

Article

Riquier–Neumann Problem for the Polyharmonic Equation in a Ball

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Abstract: The Green’s function of the Riquier–Neumann problem for the polyharmonic equation in the unit ball is constructed. Using the obtained Green’s function, an integral representation of the solution to the Riquier–Neumann problem in the unit ball is found.

Keywords: Riquier–Neumann problem; polyharmonic equation; Green’s function

MSC: 35J40; 31B30; 35J08

1. Introduction

The explicit form of Green’s functions for various boundary value problems is presented in many papers. We mention only a few of them. For example, in the two-dimensional case, in the paper [1], on the basis of the known harmonic Green’s function, Green’s functions of various biharmonic problems are obtained. The explicit form of Green’s function for the third boundary value problem is found in [2,3], and Green’s functions in the sector for the biharmonic and 3-harmonic equations are obtained in [4,5]. Studies of the Dirichlet problem for the polyharmonic Equation [6] in a ball can be found in [7,8]. In these papers, the explicit form of Green’s function is obtained. An explicit representation of Green’s function of the third boundary value problem for the Poisson equation is obtained in [9], and in [10], for the three-harmonic equation in a ball, Green’s operator acting on polynomial data is presented.

In connection with the biharmonic equation, we note the recent papers [11,12] devoted to the solvability conditions for some nonstandard problems in the ball for the biharmonic equation. As the most general results on the generalized Neumann problem containing powers of normal derivatives in boundary conditions, we note the paper [13]. In [14], for the biharmonic equation in a ball, the explicit form of Green’s functions of the Navier [15] and Riquier–Neumann problems are obtained. Green’s function is also used to study nonlocal equations. For example, in the paper [16], the solvability of four boundary value problems for one nonlocal biharmonic equation with involution is investigated. We also note some recently published papers on the construction of Green’s function for various boundary value problems [17–21]. The application of Green’s functions in problems of mechanics and physics can be found in [22–28].

The Riquier–Neumann problem [29] for the polyharmonic Equation (in [1] it is called the Neumann-2 problem) is to find the function $u \in C^{2m}(S) \cap C^{2m-1}(\bar{S})$, which is the solution of the following boundary value problem for the inhomogeneous polyharmonic equation

$$\begin{aligned} \Delta^m u(x) &= f(x), \quad x \in S, \\ \frac{\partial u}{\partial \nu} \Big|_{\partial S} &= \varphi_0(\xi), \quad \frac{\partial \Delta u}{\partial \nu} \Big|_{\partial S} = \varphi_1(\xi), \dots, \quad \frac{\partial \Delta^{m-1} u}{\partial \nu} \Big|_{\partial S} = \varphi_{m-1}(\xi), \quad \xi \in \partial S. \end{aligned} \quad (1)$$



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It is well known that Green's function of the Dirichlet problem for the Poisson equation in the unit ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$ for $n \geq 2$ has the form

$$G_2(x, \xi) = E(x, \xi) - E\left(\frac{x}{|x|}, |x|\xi\right), \quad (2)$$

where $E(x, \xi)$ is an elementary solution of the Laplace Equation [30]. In [10,31,32], elementary solutions of the biharmonic and triharmonic equations $E_4(x, \xi)$ and $E_6(x, \xi)$ are determined and Green's functions of the corresponding Dirichlet problems in S are found. In [9], Green's function of the Neumann problem for the Poisson equation in S for $n > 2$ is constructed

$$\mathcal{N}_2(x, \xi) = E_2(x, \xi) - E_0(x, \xi), \quad (3)$$

where the harmonic function $E_0(x, \xi)$ with respect to $x, \xi \in S$ is written in the form

$$E_0(x, \xi) = \int_0^1 \left(\hat{E}_2\left(\frac{x}{|x|}, t|x|\xi\right) + 1 \right) \frac{dt}{t}$$

and $\hat{E}_2(x, \xi) = \Lambda_x E_2(x, \xi)$, where denoted $\Lambda u = \sum_{i=1}^n x_i u_{x_i}$, and the index x indicates that the operator Λ is applied over the variables x . It is easy to see that $\Lambda u = \frac{\partial u}{\partial \nu}$ on ∂S . Since $\hat{E}_2(x, \xi) = -(|x|^2 - x \cdot \xi) / |x - \xi|^n$, then the function

$$\hat{E}_2\left(\frac{x}{|x|}, t|x|\xi\right) = -\frac{1 - (x \cdot \xi)t}{(1 - 2t(x \cdot \xi) + |x|^2|\xi|^2t^2)^{n/2}}$$

is symmetric, and hence the functions $E_0(x, \xi)$ and $\mathcal{N}_2(x, \xi)$ are also symmetric. The function $\mathcal{N}_2(x, \xi)$ has the properties found in [9] (Theorem 3) and [14] (Theorem 3)

$$\Lambda_x \mathcal{N}_2(x, \xi) = \Lambda_x E_2(x, \xi) - (\Lambda_x E_2)\left(\frac{x}{|x|}, |x|\xi\right) - 1, \quad x, \xi \in S, x \neq \xi, \quad (4)$$

$$\Lambda_x \mathcal{N}_2(x, \xi)|_{\xi \in \partial S} = -\frac{\partial G_2(x, \xi)}{\partial \nu_\xi} - 1, \quad x \in S,$$

and therefore the equalities

$$\begin{aligned} \int_S \frac{\partial \mathcal{N}_2(x, \xi)}{\partial \nu_x} f(\xi) d\xi|_{x \in \partial S} &= - \int_S f(\xi) d\xi, \\ \frac{1}{\omega_n} \int_{\partial S} \frac{\partial \mathcal{N}_2(x, \xi)}{\partial \nu_x} \psi(\xi) ds_\xi|_{x \in \partial S} &= \psi(x)|_{\partial S} - \frac{1}{\omega_n} \int_{\partial S} \psi(\xi) ds_\xi, \end{aligned} \quad (5)$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$ are hold true. In [14] (Theorem 3) it is shown that for $n > 2$ a solution to the Neumann problem for the Poisson equation

$$\Delta u(x) = f(x), x \in S; \quad \frac{\partial u(x)}{\partial \nu}|_{\partial S} = \psi(x), x \in \partial S$$

under the known condition $\int_{\partial S} \psi(\xi) ds_\xi = \int_S f(\xi) d\xi$ can be written as

$$u(x) = \frac{1}{\omega_n} \int_{\partial S} \mathcal{N}_2(x, \xi) \psi(\xi) ds_\xi - \frac{1}{\omega_n} \int_S \mathcal{N}_2(x, \xi) f(\xi) d\xi + C.$$

The solvability conditions of some Neumann problems for the polyharmonic equation are studied in [33–35] provides a solution to one of these problems. Proceeding from this, the goal of this work is to extend the class of Neumann-type problems for a polyharmonic equation, for which an explicit integral representation of the solution is known.

In the present paper, an elementary solution of the polyharmonic equation is defined and its properties are described in Lemmas 1 and 2. Theorem 1 gives an integral representation of functions from the class $u \in C^{2m}(D) \cap C^{2m-1}(\bar{D})$, in a bounded domain D

with piecewise smooth boundary. Next, we study the Riquier–Neumann problem [29]. Theorem 2 from Section 3 defines Green’s function of the Riquier–Neumann problem, and Theorem 3 from Section 4 finds the integral representation of the solution to this problem. Theorem 4 proves that the function found in Theorem 3 is indeed a solution to the Riquier–Neumann problem. Theorem 5 from Section 5 considers a special case of the Riquier–Neumann problem for the homogeneous equation. As an illustration of Theorem 5, Example 2 gives the solution of the Riquier–Neumann problem with simple boundary data.

2. Elementary Solution and Integral Representation

Let $m \in \mathbb{N}$. Then the set $\mathbb{N} \setminus \{1\}$ can be divided into two non-intersecting sets $\mathbb{N}_m = \{n \in \mathbb{N} : n > 2m > 1\} \cup (2\mathbb{N} + 1)$ and its complement $\mathbb{N}_m^c = \{2, 4, \dots, 2m\}$. Since the set $\mathbb{N}_m^c$ is finite, then $\mathbb{N}_m$ is infinite. It is clear that $\mathbb{N}_{m-1}^c \subset \mathbb{N}_m^c$, and therefore $\mathbb{N}_m \subset \mathbb{N}_{m-1}$. We define an elementary solution of the m -harmonic equation $\Delta^m u = 0$ as

$$E_{2m}(x, \xi) = \begin{cases} \frac{(-1)^m |x - \xi|^{2m-n}}{(2-n, 2)_m (2, 2)_{m-1}}, & n \in \mathbb{N}_m, \\ \frac{(-1)^m |x - \xi|^{2m-n}}{(2-n, 2)_m^* (2, 2)_{m-1}} \left(\ln |x - \xi| - \sum_{k=1}^{m-n/2} \frac{1}{2k} - \sum_{k=n/2}^{m-1} \frac{1}{2k} \right), & n \in \mathbb{N}_m^c. \end{cases} \quad (6)$$

where $(a, b)_k = a(a+b) \dots (a+kb-b)$ is a generalized Pochhammer symbol with the convention $(a, b)_0 = 1$. The symbol $(a, b)_k^*$ means that if there is 0 among the factors $a, (a+b), \dots, (a+kb-b)$ included in $(a, b)_k$, then it should be replaced by 1. For example, $(-2, 2)_3^* = (-2) \cdot 1 \cdot 2 = -4$. In addition, if the upper index of the sums included in (6) becomes less than the lower index, then the sum is considered to be zero. Note that $(2-n, 2)_m = (2-n)(4-n) \dots (2m-n) \neq 0$ for $n \in \mathbb{N}_m$ and hence the first part of equality (6) is well defined.

The following simple assertions are true.

Lemma 1. The function $E_{2m}(x, \xi)$ coincides with the elementary functions $E(x, \xi)$, $E_4(x, \xi)$ and $E_6(x, \xi)$ for $m = 1$, $m = 2$ and $m = 3$, respectively.

