mathematics

Article

Quasilinear Parabolic Equations Associated with Semilinear Parabolic Equations

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Abstract: We formulate a quasilinear parabolic equation describing the behavior of the global-in-time solution to a semilinear parabolic equation. We study this equation in accordance with the blow-up and quenching patterns of the solution to the original semilinear parabolic equation. This quasilinear equation is new in the theory of partial differential equations and presents several difficulties for mathematical analysis. Two approaches are examined: functional analysis and a viscosity solution.

Keywords: semilinear parabolic equation; quasilinear parabolic equation; blowup pattern

MSC: 35K68; 35K69

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Citation: Ishii, K.; Pierre, M.; Suzuki, T. Quasilinear Parabolic Equations Associated with Semilinear Parabolic Equations. *Mathematics* **2023**, *11*, 758. https://doi.org/10.3390/math11030758

Academic Editors: Peng Cao, Luc Nguyen and Bo Wang

Received: 31 December 2022
Revised: 26 January 2023
Accepted: 30 January 2023
Published: 2 February 2023

1. Introduction

The blow-up of the solution to the semilinear parabolic equation has been studied in detail. Here, we take

$$u_t - \Delta u = u^p \text{ in } \Omega \times (0,T), \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) \tag{1}$$

for $p > 1$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with the smooth boundary $\partial\Omega$, ν denotes the outer unit normal vector on $\partial\Omega$, and $0 < u_0 = u_0(x) \in C(\overline{\Omega})$. There exists a unique classical solution $u = u(x,t) > 0$ with the maximal existence time $T = T_{\max} \in (0, +\infty]$. If $T_{max} < +\infty$, the solution blows up in finite time. Hence, it holds that

$$\lim_{t \uparrow T} \|u(\cdot, t)\|_\infty = +\infty.$$

Here, we recall a few references related to the problem examined in this paper: the blow-up profile and the post-blow-up continuation of the solution. First, Masuda [1] studied the equation with the general nonlinear term

$$u_t - \Delta u = f(u) \text{ in } \Omega \times (0,T), \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x). \tag{2}$$

By accepting the complex-valued t and u, the solution continues after the blow-up time is formulated. For the cases of $f(u) = u^2$, $u^2 + u$, and e^u, it is shown that this solution becomes two-fold in $t > T$ unless u_0 is a constant.

Second, Baras–Cohen [2] and Lacey–Tzanetis [3] studied the solution $u = u(x,t) \geq 0$ to

$$u_t - \Delta u = f(u) \text{ in } \Omega \times (0,T), \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) \tag{3}$$

for the non-negative nonlinearity $f = f(u) \geq 0$. Using the least integral solution, they formulate the notion of a complete blow-up of the solution, which means, roughly, that the solution $u(\cdot, t)$ becomes $+\infty$ everywhere in Ω for $t > T$. They showed this property under the appropriate conditions on f, Ω, and u_0. Galaktionov–Vazquez [4] later refined the result such that if $N \geq 3$ and

$$f(u) = u^p, \quad p \in ((N+2)/(N-2), 1 + 6/(N-10)_+),$$

the radially symmetric unbounded L^1-solution constructed by Ni–Sacks–Tzantzis [5] blows-up in finite time with the complete blow-up profile.

Third, Sakaguchi–Suzuki [6], motivated by their previous work [7] in the one space dimension using the Backlund transformation $x = x(u, t)$ for $u = u(x, t)$, studied

$$u = u(x,t) \in C(\overline{\Omega} \times [0, T]; (-\infty, +\infty]), \tag{4}$$

satisfying

$$D(t) = \overline{\{x \in \Omega \mid u(x,t) = +\infty\}} \subset \Omega, \ 0 \leq t \leq T \tag{5}$$

and the differential inequality

$$u_t - \Delta u \geq 0 \quad \text{in} \ \bigcup_{0 \leq t \leq T} (\Omega \setminus D(t)) \times \{t\} \tag{6}$$

in the sense of distributions. They obtained that then the N-dimensional Lebesgue measure of $D(t)$ becomes zero for a.e. t. This result was later refined by Suzuki–Takahashi [8] as

$$\int_0^T \mathrm{Cap}_2(D(t))\, dt \leq \frac{L^N(\Omega)}{2},$$

where Cap_2 and L^N denote the two-capacity and N-dimensional Lebesque measure, respectively. In particular, the Hausdorff dimension of $D(t)$ is not greater than $N - 2$ for a.e.t, if $u = u(x,t) \in (-\infty, +\infty]$ satisfies (4) and (6).

Here, we study the profile at $t = T$ of the blow-up solution to (1). Our approach is based on the observation of Masuda [1], mentioned above, that the ordinary differential equation

$$\frac{du}{dt} = u^p, \ u|_{t=0} = u_0 > 0, \tag{7}$$

uses the representation formula of the solution,

$$u(t) = \left(\frac{1}{p-1}\right)^{-\frac{1}{p-1}} (T-t)|T-t|^{-\frac{1}{p-1}-1} \tag{8}$$

for

$$T = \frac{u_0^{-(p-1)}}{p-1}. \tag{9}$$

Since (8) implies that

$$\lim_{t \uparrow T} u(t) = +\infty, \quad \lim_{t \downarrow T} u(t) = -\infty, \tag{10}$$

we reach the idea of using $0 \leq v < +\infty$, defined by

$$u^2 = v^{-\frac{1}{p-1}}, \tag{11}$$

which satisfies

$$\lim_{t \to T} v(t) = 0.$$

Since it holds that

$$v=\left(\frac{1}{p-1}\right)^2(T-t)^2, \quad v|_{t=0}=v_0\equiv\left(\frac{1}{p-1}\right)^2T^2>0, \tag{12}$$

we obtain

$$\frac{dv}{dt}=-2(p-1)w,\ t\geq 0 \tag{13}$$

for

$$w=\begin{cases}\sqrt{v}, & 0\leq t\leq T\\ -\sqrt{v}, & t>T\end{cases} \tag{14}$$

and

$$T=\sup\{t>0\mid v(s)>0,\ 0<\forall s<t\}. \tag{15}$$

The solution to (13) with (14) is not uniquely global in time. It is, however, unique to $0\leq t\leq T$, as we confirm below. Hence, we can use

$$\frac{dv}{dt}=-2(p-1)\sqrt{v},\ v\geq 0, \quad t\geq 0 \tag{16}$$

with

$$v|_{t=0}=v_0>0, \tag{17}$$

instead of (13) with (15), to detect the blow-up profile of the solution $u=u(x,t)$ to (1) at $t=T$.

In fact, since the mapping

$$v\in[0,+\infty)\mapsto -2(p-1)\sqrt{v}\in\mathbb{R}$$

is non-increasing, the solution $v=v(t)\geq 0$ to (16) and (17) is unique, although it is not Lispchitz continuous at $v=0$. More precisely, if v_i, $i=1,2$, are the solutions to (16) it follows that

$$\frac{1}{2}\frac{d}{dt}\|v_1-v_2\|_2^2=(\frac{dv_1}{dt}-\frac{dv_2}{dt})\cdot(v_1-v_2)\leq 0$$

from this monotonicity.

Now we are able to recover $u=u(t)$ for $0\leq t\leq T$ using the solution $v=v(t)$ from (16) and (17) for $T>0$ defined by (15). Actually, it is given explicitly by

$$v(t)=\begin{cases}\left(\frac{1}{p-1}\right)^2(T-t)^2, & 0\leq t\leq T\\ 0, & t>T\end{cases}$$

for $T>0$, as defined by (9).

As for (1), it follows that

$$v_t-\Delta v+\frac{2(2p-1)}{p-1}|\nabla\sqrt{v}|^2+2(p-1)\sqrt{v}=0 \text{ in } \Omega\times(0,T)$$

for $v=v(x,t)>0$, as defined by (11) and $T=T_{\max}$. We thus construct

$$0\leq v=v(x,t)\in C(\overline{\Omega}\times[0,\infty)),$$

satisfying

$$v_t-\Delta v+\frac{2(2p-1)}{p-1}|\nabla\sqrt{v}|^2+2(p-1)\sqrt{v}=0 \text{ in } \Omega\times(0,\infty) \tag{18}$$

with

$$\left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0(x) > 0 \tag{19}$$

for $v_0 = u_0^{-2(p-1)}$. Once this v is obtained, the value

$$T = \sup\{t > 0 \mid \min_{\overline{\Omega}} v(\cdot, s) > 0, \ 0 < \forall s < t\}$$

coincides with the maximal existence time $T = T_{\max} \in (0, +\infty]$ of $u = u(\cdot, t)$, and the function

$$u_* = v(\cdot, T)^{-\frac{1}{2(p-1)}} \in C(\overline{\Omega}; (0, +\infty])$$

can stand for its blow-up profile at $t = T$ if $T = T_{\max} < +\infty$.

Multiplying v by a positive constant, we obtain the normal form of (18) and (19),

$$v_t - \Delta v + \gamma|\nabla\sqrt{v}|^2 + \sqrt{v} = 0, \ v \geq 0 \quad \text{in } \Omega \times (0, T) \tag{20}$$

with

$$\left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0(x) > 0 \tag{21}$$

for

$$\gamma = \frac{2(2p-1)}{p-1}, \tag{22}$$

which is the quasilinear parabolic equation studied in this paper. This equation is expected to clarify the blow-up patterns of the solution to (1). Table 1, below, summarizes several of the approaches to the blow-up problem of (1) that have been tried so far.

Table 1. Blow-up of the solution to semilinear parabolic equation.

References	Problems	Methods
[1–4]	complete blow-up	comparison theorem
[9]	post-blow-up continuation	complex function theory
this paper	blow-up patten	function analysis, theory of viscosity solutions
[5–7]	estimates of the blow-up set	method of isoperimetric inequality

Surprisingly, the problems (20) and (21) are new in the theory of partial differential equations, and several technical difficulties arise in their mathematical analysis. Here, we take two approaches: the theory of functional analysis and that of a viscosity solution.

