by specifying the spaces of functions suitable for our constructions of the GFIs and the GFDs. The spaces of this type were introduced by Dimovski in Ref. [23] in connection with his operational calculus for the hyper-Bessel differential operators. Then these spaces were extensively employed in the publications by Luchko devoted to the operational calculus for different fractional derivatives (see Ref. [24] for a survey of these results) and in his recent papers dealing with the general fractional integrals and derivatives for the functions on the real positive semi-axis ([5–11]).

Definition 1. *For $\alpha \ge -1$ and $n \in \mathbb{N}$, we define the spaces of functions*

$$C^n_\alpha(a,b] = \{f \in C_\alpha(a,b] : f^{(n)} \in C_\alpha(a,b]\},$$

$$C^n_\alpha[a,b) = \{f \in C_\alpha[a,b) : f^{(n)} \in C_\alpha[a,b)\},$$

where

$$C_\alpha(a,b] = \{f : (a,b] \to \mathbb{R} : f(t) = (t-a)^p f_1(t),\ p > \alpha,\ f_1 \in C[a,b]\},$$

$$C_\alpha[a,b) = \{f : [a,b) \to \mathbb{R} : f(t) = (b-t)^p f_1(t),\ p > \alpha,\ f_1 \in C[a,b]\},$$

and the spaces $C_\alpha(a,b]$ and $C_\alpha(a,b]$ are interpreted as $C^0_\alpha(a,b]$ and $C^0_\alpha(a,b]$, respectively.

Because the kernels of the fractional derivatives defined on a finite or infinite interval should be singular at one of the ends of the interval (see, e.g., Ref. [25]), the spaces of functions introduced in Definition 1 are very natural in the context of FC. As already mentioned, in Ref. [5–11], similar spaces were successfully employed for the development of the general FC on the semi-axis.

Another important remark is that the families of the spaces $C^n_\alpha(a,b]$ and $C^n_\alpha[a,b)$, $n = 0,1,2,\ldots$ are ordered with respect to the parameter α, i.e., for $\alpha_1 > \alpha_2 \ge -1$, the inclusions

$$C^n_{\alpha_1}(a,b] \subset C^n_{\alpha_2}(a,b] \ \text{ and } \ C^n_{\alpha_1}[a,b) \subset C^n_{\alpha_2}[a,b)$$

hold valid. This means that the spaces $C^n_{-1}(a,b]$ and $C^n_{-1}[a,b)$ are the largest in their families and contain all other spaces $C^n_\alpha(a,b]$ or $C^n_\alpha[a,b)$, respectively, as their sub-spaces. Thus, in what follows, we mainly employ the spaces $C^n_{-1}(a,b]$ and $C^n_{-1}[a,b)$. Evidently, all results derived for these spaces are also valid for the spaces $C^n_\alpha(a,b]$ and $C^n_\alpha[a,b)$ with any $\alpha \ge -1$.

In this paper, we introduce and investigate the GFIs and the GFDs on a finite interval $(a,\ b)$ with the kernels from the space $C_{-1}(0,b-a]$. Moreover, we often suppose that the kernels $\kappa \in C_{-1}(0,b-a]$ of the GFIs and the kernels $k \in C_{-1}(0,b-a]$ of the GFDs are the Sonin kernels that satisfy the Sonin condition [1]

$$(\kappa * k)(t) = \int_0^t \kappa(t-\tau)k(\tau)d\tau = \{1\},\ \ 0 < t \le b-a, \tag{1}$$

where $\{1\}$ stands for the function identically equal to one for $t \in (0,\ b-a]$. The set of such kernels will be denoted by $\mathcal{S}_f$.

In the literature, several pairs of the Sonine kernels from $\mathcal{S}_f$ were derived in terms of the elementary and special functions (see, e.g., Refs. [2,5,7,26,27] and the references therein). The most prominent example known already to Abel (see Refs. [28,29]) is a pair of the power law kernels

$$\kappa(t) = h_\alpha(t),\ k(t) = h_{1-\alpha}(t),\ 0 < \alpha < 1, \tag{2}$$

where the function h_α is given by

$$h_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)},\ \alpha > 0. \tag{3}$$

Another important example of the Sonin kernels was derived in Ref. [26]:

$$\kappa(t) = h_{1-\beta+\alpha}(t) + h_{1-\beta}(t),\ k(t) = t^{\beta-1}E_{\alpha,\beta}(-t^{\alpha}),\ \ 0 < \alpha < \beta < 1, \tag{4}$$

where

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)},\ \alpha > 0, \beta, z \in \mathbb{C}$$

is the two-parameters Mittag-Leffler function and h_α is defined by (3).