Remark 1. Sobolev S.L. [36] considered the fundamental solutions $G_{m,n}(x)$ of the polyharmonic equation, which slightly differ from the elementary solutions $E_{2m}(x, \xi)$. For example, for odd n

$$G_{m,n}(x) = \chi_{m,n} |x|^{2m-n}, \quad \chi_{m,n} = \frac{(-1)^{(n-1)/2}}{\Gamma(m)\Gamma(m-n/2+1)2^{2m}\pi^{n/2-1}},$$

where $\Gamma(m)$ is the gamma function. Taking into account that $2^{m-1}\Gamma(m) = (2, 2)_{m-1}$,

$$2^m \Gamma(m-n/2+1) = (2m-n) \dots (2-n) \Gamma(1-n/2) = (2-n, 2)_m \Gamma(1-n/2)$$

and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ one can write

$$\chi_{m,n} = \frac{1}{\omega_n} \frac{(-1)^{(n-1)/2} \pi}{(2, 2)_{m-1} (2-n, 2)_m \Gamma(n/2) \Gamma(1-n/2)}.$$

If we now notice that $\Gamma(n/2)\Gamma(1-n/2) = \frac{\pi}{\sin(n\pi/2)} = \pi(-1)^{(n-1)/2}$, then we get

$$\chi_{m,n} = \frac{1}{\omega_n} \frac{1}{(2, 2)_{m-1} (2-n, 2)_m}.$$

However, the function $E_{2m}(x, \xi)$ additionally contains the factor $(-1)^m$. This factor is additionally taken into account in Theorem 1. For $n \in \mathbb{N}_m^c$, the difference between these functions is somewhat larger and this is related to the property of the function $E_{2m}(x, \xi)$ indicated in Lemma 2 below, which $G_{m,n}(x)$ does not possess. For example, in the case $m = 2$, $n = 2 \in \mathbb{N}_2^c$ from (6) we

find $E_4(x, \xi) = \frac{|x-\xi|^2}{4} (\ln |x-\xi| - 1)$. The number -1 is chosen so that the equality $\Delta_\xi E_4(x, \xi) = \ln |x-\xi| \equiv -E_2(x, \xi)$ holds true.

Lemma 2. The symmetric function $E_{2m}(x, \xi)$ defined for $x \neq \xi$ satisfies the equalities

$$\Delta_\xi E_{2m}(x, \xi) = -E_{2(m-1)}(x, \xi), \quad \Delta_\xi E_2(x, \xi) = 0.$$

We omit the proofs of these assertions.

There is a well-known formula (see, for example, [37]) for representing the harmonic function $u(x)$ in a bounded domain $D \subset \mathbb{R}^n$ with a piecewise-smooth boundary ∂D . In the above notation, it has the form

$$u(x) = \frac{1}{\omega_n} \int_{\partial D} \left(E_2(x, \xi) \frac{\partial u}{\partial \nu} - \frac{\partial E_2(x, \xi)}{\partial \nu} u \right) ds_\xi,$$

where $\omega_n = |\partial S|$ is the surface area of the unit sphere in $\mathbb{R}^n$, ν is the outward unit normal to ∂D .

We give a similar integral representation of a function $u \in C^{2m}(D) \cap C^{2m-1}(\bar{D})$.

Theorem 1. The function $u \in C^{2m}(D) \cap C^{2m-1}(\bar{D})$ for $m \in \mathbb{N}$ has the following integral representation

$$u(x) = \frac{1}{\omega_n} \int_{\partial D} \sum_{k=0}^{m-1} (-1)^k \left(E_{2k+2}(x, \xi) \frac{\partial \Delta^k u}{\partial \nu} - \frac{\partial E_{2k+2}(x, \xi)}{\partial \nu} \Delta^k u \right) ds_\xi + \frac{(-1)^m}{\omega_n} \int_D E_{2m}(x, \xi) \Delta^m u(\xi) d\xi, \quad (7)$$

where $x \in D$.

Proof. Consider the functions $u, v \in C^{2m}(D) \cap C^{2m-1}(\bar{D})$. It is easy to see that these functions satisfy the identity in D [30]

$$\int_D (v \Delta u - u \Delta v) d\xi = \int_{\partial D} \left(v \frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu} u \right) ds_\xi. \quad (8)$$

Substituting here $\Delta^{m-k-1}u$ instead of u , and also $\Delta^k v$ instead of v , where $k = 0, \dots, m-1$ we get

$$\int_D (\Delta^k v \Delta^{m-k} u - \Delta^{k+1} v \Delta^{m-k-1} u) d\xi = \int_{\partial D} \left(\Delta^k v \frac{\partial \Delta^{m-k-1} u}{\partial \nu} - \frac{\partial \Delta^k v}{\partial \nu} \Delta^{m-k-1} u \right) ds_\xi.$$

Let us sum the resulting equality over $k = 0, \dots, m-1$

$$\begin{aligned} \int_D \sum_{k=0}^{m-1} (\Delta^k v \Delta^{m-k} u - \Delta^{k+1} v \Delta^{m-k-1} u) d\xi \\ = \int_{\partial D} \sum_{k=0}^{m-1} \left(\Delta^k v \frac{\partial \Delta^{m-k-1} u}{\partial \nu} - \frac{\partial \Delta^k v}{\partial \nu} \Delta^{m-k-1} u \right) ds_\xi. \end{aligned}$$

Since

$$\begin{aligned} \sum_{k=0}^{m-1} (\Delta^k v \Delta^{m-k} u - \Delta^{k+1} v \Delta^{m-k-1} u) &= \sum_{k=0}^{m-1} \Delta^k v \Delta^{m-k} u - \sum_{k=0}^{m-1} \Delta^{k+1} v \Delta^{m-k-1} u = \\ &= \sum_{k=0}^{m-1} \Delta^k v \Delta^{m-k} u - \sum_{k=1}^m \Delta^k v \Delta^{m-k} u = v \Delta^m u - u \Delta^m v, \end{aligned}$$

then we get

$$\int_D (v\Delta^m u - u\Delta^m v) d\xi = \int_{\partial D} \sum_{k=0}^{m-1} \left(\Delta^k v \frac{\partial \Delta^{m-k-1} u}{\partial \nu} - \frac{\partial \Delta^k v}{\partial \nu} \Delta^{m-k-1} u \right) ds_{\xi}. \quad (9)$$

Obviously, the resulting equality is true for $u, v \in C^{2m}(D) \cap C^{2m-1}(\bar{D})$. Pick out a point $x \in D$ from the domain D together with a closed ball $|\xi - x| \leq \delta$ of such a radius $0 < \delta < 1$ that this ball is in the domain D . The rest of the domain D is denoted by D_δ . Obviously, by construction, $\partial D_\delta = \partial D \cup \{\xi \in \mathbb{R}^n : |\xi - x| = \delta\}$. Let us apply the equality (9) to the domain $D = D_\delta$ in the case when $v(\xi) = E_{2m}(x, \xi)$. At the same time, we take into account that, by virtue of Lemma 2 $\Delta_\xi E_{2m}(x, \xi) = -E_{2m-2}(x, \xi)$ and $\Delta_\xi E_2(x, \xi) = 0$ when $\xi \in D_\delta$

$$\begin{aligned} & \int_{D_\delta} E_{2m}(x, \xi) \Delta^m u(\xi) d\xi \\ &= \int_{\partial D} \sum_{k=0}^{m-1} (-1)^k \left(E_{2m-2k}(x, \xi) \frac{\partial \Delta^{m-k-1} u}{\partial \nu} - \frac{\partial E_{2m-2k}(x, \xi)}{\partial \nu} \Delta^{m-k-1} u \right) ds_{\xi} \\ & \quad - \sum_{k=0}^{m-1} (-1)^k \left(\int_{|\xi-x|=\delta} E_{2m-2k}(x, \xi) \frac{\partial \Delta^{m-k-1} u}{\partial \nu} ds_{\xi} \right. \\ & \quad \left. - \int_{|\xi-x|=\delta} \frac{\partial E_{2m-2k}(x, \xi)}{\partial \nu} \Delta^{m-k-1} u ds_{\xi} \right). \quad (10) \end{aligned}$$

The minus sign before the last sum appeared due to the fact that the internal normal ν to the part of the boundary of the domain D_δ – the sphere $\{\xi : |\xi - x| = \delta\}$ is replaced by the outer normal to this sphere. Denote

$$\begin{aligned} I_k^\delta &= (-1)^{k+1} \int_{|\xi-x|=\delta} E_{2m-2k}(x, \xi) \frac{\partial \Delta^{m-k-1} u}{\partial \nu} ds_{\xi}, \\ J_k^\delta &= (-1)^k \int_{|\xi-x|=\delta} \frac{\partial E_{2m-2k}(x, \xi)}{\partial \nu} \Delta^{m-k-1} u ds_{\xi}, \end{aligned}$$

where $k = 0, \dots, m-1$. Let us estimate the dependence of I_k^δ and J_k^δ on δ .

1⁰. Let $n \in \mathbb{N}_m$. By equality (6) we have

$$E_{2m}(x, \xi)|_{|x-\xi|=\delta} = \frac{(-1)^m \delta^{2m-n}}{(2-n, 2)_m (2, 2)_{m-1}}$$

and since $n \in \mathbb{N}_m \Rightarrow n \in \mathbb{N}_{m-k}$ for $k = 0, \dots, m-1$ we find

$$\begin{aligned} |I_k^\delta| &\leq \int_{|\xi-x|=\delta} |E_{2m-2k}(x, \xi)| \left| \frac{\partial \Delta^{m-k-1} u}{\partial \nu} \right| ds_{\xi} \\ &\leq \delta^{2m-2k-n} \frac{\max_{\xi \in \bar{D}} \|\nabla \Delta^{m-k-1} u\|}{|(2-n, 2)_m| (2, 2)_{m-1}} \omega_n \delta^{n-1} \leq C_k^{(1)} \delta^{2m-2k-1} \leq C_k^{(1)} \delta, \quad (11) \end{aligned}$$

where $C_k^{(1)}$ is a constant depending on u, m , and n . Therefore $I_k^\delta \rightarrow 0$ as $\delta \rightarrow 0$.

Next, consider the value of J_k^δ for $k = 0, \dots, m-2$. It is easy to see that

$$\frac{\partial |x - \xi|^\alpha}{\partial \nu} \Big|_{|x-\xi|=\delta} = \alpha \sum_{i=1}^n \frac{\xi_i - x_i}{|x - \xi|} (\xi_i - x_i) |x - \xi|^{\alpha-2} \Big|_{|x-\xi|=\delta} = \alpha \delta^{\alpha-1}$$

and therefore

$$\begin{aligned} |J_k^\delta| &\leq \int_{|\xi-x|=\delta} \left| \frac{\partial E_{2m-2k}(x, \xi)}{\partial \nu} \right| |\Delta^{m-k-1} u| ds_\xi \\ &\leq \delta^{2m-2k-n-1} \frac{|2m-2k-n| \max_{\xi \in \bar{D}} |\Delta^{m-k-1} u|}{|(2-n, 2)_m| (2, 2)_{m-1}} \omega_n \delta^{n-1} \\ &\leq C_k^{(2)} \delta^{2m-2k-2} \leq C_k^{(2)} \delta^2, \quad (12) \end{aligned}$$

whence it follows that $J_k^\delta \rightarrow 0$ as $\delta \rightarrow 0$.