In the first approach, we use the approximate equation

$$v_{\varepsilon t} - \Delta v_\varepsilon + \gamma|\nabla\sqrt{v_\varepsilon + \varepsilon}|^2 + \sqrt{v_\varepsilon} = 0, \ v_\varepsilon \geq 0 \quad \text{in } \Omega \times (0, \infty) \tag{23}$$

with

$$\left.\frac{\partial v_\varepsilon}{\partial \nu}\right|_{\partial\Omega} = 0, \quad v_\varepsilon|_{t=0} = v_0(x), \tag{24}$$

assuming $0 < v_0 = v_0(x) \in C^\theta(\overline{\Omega})$ for $0 < \theta < 1$, and try to pass the limit as $\varepsilon \downarrow 0$ (Theorems 1 and 2).

In the second approach, we use $w = \sqrt{v}$ in (20), to obtain

$$(w^2)_t - \Delta(w^2) + \gamma|\nabla w|^2 + w = 0, \ w \geq 0 \quad \text{in } \Omega \times (0, T). \tag{25}$$

Equation (25) takes a different form from the equation studied in the standard theory of viscosity solutions, as referenced in Crandall–Ishii–Lions [10] and Koike [11], in which the comparison theorem, for example, guarantees the unique existence of the solution for the Dirichlet boundary condition

$$w|_{\partial\Omega} = 0. \tag{26}$$

We show, however, that a part of this comparison principle is valid in (3) for $\gamma \leq 2$, which ensures a profile of the quenching of the solution to (3) for $f(u) = u^p$ with $0 < p < 1$ (Theorem 5).

This paper is organized as follows: In Section 2 we derive a criterion for the convergence of the approximate solution defined by (23) and (24). Section 3 is concerned with the elliptic part of (25) and (26),

$$-\Delta(w^2) + \gamma|\nabla w|^2 + w = 0, \ w \geq 0 \quad \text{in } \Omega \tag{27}$$

with

$$w|_{\partial\Omega} = 0, \tag{28}$$

and then, Section 4 deals with (25) and (26). Section 5 is devoted to the discussion, together with several examples of the nonlinearity $f(u)$ in (2) or (3), to which our theorems are applicable.

List of Symbols

1. $C^m(\overline{\Omega})$, $m = 0, 1, 2, \cdots$; set of m-times the continuously differentiable functions, where $\Omega \subset \mathbf{R}^n$ is an open set. $C(\overline{\Omega}) = C^0(\overline{\Omega})$.
2. $C(\overline{\Omega} \times [0, T]; (-\infty, +\infty])$, $T > 0$; set of continuous functions on $\overline{\Omega} \times [0, T]$ with the value in $(-\infty, +\infty]$.
3. $W^{m,p}(\Omega)$, $1 \leq p \leq \infty$; set of measurable functions in Ω, with its distributional derivatives in $L^p(\Omega)$ up to m-th order. $H^m(\Omega) = W^{m,2}(\Omega)$.
4. $L^p_{loc}(\overline{\Omega} \times [0, T))$; set of measureable functions belonging to $L^p(K)$ for any compact set $K \subset \overline{\Omega} \times [0, T)$.
5. $C^{2+\theta, 1+\theta/2}(\overline{\Omega} \times [0, \infty))$, $0 < \theta < 1$; set of continuously differentiable functions up to the second and the first orders with respect to $x \in \overline{\Omega}$ and $t \in [0, \infty)$, with their derivatives Holder remaining continuous with the exponents θ and $\theta/2$, respectively.
6. $L^p(0, T; W^{m,q}(\Omega))$; set of L^p functions in $t \in (0, T)$ with the values in $W^{m,q}(\Omega)$.
7. $\mathcal{M}(\overline{Q}) = C'(\overline{Q})$; set of measures on $\overline{Q} = \overline{\Omega} \times [0, T]$.

2. Convergence of the Approximate Solution

This section is concerned with the convergence of the approximate solution $\{v_\varepsilon\}$ defined by (23) and (24) for $\gamma > 0$. For the moment, we use $T > 0$ arbitrarily.

To construct this solution, we use a further approximation defined for $0 < \delta \ll 1$, that is,

$$v^\delta_{\varepsilon t} - \Delta v^\delta_\varepsilon + \gamma|\nabla\sqrt{v^\delta_\varepsilon + \varepsilon}|^2 + \sqrt{(v^\delta_\varepsilon)_+ + \delta^2} = 0, \ v^\delta_\varepsilon > -\varepsilon/2 \quad \text{in } \Omega \times (0, T) \tag{29}$$

with

$$\left.\frac{\partial v^\delta_\varepsilon}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \left.v^\delta_\varepsilon\right|_{t=0} = v_0(x), \tag{30}$$

There is a local-in-time classical solution to (29) and (30) denoted by

$$-\varepsilon/2 < v^\delta_\varepsilon = v^\delta_\varepsilon(x, t) \in C^{2+\theta, 1+\theta/2}(\overline{\Omega} \times [0, T)), \tag{31}$$

for $0 < T \ll 1$, where $0 < \theta < 1$.

The spatially homogeneous part of (29) and (30), on the other hand, is described by

$$\frac{dv}{dt} = -\sqrt{v_+ + \delta^2}, \quad v|_{t=0} = v_0 > 0.$$

Its solution is given explicitly by

$$v(t) = \begin{cases} -\delta^2 + \frac{1}{4}(t_\delta - t)^2, & 0 \le t \le t_\delta - 2\delta \\ -\delta t + \delta(t_\delta - 2\delta), & t > t_\delta - 2\delta, \end{cases}$$

where

$$t_\delta = 2\sqrt{v_0 + \delta^2}.$$

Hence, we obtain

$$-\varepsilon/2 < v_-^\delta(t) \le v_\varepsilon^\delta(x,t) \le v_+^\delta(t), \quad (x,t) \in \overline{\Omega} \times [0, T_\varepsilon^\delta) \tag{32}$$

by the comparison theorem, where

$$v_\pm^\delta(t) = \begin{cases} -\delta^2 + \frac{1}{4}(t_\pm^\delta - t)^2, & 0 \le t \le t_\pm^\delta - 2\delta \\ -\delta t + \delta(t_\pm^\delta - 2\delta), & t > t_\pm^\delta - 2\delta, \end{cases} \qquad T_\varepsilon^\delta = t_-^\delta - 2\delta + \delta^{-1}\varepsilon/2,$$

and

$$t_\pm^\delta = 2\sqrt{v_0^\pm + \delta^2}, \quad v_0^+ = \max_{\overline{\Omega}} v_0, \; v_0^- = \min_{\overline{\Omega}} v_0 > 0,$$

together with the existence of $v_\varepsilon^\delta = v_\varepsilon^\delta(x,t)$ on $\overline{\Omega} \times [0, T_\varepsilon^\delta)$.

To make $\delta \downarrow 0$ with $\varepsilon > 0$ fixed, we use an estimate uniform in δ, that is,

$$\|v_\varepsilon^\delta\|_{C^{2+\theta,1+\theta/2}(\overline{\Omega}\times[0,T_\varepsilon^\delta))} \le C_\varepsilon. \tag{33}$$

This estimate follows from the standard theory of quasilinear parabolic equations (Chapters IV and IV of [12]) inside Ω and a reflextion argument on $\partial\Omega$, as in Chapter 2 of [9] and Appendix A of [13]. We thus obtain $v_\varepsilon = v_\varepsilon(x,t) \in C^{2+\theta,1+\theta/2}(\overline{\Omega} \times [0,\infty))$ satisfying

$$v_{\varepsilon t} - \Delta v_\varepsilon + \gamma|\nabla\sqrt{v_\varepsilon + \varepsilon}|^2 + \sqrt{v_{\varepsilon+}} = 0, \; v_\varepsilon \ge -\varepsilon/2 \quad \text{in } \Omega \times (0,\infty) \tag{34}$$

with

$$\left.\frac{\partial v_\varepsilon}{\partial \nu}\right|_{\partial\Omega} = 0, \quad v_\varepsilon|_{t=0} = v_0(x). \tag{35}$$

Here, the maximum principle ensures $v_\varepsilon \ge 0$, and there arises a solution

$$0 \le v_\varepsilon = v_\varepsilon(x,t) \in C^{2+\theta,1+\theta/2}(\overline{\Omega} \times [0,\infty))$$

to (23) and (24), satisfying

$$\|v_\varepsilon(\cdot,t)\|_\infty \le \|v_0\|_\infty.$$

Now we show a criterion for this $\{v_\varepsilon\}$ to converge to a solution $v = v(x,t)$ to (20) as $\varepsilon \downarrow 0$, such that

$$0 \le v \in L^\infty_{loc}(\overline{\Omega} \times [0,\infty)), \quad \sqrt{v} \in L^2(0,T;H^1(\Omega))$$

for the case of $\gamma \ge 2$.

First, Inequality (32) implies

$$\|v_\varepsilon(\cdot,t)\|_\infty \le \|v_0\|_\infty, \tag{36}$$

while

$$\frac{d}{dt}\int_\Omega v_\varepsilon \, dx + \gamma\|\nabla\sqrt{v_\varepsilon + \varepsilon}\|_2^2 \le 0$$

implies

$$\int_0^T \|\nabla\sqrt{v_\varepsilon + \varepsilon}\|_2^2 dt \le \gamma^{-1}\|v_0\|_1. \tag{37}$$

Second, the approximate solution $v_\varepsilon(x,t)$ is monotone in ε for each $(x,t) \in \overline{\Omega} \times [0,\infty)$. To see this property, we make a further approximation, using

$$\varphi_\delta^\sigma(v) \to \varphi_\delta(v) \text{ locally and uniformly in } v \in \mathbf{R}, \quad \sigma \to 0,$$

where $\varphi_\delta^\sigma \in C^1(\mathbf{R})$ and $\varphi_\delta(v) = \sqrt{v_+ + \delta^2}$, that is, $v_\varepsilon^{\sigma,\delta} = v_\varepsilon^{\sigma,\delta}(x,t)$ satisfying

$$v_{\varepsilon t}^{\sigma,\delta} - \Delta v_\varepsilon^{\sigma,\delta} + \gamma \left| \nabla \sqrt{v_\varepsilon^{\sigma,\delta} + \varepsilon} \right|^2 + \varphi_\varepsilon^{\sigma,\delta}(v_\varepsilon^{\sigma,\delta}) = 0,\ v_\varepsilon^{\sigma,\delta} > -\varepsilon/2 \quad \text{in } \Omega \times (0, T_\varepsilon^{\sigma,\delta}),$$

with

$$\left. \frac{\partial v_\varepsilon^{\sigma,\delta}}{\partial \nu} \right|_{\partial\Omega} = 0, \quad \left. v_\varepsilon^{\sigma,\delta} \right|_{t=0} = v_0(x).$$