We also mention the following pair of the Sonin kernels that was first deduced by Sonin in Ref. [1]:

$$\kappa(t) = (\sqrt{t})^{-\alpha} I_{-\alpha}(2\sqrt{t}),\ \ k(t) = (\sqrt{t})^{\alpha-1} J_{\alpha-1}(2\sqrt{t}),\ 0 < \alpha < 1, \tag{5}$$

where

$$J_\mu(t) = \sum_{n=0}^{\infty} \frac{(-1)^n (t/2)^{2n+\mu}}{n!\Gamma(n+\mu+1)},\ I_\nu(t) = \sum_{n=0}^{\infty} \frac{(t/2)^{2n+\nu}}{n!\Gamma(n+\nu)}$$

are the Bessel and the modified Bessel functions, respectively.

Definition 2. *Let a kernel κ belong to the space $C_{-1}(0, b-a]$.*

The left-sided general fractional integral (LGFI) and the right-sided general fractional integral (RGFI) are defined by the following formulas, respectively:

$$({}_{l}\mathbb{I}_{(\kappa)}\, f)(t) = (\kappa * f)(t) = \int_a^t \kappa(t-\tau) f(\tau) d\tau,\ a < t \le b, \tag{6}$$

$$({}_{r}\mathbb{I}_{(\kappa)}\, f)(t) = \int_t^b \kappa(\tau - t) f(\tau) d\tau,\ a \le t < b. \tag{7}$$

In the rest of this section, we discuss the properties of the LGFIs (6) and the RGFIs (7) that are valid any kernel κ from the space $C_{-1}(0, b-a]$. However, in the next section, where the GFDs are introduced and investigated, we suppose that κ is a Sonin kernel from the set $\mathcal{S}_f$. In particular, the power law kernel $\kappa(t) = h_\alpha(t)$ generates the well-known left- and right-sided Riemann–Liouville fractional integrals that have been extensively studied in the FC literature (see, e.g., Ref. [19] for their properties). The kernel $\kappa(t) = h_{1-\beta+\alpha}(t) + h_{1-\beta}(t)$, $0 < \alpha < \beta < 1$ from (4) leads to a sum of two left- and right-sided Riemann–Liouville fractional integrals of the orders $1 - \beta + \alpha$ and $1 - \beta$, respectively. Finally, the kernel $\kappa(t) = (\sqrt{t})^{-\alpha} I_{-\alpha}(2\sqrt{t})$, $0 < \alpha < 1$ from (5) generates the following pair of the left- and right-sided GFIs:

$$({}_{l}\mathbb{I}_{(\kappa)}\, f)(t) = \int_a^t (\sqrt{(t-\tau)})^{-\alpha} I_{-\alpha}(2\sqrt{(t-\tau)}) f(\tau) d\tau,\ a < t \le b, \tag{8}$$

$$({}_{r}\mathbb{I}_{(\kappa)}\, f)(t) = \int_t^b (\sqrt{(\tau-t)})^{-\alpha} I_{-\alpha}(2\sqrt{(\tau-t)}) f(\tau) d\tau,\ a \le t < b. \tag{9}$$

The LGFI and the RGFI introduced in Definition 2 can be studied on different spaces of functions, see e.g., Ref. [19] for the theory of the left- and right-sided Riemann–Liouville fractional integrals and derivatives on the finite intervals. In this paper, we focus on the properties of the LGFI and the RGFI on the spaces introduced in Definition 1.

Proposition 1. *Let a kernel κ belong to the space $C_{-1}(0, b-a]$.*

The LGFI (6) maps the space $C_{-1}(a, b]$ into itself:

$${}_{l}\mathbb{I}_{(\kappa)} : C_{-1}(a, b] \longrightarrow C_{-1}(a, b] \tag{10}$$

and the RGFI (7) maps the space $C_{-1}[a,b)$ into itself:

$$ {}_{r}\mathbb{I}_{(\kappa)} : C_{-1}[a,b) \longrightarrow C_{-1}[a,b). \tag{11} $$

Proof. First we prove the mapping property (10). Since $f \in C_{-1}(a,b]$, then $f(t) = (t-a)^p f_1(t), p > -1, f_1 \in C[a,b]$. Because the kernel κ belongs to the space $C_{-1}(0,b-a]$, the representation $\kappa(t) = t^q \kappa_1(t)$ with $q > -1$ and $\kappa_1(t) \in C[0,b-a]$ holds true. Then we get the relation

$$ ({}_{l}\mathbb{I}_{(\kappa)} f)(t) = \int_a^t (t-\tau)^q (\tau-a)^p \kappa_1(t-\tau) f_1(\tau) d\tau, \ a < t \le b. \tag{12} $$

Using the variables substitution $z = \frac{t-\tau}{t-a}$, we arrive at the formula

$$ ({}_{l}\mathbb{I}_{(\kappa)} f)(t) = (t-a)^{q+p+1} \int_0^1 z^q (1-z)^p \kappa_1\big(z(t-a)\big) f_1\big(t - z(t-a)\big) dz, \ a < t \le b. \tag{13} $$

Denoting the product $\kappa_1\big(z(t-a)\big) f_1\big(t-z(t-a)\big)$ by $g(t,z)$, the inclusions $\kappa_1 \in C[0,\, b-a]$, $f_1 \in C[a,b]$ mean that the function g is continuous for $0 \le z \le 1$ and $a \le t \le b$. Because the function $z^q(1-z)^p \ge 0$ is integrable, the mean value theorem for integrals yields the representation

$$ \begin{aligned} ({}_{l}\mathbb{I}_{(\kappa)} f)(t) &= (t-a)^{q+p+1} g(\hat{t}, z_0) \int_0^1 z^q (1-z)^p dz \\ &= (t-a)^{q+p+1} g(\hat{t}, z_0) B(q+1, p+1), \ \text{ for some } 0 < z_0 < 1,\ \hat{t} > a, \end{aligned} \tag{14} $$

which proves the Formula (10).