Now consider the value J_{m-1}^δ . In this case $E_{2m-2(m-1)}(x, \xi) = E_2(x, \xi)$ ($n \neq 2$ since $n \in \mathbb{N}_m$) and that means

$$\left. \frac{\partial E_2(x, \xi)}{\partial \nu} \right|_{|x-\xi|=\delta} = - \sum_{i=1}^n \frac{\xi_i - x_i}{|\xi - x|} \frac{\xi_i - x_i}{|\xi - x|^n} = -\delta^{1-n}. \quad (13)$$

Therefore we have

$$\begin{aligned} \left| J_{m-1}^\delta - (-1)^{m-1} u(x) \int_{|\xi-x|=\delta} \frac{\partial E_2(x, \xi)}{\partial \nu} ds_\xi \right| &\leq \int_{|\xi-x|=\delta} |u(\xi) - u(x)| \left| \frac{\partial E_2(x, \xi)}{\partial \nu} \right| ds_\xi \\ &\leq \delta^{1-n} \max_{|\xi-x| \leq \delta} |u(\xi) - u(x)| \omega_n \delta^{n-1} \leq \omega_f(\delta) \omega_n, \end{aligned}$$

where $\omega_f(\delta)$ is the modulus of continuity of the function f . By Cantor's theorem, $\omega_f(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, which means

$$\begin{aligned} J_{m-1}^\delta &= \left(J_{m-1}^\delta - (-1)^{m-1} u(x) \int_{|\xi-x|=\delta} \frac{\partial E_2(x, \xi)}{\partial \nu} ds_\xi \right) \\ &\quad + (-1)^m u(x) \frac{\omega_n \delta^{n-1}}{\delta^{n-1}} \rightarrow (-1)^m \omega_n u(x), \quad (14) \end{aligned}$$

as $\delta \rightarrow 0$.

2⁰. Let $n \in \mathbb{N}_m^c$. By equality (6) we have

$$E_{2m}(x, \xi) = \frac{(-1)^m |x - \xi|^{2m-n}}{(2-n, 2)_m^* (2, 2)_{m-1}} \left(\ln |x - \xi| - \sum_{k=1}^{m-n/2} \frac{1}{2k} - \sum_{k=n/2}^{m-1} \frac{1}{2k} \right).$$

Let us estimate the value of I_k^δ ($0 < \delta < 1$). If it turns out that for a given k the inclusion $n \in \mathbb{N}_{m-k}$ is true, then we have the estimate (11), otherwise

$$\begin{aligned} |I_k^\delta| &\leq \int_{|\xi-x|=\delta} |E_{2m-2k}(x, \xi)| \left| \frac{\partial \Delta^{m-k-1} u}{\partial \nu} \right| ds_\xi \\ &\leq \delta^{2m-2k-n} \frac{\max_{\xi \in \bar{D}} \|\nabla \Delta^{m-k-1} u\|}{|(2-n, 2)_m^*| (2, 2)_{m-1}} \omega_n \delta^{n-1} \left| \ln \delta - \sum_{k=1}^{m-1} \frac{1}{k} \right| \\ &\leq C_k^{(3)} \delta^{2m-2k-1} \left(-\ln \delta + \sum_{k=1}^{m-1} \frac{1}{k} \right) \leq C_k^{(3)} \delta \left(-\ln \delta + \sum_{k=1}^{m-1} \frac{1}{k} \right). \end{aligned}$$

In both cases we have $I_k^\delta \rightarrow 0$ for $\delta \rightarrow 0$, where $k = 0, \dots, m-1$.

Next, consider the value of J_k^δ for $k = 0, \dots, m-1$. It is easy to calculate that

$$\begin{aligned} & \frac{\partial}{\partial \nu} \left(|x - \xi|^\alpha (\ln |x - \xi| + C) \right) \Big|_{|x - \xi| = \delta} \\ &= \sum_{i=1}^n \frac{(\xi_i - x_i)^2}{|x - \xi|} (\alpha \ln |x - \xi| + \alpha C + 1) |x - \xi|^{\alpha-2} \Big|_{|x - \xi| = \delta} = \alpha \delta^{\alpha-1} (\ln \delta + C + 1/\alpha). \end{aligned}$$

If $k \leq m-2$ is such that $n \in \mathbb{N}_{m-k}$, then we have the estimate (12), otherwise $n \in \mathbb{N}_{m-k}^c$ and then

$$\begin{aligned} |J_k^\delta| &\leq \int_{|\xi - x| = \delta} \left| \frac{\partial E_{2m-2k}(x, \xi)}{\partial \nu} \right| |\Delta^{m-k-1} u| ds_\xi \\ &\leq \delta^{2m-2k-n-1} C_k^{(4)} |2 \ln \delta| \max_{\xi \in D} |\Delta^{m-k-1} u| \omega_n \delta^{n-1} \\ &\leq C_k^{(5)} \delta^{2m-2k-2} |\ln \delta| \leq C_k^{(5)} \delta^2 |\ln \delta| \end{aligned}$$

for a small $\delta > 0$ such that $|\ln \delta|$ is greater than the modulus of the constant from the previous inequality. This implies that $J_k^\delta \rightarrow 0$ as $\delta \rightarrow 0$.

Finally, consider the value of J_{m-1}^δ when $n \in \mathbb{N}_{m-m+1}^* = \mathbb{N}_1^* = \{2\}$. In this case $E_{2m-2(m-1)}(x, \xi) = E_2(x, \xi) = -\ln |x - \xi|$ and therefore it is easy to calculate that

$$\frac{\partial E_2(x, \xi)}{\partial \nu} \Big|_{|x - \xi| = \delta} = - \sum_{i=1}^2 \frac{\xi_i - x_i}{|\xi - x|} \frac{\xi_i - x_i}{|\xi - x|^2} = -\delta^{-1} = -\delta^{1-n}.$$

This is the same as (13). Therefore, in this case, the derivation of equality (14), which is based on the written equality, is similar, i.e., $J_{m-1}^\delta \rightarrow (-1)^m \omega_n u(x)$ for $\delta \rightarrow 0$ in this case as well.

Thus, in all considered cases $I_k^\delta \rightarrow 0$, $k = 0, 1, \dots, m-1$ and $J_k^\delta \rightarrow 0$, $k = 0, 1, \dots, m-2$, and $J_{m-1}^\delta \rightarrow (-1)^m \omega_n u(x)$ as $\delta \rightarrow 0$.

In the equality (10) we pass to the limit as $\delta \rightarrow 0$. Since $2m - n \geq 2 - n$, the singularity in the volume integral is left integrable. Therefore we have

$$\lim_{\delta \rightarrow 0} \int_{D_\delta} E_{2m}(x, \xi) f(\xi) d\xi = \int_D E_{2m}(x, \xi) f(\xi) d\xi$$

and that means

$$\begin{aligned} \int_D E_{2m}(x, \xi) \Delta^{2m} u(\xi) d\xi &= \int_{\partial D} \sum_{k=0}^{m-1} (-1)^k \left(E_{2m-2k}(x, \xi) \frac{\partial \Delta^{m-k-1} u}{\partial \nu} \right. \\ &\quad \left. - \frac{\partial E_{2m-2k}(x, \xi)}{\partial \nu} \Delta^{m-k-1} u \right) ds_\xi + (-1)^m \omega_n u(x). \end{aligned}$$

Transferring all the integrals from the right side of the resulting equality to the left side, dividing by $(-1)^m \omega_n$ and changing the summation index $k \rightarrow m - k - 1$, we get the equality (7). The theorem is proved. $\square$

Remark 2. If in equality (7) the function $u(x)$ is a harmonic function, then only one term remains in the sum at $k = 0$, the volume potential becomes 0, and we get the known equality.

Let $n \geq 3$. In the reasoning given below, we also need the following function

$$E_{2k}^r(x, \xi) = \frac{1}{\omega_n} \int_S E_{2k-2}^r(x, y) E_2(y, \xi) dy, \quad k \geq 2, \quad (15)$$

where $E_2^r(x, \xi) = E_2(x, \xi)$. For example,

$$E_4^r(x, \xi) = \frac{1}{\omega_n} \int_S E_2(x, y) E_2(y, \xi) dy, \quad E_6^r(x, \xi) = \frac{1}{\omega_n^2} \int_S E_2(x, \eta) \int_S E_2(\eta, y) E_2(y, \xi) dy d\eta.$$

Lemma 3. The function $E_{2m}^r(x, \xi)$ ($m > 1$) is defined for $\xi, x \in S$, $\xi \neq x$ and may have a singularity at $\xi = x$ such that $E_{2m}^r(x, \xi) \leq C|x - \xi|^{3-n}$, where C is some positive constant. For $\xi \neq x$ the equality $\Delta E_{2m}^r(x, \xi) = -E_{2m-2}^r(x, \xi)$ holds true.

Proof. Indeed, as shown in [14], the function $E_4^r(x, \xi)$ is defined for $\xi \neq x$ and its singularity for $\xi = x$ is at most $O(|x - \xi|^{3-n})$. Assume by induction that $E_{2m-2}^r(x, \xi)$ is defined for $\xi \neq x$ and $E_{2m-2}^r(x, \xi) \leq C_1|x - \xi|^{3-n}$. Then for $\xi \neq x$ the integral from (15) defining the function $E_{2m}^r(x, \xi)$ has two integrable singularities at $y = x$ and $y = \xi$ of orders $n - 3$ and $n - 2$, respectively. Hence $E_{2m}^r(x, \xi)$ is defined and has continuous derivatives with respect to ξ in $\bar{S}$ for $\xi \neq x \in S$ [37]. Besides

$$\begin{aligned} E_{2m}^r(x, \xi) &\leq C_2 \int_{|y|<1} |x - y|^{3-n} |y - \xi|^{2-n} dy = C_2 \int_{|\xi+\eta|<1} |x - \xi - \eta|^{3-n} |\eta|^{n-2} d\eta \\ &\leq C_2 \int_{|\eta|<2} |z - \eta|^{3-n} |\eta|^{2-n} d\eta = C_2 \left(\int_{|z|<|\eta|<2} + \int_{|\eta|<|z|} \right) |z - \eta|^{3-n} |\eta|^{2-n} d\eta \equiv I_1 + I_2, \end{aligned}$$

where $z = x - \xi$ is denoted. For the first integral I_1 , since $3 - n \leq 0$, we have

$$I_1 \leq C_2 |z|^{3-n} \int_{|\eta|<2} |z - \eta|^{3-n} |\eta|^{-1} d\eta = O(|z|^{3-n}),$$

since according to [37] the integral is bounded. Besides

$$I_2 = C_2 |z|^{2-n} |z|^n \int_{|\theta||z|<|z|} |z - \theta|^{3-n} |\theta|^{2-n} d\theta = C_2 |z|^{5-n} \int_{|\theta|<1} |\hat{z} - \theta|^{3-n} |\theta|^{2-n} d\theta,$$

where $\hat{z} = z/|z|$. Since the integral is continuous function for $\hat{z} \in \partial S$ [37], then $I_2 = O(|x - \xi|^{5-n})$. Thus, $E_{2m}^r(x, \xi) = O(|x - \xi|^{3-n})$. The induction step is proven.

