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Lie Bialgebras on the Rank Two Heisenberg–Virasoro Algebra

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Abstract: The rank two Heisenberg–Virasoro algebra can be viewed as a generalization of the twisted Heisenberg–Virasoro algebra. Lie bialgebras play an important role in searching for solutions of quantum Yang–Baxter equations. It is interesting to study the Lie bialgebra structures on the rank two Heisenberg–Virasoro algebra. Since the Lie brackets of rank two Heisenberg–Virasoro algebra are different from that of the twisted Heisenberg–Virasoro algebra and Virasoro-like algebras, and there are inner derivations (from itself to its tensor space) which are hidden more deeply in its interior algebraic structure, some new techniques and strategies are employed in this paper. It is proved that every Lie bialgebra structure on the rank two Heisenberg–Virasoro algebra is triangular coboundary.

Keywords: the rank two Heisenberg–Virasoro algebra; Lie bialgebras; Yang–Baxter equation

MSC: 17B05; 17B37; 17B62; 17B68

1. Introduction

Lie bialgebras as well as their quantizations provide important tools in searching for solutions of quantum Yang–Baxter equations and in producing new quantum groups (see, e.g., [1,2]). Thus, a number of papers were published on the structure theory of Lie bialgebras (see, e.g., [3–18]). Witt and Virasoro type Lie bialgebras were introduced in [3], which were further classified in [4]. The generalized Witt type was studied in [5]. Lie bialgebra structures on generalized Virasoro-like and the q-analog Virasoro-like algebra were considered in [6] and [7], respectively. The same problem on the twisted Heisenberg–Virasoro algebra was determined in [8]. Recently, quantizations of Lie bialgebras, duality involution, and oriented graph complexes were investigated in [19].

In this paper, we are interested in considering Lie bialgebra structures on the rank two Heisenberg–Virasoro algebra, which can be viewed as a generalization of the twisted Heisenberg–Virasoro algebra (see [20,21] for details). However, Lie brackets of these two algebras are different. The rank two Heisenberg–Virasoro algebra L is a Lie algebra spanned by elements of the form $\{t_\alpha, E_\alpha \mid \alpha \in \mathbb{Z}^2 \setminus \{(0,0)\}\}$, together with the following Lie bracket relations:

$$[t_\alpha, E_\beta] = \det\begin{pmatrix}\beta\\ \alpha\end{pmatrix} t_{\alpha+\beta},$$
$$[E_\alpha, E_\beta] = \det\begin{pmatrix}\beta\\ \alpha\end{pmatrix} E_{\alpha+\beta},\ [t_\alpha,\ t_\beta] = 0,$$

where $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, and $\det\begin{pmatrix}\beta\\ \alpha\end{pmatrix} = \beta_1\alpha_2 - \alpha_1\beta_2$. The derivation, automorphism group and central extension for L were studied in [20]. The universal Whittaker modules for L were discussed in [21], where the irreducibility of the universal Whittaker modules was determined. The Verma module structure for L was characterized in [22]. This Lie algebra L is different from Virasoro-like algebras in that L has one more type of generator $\{t_\alpha \mid \alpha \in \mathbb{Z}^2 \setminus \{(0,0)\}\}$ than Virasoro-like algebras. Thus, some

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new techniques or strategies need to be employed as can be seen in the proof of Lemma 5 (see pages 12–14 for details). This is one of our motivations. Furthermore, due to the fact that, compared L with Virasoro-like algebras, there are inner derivations (from itself to its tensor space) which are hidden more deeply in its interior algebraic structure, we must apply some new techniques (see pages 5–11 for details) to search for these deeply hidden inner derivations by thorough observations and deep considerations. Therefore, the determination of Lie bialgebra structures on the rank two Heisenberg–Virasoro algebra is attractive and more complicated compared with Virasoro-like algebras.

We introduce two degree derivations d_1 and d_2 on L, i.e.,

$$[d_i, t_\alpha] = \alpha_i t_\alpha,\ [d_i, E_\alpha] = \alpha_i E_\alpha,\ [d_1, d_2] = 0,\ \text{for } i = 1, 2.$$

Then, we arrive at the Lie algebra $\widetilde{L} = L \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$, which is still called the rank two Heisenberg–Virasoro algebra. The Lie algebra $\widetilde{L}$ has a natural $\mathbb{Z}^2$-gradation:

$$\widetilde{L} = \bigoplus_{\alpha\in\mathbb{Z}^2} \widetilde{L}_\alpha$$

where

$$\widetilde{L}_\alpha = \mathbb{C}E_\alpha \oplus \mathbb{C}t_\alpha,\ \text{for } \alpha \in \mathbb{Z}^2 \setminus \{(0,0)\},\ \widetilde{L}_0 = \mathbb{C}d_1 \oplus \mathbb{C}d_2,\ \text{for } 0 = (0,0)$$

We shall investigate Lie bialgebra structures on the Lie algebra $\widetilde{L}$ in this paper.

Now, we recall some definitions related to Lie bialgebras (see [1–3,9]). For any $\mathbb{C}$-vector space S, denote by τ the *twist map* of $S \otimes S$, namely, $\tau(a \otimes b) = b \otimes a$ and by ξ the *cyclic map* of $S \otimes S \otimes S$ cyclically permuting the coordinates, i.e., $\xi(a \otimes b \otimes c) = b \otimes c \otimes a$ for any $a, b, c \in S$. Then, the definition of a Lie algebra can be stated as follows. A *Lie algebra* is a pair (S, δ) of a vector space S and a linear map $\delta : S \otimes S \to S$ with which the following conditions are satisfied:

$$\mathrm{Ker}(1 \otimes 1 - \tau) \subset \mathrm{Ker}\delta,\ \delta(1 \otimes \delta)(1 \otimes 1 \otimes 1 + \xi + \xi^2) = 0,$$

where 1 denotes the identity map on S. The operator δ is usually called the bracket of S. Dually, a *Lie coalgebra* is a pair (S, Δ) of a vector space S and a linear map $\Delta : S \to S \otimes S$ satisfying

$$\mathrm{Im}\Delta \subset \mathrm{Im}(1 \otimes 1 - \tau),\ (1 \otimes 1 \otimes 1 + \xi + \xi^2)(1 \otimes \Delta)\Delta = 0 \tag{1}$$

The map Δ is called the cobracket of S. For a Lie algebra S, we always use the symbol "$\cdot$" to stand for the *diagonal adjoint action*:

$$a \cdot (\sum_i b_i \otimes c_i) = \sum_i ([a, b_i] \otimes c_i + b_i \otimes [a, c_i]),\ \text{for any } a, b_i, c_i \in S,$$

and in general, $[a, b] = \delta(a \otimes b)$ for any $a, b \in S$.

Definition 1 ([1–3,9])**.** *A Lie bialgebra is a triple (S, δ, Δ) such that (S, δ) is a Lie algebra, (S, Δ) is a Lie coalgebra, and the following compatible condition holds:*

$$\Delta\delta(a \otimes b) = a \cdot \Delta(b) - b \cdot \Delta(a),\ \text{for any } a, b \in S \tag{2}$$

Note that the compatibility condition (2) is equivalent to saying that Δ is a derivation.

Definition 2 ([1–3,9])**.** *A coboundary Lie bialgebra is an (S, δ, Δ, r) , where (S, δ, Δ) is a Lie bialgebra and $r \in \mathrm{Im}(1 \otimes 1 - \tau) \subset S \otimes S$ such that Δ is a coboundary of r , i.e., $\Delta = \Delta_r$. For any $a \in S$, Δ_r is defined by*

$$\Delta_r(a) = a \cdot r. \tag{3}$$

Denote by U the universal enveloping algebra of S and by 1 the identity element of U. For any $r = \sum_i a_i \otimes b_i \in S \otimes S$, define r^{ij}, $i, j = 1, 2, 3$ to be the elements of $U \otimes U \otimes U$ by

$$r^{12} = \sum_i a_i \otimes b_i \otimes 1, r^{13} = \sum_i a_i \otimes 1 \otimes b_i, r^{23} = \sum_i 1 \otimes a_i \otimes b_i,$$

and $c(r) = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]$.

Definition 3 ([1–3,9])**.** *A coboundary Lie bialgebra (S, δ, Δ, r) is called triangular if r satisfies the following classical Yang–Baxter equation (CYBE):*

$$c(r) = 0 \tag{4}$$

The main result of this paper can be formulated as follows.

Theorem 1. *Every Lie bialgebra structure on $\widetilde{L}$ is triangular coboundary.*

Throughout the paper, we denote by $\mathbb{C}$ and $\mathbb{Z}$ the sets of the complex numbers, and the integers, respectively. All vector spaces mentioned in this paper are over the complex field $\mathbb{C}$.

2. Proof of the Main Results

The aim of this section is to give a proof of Theorem 1. The first one has the following result which comes from [2].

Lemma 1 ([2]). *Let (S, δ) be a Lie algebra. Then, $\Delta = \Delta_r$ (for some $r \in \mathrm{Im}(1 \otimes 1 - \tau) \subset S \otimes S$) endows (S, δ, Δ) with a Lie bialgebra structure if and only if r satisfies the following modified Yang–Baxter equation (MYBE):*

$$a \cdot c(r) = 0, \text{ for all } a \in S \tag{5}$$

Lemma 2. *Regard $\widetilde{L} \otimes \widetilde{L} \otimes \widetilde{L}$, the tensor product of 3 copies of $\widetilde{L}$, as an -module under the adjoint diagonal action of $\widetilde{L}$. If $a \cdot c = 0$ for some $c \in \widetilde{L} \otimes \widetilde{L} \otimes \widetilde{L}$ and all $a \in \widetilde{L}$, then $c = 0$.*

Proof . It is easy to see that $V = \widetilde{L} \otimes \widetilde{L} \otimes \widetilde{L}$ is a $\mathbb{Z}^2$ -graded $\widetilde{L}$ -module under the adjoint diagonal action of $\widetilde{L}$. The gradation is given by $V = \bigoplus_{x \in \mathbb{Z}^2} V_x$, where $V_x = \sum_{\substack{\alpha,\ \beta,\ \gamma \in \mathbb{Z}^2 \\ \alpha + \beta + \gamma = x}} \widetilde{L}_\alpha \otimes \widetilde{L}_\beta \otimes \widetilde{L}_\gamma$. Write $c = \sum_{x \in \mathbb{Z}^2} c_x$ as a finite sum with $c_x \in V_x$. From $0 = d_i \cdot c = \sum_{x \in \mathbb{Z}^2} x_i c_x$, $i = 1,\ 2$, we obtain $c = c_0 \in V_0$. Now write

$$\begin{aligned} c = & \sum_{\substack{\alpha,\ \beta \in \mathbb{Z}^2 \backslash \{\mathbf{0}\} \\ X,\ Y,\ Z \in \{E,\ t\}}} \lambda_{\alpha,\beta}^{X,Y,Z} X_\alpha \otimes Y_\beta \otimes Z_{-(\alpha+\beta)} + \sum_{\substack{\alpha\ \in \mathbb{Z}^2 \backslash \{\mathbf{0}\},\ i \in \{1,\ 2\} \\ X,\ Y \in \{E,\ t\}}} \mu_{\alpha,i}^{X,Y} d_i \otimes X_\alpha \otimes Y_{-\alpha} \\ & + \sum_{\substack{\alpha \in \mathbb{Z}^2 \backslash \{\mathbf{0}\},\ i \in \{1,\ 2\} \\ X,\ Y \in \{E,\ t\}}} \eta_{\alpha,i}^{X,Y} X_\alpha \otimes d_i \otimes Y_{-\alpha} + \sum_{\substack{\alpha \in \mathbb{Z}^2 \backslash \{\mathbf{0}\},\ i \in \{1,\ 2\} \\ X,\ Y \in \{E,\ t\}}} \rho_{\alpha,i}^{X,Y} X_\alpha \otimes Y_{-\alpha} \otimes d_i, \\ & + \sum_{i,j,k \in \{1,2\}} \varepsilon_{i,j,k} d_i \otimes d_j \otimes d_k \end{aligned}$$

where all the coefficients of the tensor products are complex numbers and the sums are all finite. Fix the normal total order on $\mathbb{Z}^2$ compatible with its additive group structure. Define the total order on $\mathbb{Z}^2\backslash\{\mathbf{0}\} \times \mathbb{Z}^2\backslash\{\mathbf{0}\}$ by

$$(\alpha,\ \beta) > (\alpha\prime,\ \beta\prime) \Leftrightarrow \alpha > \alpha\prime,\ \text{or}\ \alpha = \alpha\prime,\ \beta > \beta\prime$$

