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Triple Designs: A Closer Look from Indicator Function

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Abstract: A method of tripling for a three-level design, which triples both the run size and the number of factors of the initial design, has been proposed for constructing a design that can accommodate a large number of factors by combining all possible level permutations of its initial design. Based on the link between the indicator functions of a triple design and its initial design, the close relationships between a triple design and its initial design are built from properties such as resolution and orthogonality. These theoretical results present a closer look at a triple design and provide a solid foundation for a design constructed using the tripling method, where the constructed designs have better properties, such as high resolution and orthogonality, and are recommended for application in high dimension topics of statistics or large-scale experiments.

Keywords: indicator function; level permutation; orthogonality; tripling

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1. Introduction

Fractional factorial designs are among the most popular experimental designs in various fields. The minimum aberration criterion [1] and its extension, the generalized minimum aberration criterion [2,3] are commonly used for comparing fractional factorials.

If the goal is to conduct a sensitivity analysis between inputs and outputs, computer experiments are commonly used, especially if the input-output relation of experiments is likely to have some curvature. With the rapid increase in computational power, more and more large fractional factorial designs are used in large-scale computer experiments in practice. For example, researchers at Johns Hopkins University initially employed a design with 512 runs followed by 352 additional runs to resolve the aliasing of two-factor interactions in a ballistic missile defense project [4]. The second scenario was explored using a resolution V design with 4096 runs obtained using SAS's PROC FACTEX. Another example is reported in [5] that designs with over 600 runs, and as many as 53 parameters were used in computer simulations at Los Alamos National Laboratory. Bettonvil and Kleijnen [6] discussed a case study on the CO_2 greenhouse effect using a deterministic simulation model with 281 factors [7]. Kleijnen et al. [8] applied sequential bifurcation to a practical discrete event simulation of a supply chain centered around the Ericsson company in Sweden, involving 92 factors. Motivated by practical applications, the construction method for designs with a large size (a large number of runs and/or a large number of factors while keeping the run size relatively small) is urgently needed, and it is an important issue. In particular, the construction of large-size designs from small designs has attracted more and more attention. For two-level designs, doubling plays an important role in the construction of two-level designs of resolution IV [9]. Given a two-level regular fractional factorial design of resolution IV, the method of doubling produces another design of resolution IV which doubles both the run size and the number of factors of the initial design. One can refer to [10–13] for more details about doubling.

When both the factorial main effects and some quadratic effects need to be detected, it is very necessary to apply multi-level designs for that purpose. Since three-level designs are the most commonly used designs with factor levels higher than two, the three-level fractional factorials constructed in this paper provide an alternative for this demand in practical applications, such as elemental factorial analysis of nanostructure congeners, capturing curvature or active pure-quadratic effects of quality control, and so on. For designs with more than two levels, based on the level permutation method [14–19], the doubling process of two-level designs has been naturally extended to three-level designs. A method of tripling for three-level designs, which triples both the run size and the number of factors of the initial three-level design, is proposed by combining all possible level permutations of its initial design in Ou et al. [20] and Li and Qin [21], respectively.

In this paper, we aim to explore the additional properties of triple designs using indicator functions, which provide a closer look at triple designs. The indicator function has been adopted by Fontana et al. [22] to study the two-level factorial designs. It allows us to discuss not only the regular factorial designs but also non-regular factorial designs. The indicator function has become a powerful tool for studying general two-level factorial designs; see, for example [12,23–32]. Furthermore, Cheng and Ye [14], Pistone and Rogantin [33,34], and Pang and Liu [35] established that general fractional factorial designs, three or higher levels or multilevel, can also be represented with indicator functions.

The contribution of this paper is twofold. First, the closer relationship between a triple design and its initial design is built with indicator functions. It is shown that the indicator function of a triple design is decided uniquely by one of its initial designs. The internal structure of a triple design is explored from the word characteristic of its indicator function, and a new look of triple designs is provided. Second, the properties of a triple design and its projections, such as resolution and orthogonality, are studied by the expression of the indicator function. Given a three-level fractional factorial design of resolution III (IV), we show that its triple design is a design of resolution III, and the projections of a triple design also is a design of resolution III (IV). These theoretical results provide a solid foundation for the tripling construction method for a design with a large size, in which the constructed designs have better properties, such as high resolution and orthogonality, and are recommended for use in practice. The triple designs discussed in this paper are competitive in large-scale computer experiments, such as aerospace, quantum communication, intelligent manufacturing, and so on.

The paper is organized as follows. In Section 2, some notations and preliminaries are included. In Section 3, the indicator function of a triple design is expressed based on the indicator function of its initial design, and the section provides a closer look at the internal structure of a triple design by its indicator function. In Section 4, the close relationships between a triple design and its initial design are built from properties such as resolution and orthogonality. Finally, some conclusions are given in Section 5. For clarity, we have placed all the proofs in Appendix A.

2. Notations and Preliminaries

Let $\mathcal{D}$ be a 3^s full factorial design [36] with s three-level factors, where the three levels of each factor are $w_0 = 1, w_1 = \mathrm{e}^{i\frac{2\pi}{3}}$, and $w_2 = \mathrm{e}^{i\frac{4\pi}{3}}$, i.e., evenly spaced solutions of $z^3 = 1$ on the unit circle in the complex plane $\mathbb{C}$. Accordingly, the design points of $\mathcal{D}$ are just the solutions of the polynomial system $\{x_1^3 = 1, \ldots, x_s^3 = 1\}$ on $\mathbb{C}^s$. Under this level coding strategy, the polynomial representation of the indicator function benefits from its cube-free property. An n-run unreplicated three-level fractional factorial design $\mathcal{F}$ is regarded as a subset of $\mathcal{D}$, each row of $\mathcal{F}$ corresponds to a run and each column of $\mathcal{F}$ to an experimental factor in the design. Let $\mathcal{U}(n; 3^s)$ be the set of U-type designs with n runs and s three-level factors. A design $\mathcal{F}$ in $\mathcal{U}(n; 3^s)$ can be presented as an $n \times s$ matrix with entries $0, 1, 2$ (or equivalently with entries w_0, w_1, w_2), where each entry appears equally often in each column of $\mathcal{F}$. If all the possible 3^t level combinations corresponding to any t columns of design $\mathcal{F}$ appear equally often, design $\mathcal{F}$ is called an orthogonal array [37] of strength t and denoted by $OA(n; 3^s, t)$.