Let $\xi \neq x \in S$. Denote $S_\delta = S \setminus \{y \in \mathbb{R}^n : |x - y| \leq \delta\}$ and take $\delta > 0$ so small that $\xi \in S_\delta$, i.e., $|x - \xi| > \delta$ and $\{y \in \mathbb{R}^n : |x - y| \leq \delta\} \subset S$. Then

$$E_{2m}^r(x, \xi) = \frac{1}{\omega_n} \int_{S_\delta} E_{2m-2}^r(x, y) E_2(y, \xi) dy + \frac{1}{\omega_n} \int_{|y-x| \leq \delta} E_{2m-2}^r(x, y) E_2(y, \xi) dy.$$

Since $\xi \in S_\delta$ and $x \notin S_\delta$, by the property of the volume potential

$$\frac{1}{\omega_n} \Delta_\xi \int_{S_\delta} E_{2m-2}^r(x, y) E_2(y, \xi) dy = -E_{2m-2}^r(x, \xi),$$

the second integral has no singularity under the integral with respect to ξ and hence is a harmonic function in ξ . Therefore, for $\xi \neq x$ we obtain the equality $\Delta_\xi E_{2m}^r(x, \xi) = -E_{2m-2}^r(x, \xi)$. The lemma is proved. $\square$

Remark 3. By the singularity erasure theorem [37] the function $h_{2k}(x, \xi) = E_{2k}(x, \xi) - E_{2k}^r(x, \xi)$ is k -harmonic in S with respect to ξ .

3. Green's Function of the Riquier–Neumann Problem

Now we define Green's function of the Riquier–Neumann problem for $n > 2$.

Definition 1. The function of the form

$$\mathcal{N}_{2m}(x, \xi) = E_{2m}(x, \xi) + g_{2m}^n(x, \xi), \quad m \geq 1, \quad (16)$$

where $g_{2m}^n(x, \xi)$ is an m -harmonic function in $x, \xi \in S$ such that for $x \in S$

$$\frac{\partial \mathcal{N}_{2m}(x, \xi)}{\partial v_\xi} \Big|_{\xi \in \partial S} = \dots = \frac{\partial \Delta_\xi^{m-2} \mathcal{N}_{2m}(x, \xi)}{\partial v_\xi} \Big|_{\xi \in \partial S} = 0, \quad \frac{\partial \Delta_\xi^{m-1} \mathcal{N}_{2m}(x, \xi)}{\partial v_\xi} \Big|_{\xi \in \partial S} = (-1)^m, \quad (17)$$

we call Green's function of the Riquier–Neumann problem (1).

Theorem 2. The function $\mathcal{N}_{2m}(x, \xi)$ for $\xi, x \in S$ and $m > 1$, defined recursively by the equality

$$\mathcal{N}_{2m}(x, \xi) = \frac{1}{\omega_n} \left(\int_S \mathcal{N}_{2m-2}(x, y) \mathcal{N}_2(y, \xi) dy - \frac{1}{\tau_n} \int_S \mathcal{N}_{2m-2}(x, y) dy \cdot \int_S \mathcal{N}_2(y, \xi) dy \right), \quad (18)$$

where $\tau_n = |S|$, and $\mathcal{N}_2(x, y)$ is Green's function of the Neumann problem from (3), is Green's function of the Riquier–Neumann problem (1). The function $\mathcal{N}_{2m}(x, \xi)$ has the property

$$\Lambda_x \mathcal{N}_{2k}(x, \xi) \Big|_{x \in \partial S} = 0, \quad k > 1, \quad \Lambda_x \mathcal{N}_2(x, \xi) \Big|_{x \in \partial S} = -1, \quad \xi \in S. \quad (19)$$

The function $g_{2m}^n(x, \xi)$ from the representation (16) has continuous derivatives with respect to ξ in $\bar{S}$ as $x \in S$.

Proof. Let us prove the theorem by the method of mathematical induction on m . For $m = 2$ the assertion of the theorem is proved in [14] (Theorem 4). In this case, from the representation

$$\mathcal{N}_4(x, \xi) = \frac{1}{\omega_n} \left(\int_S \mathcal{N}_2(x, y) \mathcal{N}_2(y, \xi) dy - \frac{1}{\tau_n} \int_S \mathcal{N}_2(x, y) dy \cdot \int_S \mathcal{N}_2(y, \xi) dy \right),$$

using (3) it is easy to get the equality

$$\mathcal{N}_4(x, \xi) = E_4^r(x, \xi) + \hat{g}_4^n(x, \xi),$$

where the harmonic function $\hat{g}_4^n(x, \xi)$ is defined as

$$\begin{aligned} \hat{g}_4^n(x, \xi) = & -\frac{1}{\omega_n} \left(\int_S E_2(x, y) E_0(y, \xi) dy + \int_S E_0(x, y) E_2(y, \xi) dy \right. \\ & \left. - \int_S E_0(x, y) E_0(y, \xi) dy + \frac{1}{\tau_n} \int_S \mathcal{N}_2(x, y) dy \cdot \int_S \mathcal{N}_2(y, \xi) dy \right). \end{aligned}$$

Indeed, the first and third integrals in the resulting formula are harmonic functions with respect to ξ , since the singularity in the first integral is integrable, and differentiation does not increase this singularity. The second and fourth integrals, by the property of the volume potential, are biharmonic functions with respect to ξ , since the harmonic function $E_0(x, \xi)$ has bounded derivatives with respect to ξ in $\bar{S}$ as $x \in S$. Moreover, $\hat{g}_4^n(x, \xi)$, due to the singularities of the order $|\xi - y|^{2-n}$ under the integrals, has continuous derivatives with respect to ξ in $\bar{S}$ for $x \in S$ [37]. If we recall that $E_4(x, \xi) = E_4^r(x, \xi) + h_4(x, \xi)$, then we have

$$\mathcal{N}_4(x, \xi) = E_4(x, \xi) + \hat{g}_4^n(x, \xi) - h_4(x, \xi) \equiv E_4(x, \xi) + g_4^n(x, \xi),$$

where $g_4^n(x, \xi)$, due to the properties of $\hat{g}_4^n$ and h_4 , has bounded derivatives with respect to ξ in $\bar{S}$ for $x \in S$.

Let the assertion of the theorem be true for some $m \geq 2$. Then, according to Definition 1, Green's function $\mathcal{N}_{2m-2}(x, \xi)$ has the representation (16)

$$\mathcal{N}_{2m-2}(x, \xi) = E_{2m-2}(x, \xi) + g_{2m-2}^n(x, \xi),$$

where $g_{2m-2}^n(x, \xi)$ is some $(m-1)$ -harmonic function in S with respect to ξ for $x \in S$ that has bounded derivatives with respect to ξ in $\bar{S}$. If we recall the function $E_{2k}^r(x, \xi)$ from (15) and the k -harmonic function $h_{2k}(x, \xi) = E_{2k}(x, xi) - E_{2k}^r(x, \xi)$, then we can write

$$\mathcal{N}_{2m-2}(x, \xi) = E_{2m-2}^r(x, \xi) + g_{2m-2}^n(x, \xi) + h_{2m-2}(x, \xi) \equiv E_{2m-2}^r(x, \xi) + \hat{g}_{2m-2}^n(x, \xi). \quad (20)$$

Let us show that the function

$$\mathcal{N}_{2m}(x, \xi) = \frac{1}{\omega_n} \left(\int_S \mathcal{N}_{2m-2}(x, y) \mathcal{N}_2(y, \xi) dy - \frac{1}{\tau_n} \int_S \mathcal{N}_{2m-2}(x, y) dy \cdot \int_S \mathcal{N}_2(y, \xi) dy \right)$$

for $m \geq 2$ can be represented as (16). Using (20) and keeping in mind (3) it is easy to get that

$$\begin{aligned} \mathcal{N}_{2m}(x, \xi) = E_{2m}^r(x, \xi) - \frac{1}{\omega_n} \left(\int_S E_{2m-2}^r(x, y) E_0(y, \xi) dy - \int_S \hat{g}_{2m-2}^n(x, y) E_2(y, \xi) dy \right. \\ \left. + \int_S \hat{g}_{2m-2}^n(x, y) E_0(y, \xi) dy + \frac{1}{\tau_n} \int_S \mathcal{N}_2(x, y) dy \cdot \int_S \mathcal{N}_2(y, \xi) dy \right), \end{aligned}$$

where the function $E_{2k}^r(x, \xi)$ is defined in (15). Let us estimate the integral terms in the obtained equality. The first and third integrals are harmonic functions with respect to ξ (the singularity in the first integral is integrable, and differentiation does not increase this singularity). The second integral, by the property of the volume potential, is an m -harmonic function with respect to ξ , since the $(m-1)$ -harmonic function $\hat{g}_{2m-2}^n(x, \xi)$ has bounded derivatives with respect to ξ in $\bar{S}$ as $x \in S$. The fourth integral containing the variable ξ is a biharmonic function in S . If the sum of all integral terms in this equality is denoted by $\hat{g}_{2m}^n(x, \xi)$, then due to the fact that the singularities under the integrals have order no higher than $|\xi - y|^{3-n}$ (see Lemma 3), the function $\hat{g}_{2m}^n(x, \xi)$ has continuous derivatives with respect to ξ in $\bar{S}$ as $x \in S$. Finally, if we recall that $h_{2m}(x, \xi) = E_{2m}(x, \xi) - E_{2m}^r(x, \xi)$, then we have (16) for $g_{2m}^n(x, \xi) = \hat{g}_{2m}^n(x, \xi) - h_{2m}(x, \xi)$. The equality (16) is proved.