Suppose $\lambda_{\alpha,\beta}^{X,Y,Z} \neq 0$ for some $X,\ Y,\ Z \in \{E,\ t\}$, $\alpha,\ \beta \in \mathbb{Z}^2\backslash\{\mathbf{0}\}$. Let

$$(\alpha_0,\ \beta_0) = \max\left\{(\alpha,\ \beta) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \times \mathbb{Z}^2\backslash\{\mathbf{0}\}\ \middle|\ \lambda_{\alpha,\beta}^{X,Y,Z} \neq 0\right\}$$

Choose $\gamma > 0$ such that $\det\begin{pmatrix}\alpha_0\\ \gamma\end{pmatrix} \neq 0$, $\gamma \neq \beta_0$, and $\gamma \neq -\alpha_0 - \beta_0$. Then, for some $X,\ Y,\ Z \in \{E,\ t\}$,

$$0 \neq \det\begin{pmatrix}\alpha_0\\ \gamma\end{pmatrix}\lambda_{\alpha_0,\beta_0}^{X,Y,Z} X_{\alpha_0+\gamma} \otimes Y_{\beta_0} \otimes Z_{-(\alpha_0+\beta_0)}$$

is linearly independent with other terms of $E_\gamma \cdot c$, a contradiction with the fact that $E_\gamma \cdot c = 0$. So, $\lambda_{\alpha,\beta}^{X,Y,Z} = 0$ for any $X,\ Y,\ Z \in \{E,\ t\}$, $\alpha,\ \beta \in \mathbb{Z}^2\backslash\{\mathbf{0}\}$. Similarly, we can prove that $\mu_{\alpha,i}^{X,Y} = \eta_{\alpha,i}^{X,Y} = \rho_{\alpha,i}^{X,Y} = 0$ for any $X,\ Y \in \{E,\ t\}$, $\alpha \in \mathbb{Z}^2\backslash\{\mathbf{0}\}$, $i \in \{1,\ 2\}$. Furthermore, by $E_{1,0} \cdot c = 0 = E_{0,1} \cdot c$, one can obtain $\varepsilon_{i,j,k} = 0$ for any $i,\ j,\ k \in \{1,\ 2\}$. The lemma is proved. □

As a conclusion of Lemma 2, we immediately obtain:

Corollary 1. *An elementr* $\in \mathrm{Im}(1 \otimes 1 - \tau) \subset \widetilde{L} \otimes \widetilde{L}$ *satisfies the CYBE in* (4) *if and only if it satisfies the MYBE in* (5).

Regard $W = \widetilde{L} \otimes \widetilde{L}$ as an $\widetilde{L}$-module under the adjoint diagonal action. Denote by $\mathrm{Der}(\widetilde{L},\ W)$ the set of *derivations* $D:\ \widetilde{L} \to W$, i.e., D is a linear map such that

$$D([x,\ y]) = x \cdot D(y) - y \cdot D(x) \text{ for } x,\ y \in \widetilde{L} \tag{6}$$

and $\mathrm{Inn}(\widetilde{L},\ W)$, the set consisting of the *inner derivations* u_{inn}, $u \in W$, is defined by

$$u_{\mathrm{inn}}:\ x \mapsto x \cdot u \text{ for } x \in \widetilde{L}.$$

Denote $H^1(\widetilde{L},\ W)$ as the *first cohomology group* of the Lie algebra $\widetilde{L}$ with coefficients in the $\widetilde{L}$-module W, then

$$H^1(\widetilde{L},\ W) \cong \mathrm{Der}(\widetilde{L},\ W)/\mathrm{Inn}(\widetilde{L},\ W)$$

Proposition 1. $\mathrm{Der}(\widetilde{L},\ W) = \mathrm{Inn}(\widetilde{L},\ W)$, *i.e.,*$H^1(\widetilde{L},\ W) = 0$.

Proof . It is clear that $W = \widetilde{L} \otimes \widetilde{L} = \underset{\alpha\in\mathbb{Z}^2}{\oplus} W_\alpha$ is $\mathbb{Z}^2$ -graded with

$$W_\alpha = \sum_{\substack{\beta,\gamma \in \mathbb{Z}^2\\ \beta+\gamma=\alpha}} \widetilde{L}_\beta \otimes \widetilde{L}_\gamma$$

A derivation $D \in \mathrm{Der}(\widetilde{L},\ W)$ is homogeneous of degree $\alpha \in \mathbb{Z}^2$ if $D(\widetilde{L}_\beta) \in W_{\alpha+\beta}$ for all $\beta \in \mathbb{Z}^2$. Denote by $\mathrm{Der}(\widetilde{L},\ W)_\alpha = \left\{D \in \mathrm{Der}(\widetilde{L},\ W)\ \ |\deg D = \alpha\right\}$ for $\alpha \in \mathbb{Z}^2$.

Let $D \in \mathrm{Der}(\widetilde{L}, W)$. For $\alpha \in \mathbb{Z}^2$, we define a homogeneous linear map $D_\alpha : \widetilde{L} \to W$ of degree α as follows: For any $u \in \widetilde{L}_\beta$ with $\beta \in \mathbb{Z}^2$, write $D(u) = \sum\limits_{\gamma \in \mathbb{Z}^2} w_\gamma$ with $w_\gamma \in W_\gamma$. Then, we set $D_\alpha(u) = w_{\alpha+\beta}$. It is obvious that $D_\alpha \in \mathrm{Der}(\widetilde{L}, W)_\alpha$ and we have

$$D = \sum_{\alpha \in \mathbb{Z}^2} D_\alpha, \text{ where } D_\alpha \in \mathrm{Der}(\widetilde{L}, W)_\alpha, \tag{7}$$

which holds in the sense that for every $u \in \widetilde{L}$, only finitely many $D_\alpha(u) \neq 0$, and $D(u) = \sum\limits_{\alpha \in \mathbb{Z}^2} D_\alpha(u)$ (we call such a sum in (7) summable). □

We shall divide the proof of the proposition into several lemmas.

Lemma 3. *If* $\alpha \in \mathbb{Z}^2 \setminus \{0\}$ *, then* $D_\alpha \in \mathrm{Inn}(\widetilde{L}, W)$.

Proof . Denote $T = \mathrm{Span}_{\mathbb{C}}\{d_1, d_2\}$ and define the nondegenerate bilinear map from $\mathbb{Z}^2 \times T \to \mathbb{C}$, $d(\alpha) = <\alpha, d> = a_1\alpha_1 + a_2\alpha_2$ for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$, $d = a_1 d_1 + a_2 d_2 \in T$. Now, for $\alpha \in \mathbb{Z}^2 \setminus \{0\}$, by linear algebra, one can choose $d \in T$ with $d(\alpha) \neq 0$. Denote $w = (d(\alpha))^{-1} D_\alpha(d) \in W_\alpha$. Then, for any $x \in \widetilde{L}_\beta, \beta \in \mathbb{Z}^2$, applying D_α to $[d, x] = d(\beta)x$, using $D_\alpha(x) \in W_{\alpha+\beta}$, we have

$$d(\alpha+\beta) D_\alpha(x) - x \cdot D_\alpha(d) = d(\beta) D_\alpha(x),$$

i.e., $D_\alpha(x) = w_{\mathrm{inn}}(x)$. Then, $D_\alpha = w_{\mathrm{inn}}$ is inner. □

Lemma 4. $D_0(d_1) = D_0(d_2) = 0$.

Proof . For any $x \in \widetilde{L}_\alpha$, applying D_0 to $[d_i, x] = d_i(\alpha)x$, we have

$$d_i \cdot D_0(x) - x \cdot D_0(d_i) = d_i(\alpha) D_0(x)$$

i.e., $x \cdot D_0(d_i) = 0\,(i = 1, 2)$. Thus by Lemma 2, $D_0(d_i) = 0\,(i = 1, 2)$. □

Lemma 5. *Replacing* D_0 *by* $D_0 - u_{\mathrm{inn}}$ *for some* $u \in W_0$ *, one can suppose* $D_0(\widetilde{L}) = 0$ *, i.e.,* $D_0 \in \mathrm{Inn}(\widetilde{L}, W)$.

The proof of this lemma will be carried out by two claims.

Claim 1. *By replacing* D_0 *by* $D_0 - u_{\mathrm{inn}}$ *for some* $u \in W_0$*, we can suppose* $D_0(E_{m,n}) = D_0(t_{m,n}) = 0$ *for* $m, n, m+n \in \{-1, 0, 1\}$.

Write

$$\begin{aligned} D_0(E_{0,1}) = & \sum_{m,n} \lambda^{EE}_{m,n} E_{m,n} \otimes E_{-m,1-n} + \sum_{m,n} \lambda^{Et}_{m,n} E_{m,n} \otimes t_{-m,1-n} + \sum_{m,n} \lambda^{tE}_{m,n} t_{m,n} \otimes E_{-m,1-n} \\ & + \sum_{m,n} \lambda^{tt}_{m,n} t_{m,n} \otimes t_{-m,1-n} + \lambda^{E}_{d_1} E_{0,1} \otimes d_1 + \lambda^{E}_{d_2} E_{0,1} \otimes d_2 + \lambda^{d_1}_{E} d_1 \otimes E_{0,1} \\ & + \lambda^{d_2}_{E} d_2 \otimes E_{0,1} + \lambda^{t}_{d_1} t_{0,1} \otimes d_1 + \lambda^{t}_{d_2} t_{0,1} \otimes d_2 + \lambda^{d_1}_{t} d_1 \otimes t_{0,1} + \lambda^{d_2}_{t} d_2 \otimes t_{0,1}, \end{aligned} \tag{8}$$

for some $\lambda^{EE}_{m,n}, \lambda^{Et}_{m,n}, \lambda^{tE}_{m,n}, \lambda^{tt}_{m,n}, \lambda^{E}_{d_i}, \lambda^{d_i}_{E}, \lambda^{t}_{d_i}, \lambda^{d_i}_{t} \in \mathbb{C}$, where $\{(m, n) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} \mid \lambda^{EE}_{m,n} \neq 0\}$, $\{(m, n) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} \mid \lambda^{Et}_{m,n} \neq 0\}$, $\{(m, n) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} \mid \lambda^{tE}_{m,n} \neq 0\}$, and $\{(m, n) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} \mid \lambda^{tt}_{m,n} \neq 0\}$ are finite sets. Note that the following identities hold.

$$\begin{gathered} (E_{m,n-1} \otimes E_{-m,1-n})_{\mathrm{inn}}(E_{0,1}) = m(E_{m,n} \otimes E_{-m,1-n} - E_{m,n-1} \otimes E_{-m,2-n}), \\ (E_{m,n-1} \otimes t_{-m,1-n})_{\mathrm{inn}}(E_{0,1}) = m(E_{m,n} \otimes t_{-m,1-n} - E_{m,n-1} \otimes t_{-m,2-n}), \\ (t_{m,n-1} \otimes E_{-m,1-n})_{\mathrm{inn}}(E_{0,1}) = m(t_{m,n} \otimes E_{-m,1-n} - t_{m,n-1} \otimes E_{-m,2-n}), \\ (t_{m,n-1} \otimes t_{-m,1-n})_{\mathrm{inn}}(E_{0,1}) = m(t_{m,n} \otimes t_{-m,1-n} - t_{m,n-1} \otimes t_{-m,2-n}), \\ (d_1 \otimes d_2)_{\mathrm{inn}}(E_{0,1}) = -d_1 \otimes E_{0,1},\ (d_2 \otimes d_1)_{\mathrm{inn}}(E_{0,1}) = -E_{0,1} \otimes d_1. \end{gathered}$$