The indicator function $f_{\mathcal{F}}(x)$ of design $\mathcal{F} \in \mathcal{U}(n;3^s)$, due to [22,26], is defined as a function on 3^s full factorial design $\mathcal{D}$ such that

$$f_{\mathcal{F}}(x) \quad = \quad \begin{cases} 1, & \text{if } x \in \mathcal{F}, \\ 0, & \text{if } x \in \mathcal{D} - \mathcal{F}. \end{cases}$$

Under the constraint $x_i^3 = 1, i = 1, \dots, s$, the indicator function $f_{\mathcal{F}}(x)$ of design $\mathcal{F}$ can be uniquely cube-free represented by the complex polynomial function defined on $\mathcal{D}$ as

$$f_{\mathcal{F}}(x) = \sum_{\alpha \in L} b_\alpha x^\alpha, \tag{1}$$

where L is the set of all s tuples α, that is, $L = \{\alpha = (\alpha_1, \dots, \alpha_s) | \alpha_i = 0, 1, 2, \text{ for } i = 1, \dots, s\}$, $x^\alpha = x_1^{\alpha_1} \cdots x_s^{\alpha_s}$ and

$$b_\alpha = \frac{1}{3^s} \sum_{x \in \mathcal{F}} \overline{x^\alpha}, \tag{2}$$

where $\overline{x^\alpha}$ is the conjugate of x^α. Therefore, an indicator function of design $\mathcal{F}$ has the unique cube-free polynomial representation on $\mathcal{D}$.

The coefficients b_α of $f_{\mathcal{F}}(x)$ reflect some basic information of design $\mathcal{F}$. In particular, $b_{\mathbf{0}} = n/3^s$, where n is the run size of $\mathcal{F}$. In other words, $b_{\mathbf{0}}$ is just the ratio between the number of points of $\mathcal{F}$ and the number of points of $\mathcal{D}$. The coefficients b_α of $f_{\mathcal{F}}(x)$ satisfy $|b_\alpha / b_{\mathbf{0}}| \leq 1$. A design is a regular design if and only if $|b_\alpha / b_{\mathbf{0}}| = 1$ for any $b_\alpha \neq 0$. A word of the design $\mathcal{F}$ is defined as the term with a non-zero coefficient (except the constant) in the indicator function $f_{\mathcal{F}}(x)$ of design $\mathcal{F}$. Following Li et al. [24] for two-level designs, the length of a word x^α is defined as $\|x\| = |\alpha| + (1 - |b_\alpha / b_{\mathbf{0}}|)$, where $|\alpha|$ represents the number of letters in the word x^α, i.e., the number of nonzero elements in α. The length of the shortest word of $f_{\mathcal{F}}(x)$ is called the generalized resolution of design $\mathcal{F}$.

The definition of an indicator function follows immediately from the following lemma. The proof of the lemma is straightforward and is omitted here.

Lemma 1. *Let $f_{\mathcal{A}}$ and $f_{\mathcal{B}}$ be indicator functions of two disjoint designs $\mathcal{A}$ and $\mathcal{B}$, respectively. The indicator function of design $\mathcal{A} \cup \mathcal{B}$ is then $f_{\mathcal{A} \cup \mathcal{B}} = f_{\mathcal{A}} + f_{\mathcal{B}}$.*

Following Cheng and Ye [14], the generalized word-length pattern and generalized minimum aberration criterion of three-level design $\mathcal{F}$ are defined as follows.

Definition 1. *Let $\mathcal{F}$ be an n runs s three-level factors fractional factorial design, $f_{\mathcal{F}}(x) = \sum_{\alpha \in L} b_\alpha x^\alpha$ is its indicator function. The generalized word-length pattern of $\mathcal{F}$ is defined as*

$$A_i(\mathcal{F}) = \sum_{|\alpha| = i} \left\| \frac{b_\alpha}{b_{\mathbf{0}}} \right\|^2, \quad i = 1, \dots, s, \tag{3}$$

where $\|\cdot\|$ is the complex module. The generalized minimum aberration criterion is to sequentially minimize $A_i(\mathcal{F})$ for $i = 1, \dots, s$. The resolution of $\mathcal{F}$ equals the smallest t such that $A_t(\mathcal{F}) > 0$.

Remark 1. *The definition of the generalized word-length pattern of $\mathcal{F}$ in Definition 1 is equivalent to the definition in Xu and Wu [3].*

Example 1. *Consider a 3^{5-2} regular three-level design $\mathcal{F}$ with defining relations $I = ABD^2 = AB^2CE^2$, where A, B, C, D, E (or $x_1, \dots, x_5$) are the facotor labels of design $\mathcal{F}$. Accordingly, the* *definition contrast subgroup of $\mathcal{F}$ is $I = ABD^2 = AB^2CE^2 = AC^2DE = BCDE^2$. The indicator function of $\mathcal{F}$ is*

$$f_{\mathcal{F}}(x) = \frac{1}{9}\Big(1 + x_1x_2x_4^2 + x_1^2x_2^2x_4 + x_1x_2^2x_3x_5^2 + x_1^2x_2x_3^2x_5 + x_1x_3^2x_4x_5 + x_1^2x_3x_4^2x_5^2 + x_2x_3x_4x_5^2 + x_2^2x_3^2x_4^2x_5\Big).$$

From the expression of $f_{\mathcal{F}}(x)$ given above, one can easily find that design $\mathcal{F}$ is a regular design since $|b_\alpha/b_{\mathbf{0}}| = 1$ for any $b_\alpha \neq 0$. Moreover, following Definition 1, the generalized word-length pattern of $\mathcal{F}$ is $(0,0,2,6)$.