Let us check the boundary conditions for the function $\mathcal{N}_{2m}(x, \xi)$ from Definition 1. Due to the symmetry of the function $\mathcal{N}_2(x, \xi)$, we have $\Lambda_\xi \mathcal{N}_2(x, \xi)|_{\xi \in \partial S} = -1$. Therefore

$$\begin{aligned} \frac{\partial \mathcal{N}_{2m}(x, \xi)}{\partial \nu_\xi} \Big|_{\xi \in \partial S} &= \Lambda_\xi \mathcal{N}_{2m}(x, \xi) \Big|_{\xi \in \partial S} \\ &= \frac{1}{\omega_n} \left(- \int_S \mathcal{N}_{2m-2}(x, y) dy + \frac{1}{\tau_n} \int_S \mathcal{N}_{2m-2}(x, y) dy \cdot \tau_n \right) = 0. \end{aligned}$$

For $x \in S$, by the property of the volume potential and due to the continuous differentiability of the function $\mathcal{N}_{2m-2}(x, \xi)$ with respect to $\xi \in \bar{S}$, but for $\xi \neq x$, we find

$$\Delta_\xi \mathcal{N}_{2m}(x, \xi) = -\mathcal{N}_{2m-2}(x, \xi) + \frac{1}{\tau_n} \int_S \mathcal{N}_{2m-2}(x, y) dy. \quad (21)$$

Therefore, by the induction hypothesis, for $x \in S$ we have

$$\frac{\partial \Delta_\xi^k \mathcal{N}_{2m}(x, \xi)}{\partial \nu_\xi} \Big|_{\xi \in \partial S} = - \frac{\partial \Delta_\xi^{k-1} \mathcal{N}_{2m-2}(x, \xi)}{\partial \nu_\xi} \Big|_{\xi \in \partial S} = 0, \quad k = 1, \dots, m-2$$

and

$$\frac{\partial \Delta_\xi^{m-1} \mathcal{N}_{2m}(x, \xi)}{\partial \nu_\xi} \Big|_{\xi \in \partial S} = - \frac{\partial \Delta_\xi^{m-2} \mathcal{N}_{2m-2}(x, \xi)}{\partial \nu_\xi} \Big|_{\xi \in \partial S} = -(-1)^{m-1} = (-1)^m,$$

which proves the equalities (17).

Let us prove the equality (19) for $k > 1$, since for $k = 1$ it follows from (4). Let $k = 2$, then, due to the weak singularity of the function $\mathcal{N}_2(x, \xi)$, for $x \in \partial S$ and $\xi \in S$, we write

$$\begin{aligned} \Lambda_x \mathcal{N}_4(x, \xi) &= \frac{1}{\omega_n} \int_S \Lambda_x \mathcal{N}_2(x, y) \mathcal{N}_2(y, \xi) dy - \frac{1}{\tau_n} \int_S \Lambda_x \mathcal{N}_2(x, y) dy \\ &\quad \cdot \frac{1}{\omega_n} \int_S \mathcal{N}_2(y, \xi) dy = -\frac{1}{\omega_n} \int_S \mathcal{N}_2(y, \xi) dy + \frac{1}{\omega_n} \int_S \mathcal{N}_2(y, \xi) dy = 0. \end{aligned}$$

Therefore, for $k > 2$ and $x \in \partial S$, $\xi \in S$, using the equality (18) we obtain

$$\begin{aligned} \Lambda_x \mathcal{N}_{2k}(x, \xi) &= \frac{1}{\omega_n} \int_S \Lambda_x \mathcal{N}_{2k-2}(x, y) \mathcal{N}_2(y, \xi) dy \\ &\quad - \frac{1}{\tau_n} \int_S \Lambda_x \mathcal{N}_{2k-2}(x, y) dy \cdot \frac{1}{\omega_n} \int_S \mathcal{N}_2(y, \xi) dy = 0, \end{aligned}$$

whence follows (19). The theorem is proved. $\square$

Example 1. To find a solution to the Riquier–Neumann problem (1) with polynomials in the boundary conditions or on the right side of the equation, the following formula is needed

$$\frac{1}{\omega_n} \int_S \mathcal{N}_2(x, \xi) |\xi|^{2l} d\xi = -\frac{|x|^{2l+2}}{(2l+2)(2l+n)} + \frac{1}{(2l+2)(n-2)}, \quad (22)$$

where $l \in \mathbb{N}_0$. Let us prove it. Note that in the paper [9] (remark 2) a similar formula is obtained

$$\frac{1}{\omega_n} \int_S \mathcal{N}_2(x, \xi) |\xi|^{2l} H_k(\xi) d\xi = -\frac{|x|^{2l+2} - (2l+2+k)/k H_k(x)}{(2l+2)(2l+2k+n)} H_k(x), \quad (23)$$

where $H_k(x)$ is a homogeneous harmonic polynomial of degree k , which does not work in the case under consideration, since the right-hand side of it is not defined for $k = 0$.

Let $\{H_k^{(i)}(x) : i = 1, \dots, h_k, k \in \mathbb{N}_0\}$ be the complete system of homogeneous of degree $k \in \mathbb{N}_0$ spherical harmonics orthogonal on ∂S such that $\int_{\partial S} (H_k^{(i)}(\xi))^2 d\xi = \omega_n$ [38] and $h_k = \frac{2k+n-2}{n-2} \binom{k+n-3}{n-3}$ for $n > 2$ ($h_k = 2$ for $n = 2$) is the dimension of the basis of homogeneous harmonic polynomials of degree k . In [9] (Theorem 1) it is obtained that the equality

$$\frac{1}{\omega_n} \int_S E_2(x, \xi) |\xi|^{2l} H_k(\xi) d\xi = -\frac{|x|^{2l+2} H_k(x)}{(2l+2)(2l+2k+n)} + \frac{H_k(x)}{(2l+2)(2k+n-2)},$$

holds true, where $k \in \mathbb{N}_0$ and $l \in \mathbb{N}_0$. From here we get

$$\frac{1}{\omega_n} \int_S E_2(x, \xi) |\xi|^{2l} d\xi = -\frac{|x|^{2l+2}}{(2l+2)(2l+n)} + \frac{1}{(2l+2)(n-2)}.$$

In [9] (Theorem 1) it is also proved that for $x \in S$ and $\xi \in \bar{S}$

$$E_0(x, \xi) = -\sum_{k=1}^{\infty} \frac{k+n-2}{k(2k+n-2)} \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi),$$

where the series converges uniformly with respect to ξ . Therefore we have

$$\int_S E_0(x, \xi) |\xi|^{2l} d\xi = -\sum_{k=1}^{\infty} \frac{k+n-2}{k(2k+n-2)} \sum_{i=1}^{h_k} H_k^{(i)}(x) \int_S H_k^{(i)}(\xi) |\xi|^{2l} d\xi = 0$$

and hence, taking into account (3), we obtain the formula to be proved.

4. Solution of the Riquier–Neumann Problem

Let us find an integral representation of a solution to the Riquier–Neumann problem (1) for $n > 2$.

Theorem 3. Let the function $u \in C^{2m}(S) \cap C^{2m-1}(\bar{S})$ be a solution to the Riquier–Neumann problem (1), then it can be represented as

$$u(x) = \frac{(-1)^{m-1}}{\omega_n} \int_{\partial S} \sum_{k=0}^{m-1} \left(\Delta_{\xi}^{m-k-1} \mathcal{N}_{2m}(x, \xi) \right) \varphi_k(\xi) ds_{\xi} + \frac{(-1)^m}{\omega_n} \int_S \mathcal{N}_{2m}(x, \xi) f(\xi) d\xi + C, \quad (24)$$

where C is an arbitrary constant.

Proof. Let $u \in C^{2m}(S) \cap C^{2m-1}(\bar{S})$ be a solution to the Riquier–Neumann problem (1). Let us use the equality

$$\int_D (v \Delta^m u - u \Delta^m v) d\xi = \int_{\partial D} \sum_{k=0}^{m-1} \left(\Delta^k v \frac{\partial \Delta^{m-k-1} u}{\partial \nu} - \frac{\partial \Delta^k v}{\partial \nu} \Delta^{m-k-1} u \right) ds_{\xi}$$

for $D = \{\xi : |\xi| < 1 - \varepsilon\}$, where $\varepsilon > 0$ is sufficiently small ($x \in D$) and $v(\xi) = E_{2m}(x, \xi)$. Similarly to the proof of equality (7) we find

$$\begin{aligned} & \int_D E_{2m}(x, \xi) f(\xi) d\xi \\ &= \int_{\partial D} \sum_{k=0}^{m-1} \left(\Delta_{\xi}^k E_{2m}(x, \xi) \frac{\partial \Delta^{m-k-1} u}{\partial \nu} - \frac{\partial \Delta_{\xi}^k E_{2m}(x, \xi)}{\partial \nu} \Delta^{m-k-1} u \right) ds_{\xi} + (-1)^m \omega_n u(x). \end{aligned}$$

If we now again use the previous formula for the same domain D , but for $v(\xi) = g_{2m}^n(x, \xi)$ (an m -harmonic function in S from (16)), we obtain similar equality with $g_{2m}^n(x, \xi)$ instead of $E_{2m}(x, \xi)$ and without the last term on the right. Adding these equalities, grouping the integral terms and changing the summation index $k \rightarrow m - k - 1$, we get

$$\begin{aligned} u(x) &= \frac{(-1)^m}{\omega_n} \int_{|\xi|=1-\varepsilon} \sum_{k=0}^{m-1} \left(\frac{\partial \Delta_{\xi}^{m-k-1} \mathcal{N}_{2m}(x, \xi)}{\partial \nu} \Delta^k u - \Delta_{\xi}^{m-k-1} \mathcal{N}_{2m}(x, \xi) \frac{\partial \Delta^k u}{\partial \nu} \right) ds_{\xi} \\ &+ \frac{(-1)^m}{\omega_n} \int_{|\xi|<1-\varepsilon} \mathcal{N}_{2m}(x, \xi) f(\xi) d\xi. \quad (25) \end{aligned}$$