Replacing D_0 by $D_0 - u_{\text{inn}}$, where u is a combination of some $E_{m,n-1} \otimes E_{-m,1-n}$, $E_{m,n-1} \otimes t_{-m,1-n}$, $t_{m,n-1} \otimes t_{-m,1-n}$, $d_2 \otimes d_2$, $d_1 \otimes d_2$, $d_2 \otimes d_1$, then $D_0(E_{0,1})$ can be simplified as follows

$$\begin{aligned} D_0(E_{0,1}) = & \sum_{n\neq 0,1} \lambda_{0,n}^{EE} E_{0,n} \otimes E_{0,1-n} + \sum_{m\neq 0} \lambda_{m,0}^{EE} E_{m,0} \otimes E_{-m,1} + \sum_{n\neq 0,1} \lambda_{0,n}^{Et} E_{0,n} \otimes t_{0,1-n} \\ & + \sum_{m\neq 0} \lambda_{m,0}^{Et} E_{m,0} \otimes t_{-m,1} + \sum_{n\neq 0,1} \lambda_{0,n}^{tE} t_{0,n} \otimes E_{0,1-n} + \sum_{m\neq 0} \lambda_{m,0}^{tE} t_{m,0} \otimes E_{-m,1} \\ & + \sum_{n\neq 0,1} \lambda_{0,n}^{tt} t_{0,n} \otimes t_{0,1-n} + \sum_{m\neq 0} \lambda_{m,0}^{tt} t_{m,0} \otimes t_{-m,1} + \lambda E_{0,1} \otimes d_2 \\ & + \lambda_{d_1}^{t} t_{0,1} \otimes d_1 + \lambda_{d_2}^{t} t_{0,1} \otimes d_2 + \lambda_t^{d_1} d_1 \otimes t_{0,1} + \lambda_t^{d_2} d_2 \otimes t_{0,1} \end{aligned} \tag{9}$$

Write

$$\begin{aligned} D_0(E_{0,-1}) = & \sum_{n\neq 0,1} \mu_{0,n}^{EE} E_{0,n-1} \otimes E_{0,-n} + \sum_{m\neq 0} \mu_{m,n}^{EE} E_{m,n-1} \otimes E_{-m,-n} + \sum_{n\neq 0,1} \mu_{0,n}^{Et} E_{0,n-1} \otimes t_{0,-n} \\ & + \sum_{m\neq 0} \mu_{m,n}^{Et} E_{m,n-1} \otimes t_{-m,-n} + \sum_{n\neq 0,1} \mu_{0,n}^{tE} t_{0,n-1} \otimes E_{0,-n} + \sum_{m\neq 0} \mu_{m,n}^{tE} t_{m,n-1} \otimes E_{-m,-n} \\ & + \sum_{n\neq 0,1} \mu_{0,n}^{tt} t_{0,n-1} \otimes t_{0,-n} + \sum_{m\neq 0} \mu_{m,n}^{tt} t_{m,n-1} \otimes t_{-m,-n} + \mu_{d_1}^{E} E_{0,-1} \otimes d_1 \\ & + \mu_{d_2}^{E} E_{0,-1} \otimes d_2 + \mu_E^{d_1} d_1 \otimes E_{0,-1} + \mu_E^{d_2} d_2 \otimes E_{0,-1} + \mu_{d_1}^{t} t_{0,-1} \otimes d_1 \\ & + \mu_{d_2}^{t} t_{0,-1} \otimes d_2 + \mu_t^{d_1} d_1 \otimes t_{0,-1} + \mu_t^{d_2} d_2 \otimes t_{0,-1}, \end{aligned} \tag{10}$$

for some $\mu_{0,n}^{EE}$, $\mu_{m,n}^{EE}$, $\mu_{0,n}^{Et}$, $\mu_{m,n}^{Et}$, $\mu_{0,n}^{tE}$, $\mu_{m,n}^{tE}$, $\mu_{0,n}^{tt}$, $\mu_{m,n}^{tt}$, $\mu_{d_i}^{E}$, $\mu_E^{d_i}$, $\mu_{d_i}^{t}$, $\mu_t^{d_i} \in \mathbb{C}$, where $\left\{(0,\ n),\ (m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \,\middle|\, \mu_{0,n}^{EE},\ \mu_{m,n}^{EE} \neq 0\right\}$, $\left\{(0,\ n),\ (m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \,\middle|\, \mu_{0,n}^{Et},\ \mu_{m,n}^{Et} \neq 0\right\}$, $\left\{(0,\ n),\ (m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \,\middle|\, \mu_{0,n}^{tE},\ \mu_{m,n}^{tE} \neq 0\right\}$, and $\left\{(0,\ n),\ (m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \,\middle|\, \mu_{0,n}^{tt},\ \mu_{m,n}^{tt} \neq 0\right\}$ are finite sets. Applying D_0 to $[E_{0,1},\ E_{0,-1}] = 0$, we have

$$\begin{aligned} & \sum_{m\neq 0} \mu_{m,n}^{EE} m E_{m,n} \otimes E_{-m,-n} - \sum_{m\neq 0} \mu_{m,n}^{EE} m E_{m,n-1} \otimes E_{-m,1-n} + \sum_{m\neq 0} \mu_{m,n}^{Et} m E_{m,n} \otimes t_{-m,-n} \\ & - \sum_{m\neq 0} \mu_{m,n}^{Et} m E_{m,n-1} \otimes t_{-m,1-n} + \sum_{m\neq 0} \mu_{m,n}^{tE} m t_{m,n} \otimes E_{-m,-n} - \sum_{m\neq 0} \mu_{m,n}^{tE} m t_{m,n-1} \otimes E_{-m,1-n} \\ & + \sum_{m\neq 0} \mu_{m,n}^{tt} m t_{m,n} \otimes t_{-m,-n} - \sum_{m\neq 0} \mu_{m,n}^{tt} m t_{m,n-1} \otimes t_{-m,1-n} - \mu_{d_2}^{E} E_{0,-1} \otimes E_{0,1} \\ & - \mu_E^{d_2} E_{0,1} \otimes E_{0,-1} - \mu_{d_2}^{t} t_{0,-1} \otimes E_{0,1} - \mu_t^{d_2} E_{0,1} \otimes t_{0,-1} \\ & = - \sum_{m\neq 0} \lambda_{m,0}^{EE} m E_{m,-1} \otimes E_{-m,1} + \sum_{m\neq 0} \lambda_{m,0}^{EE} m E_{m,0} \otimes E_{-m,0} - \sum_{m\neq 0} \lambda_{m,0}^{Et} m E_{m,-1} \otimes t_{-m,1} \\ & + \sum_{m\neq 0} \lambda_{m,0}^{Et} m E_{m,0} \otimes t_{-m,0} - \sum_{m\neq 0} \lambda_{m,0}^{tE} m t_{m,-1} \otimes E_{-m,1} + \sum_{m\neq 0} \lambda_{m,0}^{tE} m t_{m,0} \otimes E_{-m,0} \\ & - \sum_{m\neq 0} \lambda_{m,0}^{tt} m t_{m,-1} \otimes t_{-m,1} + \sum_{m\neq 0} \lambda_{m,0}^{tt} m t_{m,0} \otimes t_{-m,0} + \lambda E_{0,1} \otimes E_{0,-1} \\ & + \lambda_{d_2}^{t} t_{0,1} \otimes E_{0,-1} + \lambda_t^{d_2} E_{0,-1} \otimes t_{0,1}. \end{aligned}$$

Comparing the coefficients, one can deduce

$$\mu_{m,n}^{EE} = \mu_{m,n}^{Et} = \mu_{m,n}^{tE} = \mu_{m,n}^{tt} = 0, \text{ for } m \neq 0,\ n \neq 0$$
$$\mu_{m,0}^{EE} = \lambda_{m,0}^{EE},\ \mu_{m,0}^{Et} = \lambda_{m,0}^{Et},\ \mu_{m,0}^{tE} = \lambda_{m,0}^{tE},\ \mu_{m,0}^{tt} = \lambda_{m,0}^{tt}, \text{ for } m \neq 0,$$
$$\mu_E^{d_2} = -\lambda,\ \mu_{d_2}^{E} = \mu_{d_2}^{t} = \mu_t^{d_2} = \lambda_{d_2}^{t} = \lambda_t^{d_2} = 0.$$

Then, we can rewrite (10) as

$$\begin{aligned} D_0(E_{0,-1}) = & \sum_{n\neq 0,1} \mu_{0,n}^{EE} E_{0,n-1} \otimes E_{0,-n} + \sum_{m\neq 0} \lambda_{m,0}^{EE} E_{m,-1} \otimes E_{-m,0} + \sum_{n\neq 0,1} \mu_{0,n}^{Et} E_{0,n-1} \otimes t_{0,-n} \\ & + \sum_{m\neq 0} \lambda_{m,0}^{Et} E_{m,-1} \otimes t_{-m,0} + \sum_{n\neq 0,1} \mu_{0,n}^{tE} t_{0,n-1} \otimes E_{0,-n} + \sum_{m\neq 0} \lambda_{m,0}^{tE} t_{m,-1} \otimes E_{-m,0} \\ & + \sum_{n\neq 0,1} \mu_{0,n}^{tt} t_{0,n-1} \otimes t_{0,-n} + \sum_{m\neq 0} \lambda_{m,0}^{tt} t_{m,-1} \otimes t_{-m,0} + \mu_{d_1}^{E} E_{0,-1} \otimes d_1 \\ & + \mu_E^{d_1} d_1 \otimes E_{0,-1} - \lambda d_2 \otimes E_{0,-1} + \mu_{d_1}^{t} t_{0,-1} \otimes d_1 + \mu_t^{d_1} d_1 \otimes t_{0,-1} \end{aligned} \tag{11}$$

Write

$$\begin{aligned} D_0(E_{1,0}) = & \sum_{m,n} \eta_{m,n}^{EE} E_{m+1,n} \otimes E_{-m,-n} + \sum_{m,n} \eta_{m,n}^{Et} E_{m+1,n} \otimes t_{-m,-n} + \sum_{m,n} \eta_{m,n}^{tE} t_{m+1,n} \otimes E_{-m,-n} \\ & + \sum_{m,n} \eta_{m,n}^{tt} t_{m+1,n} \otimes t_{-m,-n} + \eta_{d_1}^{E} E_{1,0} \otimes d_1 + \eta_{d_2}^{E} E_{1,0} \otimes d_2 + \eta_E^{d_1} d_1 \otimes E_{1,0} \\ & + \eta_E^{d_2} d_2 \otimes E_{1,0} + \eta_{d_1}^{t} t_{1,0} \otimes d_1 + \eta_{d_2}^{t} t_{1,0} \otimes d_2 + \eta_t^{d_1} d_1 \otimes t_{1,0} + \eta_t^{d_2} d_2 \otimes t_{1,0}, \end{aligned} \tag{12}$$

for some $\eta_{m,n}^{EE}, \eta_{m,n}^{Et}, \eta_{m,n}^{tE}, \eta_{m,n}^{tt}, \eta_{d_i}^{E}, \eta_E^{d_i}, \eta_{d_i}^{t}, \eta_t^{d_i} \in \mathbb{C}$, where $\{(m,\ n) \in \mathbb{Z}^2 \backslash \{\mathbf{0}\} \mid \eta_{m,n}^{EE} \neq 0\}$, $\{(m,\ n) \in \mathbb{Z}^2 \backslash \{\mathbf{0}\} \mid \eta_{m,n}^{Et} \neq 0\}$, $\{(m,\ n) \in \mathbb{Z}^2 \backslash \{\mathbf{0}\} \mid \eta_{m,n}^{tE} \neq 0\}$, and $\{(m,\ n) \in \mathbb{Z}^2 \backslash \{\mathbf{0}\} \mid \eta_{m,n}^{tt} \neq 0\}$ are finite sets. Note that

$$\begin{aligned} (E_{0,n} \otimes E_{0,-n})_{\mathrm{inn}}(E_{1,0}) &= -n(E_{1,n} \otimes E_{0,-n} - E_{0,n} \otimes E_{1,-n}), \\ (E_{0,n} \otimes t_{0,-n})_{\mathrm{inn}}(E_{1,0}) &= -n(E_{1,n} \otimes t_{0,-n} - E_{0,n} \otimes t_{1,-n}), \\ (t_{0,n} \otimes E_{0,-n})_{\mathrm{inn}}(E_{1,0}) &= -n(t_{1,n} \otimes E_{0,-n} - t_{0,n} \otimes E_{1,-n}), \\ (t_{0,n} \otimes t_{0,-n})_{\mathrm{inn}}(E_{1,0}) &= -n(t_{1,n} \otimes t_{0,-n} - t_{0,n} \otimes t_{1,-n}), \\ (d_1 \otimes d_1)_{\mathrm{inn}}(E_{1,0}) &= -E_{1,0} \otimes d_1 - d_1 \otimes E_{1,0} \end{aligned}$$