Suppose $\mathcal{F}$ is a three-level design with n runs s three-level factors, then there are six kinds of level permutations of $\mathcal{F}$, which are listed in Table 1.

Table 1. All possible level permutations of design $\mathcal{F}$.

Permutation No.	Initial Design	Permutation Method	Image
1	$\mathcal{F}$	$(w_0, w_1, w_2) \mapsto (w_0, w_1, w_2)$	$\mathcal{F}$
2	$\mathcal{F}$	$(w_0, w_1, w_2) \mapsto (w_0, w_2, w_1)$	$\mathcal{F}_{(1)}$
3	$\mathcal{F}$	$(w_0, w_1, w_2) \mapsto (w_2, w_1, w_0)$	$\mathcal{F}_{(2)}$
4	$\mathcal{F}$	$(w_0, w_1, w_2) \mapsto (w_1, w_0, w_2)$	$\mathcal{F}_{(3)}$
5	$\mathcal{F}$	$(w_0, w_1, w_2) \mapsto (w_2, w_0, w_1)$	$\mathcal{F}_{(4)}$
6	$\mathcal{F}$	$(w_0, w_1, w_2) \mapsto (w_1, w_2, w_0)$	$\mathcal{F}_{(5)}$

Ou et al. [20] and Li and Qin [21] proposed a new concept named the tripling of three-level design $\mathcal{F}$ based on all of the possible level permutations of $\mathcal{F}$ shown in Table 1, which is defined below.

Definition 2 (Ou et al. [20])**.** *Suppose $\mathcal{F}$ is a three-level design with n runs s three-level factors, $\mathcal{F}_{(i)}, i = 1, \ldots, 5$ are the level permutations of $\mathcal{F}$ as listed in Table 1. The $3n \times 3s$ matrix*

$$T(\mathcal{F}) = \begin{pmatrix} \mathcal{F} & \mathcal{F} & \mathcal{F}_{(1)} \\ \mathcal{F} & \mathcal{F}_{(4)} & \mathcal{F}_{(2)} \\ \mathcal{F} & \mathcal{F}_{(5)} & \mathcal{F}_{(3)} \end{pmatrix}$$

is defined as **triple design** *of $\mathcal{F}$.*

3. Indicator Function of Triple Design

In this section, we aim to explore the link between a triple design $T(\mathcal{F})$ and its initial design $\mathcal{F}$ by using the tool of the indictor function, which provides a closer look at triple design $T(\mathcal{F})$.

Denote $\mathcal{A}_1 = \begin{pmatrix} \mathcal{F} \\ \mathcal{F} \\ \mathcal{F} \end{pmatrix}$, $\mathcal{A}_2 = \begin{pmatrix} \mathcal{F} \\ \mathcal{F}_{(4)} \\ \mathcal{F}_{(5)} \end{pmatrix}$ and $\mathcal{A}_3 = \begin{pmatrix} \mathcal{F}_{(1)} \\ \mathcal{F}_{(2)} \\ \mathcal{F}_{(3)} \end{pmatrix}$ as the column blocks of $T(\mathcal{F})$, and $\mathcal{B}_1 = (\mathcal{F}, \mathcal{F}, \mathcal{F}_{(1)})$, $\mathcal{B}_2 = (\mathcal{F}, \mathcal{F}_{(4)}, \mathcal{F}_{(2)})$ $\mathcal{B}_3 = (\mathcal{F}, \mathcal{F}_{(5)}, \mathcal{F}_{(3)})$ as the row blocks of $T(\mathcal{F})$. Based on the polynomial form of the indicator function of a fractional factorial design in (1), the indicator functions of $\mathcal{F}_{(i)}$ can be written as

$$f_{\mathcal{F}_{(i)}}(x) = \sum_{\alpha \in L} b_\alpha^{(i)} x_{(i)}^\alpha, \ i = 1, \ldots, 5, \tag{4}$$

where $x_{(i)}$ is the corresponding run in $\mathcal{F}_{(i)}$ for given $x \in \mathcal{F}$.

Define $L_{3s} = \{\beta = (\beta_1, \beta_2, \ldots, \beta_{3s}) | \beta_i = 0, 1, 2, \text{ for } i = 1, \ldots, 3s\}$. Similarly, the indicator function of $T(\mathcal{F})$ can be written as

$$f_{T(\mathcal{F})}(z) = \sum_{\beta \in L_{3s}} c_\beta z^\beta, \tag{5}$$

where $z^\beta = z_1^{\beta_1} z_2^{\beta_2} \cdots z_{3s}^{\beta_{3s}}$ for $z = (z_1, z_2, \ldots, z_{3s}) \in T(\mathcal{F})$, and the indicator functions of $\mathcal{B}_i$ can be written as

$$f_{\mathcal{B}_i}(z) = \sum_{\beta \in L_{3s}} c_\beta^{(i)} z^\beta, \ i = 1, 2, 3. \tag{6}$$