Let us pass to the limit as $\varepsilon \rightarrow +0$. At the same time, we take into account that the functions $g_{2k}^n(x, \xi)$, and hence $\mathcal{N}_{2m}(x, \xi)$ and its derivatives with respect to ξ , are continuous with respect to ξ in $\{\xi : 1 - \varepsilon < |\xi| \leq 1\}$ for fixed $x \in S$. Therefore, due to the properties of Green's function (17), surface integrals over $S = \{\xi : |\xi| = 1\}$ containing the functions $\frac{\partial}{\partial \nu_{\xi}} (\Delta^{m-k-1} \mathcal{N}_{2m}(x, \xi))$ for $k = 1, \dots, m-1$ turn to 0, and the function $\frac{\partial}{\partial \nu_{\xi}} (\Delta^{m-1} \mathcal{N}_{2m}(x, \xi))$ becomes $(-1)^m$. Moreover, due to the boundary conditions of the Riquier–Neumann problem, we also have $\frac{\partial}{\partial \nu} \Delta^k u(\xi) \rightarrow \varphi_k(\xi)$, $\varepsilon \rightarrow +0$ for $k = 0, \dots, m-1$. Thus, from (25) in the limit as $\varepsilon \rightarrow +0$ we obtain the representation (24), where $C = \frac{1}{\omega_n} \int_{\partial S} u(\xi) ds_{\xi}$. The theorem is proved. $\square$

Lemma 4. 1. Let $f \in C^1(\bar{S})$, then the function

$$u_f(x) = \frac{(-1)^m}{\omega_n} \int_S \mathcal{N}_{2m}(x, \xi) f(\xi) ds_{\xi} \quad (26)$$

is a solution to the following Riquier–Neumann problem

$$\Delta^m u(x) = f(x), x \in S,$$

$$\frac{\partial \Delta^k u}{\partial \nu} \Big|_{\partial S} = 0, k = 0, \dots, m-2, \frac{\partial \Delta^{m-1} u}{\partial \nu} \Big|_{\partial S} = \frac{1}{\omega_n} \int_S f(\xi) d\xi.$$

2. Let $\psi \in C^{1+\varepsilon}(\partial S)$ ($\varepsilon > 0$), then the function

$$v_\psi(x) = \frac{(-1)^{m-1}}{\omega_n} \int_{\partial S} \mathcal{N}_{2m}(x, \xi) \psi(\xi) ds_\xi \quad (27)$$

is a solution to the following Riquier–Neumann problem

$$\Delta^m v(x) = 0, x \in S,$$

$$\frac{\partial \Delta^k v}{\partial \nu} \Big|_{\partial S} = 0, k = 0, \dots, m-2, \frac{\partial \Delta^{m-1} v}{\partial \nu} \Big|_{\partial S} = \psi(x) - \frac{1}{\omega_n} \int_{\partial S} \psi(\xi) ds_\xi.$$

Proof. 1. We study the function $u_f(x)$ without specifying the smoothness of the function $f(x)$. To do this, we introduce the functions

$$\hat{\mathcal{N}}_2(x, \xi) = \mathcal{N}_2(x, \xi) - \frac{1}{\tau_n} \int_S \mathcal{N}_2(y, \xi) dy,$$

$$\hat{\mathcal{N}}_{2k}(x, \xi) = \frac{1}{\omega_n} \int_S \hat{\mathcal{N}}_{2k-2}(x, y) \hat{\mathcal{N}}_2(y, \xi) dy, \quad k > 1.$$

The function $\hat{\mathcal{N}}_2(x, \xi)$ has the property

$$\int_S \hat{\mathcal{N}}_2(x, \xi) dx = \int_S \mathcal{N}_2(x, \xi) dx - \frac{\tau_n}{\tau_n} \int_S \mathcal{N}_2(y, \xi) dy = 0.$$

It is easy to see that the equality

$$\begin{aligned} \mathcal{N}_{2m}(x, \xi) &= \frac{1}{\omega_n} \int_S \mathcal{N}_{2m-2}(x, y) \left(\mathcal{N}_2(y, \xi) - \frac{1}{\tau_n} \int_S \mathcal{N}_2(y_1, \xi) dy_1 \right) dy \\ &= \frac{1}{\omega_n} \int_S \mathcal{N}_{2m-2}(x, y) \hat{\mathcal{N}}_2(y, \xi) dy \end{aligned} \quad (28)$$

holds true. Therefore, due to the properties of the volume potential, by rearranging the integrals, we have

$$\begin{aligned} \Delta_x \frac{1}{\omega_n} \int_S \mathcal{N}_4(x, \xi) f(\xi) d\xi &= \Delta_x \frac{1}{\omega_n} \int_S \mathcal{N}_2(x, y) dy \frac{1}{\omega_n} \int_S \hat{\mathcal{N}}_2(y, \xi) f(\xi) d\xi \\ &= -\frac{1}{\omega_n} \int_S \hat{\mathcal{N}}_2(x, \xi) f(\xi) d\xi, \end{aligned}$$

whence similarly it follows

$$\begin{aligned} \Delta_x \frac{1}{\omega_n} \int_S \mathcal{N}_6(x, \xi) f(\xi) d\xi &= \Delta_x \frac{1}{\omega_n} \int_S \mathcal{N}_4(x, y) dy \frac{1}{\omega_n} \int_S \hat{\mathcal{N}}_2(y, \xi) f(\xi) d\xi \\ &= -\frac{1}{\omega_n} \int_S \hat{\mathcal{N}}_2(x, y) dy \frac{1}{\omega_n} \int_S \hat{\mathcal{N}}_2(y, \xi) f(\xi) d\xi = -\frac{1}{\omega_n} \int_S \hat{\mathcal{N}}_4(y, \xi) f(\xi) d\xi. \end{aligned}$$

Using the previous equalities by induction and changing the order of integration, we have

$$\begin{aligned}\Delta_x \frac{1}{\omega_n} \int_S \mathcal{N}_{2m}(x, \xi) f(\xi) d\xi &= \Delta_x \frac{1}{\omega_n} \int_S \mathcal{N}_{2m-2}(x, y) dy \frac{1}{\omega_n} \int_S \hat{\mathcal{N}}_2(y, \xi) f(\xi) d\xi \\ &= -\frac{1}{\omega_n} \int_S \hat{\mathcal{N}}_{2m-4}(x, y) dy \frac{1}{\omega_n} \int_S \hat{\mathcal{N}}_2(y, \xi) f(\xi) d\xi = -\frac{1}{\omega_n} \int_S \mathcal{N}_{2m-2}(y, \xi) f(\xi) d\xi. \quad (29)\end{aligned}$$

From the last equality, using the property of volume potential, it is easy to obtain

$$\begin{aligned}\Delta_x^m u_f(x) &= \Delta_x^m \frac{(-1)^m}{\omega_n} \int_S \mathcal{N}_{2m}(x, \xi) f(\xi) d\xi = -\Delta_x \frac{1}{\omega_n} \int_S \hat{\mathcal{N}}_2(x, \xi) f(\xi) d\xi \\ &= -\Delta_x \frac{1}{\omega_n} \int_S \mathcal{N}_2(x, \xi) f(\xi) d\xi = f(x). \quad (30)\end{aligned}$$

Further, due to the weak singularity of the function $\mathcal{N}_{2m}(x, \xi)$ (the singularity is the same as that of the elementary solution $E_{2m}(x, \xi)$ (16)) it can be differentiated under the integral sign, and therefore from (19) it follows that for $m > 1$

$$\frac{\partial u_f}{\partial \nu} \Big|_{\partial S} = \Lambda_x u_f(x) \Big|_{\partial S} = \frac{(-1)^m}{\omega_n} \int_S \Lambda_x \mathcal{N}_{2m}(x, \xi) \Big|_{x \in \partial S} f(\xi) d\xi = 0.$$

Moreover, due to (29) for $k = 1, \dots, m-2$ we get

$$\begin{aligned}\frac{\partial \Delta^k u_f}{\partial \nu} \Big|_{\partial S} &= \Lambda_x \Delta_x^k u_f(x) \Big|_{\partial S} = \frac{(-1)^{m-k}}{\omega_n} \int_S \Lambda_x \hat{\mathcal{N}}_{2m-2k}(x, \xi) \Big|_{x \in \partial S} f(\xi) d\xi \\ &= \frac{(-1)^{m-k}}{\omega_n} \int_S \Lambda_x \mathcal{N}_{2m-2k}(x, \xi) \Big|_{x \in \partial S} f(\xi) d\xi = 0.\end{aligned}$$

Similarly to what was done above, in accordance with (29) and (19), we find

$$\frac{\partial \Delta^{m-1} u_f}{\partial \nu} \Big|_{\partial S} = -\frac{1}{\omega_n} \int_S \Lambda_x \hat{\mathcal{N}}_2(x, \xi) \Big|_{x \in \partial S} f(\xi) d\xi = \frac{1}{\omega_n} \int_S f(\xi) d\xi.$$

This means that the function $u_f(x)$ from (26) is a solution to the Riquier–Neumann problem (1) for $\varphi_k = 0, k = 0, \dots, m-2$ and $\varphi_{m-1} = \frac{1}{\omega_n} \int_S f(\xi) d\xi$.

What smoothness is sufficient to impose on the function $f(x)$ so that the arguments made about $u_f(x)$ are valid? To satisfy the last equality from (30) for the volume potential, it is sufficient that $f \in C^1(\bar{S})$ [30]. Moreover, the volume potential with density $f(x)$ will also be in $C^1(\bar{S})$ [37], and hence the entire chain of equalities in (30) is true.

2. Consider the function $v_\psi(x)$ from (27). Substituting the value $\mathcal{N}_{2m}(x, \xi)$ from (28) and changing the order of integration, we have

$$v_\psi(x) = \frac{(-1)^{m-1}}{\omega_n} \int_S \mathcal{N}_{2m-2}(x, y) dy \frac{1}{\omega_n} \int_{\partial S} \hat{\mathcal{N}}_2(y, \xi) \psi(\xi) ds_\xi.$$

By virtue of the properties of the functions $u_f(x)$ and $\hat{\mathcal{N}}_2(x, \xi)$ obtained above, the function $v_\psi(x)$ is a solution to the Riquier–Neumann problem (1) for $m = m-1, \varphi_k = 0, k = 0, \dots, m-3,$

$$\varphi_{m-2} = \frac{1}{\omega_n} \int_S \frac{1}{\omega_n} \int_{\partial S} \hat{\mathcal{N}}_2(y, \xi) \psi(\xi) ds_\xi dy = \frac{1}{\omega_n} \int_{\partial S} \psi(\xi) ds_\xi \frac{1}{\omega_n} \int_S \mathcal{N}_2(y, \xi) dy = 0$$

and

$$f(x) = \Delta^{m-1} v(x) = \frac{1}{\omega_n} \int_{\partial S} \hat{\mathcal{N}}_2(x, \xi) \psi(\xi) ds_\xi.$$

Therefore, by the property of the surface potential

$$\Delta^m v_\psi(x) = \Delta \Delta^{m-1} v_\psi(x) = \Delta \frac{1}{\omega_n} \int_{\partial S} \mathcal{N}_2(x, \xi) \psi(\xi) ds_\xi = 0,$$

and in addition, by the property (5) of Green's function $\mathcal{N}_2(x, \xi)$, we write

$$\frac{\partial \Delta^{m-1} v_\psi}{\partial \nu} \Big|_{\partial S} = \psi(x) - \frac{1}{\omega_n} \int_{\partial S} \psi(\xi) ds_\xi.$$

Thus, the function $v_\psi(x)$ is a solution to the Riquier–Neumann problem (1) for $f = 0$, $\varphi_k = 0$, $k = 0, \dots, m-2$ and $\varphi_{m-1}(x) = \psi(x) - \frac{1}{\omega_n} \int_{\partial S} \psi(\xi) ds_\xi$, which is to be proved.