Using the above equations, by replacing D_0 by $D_0 - u_{\mathrm{inn}}$, where u is a combination of some $E_{0,n} \otimes E_{0,-n}$, $E_{0,n} \otimes t_{0,-n}$, $t_{0,n} \otimes E_{0,-n}$, $t_{0,n} \otimes t_{0,-n}$, and $d_1 \otimes d_1$ (this replacement does not affect the above Equations (9) and (11)), one can rewrite (12) as

$$\begin{aligned} D_0(E_{1,0}) = & \sum_{m\neq 0} \eta_{m,n}^{EE} E_{m+1,n} \otimes E_{-m,-n} + \sum_{m\neq 0} \eta_{m,n}^{Et} E_{m+1,n} \otimes t_{-m,-n} + \sum_{m\neq 0} \eta_{m,n}^{tE} t_{m+1,n} \otimes E_{-m,-n} \\ & + \sum_{m\neq 0} \eta_{m,n}^{tt} t_{m+1,n} \otimes t_{-m,-n} + \eta_{d_1}^{E} E_{1,0} \otimes d_1 + \eta_{d_2}^{E} E_{1,0} \otimes d_2 + \eta_E^{d_2} d_2 \otimes E_{1,0} \\ & + \eta_{d_1}^{t} t_{1,0} \otimes d_1 + \eta_{d_2}^{t} t_{1,0} \otimes d_2 + \eta_t^{d_1} d_1 \otimes t_{1,0} + \eta_t^{d_2} d_2 \otimes t_{1,0} \end{aligned} \tag{13}$$

Applying D_0 to $[E_{0,-1},\ [E_{0,1},\ E_{1,0}]] = -E_{1,0}$, one can deduce

$$
\begin{aligned}
&\sum_{m\neq 0}\eta^{EE}_{m,n}\Big[-(m+1)^2E_{m+1,n}\otimes E_{-m,-n}+m(m+1)E_{m+1,n+1}\otimes E_{-m,-1-n}+m(m+1)E_{m+1,n-1}\otimes E_{-m,1-n}\\
&-m^2E_{m+1,n}\otimes E_{-m,-n}\Big]+\sum_{m\neq 0}\eta^{Et}_{m,n}\Big[-(m+1)^2E_{m+1,n}\otimes t_{-m,-n}+m(m+1)E_{m+1,n+1}\otimes t_{-m,-1-n}\\
&+m(m+1)E_{m+1,n-1}\otimes t_{-m,1-n}-m^2E_{m+1,n}\otimes t_{-m,-n}\Big]+\sum_{m\neq 0}\eta^{tE}_{m,n}\Big[-(m+1)^2t_{m+1,n}\otimes E_{-m,-n}\\
&+m(m+1)t_{m+1,n+1}\otimes E_{-m,-1-n}+m(m+1)t_{m+1,n-1}\otimes E_{-m,1-n}-m^2t_{m+1,n}\otimes E_{-m,-n}\Big]\\
&+\sum_{m\neq 0}\eta^{tt}_{m,n}\Big[-(m+1)^2t_{m+1,n}\otimes t_{-m,-n}+m(m+1)t_{m+1,n+1}\otimes t_{-m,-1-n}+m(m+1)t_{m+1,n-1}\otimes t_{-m,1-n}\\
&-m^2t_{m+1,n}\otimes t_{-m,-n}\Big]-\eta^{E}_{d_1}E_{1,0}\otimes d_1-\eta^{E}_{d_2}E_{1,0}\otimes d_2+\eta^{E}_{d_2}E_{1,1}\otimes E_{0,-1}+\eta^{E}_{d_2}E_{1,-1}\otimes E_{0,1}\\
&+\eta^{d_2}_{E}E_{0,1}\otimes E_{1,-1}+\eta^{d_2}_{E}E_{0,-1}\otimes E_{1,1}-\eta^{d_2}_{E}d_2\otimes E_{1,0}-\eta^{t}_{d_1}t_{1,0}\otimes d_1-\eta^{t}_{d_2}t_{1,0}\otimes d_2\\
&+\eta^{t}_{d_2}t_{1,1}\otimes E_{0,-1}+\eta^{t}_{d_2}t_{1,-1}\otimes E_{0,1}-\eta^{d_1}_{t}d_1\otimes t_{1,0}+\eta^{d_2}_{t}E_{0,1}\otimes t_{1,-1}+\eta^{d_2}_{t}E_{0,-1}\otimes t_{1,1}-\eta^{d_2}_{t}d_2\otimes t_{1,0}\\
&+\sum_{n\neq 0,1}\lambda^{EE}_{0,n}[-nE_{1,n-1}\otimes E_{0,1-n}+(n-1)E_{0,n}\otimes E_{1,-n}]+\sum_{m\neq 0}\lambda^{EE}_{m,0}[-mE_{m,-1}\otimes E_{1-m,1}\\
&+(m-1)E_{m,0}\otimes E_{1-m,0}]+\sum_{n\neq 0,1}\lambda^{Et}_{0,n}[-nE_{1,n-1}\otimes t_{0,1-n}+(n-1)E_{0,n}\otimes t_{1,-n}]\\
&+\sum_{m\neq 0}\lambda^{Et}_{m,0}[-mE_{m,-1}\otimes t_{1-m,1}+(m-1)E_{m,0}\otimes t_{1-m,0}]+\sum_{n\neq 0,1}\lambda^{tE}_{0,n}[-nt_{1,n-1}\otimes E_{0,1-n}\\
&+(n-1)t_{0,n}\otimes E_{1,-n}]+\sum_{m\neq 0}\lambda^{tE}_{m,0}[-mt_{m,-1}\otimes E_{1-m,1}+(m-1)t_{m,0}\otimes E_{1-m,0}]\\
&+\sum_{n\neq 0,1}\lambda^{tt}_{0,n}[-nt_{1,n-1}\otimes t_{0,1-n}+(n-1)t_{0,n}\otimes t_{1,-n}]+\sum_{m\neq 0}\lambda^{tt}_{m,0}[-mt_{m,-1}\otimes t_{1-m,1}\\
&+(m-1)t_{m,0}\otimes t_{1-m,0}]-\lambda E_{1,0}\otimes d_2+\lambda E_{1,1}\otimes E_{0,-1}-\lambda^{t}_{d_1}t_{1,0}\otimes d_1\\
&-\lambda^{t}_{d_1}t_{0,1}\otimes E_{1,-1}-\lambda^{d_1}_{t}E_{1,-1}\otimes t_{0,1}-\lambda^{d_1}_{t}d_1\otimes t_{1,0}+\sum_{n\neq 0,1}\mu^{EE}_{0,n}[(n-1)E_{1,n}\otimes E_{0,-n}\\
&-nE_{0,n-1}\otimes E_{1,1-n}]+\sum_{m\neq 0}\lambda^{EE}_{m,0}[-(m+1)E_{m+1,0}\otimes E_{-m,0}+mE_{m,-1}\otimes E_{1-m,1}]\\
&+\sum_{n\neq 0,1}\mu^{Et}_{0,n}[(n-1)E_{1,n}\otimes t_{0,-n}-nE_{0,n-1}\otimes t_{1,1-n}]+\sum_{m\neq 0}\lambda^{Et}_{m,0}[-(m+1)E_{m+1,0}\otimes t_{-m,0}\\
&+mE_{m,-1}\otimes t_{1-m,1}]+\sum_{n\neq 0,1}\mu^{tE}_{0,n}[(n-1)t_{1,n}\otimes E_{0,-n}-nt_{0,n-1}\otimes E_{1,1-n}]\\
&+\sum_{m\neq 0}\lambda^{tE}_{m,0}[-(m+1)t_{m+1,0}\otimes E_{-m,0}+mt_{m,-1}\otimes E_{1-m,1}]+\sum_{n\neq 0,1}\mu^{tt}_{0,n}[(n-1)t_{1,n}\otimes t_{0,-n}\\
&-nt_{0,n-1}\otimes t_{1,1-n}]+\sum_{m\neq 0}\lambda^{tt}_{m,0}[-(m+1)t_{m+1,0}\otimes t_{-m,0}+mt_{m,-1}\otimes t_{1-m,1}]\\
&-\mu^{E}_{d_1}E_{1,0}\otimes d_1+\mu^{E}_{d_1}E_{0,-1}\otimes E_{1,1}+\mu^{d_1}_{E}E_{1,1}\otimes E_{0,-1}-\mu^{d_1}_{E}d_1\otimes E_{1,0}\\
&-\lambda E_{1,1}\otimes E_{0,-1}+\lambda d_2\otimes E_{1,0}-\mu^{t}_{d_1}t_{1,0}\otimes d_1+\mu^{t}_{d_1}t_{0,-1}\otimes E_{1,1}\\
&+\mu^{d_1}_{t}E_{1,1}\otimes t_{0,-1}-\mu^{d_1}_{t}d_1\otimes t_{1,0}=-D_0(E_{1,0})
\end{aligned}
$$

Comparing the coefficients of $E_{1,0}\otimes d_1$,$d_1\otimes E_{1,0}$,$E_{1,0}\otimes d_2$,$E_{1,1}\otimes E_{0,-1}$, $E_{1,-1}\otimes E_{0,1}$, $E_{0,1}\otimes E_{1,-1}$, $E_{0,-1}\otimes E_{1,1}$, $t_{1,1}\otimes E_{0,-1}$, $t_{1,-1}\otimes E_{0,1}$, $E_{0,1}\otimes t_{1,-1}$, $t_{0,1}\otimes E_{1,-1}$, $E_{1,-1}\otimes t_{0,1}$, $t_{0,-1}\otimes E_{1,1}$, $E_{1,1}\otimes t_{0,-1}$, respectively, we obtain

$$
\begin{gathered}
\mu^{E}_{d_1}=\mu^{d_1}_{E}=\lambda=0,\ \eta^{E}_{d_2}=2\lambda^{EE}_{0,2}=2\mu^{EE}_{0,-1},\ \eta^{d_2}_{E}=2\mu^{EE}_{0,2}=2\lambda^{EE}_{0,-1},\\
\eta^{t}_{d_2}=2\lambda^{tE}_{0,2}=2\mu^{tE}_{0,-1},\ \eta^{d_2}_{t}=2\mu^{Et}_{0,2}=2\lambda^{Et}_{0,-1},\ \lambda^{t}_{d_1}=-2\mu^{tE}_{0,2},\ \lambda^{d_1}_{t}=-2\mu^{Et}_{0,-1},\\
\mu^{t}_{d_1}=2\lambda^{tE}_{0,-1},\ \mu^{d_1}_{t}=2\lambda^{Et}_{0,2}
\end{gathered} \tag{14}
$$

Comparing the coefficients of $E_{0,n}\otimes E_{1,-n}$, $E_{1,n}\otimes E_{0,-n}$ for $n\neq 0,\ \pm 1$, respectively, one can deduce

$$(n-1)\lambda^{EE}_{0,n}=(n+1)\mu^{EE}_{0,n+1},\ (n+1)\lambda^{EE}_{0,n+1}=(n-1)\mu^{EE}_{0,n}$$

Since $\lambda^{EE}_{0,n}=\mu^{EE}_{0,n}=0$ for $n\gg 0$ or $n\ll 0$, the above equations force

$$\lambda^{EE}_{0,n}=\mu^{EE}_{0,n}=0 \text{ for } n\neq 0,1. \tag{15}$$

Similarly, comparing the coefficients of $E_{0,n} \otimes t_{1,-n}$, $E_{1,n} \otimes t_{0,-n}$, $t_{0,n} \otimes E_{1,-n}$, $t_{1,n} \otimes E_{0,-n}$, $t_{0,n} \otimes t_{1,-n}$, $t_{1,n} \otimes t_{0,-n}$ for $n \neq 0, \pm 1$, respectively, we can obtain

$$\lambda_{0,n}^{Et} = \mu_{0,n}^{Et} = \lambda_{0,n}^{tE} = \mu_{0,n}^{tE} = \lambda_{0,n}^{tt} = \mu_{0,n}^{tt} = 0 \text{ for } n \neq 0, 1. \tag{16}$$

By (14)–(16), we have

$$\eta_{d_2}^{E} = \eta_{E}^{d_2} = \eta_{d_2}^{t} = \eta_{t}^{d_2} = \lambda_{d_1}^{t} = \lambda_{t}^{d_1} = \mu_{d_1}^{t} = \mu_{t}^{d_1} = 0 \tag{17}$$

Comparing the coefficients of $E_{m+1,n} \otimes E_{-m,-n}$ for $m \neq 0, -1, n \neq 0$, one can deduce

$$\eta_{m,n-1}^{EE} + \eta_{m,n+1}^{EE} = 2\eta_{m,n}^{EE} \text{ for } m \neq 0, -1, n \neq 0 \tag{18}$$