For any $\mathbf{u} = (u_1, \ldots, u_s) \in L$ and $\mathbf{v} = (v_1, \ldots, v_s) \in L$, define

$$\mathbf{u} \oplus \mathbf{v} = (u_1, \ldots, u_s, v_1, \ldots, v_s).$$

For any $\beta \in L_{3s}$, there exists $\mathbf{u}, \mathbf{v}, \mathbf{y} \in L$ such that $\beta = \mathbf{u} \oplus \mathbf{v} \oplus \mathbf{y}$. Accordingly, the indicator functions of $\mathcal{B}_i$ in (6) can be rewritten as

$$f_{\mathcal{B}_i}(z) = \sum_{\mathbf{u}, \mathbf{v}, \mathbf{y} \in L} c_{\mathbf{u} \oplus \mathbf{v} \oplus \mathbf{y}}^{(i)} z^{\mathbf{u} \oplus \mathbf{v} \oplus \mathbf{y}}, \ i = 1, 2, 3, \tag{7}$$

where $c_{\mathbf{u} \oplus \mathbf{v} \oplus \mathbf{y}}^{(1)} = \frac{1}{3^{3s}} \sum_{x \in \mathcal{F}} x^{\mathbf{u}} x^{\mathbf{v}} x_{(1)}^{\mathbf{y}}$, $c_{\mathbf{u} \oplus \mathbf{v} \oplus \mathbf{y}}^{(2)} = \frac{1}{3^{3s}} \sum_{x \in \mathcal{F}} x^{\mathbf{u}} x_{(4)}^{\mathbf{v}} x_{(2)}^{\mathbf{y}}$, $c_{\mathbf{u} \oplus \mathbf{v} \oplus \mathbf{y}}^{(3)} = \frac{1}{3^{3s}} \sum_{x \in \mathcal{F}} x^{\mathbf{u}} x_{(5)}^{\mathbf{v}} x_{(3)}^{\mathbf{y}}$.

By Lemma 1, the indicator function of $T(\mathcal{F})$ in (5) can be rewritten as

$$f_{T(\mathcal{F})}(z) = \sum_{\mathbf{u}, \mathbf{v}, \mathbf{y} \in L} c_{\mathbf{u} \oplus \mathbf{v} \oplus \mathbf{y}} z^{\mathbf{u} \oplus \mathbf{v} \oplus \mathbf{y}}, \tag{8}$$

where $c_{\mathbf{u} \oplus \mathbf{v} \oplus \mathbf{y}} = c_{\mathbf{u} \oplus \mathbf{v} \oplus \mathbf{y}}^{(1)} + c_{\mathbf{u} \oplus \mathbf{v} \oplus \mathbf{y}}^{(2)} + c_{\mathbf{u} \oplus \mathbf{v} \oplus \mathbf{y}}^{(3)}$.

For any $\alpha = (\alpha_1, \ldots, \alpha_s) \in L$, define $\alpha_{[3]} = \alpha \mod 3 = (\alpha_1 \mod 3, \ldots, \alpha_s \mod 3)$, and denote $\bar{\alpha} = (\mathbf{3} - \alpha)_{[3]} = ((3 - \alpha_1) \mod 3, \ldots, (3 - \alpha_s) \mod 3)$, $\alpha_{(j)} = |\{\alpha_i | \alpha = (\alpha_1, \ldots, \alpha_s) \in L, \alpha_i = j\}|$, that is, $\alpha_{(j)}$ is the number of j in α, $j = 0, 1, 2$. For any $\mathbf{v} \in L$ and $\mathbf{y} \in L$, define $A = \mathbf{v}_{(1)} \mod 3$, $B = \mathbf{v}_{(2)} \mod 3$ and $M = \mathbf{y}_{(1)} \mod 3$, $N = \mathbf{y}_{(2)} \mod 3$. Moreover, denote $H = \{(0,0), (1,1), (2,2)\}$, $J = \{(0,1), (1,2), (2,0)\}$ and $K = \{(1,0), (2,1), (0,2)\}$.

Based on the above notations, the following three lemmas provide some properties of the term $x_{(i)}^\alpha$ in (4).

Lemma 2. *Let $\mathcal{F}$ be an n runs s three-level factors fractional factorial design. For $\mathbf{v} \in L$ and $\forall x \in \mathcal{F}$, we have*
(a) if $(A, B) \in H$, $x^{\mathbf{v}} = x_{(4)}^{\mathbf{v}} = x_{(5)}^{\mathbf{v}}$; (b) if $(A, B) \notin H$, $x^{\mathbf{v}} + x_{(4)}^{\mathbf{v}} + x_{(5)}^{\mathbf{v}} = 0$;
(c) if $(M, N) \in H$, $x_{(1)}^{\mathbf{y}} = x_{(2)}^{\mathbf{y}} = x_{(3)}^{\mathbf{y}}$; (d) if $(M, N) \notin H$, $x_{(1)}^{\mathbf{y}} + x_{(2)}^{\mathbf{y}} + x_{(3)}^{\mathbf{y}} = 0$.

Lemma 3. *Let $\mathcal{F}$ be an n runs s three-level factors fractional factorial design.*
(a) For $\mathbf{v}, \mathbf{y} \in L$ and $\forall x \in \mathcal{F}$, we have $x^{\mathbf{v}} x_{(1)}^{\mathbf{y}} = x_{(4)}^{\mathbf{v}} x_{(2)}^{\mathbf{y}} = x_{(5)}^{\mathbf{v}} x_{(3)}^{\mathbf{y}}$ when one of the following conditions satisfies:
(i) $(A, B) \in H$ and $(M, N) \in H$; (ii) $(A, B) \in J$ and $(M, N) \in K$; (iii) $(A, B) \in K$ and $(M, N) \in J$.
(b) For $\mathbf{v}, \mathbf{y} \in L$ and $\forall x \in \mathcal{F}$, we have $x^{\mathbf{v}} x_{(1)}^{\mathbf{y}} + x_{(4)}^{\mathbf{v}} x_{(2)}^{\mathbf{y}} + x_{(5)}^{\mathbf{v}} x_{(3)}^{\mathbf{y}} = 0$ when one of the following conditions satisfies:
(i) $(A, B) \in J$ and $(M, N) \in J$; (ii) $(A, B) \in K$ and $(M, N) \in K$; (iii) $(A, B) \in H$ and $(M, N) \in J$; (iv) $(A, B) \in H$ and $(M, N) \in K$; (v) $(A, B) \in J$ and $(M, N) \in H$; (vi) $(A, B) \in K$ and $(M, N) \in H$.