What smoothness of the function $\psi(x)$ is sufficient for the validity of the reasoning made above? Since the function $v_\psi(x)$ is represented in a form similar to the form of the function $u_f(x)$ with density $\frac{1}{\omega_n} \int_S \mathcal{N}_2(y, xi) \psi(\xi) ds_\xi$, then, just as in the previous case, it is sufficient that this density is in $C^1(\bar{S})$, and this is satisfied if $\psi \in C^{1+\varepsilon}(\partial S)$ [39]. The lemma is proved. $\square$

Remark 4. If $m = 1$ then for the function ψ it is sufficient to be such that $\psi \in C(\partial S)$.

Theorem 4. Let $\varphi_0 \in C(\partial S)$, $\varphi_k \in C^{1+\varepsilon}(\partial S)$ ($\varepsilon > 0$) for $k = 1, \dots, m-1$ and $f \in C^1(\bar{S})$. Then the function

$$u(x) = \frac{1}{\omega_n} \int_{\partial S} \sum_{k=0}^{m-1} (-1)^k \left(\mathcal{N}_{2k+2}(x, \xi) - \frac{1}{\tau_n} \int_S \mathcal{N}_{2k+2}(x, y) dy \right) \varphi_k(\xi) ds_\xi + \frac{(-1)^m}{\omega_n} \int_S \mathcal{N}_{2m}(x, \xi) f(\xi) d\xi + C \quad (31)$$

is a solution to the Riquier–Neumann problem (1) provided that (see [14])

$$\int_{\partial S} \varphi_{m-1}(\xi) ds_\xi = \int_S f(\xi) d\xi. \quad (32)$$

Proof. Let us prove that the function $u(x)$ defined by the Formula (24) is a solution to the problem

$$\begin{aligned} \Delta^m u(x) &= f(x), \quad x \in S, \\ \frac{\partial \Delta^k u}{\partial \nu} \Big|_{\partial S} &= 0, \quad k = 0, \dots, m-2, \\ \frac{\partial \Delta^{m-1} u}{\partial \nu} \Big|_{\partial S} &= \varphi_{m-1}(x) - \frac{1}{\omega_n} \int_{\partial S} \varphi_{m-1}(\xi) ds_\xi + \frac{1}{\omega_n} \int_S f(\xi) d\xi. \end{aligned} \quad (33)$$

It is easy to see that in this case, under the condition (32), the assertion of the theorem follows. We prove this statement by induction on m . For $m = 2$ in [14] (Theorem 5) it is proved that for a function of the form

$$\begin{aligned} u(x) &= -\frac{1}{\omega_n} \int_{\partial S} \Delta_\xi \mathcal{N}_4(x, \xi) \varphi_0(\xi) ds_\xi - \frac{1}{\omega_n} \int_{\partial S} \mathcal{N}_4(x, \xi) \varphi_1(\xi) ds_\xi \\ &\quad + \frac{1}{\omega_n} \int_S \mathcal{N}_4(x, \xi) f(\xi) d\xi + C \equiv v_1(x) + v_2(x) + v_3(x) + C \end{aligned}$$

the following equalities are true

$$\begin{aligned}\Delta^2 u(x) &= 0 + 0 + f(x) = f(x), \\ \frac{\partial u}{\partial \nu} \Big|_{x \in \partial S} &= \varphi_0(x) + 0 + 0 = \varphi_0(x), \\ \frac{\partial \Delta u}{\partial \nu} \Big|_{x \in \partial S} &= 0 + \left(\varphi_1(x) - \frac{1}{\omega_n} \int_{\partial S} \varphi_1(\xi) d\xi \right) + \frac{1}{\omega_n} \int_S f(\xi) d\xi.\end{aligned}$$

Here, in the equalities on the right, the result of applying the corresponding operators to the functions v_1 , v_2 and v_3 is indicated. This corresponds to the statement being proved for $m = 2$. Assume that the equalities (33) hold true for $m = m - 1$. Let us represent $u(x)$ from (24) as

$$u(x) = u_m(x) + u_f^{(m)}(x) + C, \quad (34)$$

where the function $u_f^{(m)}(x) \equiv u_f(x)$ is defined in (26) (the superscript m is added here to avoid confusion) and

$$u_m(x) = \frac{(-1)^{m-1}}{\omega_n} \int_{\partial S} \sum_{k=0}^{m-1} \left(\Delta_{\xi}^{m-k-1} \mathcal{N}_{2m}(x, \xi) \right) \varphi_k(\xi) ds_{\xi}.$$

Note that the function $u_f^{(m)}(x)$ for $f \in C^1(\bar{S})$ by Lemma 4 is a solution to the problem (33) for $\varphi_k = 0$, $k = 0, \dots, m-1$. Therefore, it is sufficient to prove that the function $u_m(x) = u(x) - u_f^{(m)}(x) + C$ is a solution to the problem (33) for $f = 0$. Let us transform the function $u_f^{(m)}(x)$. From the equality (21) for $k > 1$ it is easy to obtain

$$\Delta_{\xi}^k \mathcal{N}_{2m}(x, \xi) = -\Delta_{\xi}^{k-1} \mathcal{N}_{2m-2}(x, \xi),$$

and therefore

$$\begin{aligned}u_m(x) &= \frac{(-1)^{m-1}}{\omega_n} \left(\int_{\partial S} \mathcal{N}_{2m}(x, \xi) \varphi_{m-1}(\xi) ds_{\xi} + \int_{\partial S} \Delta_{\xi} \mathcal{N}_{2m}(x, \xi) \varphi_{m-2}(\xi) ds_{\xi} \right) \\ &+ \frac{(-1)^{m-2}}{\omega_n} \left(\int_{\partial S} \sum_{k=0}^{m-2} \left(\Delta_{\xi}^{m-k-2} \mathcal{N}_{2m-2}(x, \xi) \right) \varphi_k(\xi) ds_{\xi} - \int_{\partial S} \mathcal{N}_{2m-2}(x, \xi) \varphi_{m-2}(\xi) ds_{\xi} \right) \\ &= u_{m-1}(x) + \frac{(-1)^{m-1}}{\omega_n} \left(\int_{\partial S} \mathcal{N}_{2m}(x, \xi) \varphi_{m-1}(\xi) ds_{\xi} + \int_{\partial S} \Delta_{\xi} \mathcal{N}_{2m}(x, \xi) \varphi_{m-2}(\xi) ds_{\xi} \right. \\ &\quad \left. + \int_{\partial S} \mathcal{N}_{2m-2}(x, \xi) \varphi_{m-2}(\xi) ds_{\xi} \right).\end{aligned}$$

Using the equality (21) and the notation from (26) and (27) we can write

$$\begin{aligned}u_m(x) &= u_{m-1}(x) + v_{\varphi_{m-1}}(x) + \frac{(-1)^{m-1}}{\omega_n} \int_S \mathcal{N}_{2m-2}(x, y) dy \cdot \frac{1}{\tau_n} \int_{\partial S} \varphi_{m-2}(\xi) ds_{\xi} \\ &= u_{m-1}(x) + v_{\varphi_{m-1}}(x) + u_1^{(m-1)}(x) \cdot \frac{1}{\tau_n} \int_{\partial S} \varphi_{m-2}(\xi) ds_{\xi}. \quad (35)\end{aligned}$$

For the legality of using the function $v_{\varphi_{m-1}}(x)$ by Lemma 4 it is sufficient to require $\varphi_0 \in C(\partial S)$, $\varphi_k \in C^{1+\varepsilon}(\partial S)$ for $k = 1, \dots, m-1$. Since, by the induction hypothesis, the function $u_{m-1}(x)$ is $(m-1)$ -harmonic, then, according to Lemma 4, we can write

$$\Delta^m u_m(x) = 0 + 0 + \Delta \frac{1}{\tau_n} \int_{\partial S} \varphi_{m-2}(\xi) ds_{\xi} = 0.$$

Moreover, from (35) it also follows that on ∂S , by virtue of Lemma 4 and the inductive assumption, the equalities hold true

$$\begin{aligned}\frac{\partial \Delta^k u_m}{\partial \nu} &= \frac{\partial \Delta^k u_{m-1}}{\partial \nu} + 0 + 0 = \varphi_k(x), \quad k = 0, \dots, m-3, \\ \frac{\partial \Delta^{m-2} u_m}{\partial \nu} &= \frac{\partial \Delta^{m-2} u_{m-1}}{\partial \nu} + 0 + \frac{1}{\omega_n} \int_S \frac{1}{\tau_n} \int_{\partial S} \varphi_{m-2}(\xi) ds_\xi dy \\ &= \varphi_{m-2}(x) - \frac{1}{\omega_n} \int_{\partial S} \varphi_{m-2}(\xi) ds_\xi + \frac{1}{\omega_n} \int_{\partial S} \varphi_{m-2}(\xi) ds_\xi = \varphi_{m-2}(x), \\ \frac{\partial \Delta^{m-1} u_m}{\partial \nu} &= 0 + \left(\varphi_{m-1}(x) - \frac{1}{\omega_n} \int_{\partial S} \varphi_{m-1}(\xi) ds_\xi \right) + 0 \\ &= \varphi_{m-1}(x) - \frac{1}{\omega_n} \int_{\partial S} \varphi_{m-1}(\xi) ds_\xi,\end{aligned}$$

and hence the function $u_m(x)$ is a solution to the problem (33) for $f = 0$. The step of induction is proved, and hence the function $u(x)$ from (24) is a solution to the problem (1). From the equality (21) it is easy to get the equality

$$\Delta_\xi^k \mathcal{N}_{2m}(x, \xi) = (-1)^k \left(\mathcal{N}_{2m-2k}(x, \xi) - \frac{1}{\tau_n} \int_S \mathcal{N}_{2m-2k}(x, y) dy \right),$$

with the help of which the equality (24) is converted to the form (31). The theorem is proved. $\square$

Remark 5. The necessary and sufficient condition for the solvability of the Riquier–Neumann problem (32) for the polyharmonic equation is previously obtained in [29].