Replacing n by $n+k$ in (18) for $k \in \mathbb{Z}$, we obtain

$$\eta_{m,n+k}^{EE} = \begin{cases} \eta_{m,n}^{EE} + k(\eta_{m,n}^{EE} - \eta_{m,n-1}^{EE}), & k \geq 0, n > 0, \\ \eta_{m,n}^{EE} - k(\eta_{m,n}^{EE} - \eta_{m,n+1}^{EE}), & k \leq 0, n < 0, \end{cases}$$

for $m \neq 0, -1$. Note that $\{(m, n) \in \mathbb{Z}^2 \backslash \{\mathbf{0}\} \mid \eta_{m,n}^{EE} \neq 0\}$ is a finite set. We can deduce

$$\eta_{m,n}^{EE} = 0 \text{ for } m \neq 0, -1 \tag{19}$$

Similarly, comparing the coefficients of $E_{m+1,n} \otimes t_{-m,-n}$, $t_{m+1,n} \otimes E_{-m,-n}$, and $t_{m+1,n} \otimes t_{-m,-n}$ for $m \neq 0, -1, n \neq 0$, respectively, we can obtain

$$\eta_{m,n}^{Et} = \eta_{m,n}^{tE} = \eta_{m,n}^{tt} = 0 \text{ for } m \neq 0, -1. \tag{20}$$

Comparing the coefficients of $E_{m+1,0} \otimes E_{-m,0}$ for $m \neq 0, -1$, we obtain

$$m\lambda_{m+1,0}^{EE} = (m+1)\lambda_{m,0}^{EE} \text{ for } m \neq 0, -1.$$

Since $\left\{(m, 0) \in \mathbb{Z}^2 \backslash \{\mathbf{0}\} \,\middle|\, \lambda_{m,0}^{EE} \neq 0\right\}$ is a finite set, we can deduce

$$\lambda_{m,0}^{EE} = 0 \text{ for } m \neq 0 \tag{21}$$

Similarly, comparing the coefficients of $E_{m+1,0} \otimes t_{-m,0}$, $t_{m+1,0} \otimes E_{-m,0}$, and $t_{m+1,0} \otimes t_{-m,0}$ for $m \neq 0, -1$, respectively, we can obtain

$$\lambda_{m,0}^{Et} = \lambda_{m,0}^{tE} = \lambda_{m,0}^{tt} = 0 \text{ for } m \neq 0 \tag{22}$$

Now, (9), (11), and (13) become

$$\begin{gathered} D_0(E_{0,1}) = D_0(E_{0,-1}) = 0, \\ D_0(E_{1,0}) = \sum_{n\neq 0} \eta_{-1,n}^{EE} E_{0,n} \otimes E_{1,-n} + \sum_{n\neq 0} \eta_{-1,n}^{Et} E_{0,n} \otimes t_{1,-n} + \sum_{n\neq 0} \eta_{-1,n}^{tE} t_{0,n} \otimes E_{1,-n} \\ + \sum_{n\neq 0} \eta_{-1,n}^{tt} t_{0,n} \otimes t_{1,-n} + \eta_{d_1}^{E} E_{1,0} \otimes d_1 + \eta_{d_1}^{t} t_{1,0} \otimes d_1 + + \eta_{t}^{d_1} d_1 \otimes t_{1,0} \end{gathered} \tag{23}$$

Write

$$\begin{aligned} D_0(E_{-1,0}) = & \sum_{m,n} \rho_{m,n}^{EE} E_{m,n} \otimes E_{-1-m,-n} + \sum_{m,n} \rho_{m,n}^{Et} E_{m,n} \otimes t_{-1-m,-n} + \sum_{m,n} \rho_{m,n}^{tE} t_{m,n} \otimes E_{-1-m,-n} \\ & + \sum_{m,n} \rho_{m,n}^{tt} t_{m,n} \otimes t_{-1-m,-n} + \rho_{d_1}^{E} E_{-1,0} \otimes d_1 + \rho_{d_2}^{E} E_{-1,0} \otimes d_2 + \rho_{E}^{d_1} d_1 \otimes E_{-1,0} \\ & + \rho_{E}^{d_2} d_2 \otimes E_{-1,0} + \rho_{d_1}^{t} t_{-1,0} \otimes d_1 + \rho_{d_2}^{t} t_{-1,0} \otimes d_2 \\ & + \rho_{t}^{d_1} d_1 \otimes t_{-1,0} + \rho_{t}^{d_2} d_2 \otimes t_{-1,0} \end{aligned} \tag{24}$$

for some $\rho^{EE}_{m,n}$, $\rho^{Et}_{m,n}$, $\rho^{tE}_{m,n}$, $\rho^{tt}_{m,n}$, $\rho^{E}_{d_i}$, $\rho^{d_i}_{E}$, $\rho^{t}_{d_i}$, $\rho^{d_i}_{t} \in \mathbb{C}$, where $\left\{(m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \,\middle|\, \rho^{EE}_{m,n} \neq 0\right\}$, $\left\{(m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \,\middle|\, \rho^{Et}_{m,n} \neq 0\right\}$, $\left\{(m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \,\middle|\, \rho^{tE}_{m,n} \neq 0\right\}$, and $\left\{(m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \,\middle|\, \rho^{tt}_{m,n} \neq 0\right\}$ are finite sets. Applying D_0 to $[E_{0,-1},\ [E_{0,1},\ E_{-1,0}]] = -E_{-1,0}$, using (23), we obtain

$$
\begin{aligned}
&\sum_{m,n} \rho^{EE}_{m,n}\big[-m^2 E_{m,n} \otimes E_{-1-m,-n} + m(m+1)E_{m,n+1} \otimes E_{-1-m,-1-n} \\
&+m(m+1)E_{m,n-1} \otimes E_{-1-m,1-n} - (1+m)^2 E_{m,n} \otimes E_{-1-m,-n}\big]+ \\
&\sum_{m,n} \rho^{Et}_{m,n}\big[-m^2 E_{m,n} \otimes t_{-1-m,-n} + m(m+1)E_{m,n+1} \otimes t_{-1-m,-1-n} \\
&+m(m+1)E_{m,n-1} \otimes t_{-1-m,1-n} - (1+m)^2 E_{m,n} \otimes t_{-1-m,-n}\big] \\
&+\sum_{m,n} \rho^{tE}_{m,n}\big[-m^2 t_{m,n} \otimes E_{-1-m,-n} + m(m+1)t_{m,n+1} \otimes E_{-1-m,-1-n} \\
&+m(m+1)t_{m,n-1} \otimes E_{-1-m,1-n} - (1+m)^2 t_{m,n} \otimes E_{-1-m,-n}\big] \\
&+\sum_{m,n} \rho^{tt}_{m,n}\big[-m^2 t_{m,n} \otimes t_{-1-m,-n} + m(m+1)t_{m,n+1} \otimes t_{-1-m,-1-n} \\
&+m(m+1)t_{m,n-1} \otimes t_{-1-m,1-n} - (1+m)^2 t_{m,n} \otimes t_{-1-m,-n}\big] \\
&-\rho^{E}_{d_1} E_{-1,0} \otimes d_1 - \rho^{E}_{d_2} E_{-1,0} \otimes d_2 - \rho^{E}_{d_2} E_{-1,1} \otimes E_{0,-1} \\
&-\rho^{E}_{d_2} E_{-1,-1} \otimes E_{0,1} - \rho^{d_1}_{E} d_1 \otimes E_{-1,0} - \rho^{d_2}_{E} E_{0,1} \otimes E_{-1,-1} \\
&-\rho^{d_2}_{E} E_{0,-1} \otimes E_{-1,1} - \rho^{d_2}_{E} d_2 \otimes E_{-1,0} - \rho^{t}_{d_1} t_{-1,0} \otimes d_1 \\
&-\rho^{t}_{d_2} t_{-1,0} \otimes d_2 - \rho^{t}_{d_2} t_{-1,1} \otimes E_{0,-1} - \rho^{t}_{d_2} t_{-1,-1} \otimes E_{0,1} \\
&-\rho^{d_1}_{t} d_1 \otimes t_{-1,0} - \rho^{d_2}_{t} E_{0,1} \otimes t_{-1,-1} - \rho^{d_2}_{t} E_{0,-1} \otimes t_{-1,1}. \\
&-\rho^{d_2}_{t} d_2 \otimes t_{-1,0} = -D_0(E_{-1,0})
\end{aligned}
$$

Comparing the coefficients of $E_{0,1} \otimes E_{-1,-1}$, $E_{-1,-1} \otimes E_{0,1}$, $E_{0,1} \otimes t_{-1,-1}$, and $t_{-1,-1} \otimes E_{0,1}$, respectively, we obtain

$$\rho^{d_2}_{E} = \rho^{E}_{d_2} = \rho^{d_2}_{t} = \rho^{t}_{d_2} = 0$$

Comparing the coefficients of $E_{m,n} \otimes E_{-1-m,-n}$ for $m \neq 0,\ -1$, we have

$$\rho^{EE}_{m,n-1} + \rho^{EE}_{m,n+1} = 2\rho^{EE}_{m,n}$$

Since $\left\{(m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \,\middle|\, \rho^{EE}_{m,n} \neq 0\right\}$ is a finite set, we can obtain

$$\rho^{EE}_{m,n} = 0 \text{ for } m \neq 0,\ -1.$$

Similarly, comparing the coefficients of $E_{m,n} \otimes t_{-1-m,-n}$, $t_{m,n} \otimes E_{-1-m,-n}$, and $t_{m,n} \otimes t_{-1-m,-n}$ for $m \neq 0,\ -1$, respectively, we have

$$\rho^{Et}_{m,n} = \rho^{tE}_{m,n} = \rho^{tt}_{m,n} = 0 \text{ for } m \neq 0, -1.$$

Now, we can simplify (24) as

$$
\begin{aligned}
D_0(E_{-1,0}) = \ &\sum_{n\neq 0} \rho^{EE}_{0,n} E_{0,n} \otimes E_{-1,-n} + \sum_{n\neq 0} \rho^{EE}_{-1,n} E_{-1,n} \otimes E_{0,-n} + \sum_{n\neq 0} \rho^{Et}_{0,n} E_{0,n} \otimes t_{-1,-n} \\
&+\sum_{n\neq 0} \rho^{Et}_{-1,n} E_{-1,n} \otimes t_{0,-n} + \sum_{n\neq 0} \rho^{tE}_{0,n} t_{0,n} \otimes E_{-1,-n} + \sum_{n\neq 0} \rho^{tE}_{-1,n} t_{-1,n} \otimes E_{0,-n} \\
&+\sum_{n\neq 0} \rho^{tt}_{0,n} t_{0,n} \otimes t_{-1,-n} + \sum_{n\neq 0} \rho^{tt}_{-1,n} t_{-1,n} \otimes t_{0,-n} + \rho^{E}_{d_1} E_{-1,0} \otimes d_1 \\
&+\rho^{d_1}_{E} d_1 \otimes E_{-1,0} + \rho^{t}_{d_1} t_{-1,0} \otimes d_1 + \rho^{d_1}_{t} d_1 \otimes t_{-1,0}
\end{aligned} \tag{25}
$$

