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On the Approximation by Bivariate Szász–Jakimovski–Leviatan-Type Operators of Unbounded Sequences of Positive Numbers

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Abstract: In this paper, we construct the bivariate Szász–Jakimovski–Leviatan-type operators in Dunkl form using the unbounded sequences α_n, β_m and ξ_m of positive numbers. Then, we obtain the rate of convergence in terms of the weighted modulus of continuity of two variables and weighted approximation theorems for our operators. Moreover, we provide the degree of convergence with the help of bivariate Lipschitz-maximal functions and obtain the direct theorem.

Keywords: bivariate functions; weight function; Dunkl analogue; Appell polynomial; Szász operator; Szász–Jakimovski–Levitian operator; Lipschitz function

MSC: 41A25; 41A36; 33C45

1. Introduction and Preliminaries

In 1912, S. N. Bernstein [1] introduced a positive linear operator for the set of all continuous functions on $[0, 1]$, which provides the shortest proof of the well-known Weierstrass approximation theorem while, later, another positive linear operator on $[0, \infty)$, constructed by Szász in 1950, known as the Szász operator (see [2]), is given by

$$S_r(f; u_1) = e^{-ru_1} \sum_{k=0}^{\infty} \frac{(ru_1)^k}{k!} f\left(\frac{k}{r}\right), \; u_1 \geq 0, \; r \in \mathbb{N} \text{ and } f \in C[0, \infty). \tag{1}$$

Due to development of the Szász operator, another sequence of positive linear operator constructed by the mathematician Jakimovski and Leviatan in 1969 using the well-known Appell polynomial is given by (see [3])

$$L_r(f; u_1) = \frac{e^{-ru_1}}{C(1)} \sum_{k=0}^{\infty} P_k(ru_1) f\left(\frac{k}{r}\right), \tag{2}$$

where the Appell polynomial is given by $C(v_1)e^{v_1 y} = \sum_{r=0}^{\infty} P_r(y_1)v_1^r$, with the following identities $C(1) \neq 0$, $C(v_1) = \sum_{r=0}^{\infty} c_r v_1^r$, $P_r(u_1) = \sum_{j=0}^{r} c_j \frac{u_1^{r-j}}{(r-j)!}$ $(r \in \mathbb{N})$.

Recently, with the help of the exponential generating function, the Szász operators were introduced by Sucu [4]. These types of ideas are influenced by the generalized Hermite polynomials in terms of the hypergeometric functions (see [5]). Thus, for the classes of all continuous functions f on $[0, \infty)$ and a parameter $\chi \geq 0$, the Szász operators given in another form, known as Szász Dunkl operators, are given by

$$\mathcal{S}_r^*(f; u_1) := \frac{1}{e_\chi(ru_1)} \sum_{k=0}^{\infty} \frac{(ru_1)^k}{\gamma_\chi(k)} f\left(\frac{k + 2\chi\theta_k}{r}\right), \quad u_1 \geq 0, \quad r \in \mathbb{N}, \tag{3}$$

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where the exponential generating functions are given by

$$e_{\chi}(u_1) = \sum_{k=0}^{\infty} \frac{u_1^k}{\gamma_{\chi}(k)}. \tag{4}$$

$$\gamma_{\chi}(2\rho) = \frac{2^{2\rho}\rho!\Gamma\left(\rho+\chi+\frac{1}{2}\right)}{\Gamma\left(\chi+\frac{1}{2}\right)}, \tag{5}$$

$$\gamma_{\chi}(2\rho+1) = \frac{2^{2\rho+1}\rho!\Gamma\left(\rho+\chi+\frac{3}{2}\right)}{\Gamma\left(\chi+\frac{1}{2}\right)}, \tag{6}$$

and for all $\rho = 0,1,2,\dots$ a recursion γ_{χ} is defined as

$$\frac{\gamma_{\chi}(\rho+1)}{(\rho+1+2\chi\theta_{\rho+1})} = \gamma_{\chi}(\rho),$$

where

$$\theta_{\rho} = \begin{cases} 0 & \text{if } \rho = 2r, \quad r \in \{0,2,4,6,\cdots\}, \\ 1 & \text{if } \rho = 2r+1, \quad r \in \{1,3,5,7,\cdots\}. \end{cases} \tag{7}$$

Most recently, a new version of the Szász operators studied by Nasiruzzaman and Aljohani [6], and these types of Szász operators, provide the generalized version of some earlier operators, such as Szász operators, Szász–Jakimovski operators and Szász–Dunkl operators, where the operators are given by, for example, all $f \in C[0,\infty)$, $u_1 \in [0,\infty)$, $r \in \mathbb{N}$, $C(1) \neq 0$ and $\chi \geq 0$,

$$\mathcal{K}_{r,\chi}(f;u_1) = \frac{1}{C(1)e_{\chi}(ru_1)} \sum_{k=0}^{\infty} P_k(ru_1) f\left(\frac{k+2\chi\theta_k}{r}\right). \tag{8}$$

There are many published articles related to these works, for example, those by Kajla et al. [7], Mursaleen et al. [8–11], Mohiuddine et al. [12–16], Nasiruzzaman et al. [6,17–22], Özger et al. [23]. For studies on Bernstein and Szász types operators involving the idea of Chlodowsky and Charlier polynomials, we refer to [24–28].

The organization of this manuscript is as follows: in Section 2, we present a generalization of the operators defined and studied by Nasiruzzaman and Aljohani [6] in bivariate sense using the unbounded sequences α_n, β_m and ξ_m of positive numbers, such that $\lim_{n\to\infty} \frac{\alpha_n}{n} = 0$, $\lim_{m\to\infty} \frac{1}{\xi_m} = 0$ and $\frac{\beta_m}{\xi_m} = 1 + O\left(\frac{1}{\xi_m}\right)$ as $m \to \infty$, and estimation of moments and central moments. In Section 3, we discuss the rate of convergence by means of the weighted modulus of continuity and weighted approximation properties. In this section, we obtain the degree of convergence using Lipschitz-type maximal functions of two variables, as well as the direct theorem. In Section 4, we close the paper and provide conclusions.

2. Construction of New Operators and Estimation of Moments

Here, we construct the operators and prove some auxillary lemmas, which will be used to prove the approximation results.

Let $\mathcal{I}_{\alpha_n} = \{(x_1,y_1) : 0 \leq x_1 \leq \alpha_n,\ y_1 \in [0,\infty)\}$ and $C(\mathcal{I}_{\alpha_n}) = \{f : \mathcal{I}_{\alpha_n} \to \mathbb{R} \text{ is continuous}\}$, such that it satisfies the norm equipped by

$$||f||_{C(\mathcal{I}_{\alpha_n})} = \sup_{(x_1,y_1)\in\mathcal{I}_{\alpha_n}} |f(x_1,y_1)|.$$

Then, for all $f \in C(\mathcal{I}_{\alpha_n})$ and $n, m \in \mathbb{N}$, and any $\nu, \tau \geq 0$, we define

$$B_{n,m}^{\nu,\tau}(f; x_1, y_1) := \frac{1}{e_\nu\left(\frac{nx_1}{\alpha_n}\right)e_\tau(\beta_m y_1)C(1)D(1)} \sum_{k,l=0}^{\infty} P_k\left(\frac{nx_1}{\alpha_n}\right) \times Q_l(\beta_m y_1) f\left(\frac{\alpha_n(k+2\nu\theta_k)}{n}, \frac{l+2\tau\theta_l}{\xi_m}\right), \tag{9}$$

where α_n, β_m and ξ_m are the unbounded sequences of positive numbers, such that $\lim_{n\to\infty} \frac{\alpha_n}{n} = 0$, $\lim_{m\to\infty} \frac{1}{\xi_m} = 0$ and $\frac{\beta_m}{\xi_m} = 1 + O\left(\frac{1}{\xi_m}\right)$ as $m \to \infty$. Moreover, the Appell polynomial $D(u_1)$ is given by $D(u_1)e^{uy} = \sum_{r=0}^{\infty} Q_l(y_1)u_1^r$ with the following identities: $D(1) \neq 0$, $D(u_1) = \sum_{r=0}^{\infty} c_r u_1^r$, $Q_l(y_1) = \sum_{i=0}^{l} c_j \frac{y_1^{l-i}}{(l-i)!}$ $(l \in \mathbb{N})$ and $P_k(x_1)$ and $C(1) \neq 0$.

Lemma 1. *For all $y_1 \in [0, \infty)$, $P_l(y_1) \geq 0$, $\tau \geq 0$ and $D(1) \neq 0$, if we define*

$$D(\alpha)e_\tau(\alpha y_1) = \sum_{l=0}^{\infty} Q_l(y_1)\alpha^l, \tag{10}$$

then, for any unbounded sequence β_m, we have

$$\sum_{l=0}^{\infty} Q_l(\beta_m y_1) = D(1)e_\tau(\beta_m y_1), \tag{11}$$

$$\sum_{l=0}^{\infty} lQ_l(\beta_m y_1) = \Big(D'(1) + \beta_m y_1 D(1)\Big)e_\tau(\beta_m y_1), \tag{12}$$

$$\sum_{l=0}^{\infty} l^2 Q_l(\beta_m y_1) = \Big(D''(1) + (2\beta_m y_1 + 1)D'(1) + \beta_m y_1(\beta_m y_1 + 1)D(1)\Big)e_\tau(\beta_m y_1), \tag{13}$$

$$\begin{aligned}\sum_{l=0}^{\infty} l^3 Q_l(\beta_m y_1) &= \Big(D'''(1) + 3(\beta_m y_1 + 1)D''(1) + (3\beta_m^2 y_1^2 + 6\beta_m y_1 + 2)D'(1) \\ &\quad + \beta_m y_1(\beta_m^2 y_1^2 + 3\beta_m y_1 + 2)D(1)\Big)e_\tau(\beta_m y_1),\end{aligned}$$

$$\begin{aligned}\sum_{l=0}^{\infty} l^4 Q_l(\beta_m y_1) &= \Big(D''''(1) + (4\beta_m y_1 + 6)D'''(1) + (6\beta_m^2 y_1^2 + 18\beta_m y_1 + 11)D''(1) \\ &\quad + (4\beta_m^3 y_1^3 + 18\beta_m^2 y_1^2 + 22\beta_m y_1 + 6)D'(1) \\ &\quad + \beta_m y_1(\beta_m^3 y_1^3 + 6\beta_m^2 y_1^2 + 11\beta_m y_1 + 6)D(1)\Big)e_\tau(\beta_m y_1).\end{aligned}$$

Lemma 2. *For all $x_1 \in [0, \infty)$, $P_k(x_1) \geq 0$, $\nu \geq 0$ and $C(1) \neq 0$, if we define*

$$C(\beta)e_\nu(\beta x_1) = \sum_{k=0}^{\infty} P_k(x_1)\beta^k, \tag{14}$$

then, for any unbounded sequence of positive numbers α_n, we have

$$\sum_{k=0}^{\infty} P_k\left(\frac{nx_1}{\alpha_n}\right) = C(1)e_\nu\left(\frac{nx_1}{\alpha_n}\right), \tag{15}$$

$$\sum_{k=0}^{\infty} kP_k\left(\frac{nx_1}{\alpha_n}\right) = \left(C'(1) + \left(\frac{nx_1}{\alpha_n}\right)C(1)\right)e_\nu\left(\frac{nx_1}{\alpha_n}\right), \tag{16}$$

$$\begin{aligned}\sum_{k=0}^{\infty} k^2P_k\left(\frac{nx_1}{\alpha_n}\right) &= \left(C''(1) + \left(2\left(\frac{nx_1}{\alpha_n}\right) + 1\right)C'(1)\right. \\ &\left. + \left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right) + 1\right)C(1)\right)e_\nu\left(\frac{nx_1}{\alpha_n}\right),\end{aligned} \tag{17}$$

$$\begin{aligned}\sum_{k=0}^{\infty} k^3P_k\left(\frac{nx_1}{\alpha_n}\right) &= \left(C'''(1) + 3\left(\left(\frac{nx_1}{\alpha_n}\right) + 1\right)C''(1)\right. \\ &+ \left(3\left(\frac{nx_1}{\alpha_n}\right)^2 + 6\left(\frac{nx_1}{\alpha_n}\right) + 2\right)C'(1) \\ &\left. + \left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right)^2 + 3\left(\frac{nx_1}{\alpha_n}\right) + 2\right)C(1)\right)e_\nu\left(\frac{nx_1}{\alpha_n}\right),\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{\infty} k^4P_k\left(\frac{nx_1}{\alpha_n}\right) &= \left(C''''(1) + \left(4\left(\frac{nx_1}{\alpha_n}\right) + 6\right)C'''(1)\right. \\ &+ \left(6\left(\frac{nx_1}{\alpha_n}\right)^2 + 18\left(\frac{nx_1}{\alpha_n}\right) + 11\right)C''(1) \\ &+ \left(4\left(\frac{nx_1}{\alpha_n}\right)^3 + 18\left(\frac{nx_1}{\alpha_n}\right)^2 + 22\left(\frac{nx_1}{\alpha_n}\right) + 6\right)C'(1) \\ &\left. + \left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right)^3 + 6\left(\frac{nx_1}{\alpha_n}\right)^2 + 11\left(\frac{nx_1}{\alpha_n}\right) + 6\right)C(1)\right)e_\nu\left(\frac{nx_1}{\alpha_n}\right).\end{aligned}$$