Lemma 4. *Let $\mathcal{F}$ be an n runs s three-level factors fractional factorial design, $f_{\mathcal{F}}(x) = \sum_{\alpha \in L} b_\alpha x^\alpha$ and $f_{\mathcal{F}_{(1)}}(x) = \sum_{\alpha \in L} b_\alpha^{(1)} x_{(1)}^\alpha$ respectively be the indicator functions of $\mathcal{F}$ and $\mathcal{F}_{(1)}$, then $x^{\bar{\alpha}} = x_{(1)}^\alpha$.*

Based on Lemmas 2–4, the following two theorems provide the relationships between the coefficients $c^{(i)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}}$'s of $f_{\mathcal{B}_i}(z)$ given in (7) and the coefficients b_α's of the indicator function $f_{\mathcal{F}}(x)$ of $\mathcal{F}$.

Theorem 1. *Let $\mathcal{F}$ be an n runs s three-level factors fractional factorial design, $f_{\mathcal{F}}(x) = \sum_{\alpha \in L} b_\alpha x^\alpha$ be the indicator function of $\mathcal{F}$, then for any $\mathbf{u} \in L$ and $\mathbf{0} \in L$, we have*
(a) if $(A, B) \in H$ for $\mathbf{v} \in L$, $c^{(1)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{0}} = c^{(2)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{0}} = c^{(3)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{0}} = \frac{1}{3^{2s}} b_{(\mathbf{u}+\mathbf{v})_{[3]}}$;
(b) if $(A, B) \notin H$ for $\mathbf{v} \in L$, $c^{(1)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{0}} + c^{(2)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{0}} + c^{(3)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{0}} = 0$, particularly, if $\mathbf{v} = \bar{\mathbf{u}}$, $c^{(1)}_{\mathbf{u}\oplus\bar{\mathbf{u}}\oplus\mathbf{0}} = c^{(2)}_{\mathbf{u}\oplus\bar{\mathbf{u}}\oplus\mathbf{0}} = c^{(3)}_{\mathbf{u}\oplus\bar{\mathbf{u}}\oplus\mathbf{0}} = \frac{1}{3^{2s}} b_{(\mathbf{u}+\bar{\mathbf{u}})_{[3]}} = \frac{1}{3^{2s}} b_{\mathbf{0}}$;
(c) if $(M, N) \in H$ for $\mathbf{y} \in L$, $c^{(1)}_{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{y}} = c^{(2)}_{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{y}} = c^{(3)}_{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{y}} = \frac{1}{3^{2s}} b_{(\mathbf{u}+\bar{\mathbf{y}})_{[3]}}$, particularly, if $\mathbf{y} = \mathbf{u}$, $c^{(1)}_{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{u}} = c^{(2)}_{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{u}} = c^{(3)}_{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{u}} = \frac{1}{3^{2s}} b_{(\mathbf{u}+\bar{\mathbf{u}})_{[3]}} = \frac{1}{3^{2s}} b_{\mathbf{0}}$;
(d) if $(M, N) \notin H$ for $\mathbf{y} \in L$, $c^{(1)}_{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{y}} + c^{(2)}_{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{y}} + c^{(3)}_{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{y}} = 0$.

Remark 2. *Theorem 1 shows the close relationship between the initial design $\mathcal{F}$ and the subdesigns $\mathcal{B}_i$ of the triple design $T(\mathcal{F})$, $i = 1, 2, 3$. Since $c_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} = c^{(1)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} + c^{(2)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} + c^{(3)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}}$, the word $z_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{0}}$ in $f_{T(\mathcal{F})}(z)$ with coefficient $c_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{0}} \neq 0$ if and only if $(A, B) \in H$ for $\mathbf{v} \in L$, the word $z_{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{y}}$ in $f_{T(\mathcal{F})}(z)$ with coefficient $c_{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{y}} \neq 0$ if and only if $(M, N) \in H$ for $\mathbf{y} \in L$. Based on Theorem 1, one can easily obtain some properties of the projection designs of $T(\mathcal{F})$, which is given in Theorem 5.*

Theorem 2. *Let $\mathcal{F}$ be an n runs s three-level factors fractional factorial design, $f_{\mathcal{F}}(x) = \sum_{\alpha \in L} b_\alpha x^\alpha$ be the indicator function of $\mathcal{F}$, then for any $\mathbf{u}, \mathbf{v}, \mathbf{y} \in L$, we have*
(a) $c^{(1)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} = c^{(2)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} = c^{(3)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} = \frac{1}{3^{2s}} b_{(\mathbf{u}+\mathbf{v}+\bar{\mathbf{y}})_{[3]}}$ when one of the following three conditions satisfies:
(i) $(A, B) \in H$ and $(M, N) \in H$ for $\mathbf{v}, \mathbf{y} \in L$; (ii) $(A, B) \in J$ and $(M, N) \in K$ for $\mathbf{v}, \mathbf{y} \in L$;
(iii) $(A, B) \in K$ and $(M, N) \in J$ for $\mathbf{v}, \mathbf{y} \in L$.
(b) $c^{(1)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} + c^{(2)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} + c^{(3)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} = 0$ when one of the following six conditions satisfies:
(i) $(A, B) \in J$ and $(M, N) \in J$ for $\mathbf{v}, \mathbf{y} \in L$; (ii) $(A, B) \in K$ and $(M, N) \in K$ for $\mathbf{v}, \mathbf{y} \in L$;
(iii) $(A, B) \in H$ and $(M, N) \in J$ for $\mathbf{v}, \mathbf{y} \in L$; (iv) $(A, B) \in H$ and $(M, N) \in K$ for $\mathbf{v}, \mathbf{y} \in L$;
(v) $(A, B) \in J$ and $(M, N) \in H$ for $\mathbf{v}, \mathbf{y} \in L$; (vi) $(A, B) \in K$ and $(M, N) \in H$ for $\mathbf{v}, \mathbf{y} \in L$.