Applying D_0 to $[E_{-1,0}, E_{1,0}] = 0$, we have

$$\begin{aligned}
&\sum_{n\neq 0}\eta^{EE}_{-1,n}nE_{-1,n}\otimes E_{1,-n} - \sum_{n\neq 0}\eta^{EE}_{-1,n}nE_{0,n}\otimes E_{0,-n} + \sum_{n\neq 0}\eta^{Et}_{-1,n}nE_{-1,n}\otimes t_{1,-n}\\
&-\sum_{n\neq 0}\eta^{Et}_{-1,n}nE_{0,n}\otimes t_{0,-n} + \sum_{n\neq 0}\eta^{tE}_{-1,n}nt_{-1,n}\otimes E_{1,-n} - \sum_{n\neq 0}\eta^{tE}_{-1,n}nt_{0,n}\otimes E_{0,-n}\\
&+\sum_{n\neq 0}\eta^{tt}_{-1,n}nt_{-1,n}\otimes t_{1,-n} - \sum_{n\neq 0}\eta^{tt}_{-1,n}nt_{0,n}\otimes t_{0,-n} + \eta^{E}_{d_1}E_{1,0}\otimes E_{-1,0}\\
&+\eta^{t}_{d_1}t_{1,0}\otimes E_{-1,0} + \eta^{d_1}_{t}E_{-1,0}\otimes t_{1,0} + \sum_{n\neq 0}\rho^{EE}_{0,n}nE_{1,n}\otimes E_{-1,-n}\\
&-\sum_{n\neq 0}\rho^{EE}_{0,n}nE_{0,n}\otimes E_{0,-n} + \sum_{n\neq 0}\rho^{EE}_{-1,n}nE_{0,n}\otimes E_{0,-n} - \sum_{n\neq 0}\rho^{EE}_{-1,n}nE_{-1,n}\otimes E_{1,-n}\\
&+\sum_{n\neq 0}\rho^{Et}_{0,n}nE_{1,n}\otimes t_{-1,-n} - \sum_{n\neq 0}\rho^{Et}_{0,n}nE_{0,n}\otimes t_{0,-n} + \sum_{n\neq 0}\rho^{Et}_{-1,n}nE_{0,n}\otimes t_{0,-n}\\
&-\sum_{n\neq 0}\rho^{Et}_{-1,n}nE_{-1,n}\otimes t_{1,-n} + \sum_{n\neq 0}\rho^{tE}_{0,n}nt_{1,n}\otimes E_{-1,-n} - \sum_{n\neq 0}\rho^{tE}_{0,n}nt_{0,n}\otimes E_{0,-n}\\
&+\sum_{n\neq 0}\rho^{tE}_{-1,n}nt_{0,n}\otimes E_{0,-n} - \sum_{n\neq 0}\rho^{tE}_{-1,n}nt_{-1,n}\otimes E_{1,-n} + \sum_{n\neq 0}\rho^{tt}_{0,n}nt_{1,n}\otimes t_{-1,-n}\\
&-\sum_{n\neq 0}\rho^{tt}_{0,n}nt_{0,n}\otimes t_{0,-n} + \sum_{n\neq 0}\rho^{tt}_{-1,n}nt_{0,n}\otimes t_{0,-n} - \sum_{n\neq 0}\rho^{tt}_{-1,n}nt_{-1,n}\otimes t_{1,-n}\\
&+\rho^{E}_{d_1}E_{-1,0}\otimes E_{1,0} + \rho^{d_1}_{E}E_{1,0}\otimes E_{-1,0} + \rho^{t}_{d_1}t_{-1,0}\otimes E_{1,0} + \rho^{d_1}_{t}E_{1,0}\otimes t_{-1,0} = 0.
\end{aligned}$$

Comparing the coefficients of $E_{-1,0}\otimes E_{1,0}$, $E_{1,0}\otimes E_{-1,0}$, $E_{1,0}\otimes t_{-1,0}$, $E_{-1,0}\otimes t_{1,0}$, $t_{1,0}\otimes E_{-1,0}$ and $t_{-1,0}\otimes E_{1,0}$, we have

$$\rho^{E}_{d_1} = \rho^{d_1}_{t} = \eta^{d_1}_{t} = \eta^{t}_{d_1} = \rho^{t}_{d_1} = 0,\ \eta^{E}_{d_1} = -\rho^{d_1}_{E}$$

Comparing the coefficients of $E_{1,n}\otimes E_{-1,-n}$, $E_{1,n}\otimes t_{-1,-n}$, $t_{1,n}\otimes E_{-1,-n}$, $t_{1,n}\otimes t_{-1,-n}$, $E_{-1,n}\otimes E_{1,-n}$, $E_{-1,n}\otimes t_{1,-n}$, $t_{-1,n}\otimes E_{1,-n}$, $t_{-1,n}\otimes t_{1,-n}$ for $n\neq 0$, respectively, we obtain

$$\rho^{EE}_{0,n} = \rho^{Et}_{0,n} = \rho^{tE}_{0,n} = \rho^{tt}_{0,n} = 0,\ \eta^{EE}_{-1,n} = \rho^{EE}_{-1,n},\ \eta^{Et}_{-1,n} = \rho^{Et}_{-1,n}, \eta^{tE}_{-1,n} = \rho^{tE}_{-1,n},\ \eta^{tt}_{-1,n} = \rho^{tt}_{-1,n}$$

Thus, (23) and (25) become

$$\begin{aligned}
D_0(E_{1,0}) = &\sum_{n\neq 0}\eta^{EE}_{-1,n}E_{0,n}\otimes E_{1,-n} + \sum_{n\neq 0}\eta^{Et}_{-1,n}E_{0,n}\otimes t_{1,-n} + \sum_{n\neq 0}\eta^{tE}_{-1,n}t_{0,n}\otimes E_{1,-n}\\
&+\sum_{n\neq 0}\eta^{tt}_{-1,n}t_{0,n}\otimes t_{1,-n} + \eta^{E}_{d_1}E_{1,0}\otimes d_1
\end{aligned}\tag{26}$$

$$\begin{aligned}
D_0(E_{-1,0}) = &\sum_{n\neq 0}\eta^{EE}_{-1,n}E_{-1,n}\otimes E_{0,-n} + \sum_{n\neq 0}\eta^{Et}_{-1,n}E_{-1,n}\otimes t_{0,-n} + \sum_{n\neq 0}\eta^{tE}_{-1,n}t_{-1,n}\otimes E_{0,-n}\\
&+\sum_{n\neq 0}\eta^{tt}_{-1,n}t_{-1,n}\otimes t_{0,-n} - \eta^{E}_{d_1}d_1\otimes E_{-1,0}
\end{aligned}\tag{27}$$

Applying D_0 to $[E_{-1,0}, [E_{1,0}, E_{0,1}]] = -E_{0,1}$, we have

$$\begin{aligned}
&-\sum_{n\neq 0}\eta^{EE}_{-1,n}nE_{-1,n}\otimes E_{1,1-n} - \sum_{n\neq 0}\eta^{EE}_{-1,n}(1-n)E_{0,n}\otimes E_{0,1-n} - \sum_{n\neq 0}\eta^{Et}_{-1,n}nE_{-1,n}\otimes t_{1,1-n}\\
&-\sum_{n\neq 0}\eta^{Et}_{-1,n}(1-n)E_{0,n}\otimes t_{0,1-n} - \sum_{n\neq 0}\eta^{tE}_{-1,n}nt_{-1,n}\otimes E_{1,1-n} - \sum_{n\neq 0}\eta^{tE}_{-1,n}(1-n)t_{0,n}\otimes E_{0,1-n}\\
&-\sum_{n\neq 0}\eta^{EE}_{-1,n}(n+1)E_{0,n+1}\otimes E_{0,-n} + \sum_{n\neq 0}\eta^{EE}_{-1,n}nE_{-1,n}\otimes E_{1,1-n} - \sum_{n\neq 0}\eta^{Et}_{-1,n}(n+1)E_{0,n+1}\otimes t_{0,-n}\\
&+\sum_{n\neq 0}\eta^{Et}_{-1,n}nE_{-1,n}\otimes t_{1,1-n} - \sum_{n\neq 0}\eta^{tE}_{-1,n}(n+1)t_{0,n+1}\otimes E_{0,-n} + \sum_{n\neq 0}\eta^{tE}_{-1,n}nt_{-1,n}\otimes E_{1,1-n}\\
&-\sum_{n\neq 0}\eta^{tt}_{-1,n}(n+1)t_{0,n+1}\otimes t_{0,-n} + \sum_{n\neq 0}\eta^{tt}_{-1,n}nt_{-1,n}\otimes t_{1,1-n} + \eta^{E}_{d_1}E_{1,1}\otimes E_{-1,0}\\
&+\eta^{E}_{d_1}d_1\otimes E_{0,1} = 0.
\end{aligned}$$

Comparing the coefficients of $E_{0,1}\otimes d_1$, we obtain $\eta^{E}_{d_1} = 0$. Comparing the coefficients of $E_{0,n}\otimes E_{0,1-n}$, $E_{0,n}\otimes t_{0,1-n}$, $t_{0,n}\otimes E_{0,1-n}$, and $t_{0,n}\otimes t_{0,1-n}$ for $n\neq 0, 1$, respectively, we deduce

$$\eta^{EE}_{-1,n} = \eta^{Et}_{-1,n} = \eta^{tE}_{-1,n} = \eta^{tt}_{-1,n} = 0 \text{ for } n\neq 0$$

Thus, (23), (26) and (27) become

$$D_0(E_{0,1}) = D_0(E_{0,-1}) = D_0(E_{1,0}) = D_0(E_{-1,0}) = 0 \tag{28}$$

From (28), we can easily deduce that $D_0(E_{m,n}) = 0$ for $m,\ n,\ m+n \in \{-1,\ 0,\ 1\}$.
Write

$$\begin{aligned} D_0(t_{0,1}) = \ & \sum_{m,n} \pi^{EE}_{m,n} E_{m,n} \otimes E_{-m,1-n} + \sum_{m,n} \pi^{Et}_{m,n} E_{m,n} \otimes t_{-m,1-n} + \sum_{m,n} \pi^{tE}_{m,n} t_{m,n} \otimes E_{-m,1-n} \\ & + \sum_{m,n} \pi^{tt}_{m,n} t_{m,n} \otimes t_{-m,1-n} + \pi^{E}_{d_1} E_{0,1} \otimes d_1 + \pi^{E}_{d_2} E_{0,1} \otimes d_2 \\ & + \pi^{d_1}_{E} d_1 \otimes E_{0,1} + \pi^{d_2}_{E} d_2 \otimes E_{0,1} + \pi^{t}_{d_1} t_{0,1} \otimes d_1 + \pi^{t}_{d_2} t_{0,1} \otimes d_2 \\ & + \pi^{d_1}_{t} d_1 \otimes t_{0,1} + \pi^{d_2}_{t} d_2 \otimes t_{0,1}, \end{aligned} \tag{29}$$

for some $\pi^{EE}_{m,n},\ \pi^{Et}_{m,n},\ \pi^{tE}_{m,n},\ \pi^{tt}_{m,n},\ \pi^{E}_{d_i},\ \pi^{d_i}_{E},\ \pi^{t}_{d_i},\ \pi^{d_i}_{t} \in \mathbb{C}$, where $\{(m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \mid \pi^{EE}_{m,n} \neq 0\}$, $\{(m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \mid \pi^{Et}_{m,n} \neq 0\}$, $\{(m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \mid \pi^{tE}_{m,n} \neq 0\}$, and $\{(m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \mid \pi^{tt}_{m,n} \neq 0\}$ are finite sets. Applying D_0 to $[E_{0,1},\ t_{0,1}] = 0$, we have

$$\begin{aligned} & \sum_{\substack{m,n,\\ m\neq 0}} \pi^{EE}_{m,n} m E_{m,n+1} \otimes E_{-m,1-n} - \sum_{\substack{m,n,\\ m\neq 0}} \pi^{EE}_{m,n} m E_{m,n} \otimes E_{-m,2-n} + \sum_{\substack{m,n,\\ m\neq 0}} \pi^{Et}_{m,n} m E_{m,n+1} \otimes t_{-m,1-n} \\ & - \sum_{\substack{m,n,\\ m\neq 0}} \pi^{Et}_{m,n} m E_{m,n} \otimes t_{-m,2-n} + \sum_{\substack{m,n,\\ m\neq 0}} \pi^{tE}_{m,n} m t_{m,n+1} \otimes E_{-m,1-n} - \sum_{\substack{m,n,\\ m\neq 0}} \pi^{tE}_{m,n} m t_{m,n} \otimes E_{-m,2-n} \\ & + \sum_{\substack{m,n,\\ m\neq 0}} \pi^{tt}_{m,n} m t_{m,n+1} \otimes t_{-m,1-n} - \sum_{\substack{m,n,\\ m\neq 0}} \pi^{tt}_{m,n} m t_{m,n} \otimes t_{-m,2-n} - (\pi^{E}_{d_2} + \pi^{d_2}_{E}) E_{0,1} \otimes E_{0,1} \\ & - \pi^{t}_{d_2} t_{0,1} \otimes E_{0,1} - \pi^{d_2}_{t} E_{0,1} \otimes t_{0,1} = 0. \end{aligned}$$

Comparing the coefficients of $E_{0,1} \otimes E_{0,1}$, $t_{0,1} \otimes E_{0,1}$, $E_{0,1} \otimes t_{0,1}$, we obtain

$$\pi^{E}_{d_2} + \pi^{d_2}_{E} = 0,\ \pi^{t}_{d_2} = \pi^{d_2}_{t} = 0.$$

Comparing the coefficients of $E_{m,n+1} \otimes E_{-m,1-n}$, $E_{m,n+1} \otimes t_{-m,1-n}$, $t_{m,n+1} \otimes E_{-m,1-n}$, and $t_{m,n+1} \otimes t_{-m,1-n}$ for $m \neq 0$, respectively, we have

$$\pi^{EE}_{m,n} = \pi^{EE}_{m,n+1},\ \pi^{Et}_{m,n} = \pi^{Et}_{m,n+1},\ \pi^{tE}_{m,n} = \pi^{tE}_{m,n+1},\ \pi^{tt}_{m,n} = \pi^{tt}_{m,n+1} \text{ for } m \neq 0.$$