Lemma 3. *For the operators $B_{n,m}^{\nu,\tau}$, defined by (9), as demonstrated here:*

$$R_n^*(f;x_1,y_1) = \frac{1}{e_\nu\left(\frac{nx_1}{\alpha_n}\right)C(1)} \sum_{k=0}^{\infty} P_k\left(\frac{nx_1}{\alpha_n}\right) f\left(\frac{\alpha_n(k+2\nu\theta_k)}{n}, y_1\right) \tag{18}$$

$$S_m(f;x_1,y_1) = \frac{1}{e_\tau(\beta_m y_1)D(1)} \sum_{l=0}^{\infty} Q_l(\beta_m y_1) f\left(x_1, \frac{l+2\tau\theta_l}{\xi_m}\right), \tag{19}$$

then

$$B_{n,m}^{\nu,\tau}(f;x_1,y_1) = R_n^*(S_m(f;x_1,y_1)) = S_m(R_n^*(f;x_1,y_1)).$$

Proof. We can easily see that

$$\begin{aligned}R_n^*(S_m(f;x_1,y_1)) &= R_n^*\left(\frac{1}{e_\tau(\beta_m y_1)D(1)} \sum_{l=0}^{\infty} Q_l(\beta_m y_1) f\left(x_1, \frac{l+2\tau\theta_l}{\xi_m}\right)\right) \\ &= \frac{1}{e_\tau(\beta_m y_1)D(1)} \sum_{l=0}^{\infty} R_n^*\left(f\left(x_1, \frac{l+2\tau\theta_l}{\xi_m}\right)\right) Q_l(\beta_m y_1) \\ &= \frac{1}{e_\tau(\beta_m y_1)D(1)} \sum_{l=0}^{\infty} Q_l(\beta_m y_1) \\ &\quad \times \sum_{k=0}^{\infty} \frac{1}{e_\nu\left(\frac{nx_1}{\alpha_n}\right)} f\left(\frac{\alpha_n(k+2\nu\theta_k)}{n}, \frac{l+2\tau\theta_l}{\xi_m}\right) P_k\left(\frac{nx_1}{\alpha_n}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{e_\nu\left(\frac{nx_1}{\alpha_n}\right)e_\tau(\beta_m y_1)C(1)D(1)}\sum_{k,l=0}^{\infty}P_k\left(\frac{nx_1}{\alpha_n}\right)Q_l(\beta_m y_1)\\
&\quad\times f\left(\frac{\alpha_n(k+2\nu\theta_k)}{n},\frac{l+2\tau\theta_l}{\xi_m}\right)\\
&= B_{n,m}^{\nu,\tau}(f;x_1,y_1).
\end{aligned}
$$

Similarly, we prove $S_m(R_n^*(f;x_1,y_1)) = B_{n,m}^{\nu,\tau}(f;x_1,y_1)$. $\square$

Let $\mathbb{R}_+^2 = \mathbb{R}_+\times\mathbb{R}_+$ with $\mathbb{R}_+ = [0,\infty)$, and suppose γ_m is the sequence, such that $\lim_{m\to\infty}\gamma_m = \infty$ and $\mathcal{I}_{\alpha_n\gamma_m} = \{(x_1,y_1) : 0\le x_1\le\alpha_n,\ 0\le y_1\le\gamma_m\}$. We also consider the operators $T_{n,m}^*(f;x_1,y_1)$, such that

$$
T_{n,m}^*(f;x_1,y_1) = \begin{cases} B_{n,m}^{\nu,\tau}(f;x_1,y_1), & \text{for } (x_1,y_1)\in\mathcal{I}_{\alpha_n\gamma_m}\\ f(x_1,y_1), & \text{for } (x_1,y_1)\in\mathbb{R}_+^2\setminus\mathcal{I}_{\alpha_n\gamma_m}. \end{cases} \tag{20}
$$

Lemma 4. *Let $n,m\in\mathbb{N}$ and $\phi_{j,i} = u_1^j v_1^i$ for any $j,i = 0,1,2,3,4$, then, we have the following identities:*

$$
\begin{aligned}
(1)\quad B_{n,m}^{\nu,\tau}(\phi_{0,0};x_1,y_1) &= 1;\\
(2)\quad B_{n,m}^{\nu,\tau}(\phi_{1,0};x_1,y_1) &= x_1+\frac{\alpha_n}{n}\left(\frac{C'(1)}{C(1)}+2\nu\right);\\
(3)\quad B_{n,m}^{\nu,\tau}(\phi_{0,1};x_1,y_1) &= \frac{1}{\xi_m}\left(\frac{D'(1)}{D(1)}+\beta_m y_1\right)+\frac{2\tau}{\xi_m};\\
(4)\quad B_{n,m}^{\nu,\tau}(\phi_{2,0};x_1,y_1) &= x_1^2+\frac{\alpha_n}{n}\left(2\frac{C'(1)}{C(1)}+2\nu+1\right)x_1\\
&\quad+\left(\frac{\alpha_n}{n}\right)^2\left(\frac{C''(1)}{C(1)}+\frac{C'(1)}{C(1)}+4\nu^2\right);\\
(5)\quad B_{n,m}^{\nu,\tau}(\phi_{0,2};x_1,y_1) &= \frac{1}{\xi_m^2}\left[\frac{D''(1)}{D(1)}+(2\beta_m y_1+1)\frac{D'(1)}{D(1)}+\beta_m y_1(\beta_m y_1+1)\right]\\
&\quad+\frac{2\tau}{\xi_m^2}\left[\frac{D'(1)}{D(1)}+\beta_m y_1\right]+\frac{4\tau^2}{\xi_m^2};\\
(6)\quad B_{n,m}^{\nu,\tau}(\phi_{3,0};x_1,y_1) &= \frac{\alpha_n^3}{n^3}\left[\frac{C'''(1)}{C(1)}+3\left(\left(\frac{nx_1}{\alpha_n}\right)+1\right)\frac{C''(1)}{C(1)}\right.\\
&\quad+\left(3\left(\frac{nx_1}{\alpha_n}\right)^2+6\left(\frac{nx_1}{\alpha_n}\right)+2\right)\frac{C'(1)}{C(1)}\\
&\quad\left.+\left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right)^2+3\left(\frac{nx_1}{\alpha_n}\right)+2\right)\right]\\
&\quad+\frac{6\nu\alpha_n^3}{n^3}\left[\frac{C''(1)}{C(1)}+\left(2\left(\frac{nx_1}{\alpha_n}\right)+1\right)\frac{C'(1)}{C(1)}\right.\\
&\quad\left.+\left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right)+1\right)\right]\\
&\quad+\frac{12\nu^2\alpha_n^3}{n^3}\left[\frac{C'(1)}{C(1)}+\left(\frac{nx_1}{\alpha_n}\right)\right]+\frac{8\nu^3\alpha_n^3}{n^3};\\
(7)\quad B_{n,m}^{\nu,\tau}(\phi_{0,3};x_1,y_1) &= \frac{1}{\xi_m^3}\left[\frac{D'''(1)}{D(1)}+3(\beta_m y_1+1)\frac{D''(1)}{D(1)}\right.\\
&\quad+(3\beta_m^2y_1^2+6\beta_m y_1+2)\frac{D'(1)}{D(1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\beta_m y_1(\beta_m^2 y_1^2+3\beta_m y_1+2)\Big]+\frac{6\tau}{\xi_m^3}\left[\frac{D''(1)}{D(1)}\right.\\
&\left.+(2\beta_m y_1+1)\frac{D'(1)}{D(1)}+\beta_m y_1(\beta_m y_1+1)\right]\\
&+\frac{6\tau^2}{\xi_m^3}\left[\frac{D'(1)}{D(1)}+\beta_m y_1\right]+\frac{8\tau^3}{\xi_m^3};
\end{aligned}
$$

$$
\begin{aligned}
(8)\quad B_{n,m}^{\nu,\tau}(\phi_{4,0};x_1,y_1) \;=\;& \frac{\alpha_n^4}{n^4}\left[\left(\frac{C''''(1)}{C(1)}+\left(4\left(\frac{nx_1}{\alpha_n}\right)+6\right)\frac{C'''(1)}{C(1)}+\left(6\left(\frac{nx_1}{\alpha_n}\right)^2\right.\right.\right.\\
&\left.+18\left(\frac{nx_1}{\alpha_n}\right)+11\right)\frac{C''(1)}{C(1)}\\
&+\left(4\left(\frac{nx_1}{\alpha_n}\right)^3+18\left(\frac{nx_1}{\alpha_n}\right)^2+22\left(\frac{nx_1}{\alpha_n}\right)+6\right)\frac{C'(1)}{C(1)}\\
&\left.+\left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right)^3+6\left(\frac{nx_1}{\alpha_n}\right)^2+11\left(\frac{nx_1}{\alpha_n}\right)+6\right)\right]\\
&+\frac{8\nu\alpha_n^4}{n^4}\left[\left(\frac{C'''(1)}{C(1)}+3\left(\left(\frac{nx_1}{\alpha_n}\right)+1\right)\frac{C''(1)}{C(1)}\right.\right.\\
&+\left(3\left(\frac{nx_1}{\alpha_n}\right)^2+6\left(\frac{nx_1}{\alpha_n}\right)+2\right)\frac{C'(1)}{C(1)}\\
&\left.+\left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right)^2+3\left(\frac{nx_1}{\alpha_n}\right)+2\right)\right]\\
&+\frac{24\nu^2\alpha_n^4}{n^4}\left[\left(\frac{C''(1)}{C(1)}+\left(2\left(\frac{nx_1}{\alpha_n}\right)+1\right)\frac{C'(1)}{C(1)}\right.\right.\\
&\left.+\left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right)+1\right)\right]\\
&+\frac{32\nu^3\alpha_n^4}{n^4}\left[\left(\frac{C'(1)}{C(1)}+\left(\frac{nx_1}{\alpha_n}\right)\right)\right]+\frac{16\nu^4\alpha_n^4}{n^4};
\end{aligned}
$$

$$
\begin{aligned}
(9)\quad B_{n,m}^{\nu,\tau}(\phi_{0,4};x_1,y_1) \;=\;& \frac{1}{\xi_m^4}\left[\frac{D''''(1)}{D(1)}+(4\beta_m y_1+6)\frac{D'''(1)}{D(1)}\right.\\
&+(6\beta_m^2 y_1^2+18\beta_m y_1+11)\frac{D''(1)}{D(1)}\\
&+(4\beta_m^3 y_1^3+18\beta_m^2 y_1^2+22\beta_m y_1+6)\frac{D'(1)}{D(1)}\\
&\left.+\beta_m y_1(\beta_m^3 y_1^3+6\beta_m^2 y_1^2+11\beta_m y_1+6)\right]\\
&+\frac{8\tau}{\xi_m^4}\left[\frac{D'''(1)}{D(1)}+3(\beta_m y_1+1)\frac{D''(1)}{D(1)}\right.\\
&+(3\beta_m^2 y_1^2+6\beta_m y_1+2)\frac{D'(1)}{D(1)}\\
&\left.+\beta_m y_1(\beta_m^2 y_1^2+3\beta_m y_1+2)\right]\\
&+\frac{24\tau^2}{\xi_m^4}\left[\frac{D''(1)}{D(1)}+(2\beta_m y_1+1)\frac{D'(1)}{D(1)}+\beta_m y_1(\beta_m y_1+1)\right]\\
&+\frac{32\tau^3}{\xi_m^4}\left[\frac{D'(1)}{D(1)}+\beta_m y_1\right]+\frac{16\tau^4}{\xi_m^3}.
\end{aligned}
$$