From the proof of Theorem 4 it follows that if the condition (32) is not satisfied, then the function $u(x)$ from (31) is a solution to the Riquier–Neumann problem (33).

5. Particular Case

Let us consider one particular case of the boundary conditions of the Riquier–Neumann problem.

Theorem 5. Let $f = 0$, $\varphi_0 \in C(\partial S)$, $\varphi_k \in C^{1+\varepsilon}(\partial S)$ ($\varepsilon > 0$) and $\int_{\partial S} \varphi_k(\xi) ds_\xi = 0$ for $k = 1, \dots, m-1$. Then a solution of the Riquier–Neumann problem (1) exists and can be written as

$$u(x) = \sum_{k=0}^{m-1} u_k[\varphi_k](x) + C, \quad (36)$$

where C is an arbitrary constant,

$$\begin{aligned}u_k[\varphi](x) &= \frac{(-1)^k}{\omega_n} \int_{\partial S} \mathcal{N}_{2k+2}^0(x, \xi) \varphi(\xi) ds_\xi, \\ \mathcal{N}_{2k+2}^0(x, \xi) &= \frac{1}{\omega_n} \int_S \mathcal{N}_2(x, y) \mathcal{N}_{2k}^0(y, \xi) dy, \quad k > 0; \quad \mathcal{N}_2^0(x, \xi) = \mathcal{N}_2(x, \xi).\end{aligned}$$

The $(k+1)$ -harmonic functions $u_k[\varphi_k](x)$ satisfy the following conditions

$$\begin{aligned}\Delta u_k[\varphi](x) &= u_{k-1}[\varphi](x), \quad \frac{\partial u_k[\varphi]}{\partial \nu} \Big|_{\partial S} = 0, \quad k \in \mathbb{N}; \\ \Delta u_0[\varphi](x) &= 0, \quad \frac{\partial u_0[\varphi]}{\partial \nu} \Big|_{\partial S} = \varphi(x).\end{aligned} \quad (37)$$

Proof. It is easy to see that under the conditions required in the theorem, all the conditions of Theorem 4 are also satisfied, and hence a solution to the Riquier–Neumann problem exists. Let us prove that the equality (36) represents this solution. To do this, it is sufficient

to make sure that the equalities (37) hold true. According to the definition of the function $\mathcal{N}_{2k}^0(x, \xi)$ and by the property of the volume potential, for $k > 0$ we write

$$\begin{aligned}\Delta u_k[\varphi](x) &= \Delta_x \frac{(-1)^k}{\omega_n} \int_{\partial S} \mathcal{N}_{2k+2}^0(x, \xi) \varphi(\xi) ds_\xi \\ &= \frac{(-1)^{k-1}}{\omega_n} \int_{\partial S} \mathcal{N}_{2k}^0(x, \xi) \varphi(\xi) ds_\xi = u_{k-1}[\varphi](x),\end{aligned}$$

and also find $\Delta u_0[\varphi](x) = 0$. Moreover, for $k > 0$, using (5), we get

$$\begin{aligned}\frac{\partial u_k[\varphi]}{\partial \nu} \Big|_{\partial S} &= \frac{(-1)^k}{\omega_n} \int_S \Lambda_x \mathcal{N}_2(x, y) \Big|_{x \in \partial S} \frac{1}{\omega_n} \int_{\partial S} \mathcal{N}_{2k}^0(y, \xi) \varphi(\xi) ds_\xi dy \\ &= \frac{(-1)^{k-1}}{\omega_n} \int_{\partial S} \left(\int_S \mathcal{N}_{2k}^0(y, \xi) dy \right) \varphi(\xi) ds_\xi \equiv \int_{\partial S} F(\xi) \varphi(\xi) ds_\xi.\end{aligned}$$

Using the equality

$$\frac{1}{\omega_n} \int_S \mathcal{N}_{2k}^0(y_1, \xi) dy_1 = \frac{1}{\omega_n} \int_S \mathcal{N}_2(y_1, y_2) dy_1 \dots \frac{1}{\omega_n} \int_S \mathcal{N}_2(y_k, \xi) dy_k,$$

taking into account the symmetry of the function $\mathcal{N}_2(x, \xi)$ and bearing in mind the equality (22) we see that the function

$$F(\xi) = \frac{(-1)^{k-1}}{\omega_n} \int_S \mathcal{N}_{2k}^0(y, \xi) dy$$

is a polynomial of degree k in $|\xi|^2$ and hence $F(\xi)|_{\partial S} = C$. Therefore, by the assumption of the theorem

$$\frac{\partial u_k[\varphi_k]}{\partial \nu} \Big|_{\partial S} = \int_{\partial S} F(\xi) \varphi_k(\xi) ds_\xi = C \int_{\partial S} \varphi_k(\xi) ds_\xi = 0.$$

If $k = 0$, then according to (5) we get

$$\frac{\partial u_0[\varphi_0]}{\partial \nu} \Big|_{\partial S} = \frac{1}{\omega_n} \int_{\partial S} \frac{\partial \mathcal{N}_2(x, \xi)}{\partial \nu_x} \Big|_{\partial S} \varphi_0(\xi) ds_\xi = \varphi_0(x) \Big|_{\partial S} - \frac{1}{\omega_n} \int_{\partial S} \varphi_0(\xi) ds_\xi = \varphi_0(x).$$

The theorem is proved. $\square$

Example 2. Let us calculate the function $u_p[H_k](x)$ from the formula (36), where $H_k(x)$ is a homogeneous harmonic polynomial of degree $k \in \mathbb{N}$ for $p \in \mathbb{N}_0$. In this case, the conditions of Theorem 5 are satisfied because the equality $\int_{\partial S} H_k(\xi) ds_\xi = 0$ holds true.

In [9], it is established that for $x \in S$ and $\xi \in \partial S$ the equalities

$$\begin{aligned}\mathcal{N}_2(x, \xi) &= E(x, \xi) - E_0(x, \xi) = \frac{1}{n-2} \\ &+ \sum_{k=1}^{\infty} \left(\frac{1}{2k+n-2} + \frac{k+n-2}{k(2k+n-2)} \right) \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi) \\ &= \frac{1}{n-2} + \sum_{k=1}^{\infty} \frac{1}{k} \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi),\end{aligned}$$

take place, where the harmonic polynomials $H_k^{(i)}(x)$ are defined in Example 1. Therefore, due to the orthonormality of the polynomials $H_k^{(i)}(x)$ on ∂S and the uniform convergence of the series with respect to $\xi \in \partial S$, we have

$$u_0[H_k](x) = \frac{1}{\omega_n} \int_{\partial S} \mathcal{N}_2(x, \xi) H_k(\xi) ds_\xi = \frac{1}{(n-2)\omega_n} \int_{\partial S} H_k(\xi) ds_\xi \\ + \sum_{m=1}^{\infty} \frac{1}{m} \sum_{i=1}^{h_m} H_m^{(i)}(x) \frac{1}{\omega_n} \int_{\partial S} H_m^{(i)}(\xi) H_k(\xi) ds_\xi = \frac{1}{k} H_k(x).$$

Let us calculate $u_1[H_k](x)$. With the help of (23) at $l = 0$ and the previous calculations, we find

$$u_1[H_k](x) = -\frac{1}{\omega_n} \int_{\partial S} \mathcal{N}_4^0(x, \xi) H_k(\xi) ds_\xi \\ = -\frac{1}{\omega_n} \int_S \mathcal{N}_2(x, y) \frac{1}{\omega_n} \int_{\partial S} \mathcal{N}_2(y, \xi) H_k(\xi) ds_\xi dy = -\frac{1}{\omega_n} \int_S \mathcal{N}_2(x, y) u_0[H_k](y) dy \\ = -\frac{1}{k} \frac{1}{\omega_n} \int_S \mathcal{N}_2(x, y) H_k(y) dy = \frac{1}{k} \frac{|x|^2 - 1 - 2/k}{2(2k+n)} H_k(x).$$

Similarly, in the general case, for $u_p[H_k](x)$ when $p \in \mathbb{N}$ we have

$$u_p[H_k](x) = \frac{1}{\omega_n} \int_{\partial S} \mathcal{N}_{2p+2}^0(x, \xi) H_k(\xi) ds_\xi = -\frac{1}{\omega_n} \int_S \mathcal{N}_2(x, y) u_{p-1}[H_k](y) dy. \quad (38)$$

Hence, using the function $u_1[H_k](x)$ found above and the equality (23), we obtain

$$u_2[H_k](x) = \left(\frac{|x|^4 - 1 - 4/k}{8k(2k+n)(2k+2+n)} - (k+2) \frac{|x|^2 - 1 - 2/k}{4k^2(2k+n)^2} \right) H_k(x).$$

It is easy to verify that the 3-harmonic function $u_2[H_k](x)$ satisfies the condition

$$\frac{\partial u_2[H_k]}{\partial \nu} \Big|_{\partial S} = \left(\frac{(k+4)|x|^4 - k - 4}{8k(2k+n)(2k+2+n)} - (k+2) \frac{(k+2)|x|^2 - k - 2}{4k^2(2k+n)^2} \right) H_k(x) \Big|_{\partial S} = 0,$$

and since

$$\Delta u_2[H_k] = \left(\frac{|x|^2}{2k(2k+n)} - \frac{(k+2)}{2k^2(2k+n)} \right) H_k(x) = u_1[H_k],$$

we can write

$$\frac{\partial \Delta u_2[H_k]}{\partial \nu} \Big|_{\partial S} = \frac{(k+2)|x|^2 - k - 2}{2k(2k+n)} H_k(x) \Big|_{\partial S} = 0.$$

In addition, the equalities

$$\Delta^2 u_2[H_k] = \frac{1}{k} H_k(x) = u_0[H_k], \quad \frac{\partial \Delta^2 u_2[H_k]}{\partial \nu} \Big|_{\partial S} = H_k(x)$$

also hold true. Using (38) and (23) one can find any function $u_p[H_k](x)$ one by one.

6. Conclusions

The integral representation of functions from the class $u \in C^{2m}(D) \cap C^{2m-1}(\bar{D})$ obtained in Theorem 1 allows one to find the explicit form of solutions to some classes of boundary value problems for the polyharmonic equation, as it is done in Theorem 3 for the Riquier–Neumann problem. For example, the Navier problem [15] can be considered. In [1], this problem is considered for the biharmonic equation and is called the Dirichlet-2 problem.

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