Since $\{(m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \mid \pi^{EE}_{m,n} \neq 0\}$, $\{(m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \mid \pi^{Et}_{m,n} \neq 0\}$, $\{(m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \mid \pi^{tE}_{m,n} \neq 0\}$, and $\{(m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\} \mid \pi^{tt}_{m,n} \neq 0\}$ are finite sets, we deduce

$$\pi^{EE}_{m,n} = \pi^{Et}_{m,n} = \pi^{tE}_{m,n} = \pi^{tt}_{m,n} = 0 \text{ for } m \neq 0.$$

Thus, (29) becomes

$$\begin{aligned} D_0(t_{0,1}) = \ & \sum_{n\neq 0,1} \pi^{EE}_{0,n} E_{0,n} \otimes E_{0,1-n} + \sum_{n\neq 0,1} \pi^{Et}_{0,n} E_{0,n} \otimes t_{0,1-n} + \sum_{n\neq 0,1} \pi^{tE}_{0,n} t_{0,n} \otimes E_{0,1-n} \\ & + \sum_{n\neq 0,1} \pi^{tt}_{0,n} t_{0,n} \otimes t_{0,1-n} + \pi^{E}_{d_1} E_{0,1} \otimes d_1 + \pi^{E}_{d_2} E_{0,1} \otimes d_2 + \pi^{d_1}_{E} d_1 \otimes E_{0,1} \\ & - \pi^{E}_{d_2} d_2 \otimes E_{0,1} + \pi^{t}_{d_1} t_{0,1} \otimes d_1 + \pi^{d_1}_{t} d_1 \otimes t_{0,1} \end{aligned} \tag{30}$$

Applying D_0 to $[E_{0,-1},\ t_{0,1}] = 0$ and using (30), we have $\pi^{E}_{d_2} = 0$.
Write

$$\begin{aligned} D_0(t_{0,-1}) = \ & \sum_{m,n} \chi^{EE}_{m,n} E_{m,n-1} \otimes E_{-m,-n} + \sum_{m,n} \chi^{Et}_{m,n} E_{m,n-1} \otimes t_{-m,-n} + \sum_{m,n} \chi^{tE}_{m,n} t_{m,n-1} \otimes E_{-m,-n} \\ & + \sum_{m,n} \chi^{tt}_{m,n} t_{m,n-1} \otimes t_{-m,-n} + \chi^{E}_{d_1} E_{0,-1} \otimes d_1 + \chi^{E}_{d_2} E_{0,-1} \otimes d_2 \\ & + \chi^{d_1}_{E} d_1 \otimes E_{0,-1} + \pi^{d_2}_{E} d_2 \otimes E_{0,-1} + \chi^{t}_{d_1} t_{0,-1} \otimes d_1 + \chi^{t}_{d_2} t_{0,-1} \otimes d_2 \\ & + \chi^{d_1}_{t} d_1 \otimes t_{0,-1} + \chi^{d_2}_{t} d_2 \otimes t_{0,-1} \end{aligned} \tag{31}$$

for some $\chi^{EE}_{m,n}, \chi^{Et}_{m,n}, \chi^{tE}_{m,n}, \chi^{tt}_{m,n}, \chi^{E}_{d_i}, \chi^{d_i}_{E}, \chi^{t}_{d_i}, \chi^{d_i}_{t} \in \mathbb{C}$, where $\{(m, n) \in \mathbb{Z}^2 \backslash \{\mathbf{0}\} \mid \chi^{EE}_{m,n} \neq 0\}$, $\{(m, n) \in \mathbb{Z}^2 \backslash \{\mathbf{0}\} \mid \chi^{Et}_{m,n} \neq 0\}$, $\{(m, n) \in \mathbb{Z}^2 \backslash \{\mathbf{0}\} \mid \chi^{tE}_{m,n} \neq 0\}$, and $\{(m, n) \in \mathbb{Z}^2 \backslash \{\mathbf{0}\} \mid \chi^{tt}_{m,n} \neq 0\}$ are finite sets. Applying D_0 to $[E_{0,-1}, t_{0,-1}] = 0$, we have

$$\begin{aligned} & - \sum_{\substack{m,n, \\ m \neq 0}} \chi^{EE}_{m,n} m E_{m,n-2} \otimes E_{-m,-n} + \sum_{\substack{m,n, \\ m \neq 0}} \chi^{EE}_{m,n} m E_{m,n-1} \otimes E_{-m,-1-n} - \sum_{\substack{m,n, \\ m \neq 0}} \chi^{Et}_{m,n} m E_{m,n-2} \otimes t_{-m,-n} \\ & + \sum_{\substack{m,n, \\ m \neq 0}} \chi^{Et}_{m,n} m E_{m,n-1} \otimes t_{-m,-1-n} - \sum_{\substack{m,n, \\ m \neq 0}} \chi^{tE}_{m,n} m t_{m,n-2} \otimes E_{-m,-n} + \sum_{\substack{m,n, \\ m \neq 0}} \chi^{tE}_{m,n} m t_{m,n-1} \otimes E_{-m,-1-n} \\ & - \sum_{\substack{m,n, \\ m \neq 0}} \chi^{tt}_{m,n} m t_{m,n-2} \otimes t_{-m,-n} + \sum_{\substack{m,n, \\ m \neq 0}} \chi^{tt}_{m,n} m t_{m,n-1} \otimes t_{-m,-1-n} + (\chi^{E}_{d_2} + \chi^{d_2}_{E}) E_{0,-1} \otimes E_{0,-1} \\ & + \chi^{t}_{d_2} t_{0,-1} \otimes E_{0,-1} + \chi^{d_2}_{t} E_{0,-1} \otimes t_{0,-1} = 0 \end{aligned}$$

Comparing the coefficients of $E_{0,-1} \otimes E_{0,-1}$, $t_{0,-1} \otimes E_{0,-1}$, $E_{0,-1} \otimes t_{0,-1}$, we obtain

$$\chi^{E}_{d_2} + \chi^{d_2}_{E} = 0, \ \chi^{t}_{d_2} = \chi^{d_2}_{t} = 0.$$

Comparing the coefficients of $E_{m,n-2} \otimes E_{-m,-n}$, $E_{m,n-2} \otimes t_{-m,-n}$, $t_{m,n-2} \otimes E_{-m,-n}$, and $t_{m,n-2} \otimes t_{-m,-n}$ for $m \neq 0$, respectively, we obtain

$$\chi^{EE}_{m,n} = \chi^{EE}_{m,n-1}, \ \chi^{Et}_{m,n} = \chi^{Et}_{m,n-1}, \ \chi^{tE}_{m,n} = \chi^{tE}_{m,n-1}, \ \chi^{tt}_{m,n} = \chi^{tt}_{m,n-1} \text{ for } m \neq 0.$$

Note that $\{(m, n) \in \mathbb{Z}^2 \backslash \{\mathbf{0}\} \mid \chi^{EE}_{m,n} \neq 0\}$, $\{(m, n) \in \mathbb{Z}^2 \backslash \{\mathbf{0}\} \mid \chi^{Et}_{m,n} \neq 0\}$, $\{(m, n) \in \mathbb{Z}^2 \backslash \{\mathbf{0}\} \mid \chi^{tE}_{m,n} \neq 0\}$, and $\{(m, n) \in \mathbb{Z}^2 \backslash \{\mathbf{0}\} \mid \chi^{tt}_{m,n} \neq 0\}$ are finite sets, and we deduce

$$\chi^{EE}_{m,n} = \chi^{Et}_{m,n} = \chi^{tE}_{m,n} = \chi^{tt}_{m,n} = 0 \text{ for } m \neq 0.$$

Thus, (31) becomes

$$\begin{aligned} D_0(t_{0,-1}) = \ & \sum_{n \neq 0,1} \chi^{EE}_{0,n} E_{0,n-1} \otimes E_{0,-n} + \sum_{n \neq 0,1} \chi^{Et}_{0,n} E_{0,n-1} \otimes t_{0,-n} + \sum_{n \neq 0,1} \chi^{tE}_{0,n} t_{0,n-1} \otimes E_{0,-n} \\ & + \sum_{n \neq 0,1} \chi^{tt}_{0,n} t_{0,n-1} \otimes t_{0,-n} + \chi^{E}_{d_1} E_{0,-1} \otimes d_1 + \chi^{E}_{d_2} E_{0,-1} \otimes d_2 \\ & + \chi^{d_1}_{E} d_1 \otimes E_{0,-1} - \chi^{E}_{d_2} d_2 \otimes E_{0,-1} + \chi^{t}_{d_1} t_{0,-1} \otimes d_1 + \chi^{d_1}_{t} d_1 \otimes t_{0,-1} \end{aligned} \tag{32}$$

Applying D_0 to $[E_{0,1},\ t_{0,-1}] = 0$ and using (32), we obtain $\chi_{d_2}^{E} = \chi_{E}^{d_2} = 0$. Applying D_0 to $[E_{-1,0},\ [E_{1,2},\ t_{0,-1}]] = t_{0,1}$, we have

$$
\begin{aligned}
&\sum_{n\neq 0,1} \chi_{0,n}^{EE}(1-n^2)E_{0,n+1}\otimes E_{0,-n} + \sum_{n\neq 0,1} \chi_{0,n}^{EE} n(n-1)E_{1,n+1}\otimes E_{-1,-n} \\
&+ \sum_{n\neq 0,1} \chi_{0,n}^{EE} n(n-1)E_{-1,n-1}\otimes E_{1,2-n} + \sum_{n\neq 0,1} \chi_{0,n}^{EE} n(2-n)E_{0,n-1}\otimes E_{0,2-n} \\
&+ \sum_{n\neq 0,1} \chi_{0,n}^{Et}(1-n^2)E_{0,n+1}\otimes t_{0,-n} + \sum_{n\neq 0,1} \chi_{0,n}^{Et} n(n-1)E_{1,n+1}\otimes t_{-1,-n} \\
&+ \sum_{n\neq 0,1} \chi_{0,n}^{Et} n(n-1)E_{-1,n-1}\otimes t_{1,2-n} + \sum_{n\neq 0,1} \chi_{0,n}^{Et} n(2-n)E_{0,n-1}\otimes t_{0,2-n} \\
&+ \sum_{n\neq 0,1} \chi_{0,n}^{tE}(1-n^2)t_{0,n+1}\otimes E_{0,-n} + \sum_{n\neq 0,1} \chi_{0,n}^{tE} n(n-1)t_{1,n+1}\otimes E_{-1,-n} \\
&+ \sum_{n\neq 0,1} \chi_{0,n}^{tE} n(n-1)t_{-1,n-1}\otimes E_{1,2-n} + \sum_{n\neq 0,1} \chi_{0,n}^{tE} n(2-n)t_{0,n-1}\otimes E_{0,2-n} \\
&+ \sum_{n\neq 0,1} \chi_{0,n}^{tt}(1-n^2)t_{0,n+1}\otimes t_{0,-n} + \sum_{n\neq 0,1} \chi_{0,n}^{tt} n(n-1)t_{1,n+1}\otimes t_{-1,-n} \\
&+ \sum_{n\neq 0,1} \chi_{0,n}^{tt} n(n-1)t_{-1,n-1}\otimes t_{1,2-n} + \sum_{n\neq 0,1} \chi_{0,n}^{tt} n(2-n)t_{0,n-1}\otimes t_{0,2-n} \\
&+\chi_{d_1}^{E} E_{0,1}\otimes d_1 + \chi_{d_1}^{E} E_{1,1}\otimes E_{-1,0} + \chi_{d_1}^{E} E_{-1,-1}\otimes E_{1,2} \\
&-2\chi_{d_1}^{E} E_{0,-1}\otimes E_{0,2} - 2\chi_{E}^{d_1} E_{0,2}\otimes E_{0,-1} + \chi_{E}^{d_1} E_{1,2}\otimes E_{-1,-1} \\
&+\chi_{E}^{d_1} E_{-1,0}\otimes E_{1,1} + \chi_{E}^{d_1} d_1\otimes E_{0,1} + \chi_{d_1}^{t} t_{0,1}\otimes d_1 \\
&+\chi_{d_1}^{t} t_{1,1}\otimes E_{-1,0} + \chi_{d_1}^{t} t_{-1,-1}\otimes E_{1,2} - 2\chi_{d_1}^{t} t_{0,-1}\otimes E_{0,2} \\
&-2\chi_{t}^{d_1} E_{0,2}\otimes t_{0,-1} + \chi_{t}^{d_1} E_{1,2}\otimes t_{-1,-1} + \chi_{t}^{d_1} E_{-1,0}\otimes t_{1,1} \\
&+\chi_{t}^{d_1} d_1\otimes t_{0,1} = D_0(t_{0,1}).
\end{aligned}
$$