Proof. Taking $j = i = 0$ and using the operators (9), we have

$$\begin{aligned}B_{n,m}^{\nu,\tau}(\phi_{0,0};x_1,y_1) &= \frac{1}{e_\nu\left(\frac{nx_1}{\alpha_n}\right)e_\tau(\beta_m y_1)C(1)D(1)}\sum_{k,l=0}^{\infty}P_k\left(\frac{nx_1}{\alpha_n}\right)Q_l(\beta_m y_1)\\ &= 1.\end{aligned}$$

Taking $j = 1$ and $i = 0$, and using the operators (9), we have

$$\begin{aligned}B_{n,m}^{\nu,\tau}(\phi_{1,0};x_1,y_1) &= \frac{1}{e_\nu\left(\frac{nx_1}{\alpha_n}\right)e_\tau(\beta_m y_1)C(1)D(1)}\sum_{k,l=0}^{\infty}P_k\left(\frac{nx_1}{\alpha_n}\right)Q_l(\beta_m y_1)\frac{\alpha_n(k+2\nu\theta_k)}{n}\\ &= \left(\frac{\alpha_n}{ne_\nu\left(\frac{nx_1}{\alpha_n}\right)C(1)}\sum_{k=0}^{\infty}kP_k\left(\frac{nx_1}{\alpha_n}\right)\right)\left(\frac{1}{e_\tau(\beta_m y_1)D(1)}\sum_{l=0}^{\infty}Q_l(\beta_m y_1)\right)\\ &\quad+\left(\frac{1}{e_\tau(\beta_m y_1)D(1)}\sum_{l=0}^{\infty}Q_l(\beta_m y_1)\right)\\ &\quad\times\left(\frac{2\nu\alpha_n}{ne_\nu\left(\frac{nx_1}{\alpha_n}\right)C(1)}\sum_{k=2s+1}^{\infty}\theta_kP_k\left(\frac{nx_1}{\alpha_n}\right)\right),\ s\in\mathbb{N}\cup\{0\}\\ &= \frac{\alpha_n}{ne_\nu\left(\frac{nx_1}{\alpha_n}\right)C(1)}\left(C'(1)+\left(\frac{nx_1}{\alpha_n}\right)C(1)\right)e_\nu\left(\frac{nx_1}{\alpha_n}\right)+\frac{2\nu\alpha_n}{n}\\ &= x_1+\frac{\alpha_n}{n}\left(\frac{C'(1)}{C(1)}+2\nu\right).\end{aligned}$$

Taking $j = 2$ and $i = 0$, and using the operators (9), we have

$$\begin{aligned}B_{n,m}^{\nu,\tau}(\phi_{2,0};x_1,y_1) &= \frac{1}{e_\nu\left(\frac{nx_1}{\alpha_n}\right)e_\tau(\beta_m y_1)C(1)D(1)}\sum_{k,l=0}^{\infty}P_k\left(\frac{nx_1}{\alpha_n}\right)Q_l(\beta_m y_1)\frac{\alpha_n^2(k+2\nu\theta_k)^2}{n^2}\\ &= \left(\frac{1}{e_\tau(\beta_m y_1)D(1)}\sum_{l=0}^{\infty}Q_l(\beta_m y_1)\right)\left[\frac{\alpha_n^2}{n^2e_\nu\left(\frac{nx_1}{\alpha_n}\right)C(1)}\sum_{k=0}^{\infty}k^2P_k\left(\frac{nx_1}{\alpha_n}\right)\right.\\ &\quad+\frac{2\nu\alpha_n^2}{n^2e_\nu\left(\frac{nx_1}{\alpha_n}\right)C(1)}\sum_{k=2s+1}^{\infty}kP_k\left(\frac{nx_1}{\alpha_n}\right)\theta_k\\ &\quad\left.+\frac{4\nu^2\alpha_n^2}{n^2e_\nu\left(\frac{nx_1}{\alpha_n}\right)C(1)}\sum_{k=2s+1}^{\infty}P_k\left(\frac{nx_1}{\alpha_n}\right)\theta_k^2\right],\ s\in\mathbb{N}\cup\{0\}\\ &= \frac{\alpha_n^2}{n^2e_\nu\left(\frac{nx_1}{\alpha_n}\right)C(1)}\left[C''(1)+\left(2\left(\frac{nx_1}{\alpha_n}\right)+1\right)C'(1)\right.\\ &\quad\left.+\left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right)+1\right)C(1)\right]e_\nu\left(\frac{nx_1}{\alpha_n}\right)\\ &\quad+\frac{2\nu\alpha_n^2}{n^2e_\nu\left(\frac{nx_1}{\alpha_n}\right)C(1)}\left[C'(1)+\left(\frac{nx_1}{\alpha_n}\right)C(1)\right]e_\nu\left(\frac{nx_1}{\alpha_n}\right)\\ &\quad+\frac{4\nu^2\alpha_n^2}{n^2e_\nu\left(\frac{nx_1}{\alpha_n}\right)C(1)}C(1)e_\nu\left(\frac{nx_1}{\alpha_n}\right)\\ &= \frac{1}{n^2}\left[\frac{\alpha_n^2C''(1)}{C(1)}+\left(2n\alpha_nx_1+\alpha_n^2\right)\frac{C'(1)}{C(1)}+n^2x_1^2+n\alpha_nx_1\right]\\ &\quad+\frac{2\nu\alpha_n}{n^2}\left[\frac{\alpha_nC'(1)}{C(1)}+nx_1\right]+\frac{4\nu^2\alpha_n^2}{n^2}.\end{aligned}$$

Taking $j=3$ and $i=0$, and using the operators (9), we have

$$\begin{aligned}
B_{n,m}^{\nu,\tau}(\phi_{3,0};x_1,y_1) &= \frac{1}{e_\nu\left(\frac{nx_1}{\alpha_n}\right)e_\tau(\beta_m y_1)C(1)D(1)}\sum_{k,l=0}^{\infty}P_k\left(\frac{nx_1}{\alpha_n}\right)Q_l(\beta_m y_1)\frac{\alpha_n^3(k+2\nu\theta_k)^3}{n^3}\\
&= \left(\frac{1}{e_\tau(\beta_m y_1)D(1)}\frac{\alpha_n^3}{n^3 e_\nu\left(\frac{nx_1}{\alpha_n}\right)C(1)}\sum_{l=0}^{\infty}Q_l(\beta_m y_1)\right)\\
&\quad\times\left[\sum_{k=0}^{\infty}k^3P_k\left(\frac{nx_1}{\alpha_n}\right)+6\nu\sum_{k=2s+1}^{\infty}k^2P_k\left(\frac{nx_1}{\alpha_n}\right)\theta_k\right.\\
&\quad\left.+12\nu^2\sum_{k=2s+1}^{\infty}kP_k\left(\frac{nx_1}{\alpha_n}\right)\theta_k^2+8\nu^3\sum_{k=2s+1}^{\infty}P_k\left(\frac{nx_1}{\alpha_n}\right)\theta_k^3\right],\ s\in\mathbb{N}\cup\{0\}\\
&= \frac{\alpha_n^3}{n^3}\left[\frac{C'''(1)}{C(1)}+3\left(\left(\frac{nx_1}{\alpha_n}\right)+1\right)\frac{C''(1)}{C(1)}+\left(3\left(\frac{nx_1}{\alpha_n}\right)^2\right.\right.\\
&\quad\left.\left.+6\left(\frac{nx_1}{\alpha_n}\right)+2\right)\frac{C'(1)}{C(1)}+\left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right)^2+3\left(\frac{nx_1}{\alpha_n}\right)+2\right)\right]\\
&\quad+\frac{6\nu\alpha_n^3}{n^3}\left[\frac{C''(1)}{C(1)}+\left(2\left(\frac{nx_1}{\alpha_n}\right)+1\right)\frac{C'(1)}{C(1)}+\left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right)+1\right)\right]\\
&\quad+\frac{12\nu^2\alpha_n^3}{n^3}\left[\frac{C'(1)}{C(1)}+\left(\frac{nx_1}{\alpha_n}\right)\right]+\frac{8\nu^3\alpha_n^3}{n^3}.
\end{aligned}$$

Taking $j=4$ and $i=0$, and using the operators (9), we have

$$\begin{aligned}
B_{n,m}^{\nu,\tau}(\phi_{4,0};x_1,y_1) &= \frac{1}{e_\nu\left(\frac{nx_1}{\alpha_n}\right)e_\tau(\beta_m y_1)C(1)D(1)}\sum_{k,l=0}^{\infty}P_k\left(\frac{nx_1}{\alpha_n}\right)Q_l(\beta_m y_1)\frac{\alpha_n^4(k+2\nu\theta_k)^4}{n^4}\\
&= \left(\frac{1}{e_\tau(\beta_m y_1)D(1)}\frac{\alpha_n^4}{n^4 e_\nu\left(\frac{nx_1}{\alpha_n}\right)C(1)}\sum_{l=0}^{\infty}Q_l(\beta_m y_1)\right)\\
&\quad\left[\sum_{k=0}^{\infty}k^4P_k\left(\frac{nx_1}{\alpha_n}\right)+8\nu\sum_{k=2s+1}^{\infty}k^3P_k\left(\frac{nx_1}{\alpha_n}\right)\theta_k,\ s\in\mathbb{N}\cup\{0\}\right.\\
&\quad+24\nu^2\sum_{k=2s+1}^{\infty}k^2P_k\left(\frac{nx_1}{\alpha_n}\right)\theta_k^2,\ s\in\mathbb{N}\cup\{0\}\\
&\quad+32\nu^3\sum_{k=2s+1}^{\infty}kP_k\left(\frac{nx_1}{\alpha_n}\right)\theta_k^2,\ s\in\mathbb{N}\cup\{0\}\\
&\quad\left.+16\nu^4\sum_{k=2s+1}^{\infty}P_k\left(\frac{nx_1}{\alpha_n}\right)\theta_k^3,\ s\in\mathbb{N}\cup\{0\}\right]\\
&= \frac{\alpha_n^4}{n^4}\left[\frac{C''''(1)}{C(1)}+\left(4\left(\frac{nx_1}{\alpha_n}\right)+6\right)\frac{C'''(1)}{C(1)}\right.\\
&\quad+\left(6\left(\frac{nx_1}{\alpha_n}\right)^2+18\left(\frac{nx_1}{\alpha_n}\right)+11\right)\frac{C''(1)}{C(1)}\\
&\quad+\left(4\left(\frac{nx_1}{\alpha_n}\right)^3+18\left(\frac{nx_1}{\alpha_n}\right)^2+22\left(\frac{nx_1}{\alpha_n}\right)+6\right)\frac{C'(1)}{C(1)}\\
&\quad+\left(\left(\frac{nx_1}{\alpha_n}\right)^4+6\left(\frac{nx_1}{\alpha_n}\right)^3+11\left(\frac{nx_1}{\alpha_n}\right)^2+6\left(\frac{nx_1}{\alpha_n}\right)\right)\\
&\quad+8\nu\frac{\alpha_n^4}{n^4}\left[\frac{C'''(1)}{C(1)}+3\left(\left(\frac{nx_1}{\alpha_n}\right)+1\right)\frac{C''(1)}{C(1)}\right.\\
&\quad+\left(3\left(\frac{nx_1}{\alpha_n}\right)^2+6\left(\frac{nx_1}{\alpha_n}\right)+2\right)\frac{C'(1)}{C(1)}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right)^2+3\left(\frac{nx_1}{\alpha_n}\right)+2\right)\Bigg]\\
&+24\nu^2\frac{\alpha_n^4}{n^4}\left[\frac{C''(1)}{C(1)}+\left(2\left(\frac{nx_1}{\alpha_n}\right)+1\right)\frac{C'(1)}{C(1)}\right.\\
&\left.+\left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right)+1\right)\right]\\
&+2\nu^3\frac{\alpha_n^4}{n^4}\left[\frac{C'(1)}{C(1)}+\left(\frac{nx_1}{\alpha_n}\right)\right]+16\nu^4\frac{\alpha_n^4}{n^4}.
\end{aligned}
$$