Based on Theorems 1 and 2, the following theorem gives the expression of the indicator function of triple design $T(\mathcal{F})$ from the indicator function of its original design $\mathcal{F}$.

Theorem 3. *Let $\mathcal{F}$ be an n runs s three-level factors fractional factorial design, $f_{\mathcal{F}}(x) = \sum_{\alpha \in L} b_\alpha x^\alpha$ be the indicator function of $\mathcal{F}$, then the indicator function $f_{T(\mathcal{F})}(z)$ in (8) of the triple design $T(\mathcal{F})$ of $\mathcal{F}$ can be expressed as follows*

$$\begin{aligned} f_{T(\mathcal{F})}(z) \quad = \quad & \frac{1}{3^{2s-1}} \Bigg[\sum_{\mathbf{u} \in L} \sum_{\mathbf{v} \in L, (A,B) \in H} \sum_{\mathbf{y} \in L, (M,N) \in H} b_{(\mathbf{u}+\mathbf{v}+\bar{\mathbf{y}})_{[3]}} z^{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} \\ & + \sum_{\mathbf{u} \in L} \sum_{\mathbf{v} \in L, (A,B) \in J} \sum_{\mathbf{y} \in L, (M,N) \in K} b_{(\mathbf{u}+\mathbf{v}+\bar{\mathbf{y}})_{[3]}} z^{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} \\ & + \sum_{\mathbf{u} \in L} \sum_{\mathbf{v} \in L, (A,B) \in K} \sum_{\mathbf{y} \in L, (M,N) \in J} b_{(\mathbf{u}+\mathbf{v}+\bar{\mathbf{y}})_{[3]}} z^{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} \Bigg]. \end{aligned} \tag{9}$$

Remark 3. *Theorem 3 gives the analytical relationship between the indicator functions of initial design $\mathcal{F}$ and its triple design $T(\mathcal{F})$. One can easily find that the coefficients of the indicator function $f_{T(\mathcal{F})}(z)$ of $T(\mathcal{F})$ are completely decided by the coefficients of the indicator function $f_{\mathcal{F}}(x)$ of $\mathcal{F}$.*

4. Some Basic Properties of Triple Designs

In this section, some good properties of $T(\mathcal{F})$ are provided. Based on these good properties, one can easily construct large designs with resolution III or IV.

The following result, whose two-level design version of doubling can also be found in Chen and Cheng [9], reveals the crucial role played by the method of tripling in constructing designs of resolution III.

Theorem 4. *If $\mathcal{F}$ is a design of resolution III, then $T(\mathcal{F})$ is also a design of resolution III. Particularly, if $\mathcal{F}$ is a regular design of resolution III, then $T(\mathcal{F})$ is also a regular design of resolution III.*

Remark 4. *Theorem 4 shows that if the resolution of $\mathcal{F}$ is III, the resolution of triple design $T(\mathcal{F})$ remains the same as its initial design $\mathcal{F}$. Theorem 4 has no counterpart for designs of higher resolution than III. In fact, if the resolution of $\mathcal{F}$ is higher than III, the resolution of $T(\mathcal{F})$ can only achieve III. For any $\mathbf{u}, \mathbf{v}, \mathbf{y} \in L$ such that $\mathbf{u}_{(1)} + \mathbf{u}_{(2)} = 1$ and $\mathbf{v} = \bar{\mathbf{y}} = \mathbf{u}$, $(A, B) \in J$ for $\mathbf{v}$ and $(M, N) \in K$ for $\mathbf{y}$ (or $(A, B) \in K$ for $\mathbf{v}$ and $(M, N) \in J$ for $\mathbf{y}$), $(\mathbf{u} \oplus \mathbf{v} \oplus \bar{\mathbf{y}})_{(1)} + (\mathbf{u} \oplus \mathbf{v} \oplus \bar{\mathbf{y}})_{(2)} =$ 3, and the coefficient of $z^{\mathbf{u}\oplus\mathbf{v}\oplus\bar{\mathbf{y}}}$ is $b_{\mathbf{u}\oplus\mathbf{v}\oplus\bar{\mathbf{y}}} = \frac{1}{3^{2s}} b_{\mathbf{0}} \neq 0$. Hence, $T(\mathcal{F})$ must contain word(s) with a length of 3.*

In the following, the designs with resolution IV are constructed by the projections of $T(\mathcal{F})$. Denote the projection designs of $T(\mathcal{F})$ as

$$T_1(\mathcal{F}) = \begin{pmatrix} \mathcal{F} & \mathcal{F}_{(1)} \\ \mathcal{F}_{(4)} & \mathcal{F}_{(2)} \\ \mathcal{F}_{(5)} & \mathcal{F}_{(3)} \end{pmatrix}, T_2(\mathcal{F}) = \begin{pmatrix} \mathcal{F} & \mathcal{F}_{(1)} \\ \mathcal{F} & \mathcal{F}_{(2)} \\ \mathcal{F} & \mathcal{F}_{(3)} \end{pmatrix} \text{ and } T_3(\mathcal{F}) = \begin{pmatrix} \mathcal{F} & \mathcal{F} \\ \mathcal{F} & \mathcal{F}_{(4)} \\ \mathcal{F} & \mathcal{F}_{(5)} \end{pmatrix}.$$