The proof of the mapping property (11) is completely analogous and we omit it here. □

Now we proceed with formulations and proofs of other important properties of the LGFIs and the RGFIs on the spaces $C_{-1}(a,b]$ and $C_{-1}[a,b)$, respectively.

Proposition 2. *Let a kernel κ belong to the space $C_{-1}(0,b-a]$.*

For any functions $f \in C_{-1}[a,b)$ and $g \in C_{-1}(a,b]$, the formula for fractional integration by parts

$$ \int_a^b f(t)\, ({}_{l}\mathbb{I}_{(\kappa)} g)(t) dt = \int_a^b ({}_{r}\mathbb{I}_{(\kappa)} f)(t)\, g(t) dt $$

holds true.

Proof. We start with the representation

$$ \begin{aligned} \int_a^b f(t)\, ({}_{l}\mathbb{I}_{(\kappa)} g)(t) dt &= \int_a^b f(t) \int_a^t \kappa(t-\tau) g(\tau) d\tau dt \\ &= \int_a^b \int_a^t f(t) \kappa(t-\tau) g(\tau) d\tau dt. \end{aligned} \tag{15} $$

Because the integrals in the last formula are absolutely convergent, we can interchange the order of integration by Fubini's theorem and get the formula

$$ \begin{aligned} \int_a^b f(t)\, ({}_{l}\mathbb{I}_{(\kappa)} g)(t) dt &= \int_a^b \int_\tau^b f(t) \kappa(t-\tau) g(\tau) dt d\tau \\ &= \int_a^b g(\tau) \int_\tau^b f(t) \kappa(t-\tau) dt d\tau = \int_a^b g(\tau)\, ({}_{r}\mathbb{I}_{(\kappa)} f)(\tau) d\tau, \end{aligned} \tag{16} $$

which completes the proof of the theorem. □

Proposition 3. *Let κ_1 and κ_2 be two kernels from the space $C_{-1}(0,\, b-a]$.*

The LGFI (6) and the RGFI (7) possess the semi-group properties in the form

$$
\begin{aligned}
({}_l\mathbb{I}_{(\kappa_1)}\, {}_l\mathbb{I}_{(\kappa_2)} f)(t) &= ({}_l\mathbb{I}_{(\kappa_1 * \kappa_2)}\, f)(t), \ \ f \in C_{-1}(a,b], && (17)\\
({}_r\mathbb{I}_{(\kappa_1)}\, {}_r\mathbb{I}_{(\kappa_2)} f)(t) &= ({}_r\mathbb{I}_{(\kappa_1 * \kappa_2)}\, f)(t), \ \ f \in C_{-1}[a,b). && (18)
\end{aligned}
$$

As a consequence, the following commutative laws are valid:

$$
\begin{aligned}
({}_l\mathbb{I}_{(\kappa_1)}\, {}_l\mathbb{I}_{(\kappa_2)} f)(t) &= ({}_l\mathbb{I}_{(\kappa_2)}\, {}_l\mathbb{I}_{(\kappa_1)}\, f)(t), \ \ f \in C_{-1}(a,b], && (19)\\
({}_r\mathbb{I}_{(\kappa_1)}\, {}_r\mathbb{I}_{(\kappa_2)} f)(t) &= ({}_r\mathbb{I}_{(\kappa_2)}\, {}_r\mathbb{I}_{(\kappa_1)}\, f)(t), \ \ f \in C_{-1}[a,b). && (20)
\end{aligned}
$$

Proof. We start with a proof of the relation (18) and first represent its left-hand side as follows:

$$
\begin{aligned}
({}_r\mathbb{I}_{(\kappa_1)}\, {}_r\mathbb{I}_{(\kappa_2)} f)(t) &= \int_t^b \kappa_1(\tau - t)\big({}_r\mathbb{I}_{(\kappa_2)}\, f\big)(\tau)d\tau \\
&= \int_t^b \kappa_1(\tau - t) \int_\tau^b \kappa_2(y-\tau) f(y)dyd\tau \\
&= \int_t^b \int_\tau^b \kappa_1(\tau - t)\kappa_2(y-\tau) f(y)dyd\tau.
\end{aligned}
$$