Taking $j=0$ and $i=1$, and using the operators (9), we have

$$
\begin{aligned}
B_{n,m}^{\nu,\tau}(\phi_{0,1};x_1,y_1) &= \frac{1}{e_\nu\left(\frac{nx_1}{\alpha_n}\right)e_\tau(\beta_m y_1)C(1)D(1)}\sum_{k,l=0}^{\infty}P_k\left(\frac{nx_1}{\alpha_n}\right)Q_l(\beta_m y_1)\left(\frac{l+2\tau\theta_l}{\xi_m}\right)\\
&= \left(\frac{1}{e_\nu\left(\frac{nx_1}{\alpha_n}\right)C(1)}\sum_{k=0}^{\infty}P_k\left(\frac{nx_1}{\alpha_n}\right)\right)\left(\frac{1}{\xi_m e_\tau(\beta_m y_1)D(1)}\sum_{l=0}^{\infty}lQ_l(\beta_m y_1)\right)\\
&\quad+\left(\frac{1}{e_\nu\left(\frac{nx_1}{\alpha_n}\right)C(1)}\sum_{k=0}^{\infty}P_k\left(\frac{nx_1}{\alpha_n}\right)\right)\\
&\quad\times\left(\frac{2\tau}{\xi_m e_\tau(\beta_m y_1)D(1)}\sum_{l=2s+1}^{\infty}Q_l(\beta_m y_1)\right),\ s\in\mathbb{N}\cup\{0\}\\
&= \frac{1}{\xi_m}\left(\frac{D'(1)}{D(1)}+\beta_m y_1\right)+\frac{2\tau}{\xi_m}.
\end{aligned}
$$

In a similar way, other identities can be easily proved. □

With the help of the above lemma, we can calculate the central moments as follows:

Lemma 5. *Let $C(1)\neq 0$ and $D(1)\neq 0$. For all $n,m\in\mathbb{N}$, the operators $B_{n,m}^{\nu,\tau}(.\,;\,.)$ have the following central moments:*

$$
\begin{aligned}
(1)\quad B_{n,m}^{\nu,\tau}((u_1-x_1);x_1,y_1) &= \frac{\alpha_n}{n}\left(\frac{C'(1)}{C(1)}+2\nu\right);\\
(2)\quad B_{n,m}^{\nu,\tau}((v_1-y_1);x_1,y_1) &= \left(\frac{\beta_m}{\xi_m}-1\right)y_1+\frac{1}{\xi_m}\left(\frac{D'(1)}{D(1)}+2\tau\right);\\
(3)\quad B_{n,m}^{\nu,\tau}((u_1-x_1)^2;x_1,y_1) &= \frac{\alpha_n}{n}(1-2\nu)x_1+\left(\frac{\alpha_n}{n}\right)^2\left(\frac{C''(1)}{C(1)}+\frac{C'(1)}{C(1)}+4\nu^2\right);\\
(4)\quad B_{n,m}^{\nu,\tau}((v_1-y_1)^2;x_1,y_1) &= \left[\frac{\beta_m^2}{\xi_m^2}-2\frac{\beta_m}{\xi_m}+1\right]y_1^2\\
&\quad+\left[\frac{\beta_m}{\xi_m^2}+\left(\frac{2\beta_m}{\xi_m^2}-\frac{2}{\xi_m}\right)\frac{D'(1)}{D(1)}+\frac{2\tau\beta_m}{\xi_m^2}-\frac{2\tau}{\xi_m}\right]y_1\\
&\quad+\frac{1}{\xi_m^2}\left[\frac{D''(1)}{D(1)}+(1+2\tau)\frac{D'(1)}{D(1)}+4\tau^2\right];\\
(5)\quad B_{n,m}^{\nu,\tau}((u_1-x_1)^4;x_1,y_1) &= \frac{\alpha_n^4}{n^4}\left[\left(\frac{C'^{v_1}(1)}{C(1)}+\left(4\left(\frac{nx_1}{\alpha_n}\right)+6\right)\frac{C'''(1)}{C(1)}\right.\right.\\
&\quad+\left(6\left(\frac{nx_1}{\alpha_n}\right)^2+18\left(\frac{nx_1}{\alpha_n}\right)+11\right)\frac{C''(1)}{C(1)}\\
&\quad+\left(4\left(\frac{nx_1}{\alpha_n}\right)^3+18\left(\frac{nx_1}{\alpha_n}\right)^2+22\left(\frac{nx_1}{\alpha_n}\right)+6\right)\frac{C'(1)}{C(1)}
\end{aligned}
$$

$$
\begin{aligned}
& +\left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right)^3+6\left(\frac{nx_1}{\alpha_n}\right)^2+11\left(\frac{nx_1}{\alpha_n}\right)+6\right)\Bigg] \\
& +\frac{8\nu\alpha_n^4}{n^4}\left[\left(\frac{C'''(1)}{C(1)}+3\left(\left(\frac{nx_1}{\alpha_n}\right)+1\right)\frac{C''(1)}{C(1)}\right.\right. \\
& +\left(3\left(\frac{nx_1}{\alpha_n}\right)^2+6\left(\frac{nx_1}{\alpha_n}\right)+2\right)\frac{C'(1)}{C(1)} \\
& +\left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right)^2+3\left(\frac{nx_1}{\alpha_n}\right)+2\right)\Bigg] \\
& +\frac{24\nu^2\alpha_n^4}{n^4}\left[\left(\frac{C''(1)}{C(1)}+\left(2\left(\frac{nx_1}{\alpha_n}\right)+1\right)\frac{C'(1)}{C(1)}\right.\right. \\
& +\left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right)+1\right)\Bigg] \\
& +\frac{32\nu^3\alpha_n^4}{n^4}\left[\left(\frac{C'(1)}{C(1)}+\left(\frac{nx_1}{\alpha_n}\right)\right)\right]+\frac{16\nu^4\alpha_n^4}{n^4} \\
& -\frac{4\alpha_n^3x_1}{n^3}\left[\frac{C'''(1)}{C(1)}+3\left(\left(\frac{nx_1}{\alpha_n}\right)+1\right)\frac{C''(1)}{C(1)}\right. \\
& +\left(3\left(\frac{nx_1}{\alpha_n}\right)^2+6\left(\frac{nx_1}{\alpha_n}\right)+2\right)\frac{C'(1)}{C(1)} \\
& +\left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right)^2+3\left(\frac{nx_1}{\alpha_n}\right)+2\right)\Bigg] \\
& -\frac{24\nu\alpha_n^3x_1}{n^3}\left[\frac{C''(1)}{C(1)}+\left(2\left(\frac{nx_1}{\alpha_n}\right)+1\right)\frac{C'(1)}{C(1)}\right. \\
& +\left(\frac{nx_1}{\alpha_n}\right)\left(\left(\frac{nx_1}{\alpha_n}\right)+1\right)\Bigg] \\
& +\frac{12\nu^2\alpha_n^3}{n^3}\left[\frac{C'(1)}{C(1)}+\left(\frac{nx_1}{\alpha_n}\right)\right]-\frac{32\nu^3\alpha_n^3x_1}{n^3} \\
& +6x_1^4+\frac{6\alpha_n}{n}\left(2\frac{C'(1)}{C(1)}+2\nu+1\right)x_1^3 \\
& +6\left(\frac{\alpha_n}{n}\right)^2\left(\frac{C''(1)}{C(1)}+\frac{C'(1)}{C(1)}+4\nu^2\right)x_1^2 \\
& -4x_1^4-\frac{4\alpha_n}{n}\left(\frac{C'(1)}{C(1)}+2\nu\right)x_1^3+x_1^4;
\end{aligned}
$$

$$
\begin{aligned}
(6)\quad B_{n,m}^{\nu,\tau}((v_1-y_1)^4;x_1,y_1) \quad = \quad & \frac{1}{\zeta_m^4}\left[\frac{D'^{v_1}(1)}{D(1)}+(4\beta_m y_1+6)\frac{D'''(1)}{D(1)}\right. \\
& +(6\beta_m^2y_1^2+18\beta_m y_1+11)\frac{D''(1)}{D(1)} \\
& +(4\beta_m^3y_1^3+18\beta_m^2y_1^2+22\beta_m y_1+6)\frac{D'(1)}{D(1)} \\
& +\beta_m y_1(\beta_m^3y_1^3+6\beta_m^2y_1^2+11\beta_m y_1+6)\Bigg] \\
& +\frac{8\tau}{\zeta_m^4}\left[\frac{D'''(1)}{D(1)}+3(\beta_m y_1+1)\frac{D''(1)}{D(1)}+(3\beta_m^2y_1^2\right. \\
& +6\beta_m y_1+2)\frac{D'(1)}{D(1)}+\beta_m y_1(\beta_m^2y_1^2+3\beta_m y_1+2)\Bigg] \\
& +\frac{24\tau^2}{\zeta_m^4}\left[\frac{D''(1)}{D(1)}+(2\beta_m y_1+1)\frac{D'(1)}{D(1)}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\beta_m y_1(\beta_m y_1+1)\Big]+\frac{32\tau^3}{\xi_m^4}\left[\frac{D'(1)}{D(1)}+\beta_m y_1\right]+\frac{16\tau^4}{\xi_m^3}\\
&-\frac{4y_1}{\xi_m^3}\left[\frac{D'''(1)}{D(1)}+3(\beta_m y_1+1)\frac{D''(1)}{D(1)}+(3\beta_m^2 y_1^2\right.\\
&\left.+6\beta_m y_1+2)\frac{D'(1)}{D(1)}+\beta_m y_1(\beta_m^2 y_1^2+3\beta_m y_1+2)\right]\\
&-\frac{24\tau y_1}{\xi_m^3}\left[\frac{D''(1)}{D(1)}+(2\beta_m y_1+1)\frac{D'(1)}{D(1)}\right.\\
&\left.+\beta_m y_1(\beta_m y_1+1)\right]-\frac{24\tau^2 y_1}{\xi_m^3}\left[\frac{D'(1)}{D(1)}+\beta_m y_1\right]\\
&-\frac{32\tau^3 y_1}{\xi_m^3}+\frac{6y_1^2}{\xi_m^2}\left[\frac{D''(1)}{D(1)}+(2\beta_m y_1+1)\frac{D'(1)}{D(1)}\right.\\
&\left.+\beta_m y_1(\beta_m y_1+1)\right]+\frac{12\tau y_1^2}{\xi_m^2}\left[\frac{D'(1)}{D(1)}+\beta_m y_1\right]\\
&+\frac{24\tau^2 y_1^2}{\xi_m^2}-\frac{4y_1^3}{\xi_m}\left(\frac{D'(1)}{D(1)}+\beta_m y_1\right)-\frac{8\tau y_1^3}{\xi_m}+y_1^4.
\end{aligned}
$$

Lemma 6. *Let $x_1, y_1 \in \mathcal{I}_{\alpha_n}$; then, for sufficiently large $n, m \in \mathbb{N}$, we can obtain the following inequalities:*

$$
\begin{aligned}
(1)\quad B_{n,m}^{\nu,\tau}((u_1-x_1)^2;x_1,y_1) &\le O\left(\frac{1}{n}\right)(x_1+1)^2 \le M_1(x_1+1)^2 \text{ as } m,n\to\infty;\\
(2)\quad B_{n,m}^{\nu,\tau}((v_1-y_1)^2;x_1,y_1) &\le O\left(\frac{1}{\xi_m}\right)(y_1+1)^2 \le C_1(y_1+1)^2 \text{ as } m,n\to\infty;\\
(3)\quad B_{n,m}^{\nu,\tau}((u_1-x_1)^4;x_1,y_1) &\le O\left(\frac{1}{n^2}\right)(x_1+1)^4 \le M_2(x_1+1)^4 \text{ as } m,n\to\infty;\\
(4)\quad B_{n,m}^{\nu,\tau}((v_1-y_1)^4;x_1,y_1) &\le O\left(\frac{1}{\xi_m^2}\right)(y_1+1)^4 \le C_2(y_1+1)^4 \text{ as } m,n\to\infty
\end{aligned}
$$