Theorem 5. *If $\mathcal{F}$ is a design of resolution III (or IV), then $T_i(\mathcal{F})$ are also designs of resolution III (or IV) for $i = 1, 2, 3$. Particularly, If $\mathcal{F}$ is a regular design of resolution III (or IV), then $T_i(\mathcal{F})$ are also regular designs of resolution III (or IV) for $i = 1, 2, 3$.*

Remark 5. *Theorem 5 shows that if the resolution of $\mathcal{F}$ is III, the resolution of the projection designs $T_1(\mathcal{F}), T_2(\mathcal{F})$ and $T_3(\mathcal{F})$ of triple design $T(\mathcal{F})$ remains the same as its original design $\mathcal{F}$. Theorem 5 has no counterpart for designs of higher resolution than IV. In fact, if the resolution of $\mathcal{F}$ is higher than IV, the resolution of $T_1(\mathcal{F}), T_2(\mathcal{F})$ and $T_3(\mathcal{F})$ can only achieve IV. For any $\mathbf{v}, \mathbf{u} \in L$ such that $\mathbf{v}_{(1)} = \mathbf{v}_{(2)} = 1$ and $\mathbf{u} = \bar{\mathbf{v}}$, $(A, B) \in H$ for $\mathbf{v}$ and $(\mathbf{u} \oplus \mathbf{v})_{(1)} + = (\mathbf{u} \oplus \mathbf{v})_{(2)} = 4$. According to Theorem 1, the coefficient of $z^{\mathbf{u}\oplus\mathbf{v}}$ in the indicator function $f_{T_3(\mathcal{F})}(z)$ of $T_3(\mathcal{F})$ is $b_{\mathbf{u}} \oplus \mathbf{v} = \frac{1}{3^s} b_{\mathbf{0}}$, namely, $T_3(\mathcal{F})$ must contains word(s) with a length of 4. Hence, the resolution of $T_3(\mathcal{F})$ can only achieve IV. The same is true for $T_1(\mathcal{F})$ and $T_2(\mathcal{F})$.*

From Theorems 4 and 5, the following result is obvious.

Corollary 1. *If $\mathcal{F}$ is an orthogonal array of strength 2, then both $T(\mathcal{F})$ and $T_i(\mathcal{F})$ are orthogonal arrays of strength 2 for $i = 1, 2, 3$.*

5. Concluding Remarks

The additional properties of triple designs are thoroughly studied using the indicator function in this paper. The indicator function of a triple design is expressed by the indicator function of its initial design, and it is shown that the inner structure of a triple design can be effectively explored with its indicator function. The close relationships between a triple design and its initial design are built from the viewpoint of resolution and orthogonality. All the theoretical results in this paper provide a closer look at triple designs and provide a solid foundation for a design constructed by the tripling method, where the constructed designs have better properties, such as high resolution and orthogonality, and are recommended for use in practice. Using the tripling method, many good designs with a large size can be constructed from an existing small design. The construction method is effective and efficient because it does not depend on any research algorithm.

This paper focuses on the indicator function expression of a triple design and its properties. Several directions are worthy of future research as follows:

(1) The results show that the designs constructed by tripling are competitive. Therefore, a natural question is: can we extend the tripling technology to a high-level or mixed-level design? The idea of constructing triple designs can be generalized to higher levels, which will be considered in future research. Another research interest is the construction of multiple mixed-level designs such as mixed two- and three-level, mixed two- and four-level cases.

(2) Both the run sizes and columns of designs constructed in this paper are triple those of its original design and thus have some limitations. It is interesting to enhance the flexibility in run sizes of the constructed designs so that the resulting designs have good performance as well.

(3) A strong orthogonal array [38–40] is widely used in different topics of statistics. Therefore, it is of great interest to examine the performance of tripling of strong orthogonal arrays of strength 3 or strength 2+ and analyze the space-filling properties of the resulting design. This matter deserves further comprehensive and systematic exploration.

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Appendix A. Proofs

Proof of Lemma 2. Following the line of Pistone and Rogantin [34], the proof of Lemma 2 can be greatly simplified. Take item (b) of Lemma 2 as example, if $x \in \mathcal{F}, x^{\mathbf{v}} = (x_1^{v_1}, \ldots, x_s^{v_s})$, then

$$
\begin{aligned}
x_{(1)}^{\mathbf{v}} &= (x_1^{2v_1}, \ldots, x_s^{2v_s}) = x^{2\mathbf{v}} \\
x_{(2)}^{\mathbf{v}} &= ((\omega_2 x_1)^{2v_1}, \ldots, (\omega_2 x_s)^{2v_s}) = \omega_1^{(v_1+\cdots+v_s)} x^{2\mathbf{v}} = \omega_1^{(v_{(1)}+2v_{(2)})} x^{2\mathbf{v}} \\
x_{(3)}^{\mathbf{v}} &= ((\omega_1 x_1)^{2v_1}, \ldots, (\omega_1 x_s)^{2v_s}) = \omega_2^{(v_1+\cdots+v_s)} x^{2\mathbf{v}} = \omega_2^{(v_{(1)}+2v_{(2)})} x^{2\mathbf{v}} \\
x_{(4)}^{\mathbf{v}} &= ((\omega_2 x_1)^{v_1}, \ldots, (\omega_2 x_s)^{v_s}) = \omega_2^{(v_1+\cdots+v_s)} x^{2\mathbf{v}} = \omega_2^{(v_{(1)}+2v_{(2)})} x^{\mathbf{v}} \\
x_{(5)}^{\mathbf{v}} &= ((\omega_1 x_1)^{v_1}, \ldots, (\omega_1 x_s)^{v_s}) = \omega_1^{(v_1+\cdots+v_s)} x^{2\mathbf{v}} = \omega_1^{(v_{(1)}+2v_{(2)})} x^{\mathbf{v}}.
\end{aligned}
$$