It follows that $T_\varepsilon^{\sigma,\delta} \to T_\varepsilon^\delta$, and

$$v_\varepsilon^{\sigma,\delta} \to v_\varepsilon^\delta \text{ locally and uniformly on } \overline{\Omega} \times [0, T_\varepsilon^\delta), \quad \sigma \to 0.$$

There is

$$z_\varepsilon^{\sigma,\delta} = \frac{\partial v_\varepsilon^{\sigma,\delta}}{\partial \varepsilon},$$

and it holds that

$$\begin{aligned} z_{\varepsilon t}^{\sigma,\delta} - \Delta z_\varepsilon^{\sigma\delta} + 2\gamma \nabla (v_\varepsilon^{\sigma,\delta} + \varepsilon)^{1/2} \cdot \nabla \left(\frac{1}{2} (v_\varepsilon^{\sigma,\delta} + \varepsilon)^{-1/2} (z_\varepsilon^\delta + 1) \right) \\ + [(\varphi_\delta^\sigma)'(v_\varepsilon^{\sigma,\delta})] z_\varepsilon^{\sigma,\delta} = 0 \quad \text{in } \Omega \times (0, T_\varepsilon^{\sigma,\delta}) \end{aligned}$$

with

$$\left. \frac{\partial z_\varepsilon^{\sigma,\delta}}{\partial \nu} \right|_{\partial\Omega} = 0, \quad \left. z_\varepsilon^{\sigma,\delta} \right|_{t=0} = 0.$$

Then, we obtain

$$\begin{aligned} z_{\varepsilon t}^{\sigma,\delta} - \Delta z_\varepsilon^{\sigma,\delta} + \gamma \nabla (v_\varepsilon^{\sigma,\delta} + \varepsilon)^{1/2} \cdot \nabla [(v_\varepsilon^{\sigma,\delta} + \varepsilon)^{-1/2} z_\varepsilon^{\sigma,\delta}] \\ + [(\varphi_\delta^\sigma)'(v_\varepsilon^{\sigma,\delta})] z_\varepsilon^{\sigma,\delta} \geq 0 \quad \text{in } \Omega \times (0, T_\varepsilon^{\gamma,\delta}) \end{aligned}$$

by

$$\nabla (v_\varepsilon^{\sigma,\delta} + \varepsilon)^{1/2} \cdot \nabla (v_\varepsilon^{\sigma,\delta} + \varepsilon)^{-1/2} = -(v_\varepsilon^{\sigma,\delta} + \varepsilon) |\nabla (v_\varepsilon^{\sigma,\delta} + \varepsilon)^{-1/2}|^2 \leq 0,$$

which implies

$$z_\varepsilon^{\sigma,\delta} = \frac{\partial v_\varepsilon^{\sigma,\delta}}{\partial \varepsilon} \geq 0 \quad \text{on } \overline{\Omega} \times [0, T_\varepsilon^{\sigma,\delta}).$$

Then, it arises that

$$v_\varepsilon^{\sigma,\delta}(x,t) \geq v_{\varepsilon'}^{\sigma,\delta}(x,t), \quad \forall (x,t) \in \overline{\Omega} \times [0, T_{\varepsilon'}^{\sigma,\delta}),\ \forall \varepsilon > \varepsilon' > 0$$

and, hence,

$$v_\varepsilon(x,t) \geq v_{\varepsilon'}(x,t), \quad \forall (x,t) \in \overline{\Omega} \times [0,\infty),\ \forall \varepsilon > \varepsilon' > 0$$

by sending $\sigma \to 0$, and then $\delta \downarrow 0$. Thus it arises that the pointwise monotone convergence

$$\lim_{\varepsilon \downarrow 0} v_\varepsilon(x,t) = v(x,t) \geq 0, \quad \forall (x,t) \in \overline{\Omega} \times [0,T). \tag{38}$$

Letting

$$w_\varepsilon = \sqrt{v_\varepsilon + \varepsilon}, \quad Q_T = \Omega \times (0,T),$$

we obtain

$$\|w_\varepsilon\|_{L^\infty(Q_T)} + \|\nabla w_\varepsilon\|_{L^2(Q_T)} \leq C$$

via (36) and (37), and the monotone convergence

$$\lim_{\varepsilon\downarrow 0} w_\varepsilon(x,t) = w(x,t) \equiv \sqrt{v(x,t)}, \quad \forall (x,t) \in \overline{\Omega} \times [0,T) \tag{39}$$

by (38). Hence, it follows that

$$0 \leq v \in L^\infty(Q_T), \quad \sqrt{v} \in L^2(0,T;H^1(\Omega)) \tag{40}$$

and

$$w - \lim_{\varepsilon\downarrow 0} w_\varepsilon = w \quad \text{in } L^2(0,T;H^1(\Omega)), \tag{41}$$

where $w - \lim$ denotes the weak convergence. Then, we have the following theorem.

Theorem 1. *If $\gamma \geq 2$ and $w \in C(\overline{Q_T})$, the limit $v \in C(\overline{Q_T})$ in (38) is a solution to (20) and (21) in the sense of distributions.*

We begin with the following lemma.

Lemma 1. *If the convergence (41) is strong,*

$$s - \lim_{\varepsilon\downarrow 0} w_\varepsilon = w \quad \text{in } L^2(0,T;H^1(\Omega)), \tag{42}$$

the above $v = v(x,t)$ in (40) is a solution to (20) and (21) in the sense of distributions.

Proof. Assumption (42) implies

$$s - \lim_{\varepsilon\downarrow 0} v_\varepsilon = v \quad \text{in } L^1(0,T;W^{1,1}(\Omega)) \tag{43}$$

by

$$\nabla v_\varepsilon = 2 w_\varepsilon \nabla w_\varepsilon.$$

It also implies

$$s - \lim_{\varepsilon\downarrow 0} \sqrt{v_\varepsilon + \varepsilon} = \sqrt{v} \quad \text{in } L^2(0,T;H^1(\Omega)) \tag{44}$$

by

$$\nabla \sqrt{v_\varepsilon + \varepsilon} = (w_\varepsilon^2 + \varepsilon)^{-1/2} w_\varepsilon \nabla w_\varepsilon, \quad 0 \leq (w_\varepsilon^2 + \varepsilon)^{-1/2} w_\varepsilon \leq 1,$$

and the monotone convergence (39), as in

$$\begin{aligned} \nabla \sqrt{v_\varepsilon + \varepsilon} - \nabla \sqrt{v} &= (w_\varepsilon^2 + \varepsilon)^{-1/2} w_\varepsilon \nabla w_\varepsilon - \nabla w \\ &= \{(w_\varepsilon^2 + \varepsilon)^{-1/2} w_\varepsilon - 1\} \nabla w_\varepsilon + (\nabla w_\varepsilon - \nabla w). \end{aligned}$$

Given

$$\varphi \in C^2(\overline{\Omega}), \quad \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0,$$

we obtain

$$\frac{d}{dt} \int_\Omega v_\varepsilon \varphi \, dx + (\nabla v_\varepsilon, \nabla \varphi) + \int_\Omega \left(\gamma |\nabla \sqrt{v_\varepsilon + \varepsilon}|^2 + \sqrt{v_\varepsilon} \right) \varphi \, dx = 0$$

by (31), (23), and (24), where (,) denotes the L^2 inner product. Then, (42)–(44) imply

$$\frac{d}{dt} \int_\Omega v \varphi \, dx + (\nabla v, \nabla \phi) + \int_\Omega (\gamma |\nabla \sqrt{v}|^2 + \sqrt{v}) \varphi \, dx = 0$$

in the sense of distributions in t. In particular, the mapping

$$t \in [0, T) \mapsto \int_\Omega v\varphi \, dx$$

is absolutely continuous, and it holds that

$$\int_\Omega v\varphi \, dx \Big|_{t=0} = \int_\Omega v_0 \varphi \, dx.$$

Then, the result follows. □

Equation (31) with (23) implies

$$w_{\varepsilon t} - \Delta w_\varepsilon = -g_\varepsilon \quad \text{in } Q_T \tag{45}$$

with

$$\frac{\partial w_\varepsilon}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad w_\varepsilon|_{t=0} = w_0(x) > 0 \tag{46}$$

for $w_0 = v_0^{1/2}$ and

$$g_\varepsilon = 2(\gamma - 2)|\nabla \sqrt{w_\varepsilon}|^2 + \frac{\sqrt{v_\varepsilon}}{2\sqrt{v_\varepsilon + \varepsilon}} \geq 0 \tag{47}$$

if $\gamma \geq 2$. Then, we obtain

$$\frac{d}{dt} \int_\Omega w_\varepsilon \, dx = -\int_\Omega g_\varepsilon \, dx \leq 0,$$

and, therefore,

$$\|g_\varepsilon\|_{L^1(Q_T)} \leq \|v_0^{1/2}\|_1 \tag{48}$$

by $w_\varepsilon \geq 0$. Hence, there is a subsequence, denoted by the same symbol, such that

$$g_\varepsilon \rightharpoonup \mu \in \mathcal{M}(\overline{Q_T}) = C'(\overline{Q_T})$$

in the sense of measures.

Remark 1. *Inequality (48) implies*

$$w \in C([0, +\infty), L^1(\Omega))$$

according to (41) and the L^1-compactness property of the heat equation [14]. This inequality also ensures

$$s - \lim_{\varepsilon \downarrow 0} w_\varepsilon = w \text{ in } L^p(Q_T), \quad 1 \leq p < \frac{N+2}{N}$$
$$s - \lim_{\varepsilon \downarrow 0} \nabla w_\varepsilon = \nabla w \text{ in } L^q(Q_T), \quad 1 \leq q < \frac{N+2}{N+1}$$

by [15].