Interchanging the order of integration in the last double integral yields the relation

$$
\begin{aligned}
({}_r\mathbb{I}_{(\kappa_1)}\, {}_r\mathbb{I}_{(\kappa_2)} f)(t) &= \int_t^b \int_t^y \kappa_1(\tau - t)\kappa_2(y-\tau) f(y)d\tau dy \\
&= \int_t^b f(y) \int_t^y \kappa_1(\tau - t)\kappa_2(y-\tau) d\tau dy. && (21)
\end{aligned}
$$

By employing the variables substitution $\tau_1 = y - \tau$, the inner integral of the last formula can be represented in the form

$$
\int_t^y \kappa_1(\tau - t)\kappa_2(y-\tau)d\tau = \int_0^{y-t} \kappa_1(y - t - \tau_1)\kappa_2(\tau_1)d\tau_1 = (\kappa_1 * \kappa_2)(y-t)
$$

which leads to the Formula (18):

$$
({}_r\mathbb{I}_{(\kappa_1)}\, {}_r\mathbb{I}_{(\kappa_2)} f)(t) = \int_t^b f(y)(\kappa_1 * \kappa_2)(y-t)dy = ({}_r\mathbb{I}_{(\kappa_1 * \kappa_2)}\, f)(t).
$$

For $\kappa_1,\ \kappa_2 \in C_{-1}(0,\, b-a]$, the inclusion $\kappa_1 * \kappa_2 \in C_{-1}(0,\, b-a]$ is ensured by the Formula (10) from Proposition 1.

The Formula (17) is a simple consequence from the known properties of the Laplace convolution:

$$
\begin{aligned}
({}_l\mathbb{I}_{(\kappa_1)}\, {}_l\mathbb{I}_{(\kappa_2)} f)(t) &= (\kappa_1 * (\kappa_2 * f))(t) \\
&= ((\kappa_1 * \kappa_2) * f)(t) = ({}_l\mathbb{I}_{(\kappa_1 * \kappa_2)}\, f)(t).
\end{aligned}
$$

In its turn, the Formulas (19) and (20) immediately follow from the Formulas (17) and (18), respectively, because of the well-known fact that the Laplace convolution is commutative. □

It is worth mentioning that, in general, the semi-group properties presented in Proposition 3 are not valid for the GFIs with the Sonin kernels from the set $\mathcal{S}_f$ because the convolution of the two kernels from $\mathcal{S}_f$ does not always belong to $\mathcal{S}_f$. The reason is that the generalized order of the LGFI (6) and the RGFI (7) with the kernels from $\mathcal{S}_f$ is restricted to the interval $(0, 1)$. This is a direct consequence from the Sonin condition (1) because the constant function $\{1\}$ at its right-hand side corresponds to the definite integral of order one.

However, it is possible to extend the Sonin condition (1) and to define the GFIs of arbitrary order that fulfill the semi-group property (see Ref. [6] for the case of the GFIs of arbitrary order on a positive real semi-axes). This will be done elsewhere.

3. The General Fractional Derivatives on a Finite Interval

In this section, we introduce several different kinds of the GFDs on a finite interval and study their basic properties including the 1st and the 2nd fundamental theorems of FC. As in the case of the Riemann–Liouville and the Caputo fractional derivatives with the power law kernels, we define the GFD (of the Riemann–Liouville type) and the regularized GFD (of the Caputo type). Moreover, both the left- and the right-sided GFDs will be introduced and studied.

In what follows, we suppose that the kernels of the GFIs and the GFDs are the Sonin kernels from the set $\mathcal{S}_f$.

Definition 3. *Let a pair of the kernels $(\kappa,\, k)$ belong to the set $\mathcal{S}_f$.*

The left-sided general fractional derivative (LGFD) and the right-sided general fractional derivative (RGFD) are defined by the following formulas, respectively:

$$({}_l\mathbb{D}_{(k)}\, f)(t) = \frac{d}{dt}\int_a^t k(t-\tau) f(\tau)\, d\tau,\ a < t \le b, \tag{22}$$

$$({}_r\mathbb{D}_{(k)}\, f)(t) = -\frac{d}{dt}\int_t^b k(\tau-t) f(\tau)\, d\tau,\ a \le t < b. \tag{23}$$

The regularized left-sided general fractional derivative (RLGFD) and the regularized right-sided general fractional derivative (RRGFD) are defined as follows:

$$({}_{l_*}\mathbb{D}_{(k)}\, f)(t) = \int_a^t k(t-\tau) f'(\tau)\, d\tau,\ a < t \le b, \tag{24}$$

$$({}_{r_*}\mathbb{D}_{(k)}\, f)(t) = -\int_t^b k(\tau-t) f'(\tau)\, d\tau,\ a \le t < b. \tag{25}$$

A well-known example of the left- and right-sided GFDs introduced above are the Riemann–Liouville and the Caputo left- and right-sided fractional derivatives with the power law kernel $k(t) = h_{1-\alpha}(t),\ 0 < \alpha < 1$ from the Sonin pair of the kernels defined by (2).