Remark 1. *Let R_n^* and S_m be defined by* (18) *and* (19)*; then, operators $B_{n,m}^{\nu,\tau}$ satisfy $B_{n,m}^{\nu,\tau}(\phi_{0,0};x_1,y_1) = R_n^*(\phi_{0,0};x_1,y_1) = S_m(\phi_{0,0};x_1,y_1)$ and, for any $i,j = 1,2,3,4$, it follows that*

$$
\begin{aligned}
(1)\quad B_{n,m}^{\nu,\tau}(\phi_{i,0};x_1,y_1) &= R_n^*(\phi_{i,0};x_1,y_1);\\
(2)\quad B_{n,m}^{\nu,\tau}(\phi_{0,j};x_1,y_1) &= S_m(\phi_{0,j};x_1,y_1).
\end{aligned}
$$

3. Weighted Approximation and Degree of Convergence

In this section, we discuss the rate of convergence using a weighted modulus of continuity, weighted approximation results and degree of convergence for our operators (9). Let us recall the following:

Let $\Theta(x_1,y_1) = 1 + x_1^2 + y_1^2$ be a weight function defined by $B_\Theta(\mathbb{R}_+^2) = \Big\{f : |f(x_1,y_1)| \le M_f\Theta(x_1,y_1) \text{ for } M_f > 0\Big\}$. Suppose the set of r-times continuously differentiable functions on $\mathbb{R}_+^2 = \{(x_1,y_1)\in\mathbb{R}^2 : x_1,y_1 \text{ in } [0,\infty)\}$ is denoted by $C^{(r)}(\mathbb{R}_+^2)$. We can also assume the following classes of functions:

$$C_\Theta\left(\mathbb{R}_+^2\right) = \left\{f : f \in B_\Theta \cap C_\Theta\left(\mathbb{R}_+^2\right)\right\};$$

$$C_\Theta^k\left(\mathbb{R}_+^2\right) = \left\{f : f \in C_\Theta\left(\mathbb{R}_+^2\right) \text{ such that } \lim_{x_1,y_1\to\infty}\frac{f(x_1,y_1)}{\Theta(x_1,y_1)} = k_f < \infty\right\};$$

$$C_\Theta^0\left(\mathbb{R}_+^2\right) = \left\{f : f \in C_\Theta^k\left(\mathbb{R}_+^2\right) \text{ such that } \lim_{x_1,y_1\to\infty}\frac{f(x_1,y_1)}{\Theta(x_1,y_1)} = 0\right\}.$$

The norm on B_Θ is defined as $\| f \|_\Theta = \sup_{x_1,y_1 \in \mathbb{R}^2_+} \frac{|f(x_1,y_1)|}{\Theta(x_1,y_1)}$.

For all $f \in C^0_\Theta(\mathbb{R}^2_+)$ and $\delta_1, \delta_2 > 0$, the weighted modulus of continuity [29] is given as

$$\omega_\Theta(f;\delta_1,\delta_2) = \sup_{x_1,y_1 \in [0,\infty)} \sup_{0 \leq |h_1| \leq \delta_1,\, 0 \leq |h_2| \leq \delta_2} \frac{| f(x_1+h_1, y_1+h_2) - h(x_1,y_1) |}{\Theta(x_1,y_1)\Theta(h_1,h_2)} \tag{21}$$

and, for any $r_1, r_2 > 0$, the inequality

$$\omega_\Theta(f; r_1\delta_1, r_2\delta_2) \leq 4(1+r_1)(1+r_2)(1+\delta_1^2)(1+\delta_2^2)\omega_\Theta(f;\delta_1,\delta_2)$$

holds. It also follows that

$$\begin{aligned} | f(u_1,v_1) - f(x_1,y_1) | &\leq \Theta(x_1,y_1)\Theta(| u_1 - x_1 |, | v_1 - y_1 |)\omega_\Theta(f; | u_1 - x_1 |, | v_1 - y_1 |) \\ &\leq (1 + x_1^2 + y_1^2)(1 + (u_1 - x_1)^2)(1 + (v_1 - y_1)^2) \\ &\quad \times \omega_\Theta(f; | u_1 - x_1 |, | v_1 - y_1 |). \end{aligned}$$

Theorem 1. *Let $n, m \in \mathbb{N}$; then, for all $f \in C^0_\Theta(\mathbb{R}^2_+)$, it follows that*

$$\frac{| B^{\nu,\tau}_{n,m}(f; x_1,y_1) - f(x_1,y_1) |}{(1 + x_1^2 + y_1^2)^3} \leq M\left(O\left(\frac{1}{n}\right)\right)\left(O\left(\frac{1}{m}\right)\right)\omega_\Theta\left(f; O\left(\frac{1}{n}\right), O\left(\frac{1}{m}\right)\right),$$

where $\delta_n = \sqrt{B^{\nu,\tau}_{n,m}((u_1 - x_1)^2; x_1,y_1)} = O(n^{-1/2})$ and $\delta_m = \sqrt{B^{\nu,\tau}_{n,m}((v_1 - y_1)^2; x_1,y_1)} = O(m^{-1/2})$.

Proof. In the view of above inequalities for all $\delta_n, \delta_m > 0$, we see that

$$\begin{aligned} | f(u_1,v_1) - f(x_1,y_1) | &\leq 4(1 + x_1^2 + y_1^2)\left(1 + (u_1 - x_1)^2\right)\left(1 + (v_1 - y_1)^2\right) \\ &\quad \times \left(1 + \frac{| u_1 - x_1 |}{\delta_n}\right)\left(1 + \frac{| v_1 - y_1 |}{\delta_m}\right)(1 + \delta_n^2)(1 + \delta_m^2) \\ &\quad \times \omega_\Theta(f; \delta_n, \delta_m) \\ &= 4(1 + x_1^2 + y_1^2)(1 + \delta_n^2)(1 + \delta_m^2) \\ &\quad \times \left(1 + \frac{| u_1 - x_1 |}{\delta_n} + (u_1 - x_1)^2 + (u_1 - x_1)^2 \frac{| u_1 - x_1 |}{\delta_n}\right) \\ &\quad \times \left(1 + \frac{| v_1 - y_1 |}{\delta_m} + (v_1 - y_1)^2 + (v_1 - y_1)^2 \frac{| v_1 - y_1 |}{\delta_m}\right) \\ &\quad \times \omega_\Theta(f; \delta_n, \delta_m). \end{aligned}$$

Applying the operators $B^{\nu,\tau}_{n,m}$ in the light of linearity as

$$\begin{aligned} | B^{\nu,\tau}_{n,m}(f; x_1,y_1) - f(x_1,y_1) | &\leq B^{\nu,\tau}_{n,m}(| f(u_1,v_1) - f(x_1,y_1) |; x_1,y_1) 4(1 + x_1^2 + y_1^2) \\ &\quad \times B^{\nu,\tau}_{n,m}\left(1 + \frac{| u_1 - x_1 |}{\delta_n} + (u_1 - x_1)^2\right. \\ &\quad \left. + \frac{| u_1 - x_1 |}{\delta_n}(u_1 - x_1)^2; x_1, y_1\right) \\ &\quad \times B^{\nu,\tau}_{n,m}\left(1 + \frac{| v_1 - y_1 |}{\delta_m} + (v_1 - y_1)^2\right. \\ &\quad \left. + (v_1 - y_1)^2 \frac{| v_1 - y_1 |}{\delta_m}; x_1, y_1\right)(1 + \delta_n^2) \\ &\quad (1 + \delta_m^2)\omega_\Theta(f; \delta_n, \delta_m) \\ &= 4(1 + x_1^2 + y_1^2)(1 + \delta_n^2)(1 + \delta_m^2)\omega_\Theta(f; \delta_n, \delta_m) \end{aligned}$$

$$
\begin{aligned}
&\times\bigg(1+\frac{1}{\delta_n}B^{\nu,\tau}_{n,m}(\mid u_1-x_1\mid;x_1,y_1)\\
&+B^{\nu,\tau}_{n,m}((u_1-x_1)^2;x_1,y_1)\\
&+\frac{1}{\delta_n}B^{\nu,\tau}_{n,m}(\mid u_1-x_1\mid(u_1-x_1)^2;x_1,y_1\bigg)\\
&\times\bigg(1+\frac{1}{\delta_m}B^{\nu,\tau}_{n,m}(\mid v_1-y_1\mid;x_1,y_1)\\
&+B^{\nu,\tau}_{n,m}((v_1-y_1)^2;x_1,y_1)\\
&+\frac{1}{\delta_m}B^{\nu,\tau}_{n,m}(\mid v_1-y_1\mid(v_1-y_1)^2;x_1,y_1\bigg).
\end{aligned}
$$

Applying the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
\mid B^{\nu,\tau}_{n,m}(f;x_1,y_1)-f(x_1,y_1)\mid \leq\;& 4(1+x_1^2+y_1^2)(1+\delta_n^2)(1+\delta_m^2)\omega_\Theta(f;\delta_n,\delta_m)\\
&\times\bigg[1+\frac{1}{\delta_n}\sqrt{B^{\nu,\tau}_{n,m}((u_1-x_1)^2;x_1,y_1)}\\
&+B^{\nu,\tau}_{n,m}((u_1-x_1)^2;x_1,y_1)\\
&+\frac{1}{\delta_n}\sqrt{B^{\nu,\tau}_{n,m}((u_1-x_1)^2;x_1,y_1)}\sqrt{B^{\nu,\tau}_{n,m}((u_1-x_1)^4;x_1,y_1)}\bigg]\\
&\times\bigg[1+\frac{1}{\delta_m}\sqrt{B^{\nu,\tau}_{n,m}((v_1-y_1)^2;x_1,y_1)}\\
&+B^{\nu,\tau}_{n,m}((v_1-y_1)^2;x_1,y_1)\\
&+\frac{1}{\delta_m}\sqrt{B^{\nu,\tau}_{n,m}((v_1-y_1)^2;x_1,y_1)}\sqrt{B^{\nu,\tau}_{n,m}((v_1-y_1)^4;x_1,y_1)}\bigg].
\end{aligned}
$$

In the view of Lemma 6, we can obtain

$$
\begin{aligned}
\mid B^{\nu,\tau}_{n,m}(f;x_1,y_1)-f(x_1,y_1)\mid \leq\;& 4(1+x_1^2+y_1^2)(1+\delta_n^2)(1+\delta_m^2)\omega_\Theta(f;\delta_n,\delta_m)\\
&\times\Big[2+M_1(x_1+1)^2+M_2(x_1+1)^2\Big]\\
&\times\Big[2+C_1(y_1+1)^2+C_2(y_1+1)^2\Big].
\end{aligned}
$$

By choosing $\delta_n=\sqrt{B^{\nu,\tau}_{n,m}((u_1-x_1)^2;x_1,y_1)}$ and $\delta_m=\sqrt{B^{\nu,\tau}_{n,m}((v_1-y_1)^2;x_1,y_1)}$, and by using the inequality from Lemma 6, we can obtain the desired result. □

The following result can be obtained from Lemma 7 and Theorem 2, which are given by:

Lemma 7 ([30,31]). *Let $\Theta(x_1,y_1)=1+x_1^2+y_1^2$ be the weight function for all $(x_1,y_1)\in\mathbb{R}^+\times\mathbb{R}^+$, then any positive linear operator $\{J_{n,m}\}_{n,m\geq1}$ acting from $C_\Theta\to B_\Theta$ has the following property*

$$\| J_{n,m}(\Theta;x_1,y_1)\|_\Theta\leq C,$$

where $C>0$ is a real constant.