Lemma 2. *If $\gamma \geq 2$ and $w \in C(\overline{Q_T})$, it holds that*

$$\int_0^T \|\nabla(w_\varepsilon - w)\|_2^2 \, dt \leq \langle w_\varepsilon - w, \mu \rangle + \frac{1}{2}\|\sqrt{v_0 + \varepsilon} - \sqrt{v_0}\|_2^2. \tag{49}$$

Proof. We have

$$(w_\varepsilon - w_{\varepsilon'})_t - \Delta(w_\varepsilon - w_{\varepsilon'}) = -g_\varepsilon + g_{\varepsilon'} \quad \text{in } Q_T$$

with

$$\frac{\partial}{\partial \nu}(w_\varepsilon - w_{\varepsilon'})\Big|_{\partial\Omega} = 0, \quad (w_\varepsilon - w_{\varepsilon'})|_{t=0} = 0.$$

Then, it follows that

$$\frac{1}{2}\frac{d}{dt}\|w_\varepsilon - w_{\varepsilon'}\|_2^2 + \|\nabla(w_\varepsilon - w_{\varepsilon'})\|_2^2 = (w_\varepsilon - w_{\varepsilon'}, -g_\varepsilon + g_{\varepsilon'}) \leq (w_\varepsilon - w_{\varepsilon'}, g_\varepsilon)$$

for $0 < \varepsilon' < \varepsilon$, and, therefore,

$$\int_0^T \|\nabla(w_\varepsilon - w_{\varepsilon'})\|_2^2\, dt \leq \int_0^T (w_\varepsilon - w_{\varepsilon'}, g_{\varepsilon'})\, dt + \frac{1}{2}\|\sqrt{v_0+\varepsilon} - \sqrt{v_0+\varepsilon'}\|_2^2. \tag{50}$$

If $w \in C(\overline{Q_T})$, the monotone convergence (39) implies

$$\lim_{\varepsilon\downarrow 0} w_\varepsilon = w \quad \text{uniformly on } \overline{Q_T}. \tag{51}$$

by Dini's theorem. Then, it holds that

$$\lim_{\varepsilon'\downarrow 0}\int_0^T (w_\varepsilon - w_{\varepsilon'}, g_{\varepsilon'})\, dt = \langle w_\varepsilon - w, \mu\rangle,$$

by (41). Hence, (49) follows from

$$\liminf_{\varepsilon'\downarrow 0}\int_0^T \|\nabla(w_\varepsilon - w_{\varepsilon'})\|_2^2\, dt \geq \int_0^T \|\nabla(w_\varepsilon - w)\|_2^2\, dt. \tag{52}$$

□

We are now able to give the following proof.

Proof of Theorem 1. The result follows from (49) and (51). □

A variant of Theorem 1 is the following theorem.

Theorem 2. *Let $2 \leq \gamma < 4$ and assume the existence of $\hat{w}_\varepsilon, w_*, w^* \in C(\overline{Q_T})$ such that*

$$w_\varepsilon \leq \hat{w}_\varepsilon, \quad \hat{w}_\varepsilon \to w^* \text{ uniformly on } \overline{Q_T}, \tag{53}$$

and

$$w \geq w_* \text{ on } \overline{Q_T}, \quad \langle w^* - w_*, \mu\rangle = 0. \tag{54}$$

Then, it holds that (42), and hence v in (38), satisfies (12), (20), and (21) in the sense of distributions.

We use the following lemma to prove this theorem.

Lemma 3. *If $\gamma \geq 2$, it holds that*

$$\int_0^T \|\nabla(w_\varepsilon - w)\|_2^2\, dt \leq \langle w_\varepsilon, \mu\rangle - \iint_{Q_T} \frac{\gamma-2}{2}|\nabla w|^2 + \frac{w}{2}\, dxdt + \frac{1}{2}\|\sqrt{v_0+\varepsilon} - \sqrt{v_0}\|_2^2. \tag{55}$$

Proof. In (50), we have

$$\begin{aligned}(w_{\varepsilon'}, g_{\varepsilon'}) &= \int_\Omega \frac{\gamma-2}{2}|\nabla w_{\varepsilon'}|^2 + \frac{1}{2}\frac{\sqrt{v_{\varepsilon'}}}{\sqrt{v_{\varepsilon'}+\varepsilon'}} w_{\varepsilon'}\, dx \\ &\geq \int_\Omega \frac{\gamma-2}{2}|\nabla w_{\varepsilon'}|^2 + \frac{w_{\varepsilon'}}{2}\, dx\end{aligned}$$

by (47), and, therefore, (41) ensures

$$\liminf_{\varepsilon'\downarrow 0}\int_0^T (w_{\varepsilon'}, g_{\varepsilon'})\,dt \geq \iint_{Q_T} \frac{\gamma-2}{2}|\nabla w|^2 + \frac{w}{2}\,dxdt.$$

Then, we obtain (55) by (52). □

We conclude this theorem with the following proof.

Proof of Theorem 2. In (55), we have

$$\langle w_\varepsilon, \mu\rangle = \langle w_\varepsilon - \hat{w}_\varepsilon, \mu\rangle + \langle \hat{w}_\varepsilon, \mu\rangle \leq \langle \hat{w}_\varepsilon, \mu\rangle = \langle w^*, \mu\rangle + o(1) = \langle w_*, \mu\rangle + o(1).$$

It holds, furthermore, that

$$\langle w_*, \mu\rangle = \langle w_*, g_\varepsilon\rangle + o(1) \leq \langle w, g_\varepsilon\rangle + o(1)$$

and

$$\begin{aligned}\langle w, g_\varepsilon\rangle &= \iint_{Q_T} \frac{\gamma-2}{2}\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}w + \frac{1}{2}\frac{\sqrt{v_\varepsilon}}{\sqrt{v_\varepsilon+\varepsilon}}w\,dxdt \\ &\leq \iint_{Q_T} \frac{\gamma-2}{2}|\nabla w_\varepsilon|^2 + \frac{w}{2}\,dxdt + o(1)\end{aligned}$$

by (47). Then, it follows that

$$\begin{aligned}\int_0^T \|\nabla(w_\varepsilon - w)\|_2^2\,dt &\leq \frac{\gamma-2}{2}\iint_{Q_T} |\nabla w_\varepsilon|^2 - |\nabla w|^2\,dxdt + o(1) \\ &= \frac{\gamma-2}{2}\int_0^T \|\nabla(w_\varepsilon - w)\|_2^2\,dt + o(1)\end{aligned}$$

from (41), which implies (42) by $\gamma < 4$. □

3. Comparison Theorem for the Elliptic Equation

In this section, we study the viscosity solutions to (27) with (28). To introduce this notion, let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let $USC(\Omega)$ (resp., $LSC(\Omega)$) be the set of upper semi-continuous (resp., lower semi-continuous) functions in Ω. Given $w : \Omega \to \mathbb{R}$, we define w^* and w_* by

$$w^*(x) = \limsup_{y\to x} w(y), \ w_*(x) = \liminf_{y\to x} w(y)$$

for $x \in \Omega$. If $w \in C(\Omega)$, it holds that $w = w_* = w^*$.

Remark 2. *We have $w^* \in USC(\Omega)$ if and only if $w^*(x) < \infty$ for any $x \in \Omega$. Similarly, we have $w_* \in LSC(\Omega)$ if and only if $w_*(x) > -\infty$ for any $x \in \Omega$.*

Remark 3. *If $w : \overline{\Omega} \to \mathbb{R}$, the above w^* and w_* are extended as*

$$w^*(x) = \limsup_{y\in\overline{\Omega}\to x} w(x), \quad w_*(x) = \liminf_{y\in\overline{\Omega}\to x} w(y)$$

for $x \in \overline{\Omega}$. The same properties as in Remark 2 are valid for w^* and w_*.

Using

$$F(r,p,X) = -\mathrm{tr}X + \gamma|p|^2 + r, \quad r \in \mathbb{R},\ p \in \mathbb{R}^N,\ X \in \mathbb{S}^N(\mathbf{R}), \tag{56}$$

we write (27) as

$$F(w, \nabla w, \nabla^2 w^2) = 0 \text{ in } \Omega, \tag{57}$$

where

$$\mathbb{S}^N = \{A \in M_N(\mathbf{R}) \mid A^T = A\},$$

$M_N(\mathbf{R})$ is the set of $N \times N$ matrix with real entries, and A^T denotes the transpose matrix of A.

Since Equation (57) takes a different form when treated with the standard theory of visicosity solution,

$$F(w, \nabla w, \nabla^2 w) = 0 \text{ in } \Omega,$$

we begin with the definition of its viscosity solution.

Definition 1. *Let $w = w(x)$ be a function defined in Ω.*

(1) *We say that if w in $w^* \in USC(\Omega)$ is a viscosity subsolution to (27) if $w^* - \varphi$ attains a local maximum 0 at $x \in \Omega$ for $\varphi \in C^2(\Omega)$, then it holds that*

$$F(\varphi, \nabla\varphi, \nabla^2\varphi^2) \leq 0 \quad \textit{at } x.$$

(2) *We say that if w in $w_* \in LSC(\Omega)$ is a viscosity supersolution to (27) if $w_* - \varphi$ attains a local minimum 0 at $x_0 \in \Omega$ for $\varphi \in C^2(\Omega)$, then it holds that*

$$F(\varphi, \nabla\varphi, \nabla^2\varphi^2) \geq 0 \quad \textit{at } x_0.$$

(3) *We say that if w is a viscosity solution to (27) if it is a viscosity subsolution and a viscosity supersolution to (27).*

Remark 4. *It is obvious that if $w \in C^2(\Omega)$ is a viscosity subsolution to (27), then it holds that*

$$F(w, \nabla w, \nabla^2 w^2) \leq 0 \quad \textit{in } \Omega,$$

and hence, it is a classical subsolution. Similarly, if $w \in C^2(\Omega)$ is a viscosity supersolution (resp., solution) to (27), it is a classical supersolution (resp., solution).

Even if $w \in C^2(\Omega)$ is a classical solution to (27),

$$F(w, \nabla w, \nabla^2 w^2) = 0 \quad \text{in } \Omega. \tag{58}$$

On the other hand, it is not necessarily a viscosity subsolution, nor a viscosity supersolution, although F in (56) is elliptic:

$$X, Y \in \mathbb{S}^N,\ X \leq Y \quad \Rightarrow \quad F(r, p, Y) \leq F(r, p, X),\ \forall (r, p) \in \mathbb{R} \times \mathbb{R}^N. \tag{59}$$

Despite this, we have the following fact.

Proposition 1. *If $0 \leq w = w(x) \in C^2(\Omega)$ is a classical subsolution to (27), it is a viscosity subsolution. If $0 < w = w(x) \in C^2(\Omega)$ is a classical supersolution to (27), it is a viscosity supersolution.*

To prove the above fact, we recall, for the moment, a fundamental fact used in the theory of viscosity solutions.