Another important example is generated by the kernel $k(t) = t^{\beta-1}E_{\alpha,\beta}(-t^\alpha),\ 0 < \alpha < \beta < 1$ from the Sonin pair (4). For this kernel, the left-sided GFD and the left-sided regularized GFD on the positive real semi-axes have been already defined and investigated (see, e.g., Refs. [5,30]). However, to the best of our knowledge, the right-sided GFDs on a finite interval are introduced here for the first time in the FC literature:

$$\begin{aligned}
({}_r\mathbb{D}_{(k)}\, f)(t) &= -\frac{d}{dt}\int_t^b (\tau-t)^{\beta-1}E_{\alpha,\beta}(-(\tau-t)^\alpha) f(\tau) d\tau,\ a \le t < b,\\
({}_{r_*}\mathbb{D}_{(k)}\, f)(t) &= -\int_t^b (\tau-t)^{\beta-1}E_{\alpha,\beta}(-(\tau-t)^\alpha) f'(\tau) d\tau,\ a \le t < b.
\end{aligned}$$

Finally, we mention the right-sided GFDs on a finite interval with the Sonin kernel $k(t) = (\sqrt{t})^{-\alpha}I_{-\alpha}(2\sqrt{t}),\ 0 < \alpha < 1$ from the Sonin pair (5):

$$\begin{aligned}
({}_r\mathbb{D}_{(k)}\, f)(t) &= -\frac{d}{dt}\int_t^b (\sqrt{\tau-t})^{-\alpha}I_{-\alpha}(2\sqrt{\tau-t}) f(\tau) d\tau,\ a \le t < b,\\
({}_{r_*}\mathbb{D}_{(k)}\, f)(t) &= -\int_t^b (\sqrt{\tau-t})^{-\alpha}I_{-\alpha}(2\sqrt{\tau-t}) f'(\tau) d\tau,\ a \le t < b.
\end{aligned}$$

In what follows, we consider the left- and right-sided GFDs introduced above on the spaces $C^1_{-1}(a,b]$ and $C^1_{-1}[a,b)$, respectively (see Definition 1). First, a connection between the GFDs and the regularized GFDs on a finite interval is established.

Proposition 4. *For any functions $f \in C^1_{-1}(a,b]$ and $g \in C^1_{-1}[a,b)$, the relations*

$$({}_l\mathbb{D}_{(k)}\, f)(t) \;=\; f(a)k(t-a) + ({}_{l_*}\mathbb{D}_{(k)}\, f)(t), \;\; a < t \le b \tag{26}$$

and

$$({}_r\mathbb{D}_{(k)}\, g)(t) \;=\; g(b)k(b-t) + ({}_{r_*}\mathbb{D}_{(k)}\, g)(t), \;\; a \le t < b, \tag{27}$$

hold true, respectively.

Proof. To prove the Formula (27), let us introduce an auxiliary function $\hat{k}(t) = \int_0^t k(s)ds$. Then we have the relations

$$\hat{k}(0) = 0 \text{ and } \frac{d}{dt}\hat{k} = k(t), t > 0.$$

Integration by parts yields

$$\begin{aligned}
({}_r\mathbb{D}_{(k)}\, g)(t) &= -\frac{d}{dt}\int_t^b k(\tau-t)g(\tau)d\tau \\
&= -\frac{d}{dt}\left(\left[g(\tau)\hat{k}(\tau-t)\right]_t^b\right) + \frac{d}{dt}\int_t^b \hat{k}(\tau-t)g'(\tau)d\tau \\
&= -\frac{d}{dt}\left(g(b)\hat{k}(b-t)\right) - \int_t^b k(\tau-t)g'(\tau)d\tau \\
&= g(b)k(b-t) + ({}_{r_*}\mathbb{D}_{(k)}\, g)(t),
\end{aligned}$$

which completes the proof of the Formula (27). The Formula (26) can be derived using analogous steps and we omit its proof here. □

The next result concerns different kinds of integration by parts formulas for the GFDs introduced above. It is well known that such formulas play a very important role, say, in the fractional calculus of variations involving the functionals that depend on the fractional derivatives.