Theorem 2 ([30,31]). *For any positive linear operator $\{J_{n,m}\}_{n,m\geq1}$ acting from $C_\Theta\to B_\Theta$ and satisfying conditions*

$$
\begin{aligned}
(1)\quad &\lim_{n,m\to\infty}\| J_{n,m}(1;x_1,y_1)-1\|_\Theta &=&\; 0;\\
(2)\quad &\lim_{n,m\to\infty}\| J_{n,m}(u_1;x_1,y_1)-x_1\|_\Theta &=&\; 0;
\end{aligned}
$$

$$(3)\quad \lim_{n,m\to\infty} \parallel J_{n,m}(v_1;x_1,y_1) - y_1 \parallel_\Theta \;=\; 0;$$
$$(4)\quad \lim_{n,m\to\infty} \parallel J_{n,m}((u_1^2+v_1^2);x_1,y_1) - (x_1^2+y_1^2) \parallel_\Theta \;=\; 0;$$

we can obtain, for each $f \in C_\Theta^0$, that

$$\lim_{n,m\to\infty} \parallel J_{n,m}(f) - f \parallel_\Theta = 0,$$

and there is another function $g \in C_\Theta \setminus C_\Theta^0$, such that

$$\lim_{n,m\to\infty} \parallel J_{n,m}(g) - g \parallel_\Theta \geq 1.$$

Theorem 3. *For all $f \in C_\Theta^0(\mathbb{R}_+^2)$, the operators $\{T_{n,m}^*\}_{n,m\geq 1}$ defined by (20) satisfy*

$$\parallel T_{n,m}^* f - f \parallel_\Theta = 0.$$

Proof. Write

$$\begin{aligned}
\parallel T_{n,m}^*(\Theta;x_1,y_1) \parallel_\Theta &= \sup_{(x_1,y_1)\in\mathbb{R}_+^2} \frac{\mid T_{n,m}^*(1+u_1^2+v_1^2;x_1,y_1) \mid}{1+x_1^2+y_1^2} \\
&\leq 1 + \sup_{(x_1,y_1)\in\mathcal{I}_{\alpha_n\gamma_m}} \left[\frac{1}{1+x_1^2+y_1^2}\left|\left(B_{n,m}^{\nu,\tau}(u_1^2;x_1,y_1) + B_{n,m}^{\nu,\tau}(v_1^2;x_1,y_1)\right)\right|\right] \\
&= 1 + \sup_{(x_1,y_1)\in\mathcal{I}_{\alpha_n\gamma_m}} \left[x_1^2 + \frac{\alpha_n}{n}\left(2\frac{C'(1)}{C(1)} + 2\nu + 1\right)x_1 + \left(\frac{\alpha_n}{n}\right)^2\left(\frac{C''(1)}{C(1)} + \frac{C'(1)}{C(1)} + 4\nu^2\right)\right. \\
&\quad \left. + \frac{1}{\xi_m^2}\left\{\frac{D''(1)}{D(1)} + (2\beta_m y_1 + 1)\frac{D'(1)}{D(1)} + \beta_m y_1(\beta_m y_1 + 1)\right\} + \frac{2\tau}{\xi_m^2}\left\{\frac{D'(1)}{D(1)} + \beta_m y_1\right\} + \frac{4\tau^2}{\xi_m^2}\right] \\
&\leq 1 + \max_{0\leq x_1\leq\alpha_n}\left|\xi_{n,m}^*(x_1)\right| + \max_{0\leq y_1\leq\gamma_m}\left|\zeta_{n,m}^*(y_1)\right|,
\end{aligned}$$

where

$$\xi_{n,m}^*(x_1) = \alpha_n^2 + \frac{\alpha_n}{n}\left(2\frac{C'(1)}{C(1)} + 2\nu + 1\right)\alpha_n + \left(\frac{\alpha_n}{n}\right)^2\left(\frac{C''(1)}{C(1)} + \frac{C'(1)}{C(1)} + 4\nu^2\right)$$

and

$$\begin{aligned}
\zeta_{n,m}^*(y_1) &= \frac{1}{\xi_m^2}\left\{\frac{D''(1)}{D(1)} + (2\beta_m y_1 + 1)\frac{D'(1)}{D(1)} + \beta_m y_1(\beta_m y_1 + 1)\right\} \\
&\quad + \frac{2\tau}{\xi_m^2}\left\{\frac{D'(1)}{D(1)} + \beta_m y_1\right\} + \frac{4\tau^2}{\xi_m^2}.
\end{aligned}$$

Therefore, if $n,m \to \infty$, $0 \leq x_1 \leq \alpha_n$ and $0 \leq y_1 \leq \gamma_m$ then $\lim_{n,m\to\infty} \xi_{n,m}^*(x_1) = \alpha_n^2$ and $\lim_{n,m\to\infty} \zeta_{n,m}^*(y_1) = 0$. Thus, for all $n,m \in \mathbb{N}$, there is a positive number C such that $\xi_{n,m}^*(\alpha_n) + \zeta_{n,m}^*(\gamma_m) < C$. Finally, we arrived at

$$\parallel T_{n,m}^*(\Theta;x_1,y_1) \parallel_\Theta \leq M.$$

From Lemma 7, we have $T_{n,m}^* : C_\Theta(\mathbb{R}_+^2) \to B_\Theta(\mathbb{R}_+^2)$. If we can show that the conditions of Theorem 2 are satisfied; then, proof of Theorem 3 is completed. Hence, by the use of Lemma 4 we can obtain: $\lim_{n,m\to\infty} \parallel T_{n,m}^*(1;x_1,y_1) - 1 \parallel_\Theta = 0$, $\lim_{n,m\to\infty} \parallel$

$T^*_{n,m}(u_1; x_1, y_1) - x_1 \parallel_{\Theta} = 0$, and $\lim_{n,m\to\infty} \parallel T^*_{n,m}(v_1; x_1, y_1) - y_1 \parallel_{\Theta} = 0$. Finally, using Lemma 4, we can obtain

$$\begin{aligned}
\lim_{n,m\to\infty} &\parallel T^*_{n,m}((u_1^2+v_1^2); x_1, y_1) - (x_1^2+y_1^2) \parallel_{\Theta} \\
&\leq \sup_{(x_1,y_1)\in\mathcal{I}_{\alpha_n\gamma_m}} \left[\frac{1}{\Theta} \left| \left(T^*_{n,m}(u_1^2; x_1, y_1) + T^*_{n,m}(v_1^2; x_1, y_1) - (x_1^2+y_1^2) \right) \right| \right] \\
&\leq \max_{0\leq x_1\leq\alpha_n} \left| B^{\nu,\tau}_{n,m}(\phi_{2,0}; x_1, y_1) - x_1^2 \right| + \max_{0\leq y_1\leq\gamma_m} \left| B^{\nu,\tau}_{n,m}(\phi_{0,2}; x_1, y_1) - y_1^2 \right|,
\end{aligned}$$

which allow us

$$\lim_{n,m\to\infty} \left\{ \max_{0\leq x_1\leq\alpha_n} \left| B^{\nu,\tau}_{n,m}(\phi_{2,0}; x_1, y_1) - x_1^2 \right| + \max_{0\leq y_1\leq\gamma_m} \left| B^{\nu,\tau}_{n,m}(\phi_{0,2}; x_1, y_1) - y_1^2 \right| \right\} = 0.$$

Therefore, $\lim_{n,m\to\infty} \parallel T^*_{n,m}((u_1^2+v_1^2); x_1, y_1) - (x_1^2+y_1^2) \parallel_{\Theta} = 0$, which completes the proof. □

Theorem 4 ([30,31])**.** *Let the sequence of positive linear operators $\{K_{n,m}\}_{n,m\geq1}$ acting from $C_{\Theta}(\mathbb{R}^2_+) \to B_{\Theta}(\mathbb{R}^2_+)$ be defined as before, and $\Theta_1(x_1, y_1) \geq 1$ be the continuous function, such that*

$$\lim_{|x_1,y_1|\to\infty} \frac{\Theta(x_1, y_1)}{\Theta_1(x_1, y_1)} = 0.$$

If $K_{n,m}$ satisfy all conditions of Theorem 2; then, for all $f \in C_{\Theta}(\mathbb{R}^2_+)$

$$\parallel K_{n,m} f - f \parallel_{\Theta_1} = 0.$$

Theorem 5. *Let $\{T^*_{n,m}\}_{n,m\geq1} : C_{\Theta}(\mathbb{R}^2_+) \to B_{\Theta}(\mathbb{R}^2_+)$ and $\Theta_1(x_1, y_1) \geq 1$ be the continuous function, such that $\lim_{|x_1,y_1|\to\infty} \frac{\Theta(x_1,y_1)}{\Theta_1(x_1,y_1)} = 0$. Then, for any $f \in C_{\Theta}(\mathbb{R}^2_+)$ we can obtain the equality*

$$\parallel T^*_{n,m} f - f \parallel_{\Theta_1} = 0.$$

Proof. We prove our results using Theorem 3 and Theorem 4. It is easy to obtain the operators $\{T^*_{n,m}\}_{n,m\geq1}$ acting from $C_{\Theta_1}(\mathbb{R}^2_+) \to B_{\Theta_1}(\mathbb{R}^2_+)$. Using Lemma 4 we can see

$$\begin{aligned}
\parallel T^*_{n,m}(\Theta; x_1, y_1) \parallel_{\Theta_1} \leq\ & 1 + \sup_{(x_1,y_1)\in\mathcal{I}_{\alpha_n\gamma_m}} \frac{x_1^2}{\Theta_1(x_1, y_1)} \\
&+ \frac{\alpha_n}{n} \left(2\frac{C'(1)}{C(1)} + 2\nu + 1 \right) \sup_{(x_1,y_1)\in\mathcal{I}_{\alpha_n\gamma_m}} \frac{x_1}{\Theta_1(x_1, y_1)} \\
&+ \left(\frac{\alpha_n}{n} \right)^2 \left(\frac{C''(1)}{C(1)} + \frac{C'(1)}{C(1)} + 4\nu^2 \right) \sup_{(x_1,y_1)\in\mathcal{I}_{\alpha_n\gamma_m}} \frac{1}{\Theta_1(x_1, y_1)} \\
&+ \left(\frac{\beta_m}{\xi_m} \right)^2 \sup_{(x_1,y_1)\in\mathcal{I}_{\alpha_n\gamma_m}} \frac{y_1^2}{\Theta_1(x_1, y_1)} \\
&+ \left(\frac{\beta_m}{\xi_m^2} \right) \left(1 + 2\frac{D'(1)}{D(1)} + 2\tau \right) \sup_{(x_1,y_1)\in\mathcal{I}_{\alpha_n\gamma_m}} \frac{y_1}{\Theta_1(x_1, y_1)} \\
&+ \left(\frac{1}{\xi_m^2} \right) \left(1 + 2\frac{D''(1)}{D(1)} + (2\tau+1)\frac{D'(1)}{D(1)} + 4\tau^2 \right) \\
&\times \sup_{(x_1,y_1)\in\mathcal{I}_{\alpha_n\gamma_m}} \frac{1}{\Theta_1(x_1, y_1)}
\end{aligned}$$

$$
\begin{aligned}
&\leq 1+\max_{\substack{0\leq x_1\leq\alpha_n\\0\leq y_1\leq\gamma_m}}\frac{x_1^2}{\Theta_1(x_1,y_1)}+\max_{\substack{0\leq x_1\leq\alpha_n\\0\leq y_1\leq\gamma_m}}\frac{y_1^2}{\Theta_1(x_1,y_1)}\\
&= 1+\mu_{n,m}+\nu_{n,m},
\end{aligned}
$$

where, clearly, $\mu_{n,m}=1+\max_{\substack{0\leq x_1\leq\alpha_n\\0\leq y_1\leq\gamma_m}}\frac{x_1^2}{\Theta_1(x_1,y_1)}$ and $\nu_{n,m}=\max_{\substack{0\leq x_1\leq\alpha_n\\0\leq y_1\leq\gamma_m}}\frac{y_1^2}{\Theta_1(x_1,y_1)}$; then, for any $n,m\in\mathbb{N}$ there is a positive real number C, such that $\mu_{n,m}+\nu_{n,m}<C$. Therefore, we have

$$\| T^*_{n,m}(\Theta;x_1,y_1)\|_{\Theta_1}\leq 1+C.$$

From Lemma 7, it is obvious that the operators $\{T^*_{n,m}\}_{n,m\geq1}$ acting $C_{\Theta_1}(\mathbb{R}^2_+)\to B_{\Theta_1}(\mathbb{R}^2_+)$. If $n,m\to\infty$, then from Theorem 1, it is easy to obtain $\| T^*_{n,m}(\phi_{0,0};x_1,y_1)-1\|_{\Theta_1}=0$, $\| T^*_{n,m}(\phi_{1,0};x_1,y_1)-x_1\|_{\Theta_1}=0$, $\| T^*_{n,m}(\phi_{0,1};x_1,y_1)-y_1\|_{\Theta_1}=0$, and

$$\| T^*_{n,m}(\phi_{2,0}+\phi_{0,2};x_1,y_1)-(x_1^2+y_1^2)\|_{\Theta_1}=0.$$

Hence, operators $T^*_{n,m}$ satisfy all the conditions of Theorem 1. Therefore, Theorem 4 implies that $\| T^*_{n,m}f-f\|_{\Theta_1}=0$. This completes our proof. □

For our operators $B^{\nu,\tau}_{n,m}$, we obtain the degree of convergence; therefore, we let $C(\mathcal{I}_{bc})$ denote the set of all continuous functions on $\mathcal{I}_{bc}=[0,b]\times[0,c]\subset\mathcal{I}_{\alpha_n}$, which endowed the sup-norm $\sup_{(x_1,y_1)\in\mathcal{I}_{bc}}|f(x_1,y_1)|$.