Given $w : \Omega \to \mathbb{R}$ and $x \in \Omega$, we thus put

$$\begin{aligned} J^{2,+}w(x) = \{(p, X) \in \mathbb{R}^N \times \mathbb{S}^N \mid\ & w^*(x+h) - w^*(x) \\ & \leq \langle p, h\rangle + \frac{1}{2}\langle Xh, h\rangle + o(|h|^2) \text{ as } h \to 0\}. \end{aligned}$$

Given $0 < r \ll 1$ and $(p, X) \in \mathbb{R}^N \times \mathbb{S}^N$, furthermore, we put

$$\begin{aligned}\Phi_r^+(x, p, X, w) = \{\varphi \in C^2(B_r(x)) \mid\ & w^* - \varphi \text{ attains the maximum } 0 \text{ at } x, \\ & \nabla\varphi(x) = p,\ \nabla^2\varphi(x) = X\}.\end{aligned}$$

Let

$$\Phi^+(x, p, X; w) = \bigcup_{0<r\ll 1} \Phi_r^+(x, p, X; w).$$

The following fact is proven in Koike's work ([11], Proposition 2.6).

Lemma 4. *Any $x \in \Omega$ and $(p, X) \in J^{2,+}w(x)$ admit $\varphi \in \Phi^+(x, p, X; w)$.*

Remark 5. *We define $\Phi^-(x, p, X; w)$ similarly, using*

$$\begin{aligned}J^{2,-}w(x) = \{(p, X) \in \mathbb{R}^N \times \mathbb{S}^N \mid\ & w_*(x + h) - w_*(x) \\ & \geq \langle p, h\rangle + \frac{1}{2}\langle Xh, h\rangle + o(h^2),\ h \to 0\}.\end{aligned}$$

Then, an analogous result to the above lemma holds.

An immediate consequence is the following fact, analogous to the work of Koike ([11], Corollary 2.3), in which

$$j^{2,+}w(x) = \left\{(\nabla\varphi(x), \nabla^2\varphi(x)^2) \mid \exists (p, X) \in J^{2,+}w(x),\ \varphi \in \Phi^+(x, p, X; w)\right\}.$$

Corollary 1. *The function w is a viscosity subsolution to (27) if and only if*

$$F(w^*(x), q, Y) \leq 0, \quad \forall x \in \Omega,\ \forall (q, Y) \in j^{2,+}w(x).$$

We are ready to give the following proof. Note that (60) does not imply (61) without $w(y) \geq 0$, and, similarly, (62) does not imply (63) without $\varphi(y) \geq 0$.

Proof of Proposition 1. Let $0 \leq w \in C^2(\Omega)$ be a classical subsolution:

$$F(w, \nabla w, \nabla^2 w) \leq 0 \quad \text{in } \Omega.$$

Observe that $w = w^* = w_*$ in Ω.

Fix $x \in \Omega$ and use $(q, Y) \in j^{2,+}w(x)$ arbitrarily. Then, there are $(p, X) \in J^{2,+}w(x)$ and $\varphi \in \Phi(x, p, X; w)$ such that

$$q = \nabla\varphi(x), \quad Y = \nabla^2\varphi(x)^2.$$

Next, we obtain

$$w(y) - \varphi(y) \leq w(x) - \varphi(x) = 0, \quad |y - x| \ll 1, \tag{60}$$

and hence

$$w(y)^2 \leq \varphi(y)^2,\ |y - x| \ll 1, \quad w(x)^2 = \varphi(x)^2 \tag{61}$$

by $w(y) \geq 0$. It follows, therefore,

$$\nabla(w(x) - \varphi(x)) = 0, \quad \nabla^2(w(x)^2 - \varphi(x)^2) \leq 0,$$

which implies

$$F(w(x), q, Y) = F(\varphi(x), \nabla\varphi(x), \nabla^2\varphi(x)^2) \leq F(w(x), \nabla w(x), \nabla^2 w(x)^2) \leq 0$$

by (59). Hence, $0 \leq w = w(x) \in C^2(\Omega)$ is a viscosity subsolution.

The proof of the latter part is similar. In fact, if $0 < w \in C^2(\Omega)$ and $\varphi \in C^2(\Omega)$ satisfy

$$w(y) - \varphi(y) \geq w(x) - \varphi(x) = 0, \quad |y - x| \ll 1 \tag{62}$$

then it arises that

$$w(y)^2 \geq \varphi(y)^2, \ |y - x| \ll 1, \quad w(x)^2 = \varphi(x)^2 \tag{63}$$

by $w(x) > 0$. Hence, if $0 < w = w(x) \in C^2(\Omega)$ is a classical supersolution, then it is a viscosity supersolution. □

Here, we show the following fact.

Theorem 3. *Assume $\gamma \leq 2$. Let $w \geq 0$ (resp., $z \geq 0$) be a viscosity subsolution (resp., supersolution) to (27) defined on $\overline{\Omega}$, and put*

$$\Omega_0 = \{x \in \Omega \mid z_*(x) > 0\}.$$

Then, if

$$\sup_{\partial\Omega_0}(w^* - z_*) \leq 0, \quad \sup_{\overline{\Omega}_0} z < +\infty$$

it holds that

$$\sup_{\overline{\Omega}_0}(w^* - z_*) \leq 0.$$

Remark 6. *The set Ω_0 is open in Ω because z_* is lower semicontinuous on $\overline{\Omega}$ according to the same reason $z_* = 0$ in $\partial\Omega_0 \setminus \partial\Omega$.*

For the proof of this theorem, we use the following notations used in the theory of viscosity solutions; namely, given $w : \Omega \to \mathbb{R}$, let

$$\begin{aligned}\overline{J}^{2,+}w(x) = \{&(p, X) \in \mathbb{R}^N \times \mathbb{S}^N \mid \exists (x_k, p_k, X_k) \in \Omega \times \mathbb{R}^N \times \mathbb{S}^N, \ k = 1, 2, \cdots, \\ &\lim_{k\to\infty}(x_k, w^*(x_k), p, X) = (x, w^*(x), p, X), \ (p_k, X_k) \in J^{2,+}w(x_k)\},\end{aligned}$$

and

$$\begin{aligned}\overline{J}^{2,-}w(x) = \{&(p, X) \in \mathbb{R}^N \times \mathbb{S}^N \mid \exists (x_k, p_k, X_k) \in \Omega \times \mathbb{R}^N \times \mathbb{S}^N, \ k = 1, 2, \cdots, \\ &\lim_{k\to\infty}(x_k, w_*(x_k), p, X) = (x, w_*(x), p, X), \ (p_k, X_k) \in J^{2,-}w(x_k)\}\end{aligned}$$

Proof of Theorem 3. Assume the contrary,

$$\sup_{\overline{\Omega}_0}(w^* - z_*) = \theta > 0.$$

We take $0 < \rho < 1$, satisfying

$$\sup_{\partial\Omega_0}(w^* - \rho z_*) \leq (1 - \rho) \sup_{\partial\Omega_0} z_* \leq (1 - \rho) \sup_{\overline{\Omega}_0} z_* \leq \frac{\theta}{3},$$

and put

$$\tilde{\theta} = \sup_{\overline{\Omega}_0}(w^* - \rho z_*) \geq \theta.$$

Let $\varepsilon > 0$ and $(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \in \overline{\Omega}_0 \times \overline{\Omega}_0$ be the maximum point of

$$\Phi_\varepsilon(x, y) = w^*(x) - \rho z_*(y) - \frac{1}{2\varepsilon}|x - y|^2, \quad (x, y) \in \overline{\Omega}_0 \times \overline{\Omega}_0.$$

Since

$$\Phi_\varepsilon(\hat{x}_\varepsilon, \hat{x}_\varepsilon) \le \Phi_\varepsilon(\hat{x}_\varepsilon, \hat{y}_e), \quad z_*(\hat{y}_\varepsilon) \ge 0$$

it holds that

$$\frac{1}{2\varepsilon}|\hat{x}_\varepsilon - \hat{y}_\varepsilon|^2 \le \rho z_*(\hat{x}_\varepsilon) \le \rho \sup_{\overline{\Omega}_0} z_*,$$

and hence,

$$\lim_{\varepsilon\to 0} |\hat{x}_\varepsilon - \hat{y}_\varepsilon| = 0.$$

A subsequence, therefore, admits $x_0 \in \overline{\Omega}_0$ such that

$$\lim_{\varepsilon\to 0} \hat{x}_\varepsilon = \lim_{\varepsilon\to 0} \hat{y}_\varepsilon = x_0.$$

Since $\Phi_\varepsilon(x_0, x_0) \le \Phi_\varepsilon(\hat{x}_\varepsilon, \hat{y}_\varepsilon)$, it holds that

$$\rho(z_*(\hat{y}_\varepsilon) - z_*(x_0)) \le w(\hat{x}_\varepsilon) - w^*(x_0),$$

and therefore,

$$\begin{aligned} 0 &\le \liminf_{\varepsilon\to 0} \rho(z_*(\hat{y}_\varepsilon) - z_*(x_0)) \le \liminf_{\varepsilon\to 0}(w^*(\hat{x}_\varepsilon) - w^*(x_0)) \\ &\le \limsup_{\varepsilon\to 0}(w^*(\hat{x}_\varepsilon) - w^*(x_0)) \le 0 \end{aligned}$$

by the semicontinuity. We thus obtain

$$\lim_{\varepsilon\to 0} w^*(\hat{x}_\varepsilon) = w^*(x_0), \quad \lim_{\varepsilon\to 0} z_*(\hat{y}_\varepsilon) = z_*(x_0), \tag{64}$$

which implies

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}|\hat{x}_\varepsilon - \hat{y}_\varepsilon|^2 = 0 \tag{65}$$

by $\Phi_\varepsilon(x_0, x_0) \le \Phi_\varepsilon(\hat{x}_\varepsilon, \hat{y}_\varepsilon)$.