Proposition 5. *The following integration by parts formulas hold true*

$$\begin{aligned}
\int_a^b f(t)({}_l\mathbb{D}_{(k)}\, g)(t)dt &= \int_a^b g(t)({}_{r_*}\mathbb{D}_{(k)}\, f)(t)dt + \left[f(t)({}_l\mathbb{I}_{(k)}g)(t)\right]_a^b, \\
& \quad f \in C^1_{-1}[a,b), g \in C^1_{-1}(a,b], && (28) \\
\int_a^b f(t)({}_r\mathbb{D}_{(k)}\, g)(t)dt &= \int_a^b g(t)({}_{l_*}\mathbb{D}_{(k)}\, f)(t)dt - \left[f(t)({}_r\mathbb{I}_{(k)}g)(t)\right]_a^b, \\
& \quad f \in C^1_{-1}(a,b], g \in C^1_{-1}[a,b), && (29) \\
\int_a^b f(t)({}_{l_*}\mathbb{D}_{(k)}\, g)(t)dt &= \int_a^b g(t)({}_r\mathbb{D}_{(k)}\, f)(t)dt + \left[g(t)({}_r\mathbb{I}_{(k)}f)(t)\right]_a^b, \\
& \quad f \in C^1_{-1}[a,b), g \in C^1_{-1}(a,b], && (30) \\
\int_a^b f(t)({}_{r_*}\mathbb{D}_{(k)}\, g)(t)dt &= \int_a^b g(t)({}_l\mathbb{D}_{(k)}\, f)(t)dt - \left[g(t)({}_l\mathbb{I}_{(k)}f)(t)\right]_a^b, \\
& \quad f \in C^1_{-1}(a,b], g \in C^1_{-1}[a,b). && (31)
\end{aligned}$$

Proof. We start with a proof of the Formula (28) and first represent its left-hand side in the form

$$\int_a^b f(t)({}_l\mathbb{D}_{(k)}\, g)(t)dt = \int_a^b f(t)\left(\frac{d}{dt}\int_a^t k(t-\tau)g(\tau)d\tau\right)dt.$$

Integration by parts in the last integral yields a chain of the relations

$$\begin{aligned}
\int_a^b f(t)({}_l\mathbb{D}_{(k)}\, g)(t)dt &= \left[f(t)\int_a^t k(t-\tau)g(\tau)d\tau\right]_a^b - \int_a^b f'(t)\int_a^t k(t-\tau)g(\tau)d\tau dt \\
&= \left[f(t)({}_l\mathbb{I}_{(k)}g)(t)\right]_a^b - \int_a^b\int_a^t f'(t)k(t-\tau)g(\tau)d\tau dt \\
&= \left[f(t)({}_l\mathbb{I}_{(k)}g)(t)\right]_a^b - \int_a^b g(\tau)\int_\tau^b f'(t)k(t-\tau)dt d\tau \\
&= \left[f(t)({}_l\mathbb{I}_{(k)}g)(t)\right]_a^b + \int_a^b g(\tau)({}_{r_*}\mathbb{D}_{(k)}\, f)(\tau)d\tau,
\end{aligned}$$

which completes the proof the Formula (28). The Formula (29) is proved by following the exact same lines, whereas the Formula (30) immediately follows from (29) and the Formula (31) is a direct consequence from the Formula (28). □

In the rest of this section, we formulate and prove the 1st and the 2nd fundamental theorems of FC for the GFIs and the GFDs introduced above.

Theorem 1. *(1st Fundamental Theorem of FC)*
Let a pair of the kernels $(\kappa,\, k)$ belong to the set $\mathcal{S}_f$.
The left- and the right-sided GFDs are the left-inverse operators to the corresponding GFIs:

$$({}_l\mathbb{D}_{(k)}\,{}_l\mathbb{I}_{(\kappa)}f)(t) = f(t),\ f\in C_{-1}^1(a,b],\ a<t\le b, \tag{32}$$

$$({}_{l_*}\mathbb{D}_{(k)}\,{}_l\mathbb{I}_{(\kappa)}f)(t) = f(t),\ f\in C_{-1}^1(a,b],\ a<t\le b, \tag{33}$$

$$({}_r\mathbb{D}_{(k)}\,{}_r\mathbb{I}_{(\kappa)}f)(t) = f(t),\ f\in C_{-1}^1[a,b),\ a\le t<b, \tag{34}$$

$$({}_{r_*}\mathbb{D}_{(k)}\,{}_r\mathbb{I}_{(\kappa)}f)(t) = f(t),\ f\in C_{-1}^1[a,b),\ a\le t<b. \tag{35}$$

Proof. We start with a proof of the Formula (32). By definition, the left-hand side of (32) takes the form

$$\begin{aligned}
({}_l\mathbb{D}_{(k)}\,{}_L\mathbb{I}_{(\kappa)}f)(t) &= \frac{d}{dt}\int_a^t k(t-\tau)({}_l\mathbb{I}_{(\kappa)}f)(\tau)d\tau \\
&= \frac{d}{dt}\int_a^t k(t-\tau)\int_a^\tau \kappa(\tau-y)f(y)dyd\tau.
\end{aligned}$$

Interchanging the order of integration in the last integral yields

$$({}_l\mathbb{D}_{(k)}\,{}_l\mathbb{I}_{(\kappa)}f)(t) = \frac{d}{dt}\int_a^t f(y)\int_y^t k(t-\tau)\kappa(\tau-y)d\tau dy. \tag{36}$$

Due to the relation

$$\int_y^t k(t-\tau)\kappa(\tau-y)d\tau = \int_0^{t-y} k(\tau)\kappa(t-y-\tau)d\tau = (k*\kappa)(t-y) = 1,\ 0<t-y\le b-a, \tag{37}$$

the representation (36) immediately leads to the formula

$$({}_l\mathbb{D}_{(k)}\,{}_l\mathbb{I}_{(\kappa)}f)(t) = \frac{d}{dt}\int_a^t f(y)dy = f(t),\ t>a,$$

which completes the proof of (32).