Comparing the coefficients of $E_{1,1}\otimes E_{-1,0}$, $E_{-1,0}\otimes E_{1,1}$, $t_{1,1}\otimes E_{-1,0}$, $E_{-1,0}\otimes t_{1,1}$, $E_{0,1}\otimes d_1$, $d_1\otimes E_{0,1}$, $t_{0,1}\otimes d_1$, $d_1\otimes t_{0,1}$, we obtain

$$\chi_{d_1}^{E} = \chi_{E}^{d_1} = \chi_{d_1}^{t} = \chi_{t}^{d_1} = \pi_{d_1}^{E} = \pi_{E}^{d_1} = \pi_{d_1}^{t} = \pi_{t}^{d_1} = 0$$

Comparing the coefficients of $E_{1,n+1}\otimes E_{-1,-n}$, $E_{1,n+1}\otimes t_{-1,-n}$, $t_{1,n+1}\otimes E_{-1,-n}$, $t_{1,n+1}\otimes t_{-1,-n}$, we have

$$\chi_{0,n}^{EE} = \chi_{0,n}^{Et} = \chi_{0,n}^{tE} = \chi_{0,n}^{tt} = 0 \text{ for } n\neq 0,\ 1$$

Comparing the coefficients of $E_{0,n}\otimes E_{0,1-n}$, $E_{0,n}\otimes t_{0,1-n}$, $t_{0,n}\otimes E_{0,1-n}$, $t_{0,n}\otimes t_{0,1-n}$, we deduce

$$\pi_{0,n}^{EE} = \pi_{0,n}^{Et} = \pi_{0,n}^{tE} = \pi_{0,n}^{tt} = 0 \text{ for } n\neq 0,\ 1.$$

Thus, (30) and (32) become

$$D_0(t_{0,1}) = D_0(t_{0,-1}) = 0 \tag{33}$$

Using

$$D_0(t_{1,0}) = -D_0([E_{1,-1},\ t_{0,1}]) = 0,\ D_0(t_{-1,0}) = -D_0([E_{-1,1},\ t_{0,-1}]) = 0, \tag{34}$$

and (33), we deduce that $D_0(t_{m,n}) = 0$ for $m,\ n,\ m+n \in \{-1,\ 0,\ 1\}$. Then, Claim 1 is proved.

Claim 2. $D_0(E_{m,n}) = D_0(t_{m,n}) = 0$ *for* $(m,\ n)\in \mathbb{Z}^2\backslash\{0\}$.

Note that $E_{m,n}$, $t_{m,n}$ with $(m,\ n) \in \mathbb{Z}^2\backslash\{\mathbf{0}\}$ can be generated by $\{E_{k,l},\ t_{k,l} \mid k,\ l,\ k+l\in\{0,\ \pm1\}\}$. From Claim 1, we can easily deduce that $D_0(E_{m,n}) = D_0(t_{m,n}) = 0$ for $(m,\ n)\in\mathbb{Z}^2\backslash\{\mathbf{0}\}$.

By Lemma 4 and Claims 1 and 2, Lemma 5 is proved.

Remark 1. *We use the convention that if an undefined notation appears in an expression, we always treat it as zero. For instance,* $E_{0,0} = t_{0,0} = 0$.

Lemma 6. *For any* $D \in \mathrm{Der}(\widetilde{L}, W)$ *,* $D = \sum_{\alpha \in \mathbb{Z}^2} D_\alpha$ *is a finite sum, where* $D_\alpha \in \mathrm{Der}(\widetilde{L}, W)_\alpha$.

Proof . By Lemmas 9, 10, and 11, one can suppose $D_\alpha \in (w_\alpha)_{\mathrm{inn}}$ for some $w_\alpha \in W_\alpha$ and $\alpha \in \mathbb{Z}^2$. If $\Gamma = \{\alpha \in \mathbb{Z}^2 \backslash \{0\} \mid w_\alpha \neq 0\}$ is an infinite set, by linear algebra, there exists $d \in T$ such that $d(\alpha) \neq 0$ for $\alpha \in \Gamma$. Then, $D(d) = \sum_{\alpha \in \Gamma \cup \{0\}} D_\alpha(d) = \sum_{\alpha \in \Gamma} d(\alpha) w_\alpha$ is an infinite sum, which is not an element in W. This is a contradiction with the fact that $D \in \mathrm{Der}(\widetilde{L}, W)$. This proves the lemma. □

The proof of Proposition 1 is finally completed.

Lemma 7. *Suppose* $w \in W$ *such that* $x \cdot w \in \mathrm{Im}(1 \otimes 1 - \tau)$ *for all* $x \in \widetilde{L}$*. Then,* $w \in \mathrm{Im}(1 \otimes 1 - \tau)$.

Proof . It is obvious that $\widetilde{L} \cdot \mathrm{Im}(1 \otimes 1 - \tau) \subset \mathrm{Im}(1 \otimes 1 - \tau)$. We shall prove that after a number of steps, by replacing w with w-v for some $v \in \mathrm{Im}(1 \otimes 1 - \tau)$, the zero element is obtained, thus proving that $w \in \mathrm{Im}(1 \otimes 1 - \tau)$. Write $w = \sum_{\alpha \in \mathbb{Z}^2} w_\alpha$, where $w_\alpha \in W_\alpha$. It is clear that

$$w \in \mathrm{Im}(1 \otimes 1 - \tau) \Leftrightarrow w_\alpha \in \mathrm{Im}(1 \otimes 1 - \tau) \text{ for all } \alpha \in \mathbb{Z}^2 \tag{35}$$

For any $\alpha \in \mathbb{Z}^2 \backslash \{\mathbf{0}\}$, choose $d \in T$ such that $d(\alpha) \neq 0$. Then,

$$d \cdot w = \sum_{\alpha \in \mathbb{Z}^2 \backslash \{0\}} d(\alpha) w_\alpha \in \mathrm{Im}(1 \otimes 1 - \tau)$$

Thus, (35) gives $w_\alpha \in \mathrm{Im}(1 \otimes 1 - \tau)$. Replacing w by $w - \sum_{\alpha \neq 0} w_\alpha$, we can suppose $w = w_0 \in W_0$. Now, we can write

$$\begin{aligned} w = \;& \sum_{m,n} \lambda_{m,n} E_{m,n} \otimes E_{-m,-n} + \sum_{m,n} \mu_{m,n} E_{m,n} \otimes t_{-m,-n} + \sum_{m,n} \rho_{m,n} t_{m,n} \otimes E_{-m,-n} \\ & + \sum_{m,n} \eta_{m,n} t_{m,n} \otimes t_{-m,-n} + \lambda_1 d_1 \otimes d_1 + \lambda_1' d_1 \otimes d_2 + \lambda_2 d_2 \otimes d_1 + \lambda_2' d_2 \otimes d_2, \end{aligned}$$

for some $\lambda_{m,n},\ \mu_{m,n},\ \rho_{m,n},\ \eta_{m,n},\ \lambda_i,\ \lambda_i{}' \in \mathbb{C}$. Since $v^{EE}_{m,n} := E_{m,n} \otimes E_{-m,-n} - E_{-m,-n} \otimes E_{m,n}$,$v^{Et}_{m,n} :=E_{m,n} \otimes t_{-m,-n} - t_{-m,-n} \otimes E_{m,n}$, $v^{tE}_{m,n} := t_{m,n} \otimes E_{-m,-n} - E_{-m,-n} \otimes t_{m,n}$, $v^{tt}_{m,n} := t_{m,n} \otimes t_{-m,-n} - t_{-m,-n} \otimes t_{m,n}$ are all in $\mathrm{Im}(1 \otimes 1 - \tau)$, by replacing w by $w - v$, where v is a combination of some $v^{EE}_{m,n}$, $v^{Et}_{m,n}$, $v^{tE}_{m,n}$ and $v^{tt}_{m,n}$, one can suppose

$$\lambda_{m,n},\ \mu_{m,n},\ \rho_{m,n},\ \eta_{m,n} \neq 0 \Rightarrow m > 0 \text{ or } m = 0,\ n > 0 \tag{36}$$

First, assume that $\lambda_{m,n} \neq 0$ for some $m > 0$ or $m = 0$, $n > 0$. Choose $k, l > 0$ such that $ml - nk \neq 0$. Then, we see that the term $E_{m+k,n+l} \otimes E_{-m,-n}$ appears in $E_{k,l} \cdot w$, but (36) implies that the term $E_{-m,-n} \otimes E_{m+k,n+l}$ does not appear in $E_{k,l} \cdot w$, a contradiction with the fact that $E_{k,l} \cdot w \in \mathrm{Im}(1 \otimes 1 - \tau)$. Thus, we can further suppose that $\lambda_{m,n} = 0$ for any $m > 0$ or $m = 0$, $n > 0$. Similarly, we can also suppose that $\mu_{m,n} = \rho_{m,n} = \eta_{m,n} = 0$ for any $m > 0$ or $m = 0$, $n > 0$. Then, we can simplify w as

$$w = \lambda_1 d_1 \otimes d_1 + \lambda_1' d_1 \otimes d_2 + \lambda_2 d_2 \otimes d_1 + \lambda_2' d_2 \otimes d_2$$

Since

$$\begin{aligned} E_{1,0} \cdot w &= -\lambda_1 E_{1,0} \otimes d_1 - \lambda_1 d_1 \otimes E_{1,0} - \lambda_1' E_{1,0} \otimes d_2 - \lambda_2 d_2 \otimes E_{1,0} \in \mathrm{Im}(1 \otimes 1 - \tau), \\ E_{0,1} \cdot w &= -\lambda_1' d_1 \otimes E_{0,1} - \lambda_2 E_{0,1} \otimes d_1 - \lambda_2' E_{0,1} \otimes d_2 - \lambda_2' d_2 \otimes E_{0,1} \in \mathrm{Im}(1 \otimes 1 - \tau), \end{aligned}$$

we have that $\lambda_1 = \lambda'_2 = 0$, $\lambda'_1 + \lambda_2 = 0$. Thus, $w = \lambda'_1(d_1 \otimes d_2 - d_2 \otimes d_1) \in \mathrm{Im}(1 \otimes 1 - \tau)$. Replacing w by $w - \lambda'_1(d_1 \otimes d_2 - d_2 \otimes d_1)$, then $w = 0$. The lemma is proved. □

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Let $(\widetilde{L}, [\cdot, \cdot], \Delta)$ be a Lie bialgebra structure on $\widetilde{L}$. By (2), (6), and Proposition 1, $\Delta = \Delta_r$ is defined by (3) for some $r \in W$. By (1), $\mathrm{Im}\Delta \subset \mathrm{Im}(1 \otimes 1 - \tau)$. Thus, by Lemma 7, $r \in \mathrm{Im}(1 \otimes 1 - \tau)$. Then, Lemma 1 and Corollary 1 show that $c(r) = 0$. Therefore, Definitions 2 and 3 imply that $(\widetilde{L}, [\cdot, \cdot], \Delta)$ is a triangular coboundary Lie bialgebra. □

3. Conclusions

Lie bialgebras are closely related to solutions of quantum Yang–Baxter equations. The rank two Heisenberg–Virasoro algebra is a generalization of the twisted Heisenberg–Virasoro algebra. In this paper, all Lie bialgebra structures on the rank two Heisenberg–Virasoro algebra are determined. It is proved that all such Lie bialgebras are triangular coboundary. This result makes sense since dualizing a triangular coboundary Lie bialgebra may produce new Lie algebras (see, e.g., [16]). This will be studied in a sequel.

Lie bialgebra has some applications in physics. For example, the quantum open-closed homotopy algebra was described in the framework of homotopy involutive Lie bialgebras, as a morphism from the loop homotopy Lie algebra of closed strings to the involutive Lie bialgebra on the Hochschild complex of open strings (see [23] for details).

Lie algebras not only have applications in pure mathematics and physics, but also may have applications in other fields, such as the field of fuzzy sets. The reader can refer to [24,25]. Perhaps the concept of Lie bialgebra can also be extended to the field of fuzzy sets, which may be our work to study in the future.

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