Since

$$\sup_{\partial\Omega_0}(w^* - \rho z_*) \le \theta/3 < \tilde{\theta} \le \Phi_\varepsilon(\hat{x}_\varepsilon, \hat{y}_\varepsilon),$$

Equalities (64) and (65) imply $x_0 \in \Omega_0$, which means $z_*(x_0) > 0$. Noting

$$0 < \tilde{\theta} \le \Phi_\varepsilon(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \le w^*(\hat{x}_\varepsilon) - \rho z_*(\hat{y}_\varepsilon),$$

thus we obtain

$$w^*(\hat{x}_\varepsilon) > \rho z_*(\hat{y}_\varepsilon) > 0, \quad 0 < \varepsilon \ll 1. \tag{66}$$

Then, H. Ishii's lemma, as referenced in Koike ([11], Appendix A), guarantees $\hat{X}_\varepsilon, \hat{Y}_\varepsilon \in \mathbb{S}^N$ satisfying

$$(\hat{p}_\varepsilon, \hat{X}_\varepsilon) \in \overline{J}^{2,+} w(\hat{x}_\varepsilon), \ (\hat{p}_\varepsilon, -\hat{Y}_\varepsilon) \in \overline{J}^{2,-} \rho z(\hat{y}_\varepsilon) \tag{67}$$

and

$$\frac{-3}{\varepsilon}\begin{pmatrix} I & O \\ O & I \end{pmatrix} \le \begin{pmatrix} \hat{X}_\varepsilon & O \\ O & \hat{Y}_\varepsilon \end{pmatrix} \le \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{68}$$

for $\hat{p}_\varepsilon = (\hat{x}_\varepsilon - \hat{y}_\varepsilon)/\varepsilon$. Note that the second relation of (67) means

$$(\hat{p}_\varepsilon/\rho, -\hat{Y}_\varepsilon/\rho) \in \overline{J}^{2,-} z(\hat{y}_\varepsilon).$$

Then, Lemma 4 and Remark 5 ensure

$$(x_k, p_k, X_k, \phi_k), \ (y_k \, q_k, Y_k, \psi_k) \in \Omega \times \mathbb{R}^N \times \mathbb{S}^N, \quad k = 1, 2, \cdots,$$

satisfying

$$\phi_k \in \Phi^+(x_k, p_k, X_k; w), \ \psi_k \in \Phi^-(y_k, q_k, -Y_k, z) \tag{69}$$

and

$$\lim_{k\to\infty}(x_k, w^*(x_k), p_k, X_k) = (\hat{x}_\varepsilon, w^*(\hat{x}_\varepsilon), \hat{p}_\varepsilon, \hat{X}_\varepsilon), \tag{70}$$

$$\lim_{k\to\infty}(y_k, z_*(y_k), q_k, Y_k) = (\hat{y}_\varepsilon, z_*(\hat{y}_\varepsilon), \hat{p}_\varepsilon/\rho, -\hat{Y}_\varepsilon/\rho). \tag{71}$$

Since w (resp., z) is a viscosity subsolution (resp., supersolution) to (27), it holds that

$$F(\phi_k, \nabla\phi_k, \nabla^2\phi_k^2) \le 0 \text{ at } x_k, \quad F(\psi_k, \nabla\psi_k, \nabla^2\psi_k^2) \ge 0 \text{ at } y_k.$$

Here , we use

$$\mathrm{tr}\nabla^2\phi^2 = \Delta\phi^2 = 2|\nabla\phi|^2 + 2\phi\Delta\phi$$

valid for $\phi \in C^2(\Omega)$, to obtain

$$\begin{aligned} &-2\phi_k \mathrm{tr} X_k + (\gamma-2)|p_k|^2 + \phi_k \le 0 \quad \text{at } x_k, \\ &-2\psi_k \mathrm{tr}(-Y_k) + (\gamma-2)|q_k|^2 + \psi_k \ge 0 \quad \text{at } y_k \end{aligned}$$

by (69). Sending $k \to \infty$, it arises that

$$\begin{aligned} &-2w^*(\hat{x}_\varepsilon)\mathrm{tr}\hat{X}_\varepsilon + (\gamma-2)|\hat{p}_\varepsilon|^2 + w^*(\hat{x}_\varepsilon) \le 0, \\ &-2z_*(\hat{y}_\varepsilon)\mathrm{tr}(-\frac{\hat{Y}_\varepsilon}{\rho}) + \frac{\gamma-2}{\rho^2}|\hat{p}_\varepsilon|^2 + z_*(\hat{y}_\varepsilon) \ge 0, \end{aligned}$$

from (70) and (71), which implies

$$-2\mathrm{tr}\hat{X}_\varepsilon + \frac{\gamma-2}{w^*(\hat{x}_\varepsilon)}|\hat{p}_\varepsilon|^2 + 1 \le 0, \quad -2\mathrm{tr}(-\hat{Y}_\varepsilon) + \frac{\gamma-2}{\rho z_*(\hat{y}_\varepsilon)}|\hat{p}_\varepsilon|^2 + \rho \ge 0$$

by (66).

Since (68) implies

$$\mathrm{tr}(\hat{X}_\varepsilon + \hat{Y}_\varepsilon) \le 0,$$

we obtain

$$\left(\frac{1}{w^*(\hat{x}_\varepsilon)} - \frac{1}{\rho z_*(\hat{y}_\varepsilon)}\right)(\gamma-2)|\hat{p}_\varepsilon|^2 \le 2\mathrm{tr}(\hat{X}_\varepsilon + \hat{Y}_\varepsilon) + \rho - 1 < 0,$$

which is in contradiction to $\gamma \le 2$ and (66). Thus, the result follows. □

A direct consequence of Theorem 3 is the following fact based on its uniqueness and regularity.

Corollary 2. *Assume $\gamma \le 2$. Let $w \ge 0$ and $z \ge 0$ be bounded viscosity solutions to (27), defined on $\overline{\Omega}$, and put*

$$\Omega_0 = \{x \in \Omega \mid w_*(x) > 0\} \cap \{x \in \Omega \mid z_*(x) > 0\}.$$

Then , if $w^* = w_* = z^* = z_*$ on $\partial\Omega_0$, it follows that $u = v \in C(\overline{\Omega}_0)$.

Proof. This result is similar to the standard case. In fact, the assumption implies

$$w^* \le z_* \le z^* \le w_* \le w^* \quad \text{on } \overline{\Omega}_0.$$

□

The following theorem is obvious if $w \in C(\overline{\Omega})$.

Theorem 4. *Let $w \geq 0$ be a bounded viscosity solution to (27), defined on $\overline{\Omega}$ with $\gamma \leq 2$ satisfying*

$$w^* = w_* = 0 \quad \text{on } \partial(\Omega \backslash \Omega_0), \tag{72}$$

and put

$$\Omega_0 = \{x \in \Omega \mid w_*(x) > 0\}. \tag{73}$$

Then, it follows that

$$w^* = w_* = 0 \quad \text{on } \overline{\Omega \backslash \Omega_0}. \tag{74}$$

Proof. Assume the contrary,

$$\sup_{\overline{\Omega \backslash \Omega_0}} w^* > 0,$$

and let $x_0 \in \overline{\Omega \backslash \Omega_0}$ be a maximum point of w^* on $\overline{\Omega \backslash \Omega_0}$. Then, we obtain $x_0 \in \Omega \backslash \overline{\Omega}_0$ by (72). Since $\Omega \backslash \overline{\Omega}_0$ is open, we find $r_0 > 0$ such that $w^*(x) \leq w^*(x_0)$ for all $x \in B_{r_0}(x_0)$.

Let $\varphi(x) = w^*(x_0)$. It is obvious that $w^* - \varphi$ attains a local maximum at x_0. Since $w \geq 0$ is a viscosity subsolution to (27), it arises that

$$0 \geq F(\varphi, \nabla \varphi, \nabla^2 \varphi^2) = w^*(x_0) > 0 \quad \text{at } x_0,$$

which is a contradiction. Thus, $w^* \leq 0$ on $\overline{\Omega \backslash \Omega_0}$. Since $w \geq 0$, it follows (74). □

4. Comparison Theorem to (25)

The viscosity solutions to (25) are treated similarly. Recall $F(r, p, X)$ for $(r, p, X) \in \mathbf{R} \times \mathbf{R}^n \times \mathbb{S}^N$ in (56).

Definition 2. *Let $w : Q_T = \Omega \times (0, T) \to \mathbf{R}$.*

(1) *We say that w with $w^* < +\infty$ in Q_T is a viscosity subsolution to (25), provided that if $w^* - \varphi$ attains a local maximum 0 at $(x, t) \in Q_T$ for $\varphi \in C^2(Q_T)$, then it holds that*

$$(\varphi^2)_t + F(\varphi, \nabla \varphi, \nabla^2 \varphi^2) \leq 0 \quad \text{at } (x_0, t_0).$$

(2) *We say that w in $w_* > -\infty$ in Q_T is a viscosity supersolution to (25), provided that if $w_* - \varphi$ attains a local minimum 0 at $(x, t) \in Q_T$ for $\varphi \in C^2(Q_T)$, then it holds that*

$$(\varphi^2)_t + F(\varphi, \nabla \varphi, \nabla^2 \varphi^2) \geq 0 \quad \text{at } (x, t).$$

(3) *We say that w is a viscosity solution to (25) if it is a viscosity subsolution and a viscosity supersolution to (25).*

Similarly to Remark 4 and Proposition 1, we obtain the following fact.