Now we verify the relation (33) and first show that

$$({}_l\mathbb{I}_{(\kappa)}f)(a) = 0 \tag{38}$$

for any $f \in C^1_{-1}(a,b]$.

Indeed, the inclusion $\kappa \in C_{-1}(0, b-a]$ leads to the representation $\kappa(t) = t^{p-1}h(t)$, $p > 0, h \in C[0, b-a]$. Since $f \in C^1_{-1}(a,b]$, the same arguments that were employed in [31] for the space $C^1_{-1}(0,+\infty)$ lead to the inclusion $f \in C[a,b]$ and it holds that

$$({}_l\mathbb{I}_{(\kappa)}f)(t) = \int_a^t (t-\tau)^{p-1} h(t-\tau) f(\tau) d\tau.$$

Using the substitution $z = \frac{t-\tau}{t-a}$ in the last integral, we arrive at the representation

$$({}_l\mathbb{I}_{(\kappa)}f)(t) = (t-a)^p \int_0^1 z^{p-1} h(z(t-a)) f(t - z(t-a)) dz. \tag{39}$$

Now let us introduce an auxiliary function as follows: $g(t,z) = h(z(t-a))f(t-z(t-a))$. Because the functions f and h are continuous, the function g is continuous in z on the interval $[0,1]$. Then we can apply the mean value theorem (the function $z^{p-1} \geq 0$ is integrable) to the integral at the right-hand side of the Formula (39) and thus obtain the representation

$$\begin{aligned} ({}_l\mathbb{I}_{(\kappa)}f)(t) &= (t-a)^p g(\hat{t}, z_0) \int_0^1 z^{p-1} dz \\ &= (t-a)^p g(\hat{t}, z_0) \frac{1}{p},\ 0 < z_0 < 1,\ \hat{t} > a,\ p > 0, \end{aligned}$$

which completes the proof of the Formula (38).

Because $({}_l\mathbb{I}_{(\kappa)}f)(a) = 0$, the formula (33) follows by combining the results provided in the Equations (26), (32), and (38).

The proof of the Formula (34) is completely analogous to the proof of (32) and the proof of (35) follows the lines of the proof of (33). □

Remark 1. *Applying the methods used in Ref. [5] for the GFI and the GFDs on the real positive semi-axis, the relations (32)–(35) can be proved on the following spaces of functions that are larger than $C^1_{-1}(a,b]$ and $C^1_{-1}[a,b)$, respectively (see Ref. [5] for details):*

$$C_{-1,(\kappa)}(a,b] = \{f : {}_l\mathbb{I}_{(\kappa)}f \in C^1_{-1}(a,b], ({}_l\mathbb{I}_{(\kappa)}f)(a) = 0\},$$

$$C_{-1,(\kappa)}[a,b) = \{f : {}_r\mathbb{I}_{(\kappa)}f \in C^1_{-1}[a,b), ({}_r\mathbb{I}_{(\kappa)}f)(b) = 0\}.$$

Now we formulate and prove the 2nd fundamental theorem of FC for the left- and right-sided GFIs and GFDs.

Theorem 2. *(2nd Fundamental Theorem of FC)*

Let a pair of the kernels (κ, k) belong to the set $\mathcal{S}_f$.

The compositions of the left- and the right-sided GFIs and the corresponding GFDs take the following form:

$$({}_l\mathbb{I}_{(\kappa)}\, {}_{l_*}\mathbb{D}_{(k)}f)(t) = f(t) - f(a),\ f \in C^1_{-1}(a,b], \tag{40}$$

$$({}_l\mathbb{I}_{(\kappa)}\, {}_l\mathbb{D}_{(k)}f)(t) = f(t),\ f \in C^1_{-1}(a,b], \tag{41}$$

$$({}_r\mathbb{I}_{(\kappa)}\, {}_{r_*}\mathbb{D}_{(k)}f)(t) = f(b) - f(t),\ f \in C^1_{-1}[a,b), \tag{42}$$

$$({}_r\mathbb{I}_{(\kappa)}\, {}_{r_*}\mathbb{D}_{(k)}f)(t) = f(t),\ f \in C^1_{-1}[a,b). \tag{43}$$

Proof. We start with a proof of the Formula (40). By definition, its left-hand side can be represented as follows:

$$\begin{aligned}({}_l\mathbb{I}_{(\kappa)}\, {}_{l_*}\mathbb{D}_{(k)} f)(t) &= \int_a^t \kappa(t-\tau)({}_{l_*}\mathbb{D}_{(k)} f)(\tau)d\tau \\ &= \int_a^t \kappa(t-\tau)\int_a^\tau k(\tau-y)f'(y)dyd\tau.\end{aligned}$$

Interchanging the order of integration in the last integral and using the Formula (37) yields

$$\begin{aligned}({}_l\mathbb{I}_{(\kappa)}\, {}_{l_*}\mathbb{D}_{(k)} f)(t) &= \int_a^t f'(y)\int_y^t \kappa(t-\tau)k(\tau-y)d\tau dy \\ &= \int_a^t f'(y)dy = f(t)-f(a),\end{aligned} \tag{44}$$

which proves the Formula (40).