Theorem 6. *For any $\varphi\in C(\mathcal{I}_{bc})$ we can obtain the inequality*

$$|B^{\nu,\tau}_{n,m}(\varphi;x_1,y_1)-\varphi(x_1,y_1)|\leq 2\big(\omega_1(\varphi;\delta_{n,x_1})+\omega_2(\varphi;\delta_{m,y_1})\big).$$

Proof. To prove the result, we use the Cauchy–Schwarz inequality and obtain

$$
\begin{aligned}
|B^{\nu,\tau}_{n,m}(\varphi;x_1,y_1)-\varphi(x_1,y_1)| &\leq B^{\nu,\tau}_{n,m}(|\varphi(u_1,v_1)-\varphi(x_1,y_1)|;x_1,y_1)\\
&\leq B^{\nu,\tau}_{n,m}(|\varphi(u_1,v_1)-\varphi(x_1,s)|;x_1,y_1)\\
&\quad+B^{\nu,\tau}_{n,m}(|\varphi(x_1,s)-\varphi(x_1,y_1)|;x_1,y_1)\\
&\leq B^{\nu,\tau}_{n,m}(\omega_1(\varphi;|u_1-x_1|);x_1,y_1)\\
&\quad+B^{\nu,\tau}_{n,m}(\omega_2(\varphi;|v_1-y_1|);x_1,y_1)\\
&\leq \omega_1(\varphi;\delta_n)\left(1+\frac{1}{\delta_n}B^{\nu,\tau}_{n,m}(|u_1-x_1|;x_1,y_1)\right)\\
&+ \omega_2(\varphi;\delta_m)\left(1+\frac{1}{\delta_m}B^{\nu,\tau}_{n,m}(|v_1-y_1|;x_1,y_1)\right)\\
&\leq \omega_1(\varphi;\delta_n)\left(1+\frac{1}{\delta_n}\sqrt{B^{\nu,\tau}_{n,m}((u_1-x_1)^2;x_1,y_1)}\right)\\
&+ \omega_2(f;\delta_m)\left(1+\frac{1}{\delta_m}\sqrt{B^{\nu,\tau}_{n,m}((v_1-y_1)^2;x_1,y_1)}\right),
\end{aligned}
$$

by puting $\delta_n^2=B^{\nu,\tau}_{n,m}((u_1-x_1)^2;x_1,y_1)=\delta^2_{n,x_1}$ and $\delta_m^2=B^{\nu,\tau}_{n,m}((v_1-y_1)^2;x_1,y_1)=\delta^2_{m,y_1}$, we get the desired result. □

To obtain our next result, we suppose that, for a positive real number $\mathcal{K}$ and any $0<\rho_1,\rho_2\leq1$, the Lipschitz maximal function $\mathcal{L}$ on space $\mathcal{J}\times\mathcal{J}\subset\mathbb{R}^2_+$ is defined by

$$
\begin{aligned}
\mathcal{L}_{\rho_1,\rho_2}(\mathcal{J}\times\mathcal{J}) &= \Big\{\varphi:\sup(1+u_1)^{\rho_1}(1+v_1)^{\rho_2}\big(\varphi_{\rho_1,\rho_2}(u_1,v_1)-\varphi_{\rho_1,\rho_2}(x_1,y_1)\big)\\
&\leq K\frac{1}{(1+x_1)^{\rho_1}}\frac{1}{(1+y_1)^{\rho_2}}\Big\},
\end{aligned}
$$

$$\varphi_{\rho_1,\rho_2}(u_1,v_1)-\varphi_{\rho_1,\rho_2}(x_1,y_1)=\frac{\mid\varphi(u_1,v_1)-\varphi(x_1,y_1)\mid}{\mid u_1-x_1\mid^{\rho_1}\mid v_1-y_1\mid^{\rho_2}};\quad (u_1,v_1),(x_1,y_1)\in\mathcal{I}_{bc},\tag{22}$$

where φ is a continuous and bounded function defined on $\mathbb{R}^2_+$.

Theorem 7. *For any $\varphi\in\mathcal{L}_{\rho_1,\rho_2}(\mathcal{J}\times\mathcal{J})$, there is a real positive number K satisfying*

$$\begin{aligned}&\mid B^{\nu,\tau}_{n,m}(f;x_1,y_1)-f(x_1,y_1)\mid\\&\quad\leq K\bigg\{\bigg((d(x_1,\mathcal{J}))^{\rho_1}+\Big(\delta^2_{n,x_1}\Big)^{\frac{\rho_1}{2}}\bigg)\bigg((d(y_1,\mathcal{J}))^{\rho_2}+\Big(\delta^2_{m,y_1}\Big)^{\frac{\rho_2}{2}}\bigg)\\&\qquad+(d(x_1,\mathcal{J}))^{\rho_1}(d(y_1,\mathcal{J}))^{\rho_2}\bigg\},\end{aligned}$$

where $0<\rho_1,\rho_2\leq 1$, $(x_1,y_1)\in\mathcal{I}_{bc}$ and δ_{n,x_1}, δ_{m,y_1} are defined in Theorem 6.

Proof. Take $(x_1,y_1)\in\mathcal{I}_{bc}$; then, for any fixed $(x_0,y_0)\in\mathcal{J}\times\mathcal{J}$ suppose $\mid x_1-x_0\mid=d(x_1,\mathcal{J})$ and $\mid y_1-y_0\mid=d(y_1,\mathcal{J})$, where $d(x_1,\mathcal{J})=\inf\{\mid x_1-y_1\mid: y_1\in\mathcal{J}\}$; thus, we can write

$$\mid\varphi(u_1,v_1)-\varphi(x_1,y_1)\mid\leq K\mid\varphi(u_1,v_1)-\varphi(x_0,y_0)\mid+\mid\varphi(x_0,y_0)-\varphi(x_1,y_1)\mid.\tag{23}$$

Applying $B^{\nu,\tau}_{n,m}$ on both sides

$$\begin{aligned}\mid B^{\nu,\tau}_{n,m}(\varphi;x_1,y_1)-\varphi(x_1,y_1)\mid&\leq B^{\nu,\tau}_{n,m}(\mid\varphi(u_1,v_1)-\varphi(x_0,y_0)\mid+\mid\varphi(x_0,y_0)-\varphi(x_1,y_1)\mid)\\&\leq KB^{\nu,\tau}_{n,m}(\mid u_1-x_0\mid^{\rho_1}\mid v_1-y_0\mid^{\rho_2};x_1,y_1)\\&\quad+K\mid x_1-x_0\mid^{\rho_1}\mid y_1-y_0\mid^{\rho_2}.\end{aligned}$$

For any $p,q\geq 0$ and $0\leq\rho\leq 1$, we know the inequality $(p+q)^\rho\leq p^\rho+q^\rho$; thus, we obtain

$$\mid u_1-x_0\mid^{\rho_1}\leq\mid u_1-x_1\mid^{\rho_1}+\mid x_1-x_0\mid^{\rho_1}$$

and

$$\mid v_1-y_0\mid^{\rho_2}\leq\mid v_1-y_1\mid^{\rho_2}+\mid y_1-y_0\mid^{\tau_2}.$$

Therefore,

$$\begin{aligned}\mid B^{\nu,\tau}_{n,m}(\varphi;x_1,y_1)-\varphi(x_1,y_1)\mid&\leq KB^{\nu,\tau}_{n,m}(\mid u_1-x_1\mid^{\rho_1}\mid v_1-y_1\mid^{\rho_2};x_1,y_1)\\&\quad+K\mid x_1-x_0\mid^{\rho_1}B^{\nu,\tau}_{n,m}(\mid v_1-y_1\mid^{\rho_2};x_1,y_1)\\&\quad+K\mid y_1-y_0\mid^{\rho_2}B^{\nu,\tau}_{n,m}(\mid u_1-x_1\mid^{\rho_1};x_1,y_1)\\&\quad+2K\mid x_1-x_0\mid^{\rho_1}\mid y_1-y_0\mid^{\rho_2}B^{\nu,\tau}_{n,m}(\phi_{0,0};x_1,y_1).\end{aligned}$$

Using Hölder inequality, we obtain

$$\begin{aligned}&B^{\nu,\tau}_{n,m}(\mid u_1-x_1\mid^{\rho_1}\mid v_1-y_1\mid^{\rho_2};x_1,y_1)\\&\quad=R^*_n(\mid u_1-x_1\mid^{\rho_1};x_1,y_1)S_m(\mid v_1-y_1\mid^{\rho_2};x_1,y_1)\\&\quad\leq\Big(B^{\nu,\tau}_{n,m}(\mid u_1-x_1\mid^2;x_1,y_1)\Big)^{\frac{\rho_1}{2}}\big(B^{\nu,\tau}_{n,m}(\phi_{0,0};x_1,y_1)\big)^{\frac{2-\rho_1}{2}}\\&\qquad\times\Big(B^{\nu,\tau}_{n,m}(\mid v_1-y_1\mid^2;x_1,y_1)\Big)^{\frac{\rho_2}{2}}\big(B^{\nu,\tau}_{n,m}(\phi_{0,0};x_1,y_1)\big)^{\frac{2-\rho_2}{2}},\end{aligned}$$

thus, we have

$$\mid B^{\nu,\tau}_{n,m}(\varphi;x_1,y_1)-\varphi(x_1,y_1)\mid\quad\leq\quad K\Big(\delta^2_{n,x_1}\Big)^{\frac{\rho_1}{2}}\Big(\delta^2_{m,y_1}\Big)^{\frac{\rho_2}{2}}+2K(d(x_1,\mathcal{J}))^{\rho_1}(d(y_1,\mathcal{J}))^{\rho_2}$$

$$+K(d(x_1,\mathcal{J}))^{\rho_1}\left(\delta^2_{m,y_1}\right)^{\frac{\rho_2}{2}}+K(d(y_1,\mathcal{J}))^{\rho_2}\left(\delta^2_{n,x_1}\right)^{\frac{\rho_1}{2}},$$

which completes the proof. □

Theorem 8. *Suppose $(x_1,y_1)\in\mathcal{I}_{bc}$. Then, for any function $\psi(x_1,y_1)\in C'(\mathcal{I}_{bc})$, the operators $B^{\nu,\tau}_{n,m}$ have the inequality*

$$|\, B^{\nu,\tau}_{n,m}(\psi;x_1,y_1)-\psi(x_1,y_1)\,|\leq\|\,\psi'_{x_1}\,\|_{C(\mathcal{I}_{bc})}\left(\delta^2_{n,x_1}\right)^{\frac{1}{2}}+\|\,\psi'_{y_1}\,\|_{C(\mathcal{I}_{bc})}\left(\delta^2_{m,y_1}\right)^{\frac{1}{2}},$$

where δ_{n,x_1}, δ_{m,y_1} are given in Theorem 6 and $\psi'_{x_1}=\frac{\partial\psi(x_1,y_1)}{\partial x_1}$, $\psi'_{y_1}=\frac{\partial\psi(x_1,y_1)}{\partial y_1}$.