Proposition 2. *If $w \in C^2(Q_T)$ is a viscosity subsolution (resp., supersolution) to (25), it is a classical subsolution (resp., supersolution). If $0 \leq w \in C^2(Q_T)$ is a classical subsolution to (25), conversely, it is a viscosity subsolution. If $0 < w \in C^2(Q_T)$ is classical supersolution to (25), finally, it is a viscosity supersolution.*

Given $(x, t) \in Q_T$ and $r > 0$, we use the following sets, where $Q_r(x, t) = B_r(x) \times (t - r, t + r)$:

$$\begin{aligned} P^{2,+}w(x,t) = \{(p, \tau, X) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \mid\ & w^*(x+h, t+k) - w^*(x,t) \\ & \leq \tau k + \langle p, h \rangle + \frac{1}{2}\langle Xh, h \rangle + o(|k| + |h|^2) \text{ as } (h,k) \to (0,0)\}. \end{aligned}$$

$$
\begin{aligned}
P^{2,-}w(x,t) = \{&(p,\tau,X) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{S}^N \mid w_*(x+h,t+k) - w_*(x,t) \\
&\geq \tau k + \langle p,h\rangle + \frac{1}{2}\langle Xh,h\rangle + o(|k|+|h|^2) \text{ as } (k,h)\to(0,0)\}.
\end{aligned}
$$

$$
\begin{aligned}
&\overline{P}^{2,+}w(x,t) = \{(p,\tau,X) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{S}^N \mid \exists(x_k,p_k,\tau_k,X_k)\in Q_T\times\mathbb{R}^N\times\mathbb{S}^N, \\
&\quad k=1,2,\cdots,\ \lim_{k\to\infty}(x_k,\tau_k,w^*(x_k,\tau_k),p_k,X_k) = (x,\tau,w^*(x,\tau),p,X), \\
&\quad (p_k,\tau_k,X_k)\in P^{2,+}w(x_k,t_k)\}.
\end{aligned}
$$

$$
\begin{aligned}
&\overline{P}^{2,-}w(x,t) = \{(p,\tau,X) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{S}^N \mid \exists(x_k,\tau_k,p_k,X_k)\in Q_T\times\mathbb{R}^N\times\mathbb{S}^N, \\
&\quad k=1,2,\cdots,\ \lim_{k\to\infty}(x_k,\tau_k,w_*(x_k,t_k),p_k,X_k) = (x,\tau,w_*(x,\tau),p,X), \\
&\quad (p_k,\tau_k,X_k)\in P^{2,-}w(x_k,t_k)\}.
\end{aligned}
$$

$$
\begin{aligned}
&\Phi_r^+(x,t,p,\tau,X;w) = \{\varphi\in C^2(Q_r(x,t))) \mid w^*-\varphi \text{ attains its maximum 0 at } (x,t), \\
&\quad \nabla\varphi(x,t)=p,\ \varphi_t(x,t)=\tau,\ \nabla^2\varphi(x,t)=X\}.
\end{aligned}
$$

$$
\begin{aligned}
&\Phi_r^-(x,t,p,\tau,p,X;w) = \{\varphi\in C^2(Q_r(x,t)) \mid w_*-\varphi \text{ attains its minimum 0 at } (x,t), \\
&\quad \nabla\varphi(x,t)=p,\ \varphi_t(x,t)=\tau,\ \nabla^2\varphi(x,t)=X\}.
\end{aligned}
$$

$$
\Phi^\pm(x,t,p,\tau,p,X;w) = \bigcup_{0<r\ll 1}\Phi_r^\pm(x,t,p,\tau,X;w).
$$

$$
\begin{aligned}
p^{2,\pm}w(x,t) = \{&(\nabla\varphi(x,t),\varphi_t(x,t),\nabla^2\varphi(x,t)^2) \\
&\mid \varphi\in\Phi^\pm(x,t,p,\tau,p,X;w),\ (p,\tau,X)\in P^{2,+}w(x,t)\}.
\end{aligned}
$$

Similarly to Lemma 4, we obtain the following fact.

Lemma 5. *If w is locally bounded in Q_T, each $(x,t)\in Q_T$ and $(p,\tau,X)\in P^{2,\pm}w(x,t)$ admits $\varphi\in\Phi^\pm(x,t,p,\tau,X;w)$.*

The comparison principle for the viscosity solution to (25) is described as follows. Although it is proven similarly to Theorem 3, we show the proof for completeness.

Theorem 5. *Assume $\gamma\leq 2$. Let $w\geq 0$ (resp., $z\geq 0$) be a viscosity subsolution (resp., supersolution) to (25) defined on $\overline{Q_T}$, and put*

$$
Q_0 = \{(x,t)\in Q_T \mid z_*(x,t)>0\}.
$$

Then, if

$$
\sup_{\partial Q_0\setminus(\overline{\Omega}\times\{T\})}(w^*-z_*)\leq 0, \quad \sup_{\overline{Q_T}} w<\infty,
$$

it holds that

$$
\sup_{\overline{Q_0}\setminus(\overline{\Omega}\times\{T\})}(w^*-z_*)\leq 0.
$$

Proof. Assuming the contrary,

$$
\sup_{\overline{Q_0}}(w^*-z_*) = \theta > 0,
$$

we take $0 < \rho, \gamma < 1$, satisfying

$$\sup_{\partial Q_0 \setminus (\overline{\Omega} \times \{T\})} (w^* - \rho z_* - \frac{2\gamma}{T-t}) \leq \frac{\theta}{3}. \tag{75}$$

It is obvious that

$$\sup_{\overline{Q_0} \setminus (\overline{\Omega} \times \{T\})} (w^* - \rho z_* - \frac{2\gamma}{T-t}) \equiv \tilde{\theta} \geq \theta. \tag{76}$$

Let

$$\Phi_\varepsilon(x,t,y,s) = w^*(x,t) - \rho z_*(y,s) - \frac{1}{2\varepsilon}\{|x-y|^2 + (t-s)^2\} - \frac{\gamma}{T-t} - \frac{\gamma}{T-s}$$

be defined for $(x,t,y,s) \in \tilde{Q}_0 \times \tilde{Q}_0$, where $\varepsilon > 0$ and

$$\tilde{Q}_0 = \overline{Q_0} \setminus (\overline{\Omega} \times \{T\}).$$

Let, furthermore, $(\hat{x}_\varepsilon, \hat{t}_\varepsilon, \hat{y}_\varepsilon, \hat{s}_\varepsilon) \in \tilde{Q}_0 \times \tilde{Q}_0$ be a maximum point of Φ_ε. Then, the inequality

$$\Phi_\varepsilon(\hat{x}_\varepsilon, \hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{t}_\varepsilon) + \Phi_\varepsilon(\hat{y}_\varepsilon, \hat{s}_\varepsilon, \hat{y}_\varepsilon, \hat{s}_\varepsilon) \leq 2\Phi_\varepsilon(\hat{x}_\varepsilon, \hat{t}_\varepsilon, \hat{y}_\varepsilon, \hat{s}_\varepsilon)$$

implies

$$\lim_{\varepsilon \to 0}\{(\hat{t}_\varepsilon - \hat{s}_\varepsilon)^2 + |\hat{x}_\varepsilon - \hat{y}_\varepsilon|^2\} = 0,$$

and, furthermore, it arises that

$$\begin{aligned}
&\lim_{\varepsilon \to 0}(\hat{x}_\varepsilon, \hat{t}_\varepsilon) = \lim_{\varepsilon \to 0}(\hat{y}_\varepsilon, \hat{s}_\varepsilon) = (x_0, t_0) \in \overline{Q_0} \\
&\lim_{\varepsilon \to 0} w^*(\hat{x}_\varepsilon, \hat{t}_\varepsilon) = w^*(x_0, t_0), \quad \lim_{\varepsilon \to 0} z_*(\hat{y}_\varepsilon, \hat{s}_\varepsilon) = z_*(x_0, t_0), \\
&\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\{(\hat{t}_\varepsilon - \hat{s}_\varepsilon)^2 + |\hat{x}_\varepsilon - \hat{y}_\varepsilon|^2\} = 0.
\end{aligned}$$

as in Theorem 3. Using the fact that (x_0, t_0) is also a maximum point of

$$w^*(x,t) - \rho z_*(x,t) - \frac{2\gamma}{T-t} \quad \text{on } \tilde{Q}_0,$$

we obtain $(x_0, t_0) \in Q_0$ and also

$$w^*(\hat{x}_\varepsilon, \hat{t}_\varepsilon) > \rho z_*(\hat{y}_\varepsilon, \hat{s}_\varepsilon) > 0, \quad 0 < \varepsilon \ll 1.$$

Then, H. Ishii's lemma guarantees $\hat{X}_\varepsilon, \hat{Y}_\varepsilon \in \mathbb{S}^N$ such that

$$(\hat{p}_\varepsilon, \tau_\varepsilon + \frac{\gamma}{(T-\hat{t}_\varepsilon)^2}, \hat{X}_\varepsilon) \in \overline{P}^{2,+} w(\hat{x}_\varepsilon, \hat{t}_\varepsilon), \tag{77}$$

$$\rho^{-1}(\hat{p}_\varepsilon, \tau_\varepsilon - \frac{\gamma}{(T-\hat{s}_\varepsilon)^2}, -\hat{Y}_\varepsilon) \in \overline{P}^{2,-} z(\hat{y}_\varepsilon, \hat{s}_\varepsilon), \tag{78}$$

and

$$\frac{-3}{\varepsilon}\begin{pmatrix} I & O \\ O & I \end{pmatrix} \leq \begin{pmatrix} \hat{X}_\varepsilon & O \\ O & \hat{Y}_\varepsilon \end{pmatrix} \leq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{79}$$

for

$$\tau_\varepsilon = (\hat{t}_\varepsilon - \hat{s}_\varepsilon)/\varepsilon, \quad \hat{p}_\varepsilon = (\hat{x}_\varepsilon - \hat{y}_\varepsilon)/\varepsilon.$$

By the definition of $\overline{P}^{2,\pm}$ and Lemma 5, we have

$$(x_k, t_k, p_k, X_k, \phi_k), \ (y_k, s_k, q_k, Y_k, \psi_k), \quad k = 1, 2, \cdots,$$

satisfying

$$\phi_k \in \Phi^+(x_k, t_k, p_k, \tau_k, X_k; w), \quad \psi_k \in \Phi^-(y_k, s_k, q_k, \kappa_k, -Y_k; z) \tag{80}$$

and

$$\lim_{k\to\infty}(x_k, t_k, w^*(x_k, t_k), p_k, \tau_k, X_k) = (\hat{x}_\varepsilon, \hat{t}_\varepsilon, w^*(\hat{x}_\varepsilon, \hat{t}_\varepsilon), p_\varepsilon, \tau_\varepsilon + \frac{\gamma}{(T-\hat{t}_\varepsilon)^2}, \hat{X}_\varepsilon), \tag{81}$$

$$\lim_{k\to\infty}(y_k, s_k, z_*(y_k, s_k), q_k, \kappa_k, Y_k) = (\hat{y}_\varepsilon, \hat{s}_\varepsilon, z_*(\hat{y}_\varepsilon, \hat{s}_\varepsilon), \frac{p_\varepsilon}{\rho}, \frac{\hat{\tau}_\varepsilon}{\rho} - \frac{\gamma}{\rho(T-\hat{s}_\varepsilon)^2}, \frac{-\hat{Y}_\varepsilon}{\rho}), \tag{82}$$

which implies

$$\begin{aligned}(\varphi_k^2)_t + F(\phi_k^2, \nabla\phi_k, \nabla^2\phi_k^2) \le 0 \quad \text{at } (x_k, t_k),\\ (\psi_k^2)_t + F(\psi_k, \nabla\psi_k, \nabla^2(\psi_k)^2) \ge 0 \quad \text{at } (y_k, s_k),\end{aligned}$$

because w (resp., z) is a viscosity subsolution (resp., supersolution) to (27). Then, there holds that

$$\begin{aligned}2\phi_k\tau_k - 2\phi_k \mathrm{tr} X_k + (\gamma-2)|p_k|^2 + \phi_k \le 0 \quad \text{at } (x_k, t_k),\\ 2\psi_k\kappa_k - 2\psi_k \mathrm{tr}(-Y_k) + (\gamma-2)|q_k|^2 + \psi_k \ge 0 \quad \text{at } (s_k, y_k).\end{aligned}$$

by (80).