To prove the Formula (41), we first show that

$$\big({}_l\mathbb{I}_{(\kappa)}\, k(\tau-a)\big)(t) = \{1\},\ a<t\le b. \tag{45}$$

Indeed, we have the following chain of relations:

$$\begin{aligned}\big({}_l\mathbb{I}_{(\kappa)}\, k(\tau-a)\big)(t) &= \int_a^t \kappa(t-\tau)k(\tau-a)d\tau = \int_0^{t-a}\kappa(t-a-\tau)k(\tau)d\tau \\ &= (\kappa * k)(t-a) = \{1\},\ a<t\le b.\end{aligned}$$

Because of the inclusion $f\in C_{-1}^1(a,b]$, the function f' is from the space $C_{-1}(a,b]$. Now we can employ the representation (26) and get the following relations:

$$\begin{aligned}({}_l\mathbb{I}_{(\kappa)}\, {}_l\mathbb{D}_{(k)} f)(t) &= \big({}_l\mathbb{I}_{(\kappa)}\, ({}_{l_*}\mathbb{D}_{(k)} f)(\tau) + f(a)k(\tau-a)\big)(t) \\ &= \big({}_l\mathbb{I}_{(\kappa)}\, {}_{l_*}\mathbb{D}_{(k)} f\big)(t) + f(a)\big({}_l\mathbb{I}_{(\kappa)} k(\tau-a)\big)(t) \\ &= f(t)-f(a)+f(a) = f(t),\end{aligned}$$

which completes the proof of the Formula (41).

The proof of the Formula (42) follows the lines of the proof of (40) and the proof of (43) is completely analogous to the proof of (41). □

4. Conclusions and Directions for Further Research

In this paper, to the best of authors' knowledge, the left- and right-sided GFIs and GFDs on a finite interval have been introduced for the first time in the FC literature. We also provided some of their basic properties on the spaces of functions that are continuous of the finite open intervals, but can have an integrable singularity of a power function type at one of its end points and on their suitable sub-spaces.

In particular, we derived the formulas that connect the GFDs (of the Riemann–Liuville type) and the regularized GFDs (of the Caputo type) as well as several integration by parts formulas for the right- and left-sided GFIs and GFDs. The formulas of this type are especially important while dealing with the fractional variation calculus for the functionals that involve the left- and right-sided fractional derivatives.

The main results presented in the paper are the 1st and the 2nd fundamental theorems of FC formulated for the left- and right-sided GFIs and GFDs on a finite interval. These theorems allow for an interpretation of the operators introduced in the paper as some FC operators. In particular, we showed that the GFDs are the left-inverse operators to the corresponding GFIs. In fact, this property can be interpreted as a definition of the fractional derivatives as soon as the notion of the fractional integrals is fixed (see Ref. [32] for a discussion of properties of the FC operators).

The research line initiated in this paper can be extended in several directions. The first important topic for further research would be to develop a theory of the left- and right-sided GFIs and GFDs on a finite interval on other classical spaces of functions, say, on the Hölder spaces or the weighted L_p-spaces (see Ref. [19] for a theory of the Riemann–Liouville fractional integrals and derivatives on a finite interval on these spaces of functions).

In this paper, we dealt only with the GFIs and the GFDs on a finite interval of the "generalized order" less or equal to one (see the Sonin condition (1)). However, in Ref. [6], the Sonin condition has been extended in a manner that allows defining GFIs and the GFDs of arbitrary order on the positive real semi-axis. It would be worth to develop similar constructions for the case of the GFIs and the GFDs on a finite interval.

In a recent paper [11], the so-called 1st level GFDs on the positive real semi-axis were introduced and investigated. These derivatives contain the GFDs (of the Riemann–Liouville type) and the regularized GFDs (of the Caputo type) as their particular cases. Thus, any result that concerns the 1st level GFDs covers the corresponding results for the GFDs and for the regularized GFDs. Similarly to the case of the positive real semi-axis, a general construction of the GFDs on a finite interval that covers both the GFDs (of the Riemann–Liouville type) and the regularized GFDs (of the Caputo type) introduced in this paper would be useful.

Finally, we mention here the open problems related to the ordinary and partial fractional differential equations with the GFDs. In Ref. [2], this line of research was initiated by Kochubei for the GFDs on the positive real semi-axis with the Sonin kernels from the Kochubei set. In the meantime, many other papers devoted to this topic have been published, see, e.g., Refs. [33–41] and the recent survey in Ref. [3]. The fractional differential equations with the left- and right-sided GFDs on a finite interval introduced in this paper would be another important topic for further research.

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