Proof. Take $\psi\in C'(\mathcal{I}_{bc})$ and $(x_1,y_1)\in\mathcal{I}_{bc}$. Then, for any fixed $(u_1,v_1)\in\mathcal{I}_{bc}$, we see that

$$\psi(u_1,v_1)-\psi(x_1,y_1)\;=\;\int_{x_1}^{u_1}\psi'_{v}(v,v_1)\mathrm{d}v+\int_{y_1}^{v_1}\psi'_{\varsigma}(x_1,\varsigma)\mathrm{d}\varsigma$$

which implies

$$\begin{aligned}B^{\nu,\tau}_{n,m}(\psi(u_1,v_1);x_1,y_1)-\psi(x_1,y_1)\;&=\;B^{\nu,\tau}_{n,m}\left(\int_{x_1}^{u_1}\psi'_{v}(v,v_1)\mathrm{d}t;x_1,y_1\right)\\&\quad+B^{\nu,\tau}_{n,m}\left(\int_{y_1}^{v_1}\psi'_{\varsigma}(x_1,\varsigma)\mathrm{d}\varsigma;x_1,y_1\right).\end{aligned}\tag{24}$$

Using the equipped sup-norm $\mathcal{I}_{bc}$, it is easy to obtain

$$\left|\int_{x_1}^{u_1}\psi'_{v}(v,v_1)\mathrm{d}v\right|\leq\int_{x_1}^{u_1}|\,\psi'_{v}(v,v_1)\mathrm{d}v\,|\leq\|\,\psi'_{x_1}\,\|_{C(\mathcal{I}_{bc})}|\,u_1-x_1\,|\tag{25}$$

and

$$\left|\int_{y_1}^{v_1}C'_{\varsigma}(x_1,\varsigma)\mathrm{d}\varsigma\right|\leq\int_{y_1}^{v_1}|\,\psi'_{\varsigma}(x_1,\varsigma)\mathrm{d}\varsigma\,|\leq\|\,\psi'_{y_1}\,\|_{C(\mathcal{I}_{bc})}|\,v_1-y_1\,|\,.\tag{26}$$

In light of (24), (25) and (26), we obtain

$$\begin{aligned}&|\, B^{\nu,\tau}_{n,m}(\psi(u_1,v_1);x_1,y_1)-\psi(x_1,y_1)\,|\\&\quad\leq B^{\nu,\tau}_{n,m}\left(\left|\int_{x_1}^{u_1}\psi'_{v}(v,v_1)\mathrm{d}v\right|;x_1,y_1\right)+B^{\nu,\tau}_{n,m}\left(\left|\int_{y_1}^{v_1}\psi'_{\varsigma}(x_1,\varsigma)\mathrm{d}\varsigma\right|;x_1,y_1\right)\\&\quad\leq\|\,\psi'_{x_1}\,\|_{C(\mathcal{I}_{bc})}\,B^{\nu,\tau}_{n,m}(|\,u_1-x_1\,|;x_1,y_1)+\|\,\psi'_{y_1}\,\|_{C(\mathcal{I}_{bc})}\,B^{\nu,\tau}_{n,m}(|\,v_1-y_1\,|;x_1,y_1)\\&\quad\leq\|\,\psi'_{x_1}\,\|_{C(\mathcal{I}_{bc})}\left(B^{\nu,\tau}_{n,m}((u_1-x_1)^2;x_1,y_1)B^{\nu,\tau}_{n,m}(1;x_1,y_1)\right)^{\frac{1}{2}}\\&\qquad+\|\,\psi'_{y_1}\,\|_{C(\mathcal{I}_{bc})}\left(B^{\nu,\tau}_{n,m}((v_1-y_1)^2;x_1,y_1)B^{\nu,\tau}_{n,m}(1;x_1,y_1)\right)^{\frac{1}{2}}\\&\quad=\|\,\psi'_{x_1}\,\|_{C(\mathcal{I}_{bc})}\left(\delta^2_{n,x_1}\right)^{\frac{1}{2}}+\|\,\psi'_{y_1}\,\|_{C(\mathcal{I}_{bc})}\left(\delta^2_{m,y_1}\right)^{\frac{1}{2}}.\end{aligned}$$

□

Theorem 9. *Suppose $B^{\nu,\tau}_{n,m}(\phi_{0,1};x_1,y_1)$ is as defined by Lemma 4. If, for any $g\in C(\mathcal{I}_{bc})$, we define the auxiliary operators $R^*_{n,m}$ such that*

$$R^*_{n,m}(g;x_1,y_1)=B^{\nu,\tau}_{n,m}(g;x_1,y_1)+g(x_1,y_1)-g\big(x_1,B^{\nu,\tau}_{n,m}(\phi_{0,1};x_1,y_1)\big),\tag{27}$$

then, for an arbitrary function, $\varphi\in C'(\mathcal{I}_{bc})$, we obtain the following inequality

$$R^*_{n,m}(\varphi;x_1,y_1)-\varphi(x_1,y_1)$$

$$\leq \left[\delta^2_{n,x_1} + \delta^2_{m,y_1} + \left\{\left(\frac{\beta_m}{\xi_m} - 1\right)y_1 + \frac{1}{\xi_m}\left(\frac{D'(1)}{D(1)} + 2\tau\right)\right\}^2\right] \parallel \varphi \parallel_{C^2(\mathcal{I}_{bc})}.$$

Proof. From Lemma 4 and (27), we have $R^*_{n,m}(1;x_1,y_1) = 1$, $R^*_{n,m}(u_1 - x_1; x_1, y_1) = 0$ and $R^*_{n,m}(v_1 - y_1; x_1, y_1) = 0$. For all $\varphi \in C'(\mathcal{I}_{bc})$, the Taylor series expansion gives us

$$\begin{aligned} \varphi(u_1,v_1) - \varphi(x_1,y_1) &= \frac{\partial \varphi(x_1,y_1)}{\partial x_1}(u_1 - x_1) + \int_{x_1}^{t} (u_1 - \kappa)\frac{\partial^2 \varphi(\kappa,y_1)}{\partial \kappa^2} d\kappa \\ &\quad + \frac{\partial \varphi(x_1,y_1)}{\partial y_1}(v_1 - y_1) + \int_{y_1}^{s} (v_1 - \varrho)\frac{\partial^2 \varphi(x_1,\varrho)}{\partial \varrho^2} d\varrho, \end{aligned}$$

by applying $R^*_{n,m}$ defined by (27), we can obtain

$$\begin{aligned} R^*_{n,m}(\varphi(u_1,v_1);x_1,y_1) - R^*_{n,m}(\varphi(x_1,y_1)) &= R^*_{n,m}\left(\int_{x_1}^{t}(u_1-\kappa)\frac{\partial^2\varphi(\kappa,y_1)}{\partial\kappa^2}d\kappa; x_1,y_1\right) \\ &\quad + R^*_{n,m}\left(\int_{y_1}^{s}(v_1-\varrho)\frac{\partial^2\varphi(x_1,\varrho)}{\partial\varrho^2}d\varrho; x_1,y_1\right) \\ &= B^{\nu,\tau}_{n,m}\left(\int_{x_1}^{t}(u_1-\kappa)\frac{\partial^2\varphi(\kappa,y_1)}{\partial\kappa^2}d\kappa; x_1,y_1\right) \\ &\quad - \int_{x_1}^{x_1}(x_1-\kappa)\frac{\partial^2\varphi(\kappa,y_1)}{\partial\kappa^2}d\kappa \\ &\quad + B^{\nu,\tau}_{n,m}\left(\int_{y_1}^{s}(v_1-\varrho)\frac{\partial^2\varphi(x_1,\varrho)}{\partial\varrho^2}d\varrho; x_1,y_1\right) \\ &\quad - \int_{y_1}^{\frac{\beta_m}{\xi_m}y_1}\left(\frac{\beta_m}{\xi_m}y_1 - \varrho\right)\frac{\partial^2\varphi(x_1,\varrho)}{\partial\varrho^2}d\varrho. \end{aligned}$$

Therefore,

$$\begin{aligned} &\mid R^*_{n,m}(\varphi;x_1,y_1) - R^*_{n,m}(\varphi(x_1,y_1)) \mid \\ &= B^{\nu,\tau}_{n,m}\left(\left|\int_{x_1}^{t}(u_1-\kappa)\frac{\partial^2\varphi(\kappa,y_1)}{\partial\kappa^2}d\kappa\right|; x_1,y_1\right) \\ &+ B^{\nu,\tau}_{n,m}\left(\left|\int_{y_1}^{s}(v_1-\varrho)\frac{\partial^2\varphi(x_1,\varrho)}{\partial\varrho^2}d\varrho\right|; x_1,y_1\right) \\ &\quad - \left|\int_{y_1}^{B^{\nu,\tau}_{n,m}(\phi_{0,1};x_1,y_1)}(B^{\nu,\tau}_{n,m}(\phi_{0,1};x_1,y_1) - \varrho)\frac{\partial^2\varphi(x_1,\varrho)}{\partial\varrho^2}d\varrho\right|. \end{aligned}$$

Therefore, we can easily conclude that

$$\left|\int_{x_1}^{t}(u_1-\kappa)\frac{\partial^2\varphi(\kappa,y_1)}{\partial\kappa^2}d\kappa\right| \leq \int_{x_1}^{t}\left|(u_1-\kappa)\frac{\partial^2\varphi(\kappa,y_1)}{\partial\kappa^2}d\kappa\right| \leq \parallel \varphi \parallel_{C^2(\mathcal{I}_{bc})} (u_1 - x_1)^2, \quad (28)$$

$$\left|\int_{y_1}^{s}(v_1-\varrho)\frac{\partial^2\varphi(x_1,\varrho)}{\partial\varrho^2}d\varrho\right| \leq \int_{y_1}^{s}\left|(v_1-\varrho)\frac{\partial^2\varphi(x_1,\varrho)}{\partial\varrho^2}d\varrho\right| \leq \parallel \varphi \parallel_{C^2(\mathcal{I}_{bc})} (v_1 - y_1)^2, \quad (29)$$

and

$$\begin{aligned} &\left|\int_{y_1}^{B^{\nu,\tau}_{n,m}(\phi_{0,1};x_1,y_1)}(B^{\nu,\tau}_{n,m}(\phi_{0,1};x_1,y_1) - \varrho)\frac{\partial^2\varphi(x_1,\varrho)}{\partial\varrho^2}d\varrho\right| \\ &\quad \leq \int_{y_1}^{B^{\nu,\tau}_{n,m}(\phi_{0,1};x_1,y_1)}\left|(B^{\nu,\tau}_{n,m}(\phi_{0,1};x_1,y_1) - \varrho)\frac{\partial^2\varphi(x_1,\varrho)}{\partial\varrho^2}d\varrho\right| \end{aligned}$$

$$\leq \left(B^{\nu,\tau}_{n,m}(\phi_{0,1};x_1,y_1) - y_1\right)^2 \parallel \varphi \parallel_{C^2(\mathcal{I}_{bc})} . \tag{30}$$

Hence, using equalities (28), (29) and (30), we can obtain

$$\begin{aligned} \mid R^{*}_{n,m}(\varphi;x_1,y_1) - \varphi(x_1,y_1) \mid \quad \leq \quad & \Big\{ B^{\nu,\tau}_{n,m}((u_1-x_1)^2;x_1,y_1) + B^{\nu,\tau}_{n,m}((v_1-y_1)^2;x_1,y_1) \\ & + \left(B^{\nu,\tau}_{n,m}(\phi_{0,1};x_1,y_1) - y_1\right)^2 \Big\} \parallel \varphi \parallel_{C^2(\mathcal{I}_{bc})}, \end{aligned}$$

this completes the results. □

4. Conclusions

The motivation for this research article was to introduce the bivariate Szász–Jakimovski–Leviatan operators using unbounded sequences of positive numbers. We also discussed the rate of convergence and weighted approximation theorems for these operators. With the help of bivariate Lipschitz-maximal functions, we obtained the degree of convergence, as well as the direct theorem for our operators. These results meant that more attention was paid to these researchers, providing them with a new research path in bivariate sense. Moreover, our newly constructed operators generalized some existing operators in the literature (see [2,3,6]). For further research on the above operators, one can study the approximation results using the idea of convergence given in [32–34], and also extend our bivariate operators for more than two variables and study their approxmation properties.

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