Sending $k \to \infty$, we obtain

$$\begin{aligned}&2w^*(\hat{x}_\varepsilon, \hat{t}_\varepsilon)\{\hat{\tau}_\varepsilon + \frac{\gamma}{(T-\hat{t}_\varepsilon)^2}\} - 2w^*(\hat{x}_\varepsilon, \hat{t}_\varepsilon)\mathrm{tr}\hat{X}_\varepsilon + (\gamma-2)|\hat{p}_\varepsilon|^2 + w^*(\hat{x}_\varepsilon, \hat{t}_\varepsilon) \le 0,\\ &2z_*(\hat{y}_\varepsilon, \hat{s}_\varepsilon)\{\frac{\hat{\tau}_\varepsilon}{\rho} - \frac{\gamma}{\rho(T-\hat{t}_\varepsilon)^2}\} - 2z_*(\hat{y}_\varepsilon, \hat{s}_\varepsilon)\mathrm{tr}(-\frac{\hat{Y}_\varepsilon}{\rho}) + \frac{\gamma-2}{\rho^2}|\hat{p}_\varepsilon|^2 + v(\hat{y}_\varepsilon, \hat{s}_\varepsilon) \ge 0.\end{aligned}$$

by (81) and (82), which implies

$$\begin{aligned}2\{\hat{\tau}_\varepsilon + \frac{\gamma}{(T-\hat{t}_\varepsilon)^2}\} - 2\mathrm{tr}\hat{X}_\varepsilon + \frac{\gamma-2}{w^*(\hat{x}_\varepsilon, \hat{t}_\varepsilon)}|\hat{p}_\varepsilon|^2 + 1 \le 0,\\ 2\{\hat{\tau}_\varepsilon - \frac{\gamma}{(T-\hat{s}_\varepsilon)^2}\} - 2\mathrm{tr}(-\hat{Y}_\varepsilon) + \frac{\gamma-2}{\rho z_*(\hat{y}_\varepsilon, \hat{s}_\varepsilon)}|\hat{p}_\varepsilon|^2 + \rho \ge 0.\end{aligned}$$

Using $\mathrm{tr}(\hat{X}_\varepsilon + \hat{Y}_\varepsilon) \le 0$, derived from (79), now we reach

$$\begin{aligned}&2\gamma\{\frac{1}{(T-\hat{t}_\varepsilon)^2} + \frac{1}{(T-\hat{s}_\varepsilon)^2}\} + (\frac{1}{w^*(\hat{x}_\varepsilon, \hat{t}_\varepsilon)} - \frac{1}{\rho z_*(\hat{y}_\varepsilon, \hat{s}_\varepsilon)})(\gamma-2)|\hat{p}_\varepsilon|^2\\ &\le 2\mathrm{tr}(\hat{X}_\varepsilon + \hat{Y}_\varepsilon) + \rho - 1 < 0,\end{aligned}$$

which is a contradiction. □

The following corollary is a direct consequence of Theorem 5.

Corollary 3. *Assume $\gamma \le 2$, and let w and z be bounded viscosity solutions to (25) defined on $\overline{Q_T}$. Let*

$$Q_0 = \{(x,t) \in Q_T \mid w_*(x,t) > 0\} \cap \{(x,t) \in Q_T \mid z_*(x,t) > 0\}.$$

Then, if $w^* = w_* = z^* = z_*$ on $\partial Q_0 \backslash (\overline{\Omega} \times \{T\}$, it follows that $w = z \in C(\overline{Q_0} \backslash (\overline{\Omega} \times \{T\}))$.

An analogous result to Theorem 4 is the following theorem.

Theorem 6. *Let $w \geq 0$, defined on $\overline{Q_T}$, be a bounded viscosity solution to (25) with $\gamma \leq 2$, and put*

$$Q_0 = \{(x,t) \in Q_T \mid w_*(x,t) > 0\}, \quad \tilde{Q} = Q_T \backslash Q_0.$$

Then, if

$$w^* = w_* = 0 \quad on\ \partial\tilde{Q} \backslash (\overline{\Omega} \times \{T\}), \tag{83}$$

it follows that

$$w^* = w_* = 0 \quad in\ \tilde{Q}. \tag{84}$$

Proof. Assuming

$$\sup_{\tilde{Q}} w^* > 0,$$

we obtain $0 < \gamma \ll 1$ and $(x_0, t_0) \in Q_T \setminus \overline{Q_0}$, satisfying

$$\sup_{\tilde{Q}} \left(w^* - \frac{\gamma}{T-t} \right) = w^*(x_0, t_0) - \frac{\gamma}{T-t_0} > 0$$

by (83). Then, there is $r_0 > 0$ such that

$$w^*(x_0, t_0) - \frac{\gamma}{T-t_0} \geq w^*(x,t) - \frac{\gamma}{T-t}, \quad \forall (x,t) \in Q_{r_0}(x_0, t_0).$$

Put

$$\varphi(x,t) = \frac{\gamma}{T-t} + w^*(x_0, t_0) - \frac{\gamma}{T-t_0},$$

and see that that $w^* - \varphi$ attains a local maximum 0 at (x_0, t_0). Then, we obtain

$$\begin{aligned} 0 \quad &\geq \quad (\varphi^2)_t + F(\varphi, \nabla\varphi, \nabla^2\varphi^2) \\ &= \quad 2w^*(x_0, t_0)\frac{\gamma}{(T-t_0)^2} + w^*(x_0, t_0) > 0 \quad \text{at } (x_0, t_0), \end{aligned}$$

which is a contradiction.

It thus follows that $w^* \leq 0$ in $\tilde{Q}$, and hence in (84), from $w \geq 0$. □

5. Conclusions

In this paper, we study the quasilinear parabolic Equation (20) derived from the semilinear parabolic equation (1). Here, we review our results and discuss them in the relation to (1)–(3). To begin with, we note two facts regarding the relation between the exponents p and γ.

First, the condition $p > 1$ is equivalent to $4 < \gamma < \infty$ in (22). The case $\gamma = 4$ arises if $f(u) = e^u$ in (2) by taking $v = e^{-2u}$. Several results of (20) obtained in this paper, however, are beyond this range of γ. The case $\gamma < 2$ means $0 < p < 1$ for $f(u) = u^p$, and then, $v = 0$ is equivalent to $u = 0$. Hence, these results are associated with the quenching of the sub-linear parabolic equation. The other case of $2 < \gamma < 4$ means $p < 0$, and then $v = 0$ if and only if $u = 0$. Thus, several properties of (20) presented in this paper are associated with the blow-up or quenching profiles of the solution to (1).

Second, Equation (1) for $\Omega = \mathbb{R}^N$, $N \geq 2$, admits a singular stationary solution

$$u = C|x|^{-\frac{2}{p-1}}$$

with a constant $C > 0$ if $p > \frac{N}{N-2}$. This exponent of p corresponds to $2 < \gamma < N+2$, where $v = v(x) \geq 0$ is realized by

$$\sqrt{v} = C^{-(p-1)}|x|^2.$$

The role of the third exponent of this $\gamma = N+2$, other than $\gamma = 2$ and $\gamma = 4$, however, has not been clarified yet.

Now we examine the results obtained in this paper in detail, in accordance with the theory of semilinear parabolic equations. Section 2 was devoted to the case of $\gamma \geq 2$, in which the convergence of a family of approximate solutions is discussed.

First, Theorems 1 and 2 are valid to $2 \leq \gamma < 4$, with the non-trivial case $2 < \gamma < 4$ corresponding to $p < 0$. Since $u^2 = 0$ if and only if $v = 0$ in (11) for this p, this family of approximate solutions to (20) converges if a continuous quenching profile of the solution $u = u(x,t)$ to (1) arises for $p < 0$.

Second, Theorem 2 for $\gamma = 4$ and $\gamma > 4$ is associated with (2) for $f(u) = e^u$ and $f(u) = u^p$ with $p > 1$, respectively. Since $u^2 = +\infty$ if and only if $v = 0$ in this case, a similar convergence is assured if there is a blow-up profile of the solution $u = u(x,t)$ to (2) for these nonlinearities, which are continuous on $\overline{\Omega}$ with the value $+\infty$ admitted.

Third, Theorem 6 is concerned with the case of $\gamma \leq 2$, with the non-trivial case $\gamma < 2$ corresponding to $0 < p < 1$. There, we obtain $w \equiv \sqrt{v} = 0$ if and only if $u = 0$, and therefore, this theorem is associated with the quenching profiles of the solution $u = u(x,t)$ to (3) for $f(u) = u^p$ and $0 < p < 1$. In more detail, the quenching region

$$K = \{x \in \Omega \mid u(x,T) = 0\}$$

determines the value of $u = u(x,T)$ on the residual domain, $\Omega \setminus K$.

The theorems obtained in this paper are thus strongly related to the results regarding the blow-up and quenching of the solution to semilinear equations. As we observe, several critical exponents on p are detected for these semilinear problems, which should lead to those of γ to (20) in accordance with the profile of $v(\cdot, T)$.

We observe, also, that the theory of viscosity solutions guarantees the existence of the solution under the presence of the comparison theorem via the method of Perron [10]. There may be a challenge when taking this approach via Theorem 3 or Theorem 5, regarding their incompleteness, as compared to the standard case in which the positive region of the super-solution is not involved. We will come back to these problems in the future.

Author Contributions: Methodology, K.I., M.P. and T.S.; Writing—original draft, T.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors have no conflict of interest directly relevant to the content of this article.

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