Thus, $\omega_2^{(v_{(1)}+2v_{(2)})} = 1$ iff $v_{(1)} = v_{(2)}$ and

$$
x^{\mathbf{v}} + x_{(4)}^{\mathbf{v}} + x_{(5)}^{\mathbf{v}} = \left(1 + \omega_2^{(v_{(1)}+2v_{(2)})} + \omega_1^{(v_{(1)}+2v_{(2)})}\right) x^{\mathbf{v}}.
$$

The coefficient is $(1 + \omega_2 + \omega_1)$ if $v_{(1)} + 2v_{(2)} = 1$ and $(1 + \omega_2 + \omega_1)$ if $v_{(1)} + 2v_{(2)} = 2$. Thus $x^{\mathbf{v}} + x_{(4)}^{\mathbf{v}} + x_{(5)}^{\mathbf{v}} = 0$ for $(A, B) \notin H$, and item (b) of Lemma 2 is true. $\square$

Proof of Lemma 3. The proof is omitted since it is similar to the proof of Lemma 2. □

Proof of Lemma 4. For any $\alpha = (\alpha_1, \ldots, \alpha_s) \in L$, let $\alpha_{(1)} + \alpha_{(2)} = k$ and $\alpha_{(2)} = a + b + c$. Then x^α contains k factors of $\mathcal{F}$, where $\alpha_{(1)}$ factors with power 1 and $\alpha_{(2)}$ factors with power 2. Without loss of generality, let the k factors be $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$, where the $\alpha_{(2)}$ factors $x_{j_1}, x_{j_2}, \ldots, x_{j_{(a+b+c)}}$ with power 2. Moreover, let the elements in the j-th row and the columns group $\{i_1, i_2, \ldots, i_k\}$ of $\mathcal{F}$ consist of k_1 elements with entry w_1, k_2 elements with entry w_2 and the other $k - k_1 - k_2$ elements with entry w_0, let the elements in the j-th row and the columns group $\{j_1, j_2, \ldots, j_{(a+b+c)}\}$ of $\mathcal{F}$ consist of a elements with entry w_0, b elements with entry w_1 and the other c elements with entry w_2. Therefore, for $\mathbf{x}_j \in \mathcal{F}, j = 1, \ldots, n$, $\mathbf{x}_j^\alpha = w_1^{k_1-b} w_2^{k_2-c} w_1^{2b} w_2^{2c}$, and $\mathbf{x}_j^{\bar{\alpha}} = w_1^{2k_1-2b} w_2^{2k_2-2c} w_1^{b} w_2^{c} = w_1^{2k_1-b} w_2^{2k_2-c} = w_1^{2k_1+2b} w_2^{2k_2+2c} = w_1^{2k_1+k_2+2b+c}$. On the other hand, $\mathcal{F}_{(1)}$ comes from $\mathcal{F}$ based on the level permutation $\{w_0, w_1, w_2\} \to \{w_0, w_2, w_1\}$, thus $(\mathbf{x}_j)^\alpha_{(1)} = w_1^{k_2-c} w_2^{k_1-b} w_1^{2c} w_2^{2b} = w_1^{k_2+c} w_2^{k_1+b}$ $= w_1^{2k_1+k_2+2b+c}$. Hence proved. □

Proof of Theorem 1. According to the definition of $c^{(i)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}}$ in (7) for $i = 1, 2, 3$, we have

$$
\begin{aligned}
c^{(1)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{0}} &= \frac{1}{3^{3s}} \sum_{x \in \mathcal{F}} x^{\mathbf{u}} x^{\mathbf{v}} = \frac{1}{3^{3s}} \sum_{x \in \mathcal{F}} x^{(\mathbf{u}+\mathbf{v})_{[3]}} = \frac{1}{3^{2s}} b_{(\mathbf{u}+\mathbf{v})_{[3]}}, \\
c^{(2)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{0}} &= \frac{1}{3^{3s}} \sum_{x \in \mathcal{F}} x^{\mathbf{u}} x^{\mathbf{v}}_{(4)}, c^{(3)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{0}} = \frac{1}{3^{3s}} \sum_{x \in \mathcal{F}} x^{\mathbf{u}} x^{\mathbf{v}}_{(5)}.
\end{aligned}
$$

From Lemma 2, if $(A, B) \in H$ for $\mathbf{v} \in L$, $c^{(1)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{0}} = c^{(2)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{0}} = c^{(3)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{0}} = \frac{1}{3^{2s}} b_{(\mathbf{u}+\mathbf{v})_{[3]}}$ since $x^{\mathbf{v}} = x^{\mathbf{v}}_{(4)} = x^{\mathbf{v}}_{(5)}$, if $(A, B) \notin H$ for $\mathbf{v} \in L$, $c^{(1)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{0}} + c^{(2)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{0}} + c^{(3)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{0}} = 0$ since $x^{\mathbf{v}} + x^{\mathbf{v}}_{(4)} + x^{\mathbf{v}}_{(5)}=0$.

Similarly, from Lemmas 2 and 4, if $(M, N) \in H$ for $\mathbf{y} \in L$, $c^{(1)}_{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{y}} = c^{(2)}_{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{y}} = c^{(3)}_{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{y}} = \frac{1}{3^{2s}} b_{(\mathbf{u}+\bar{\mathbf{y}})_{[3]}}$ since $x^{\mathbf{y}}_{(1)} = x^{\mathbf{y}}_{(2)} = x^{\mathbf{y}}_{(3)}$ and $x^{\mathbf{y}}_{(1)} = x^{\bar{\mathbf{y}}}$, if $(M, N) \notin H$ for $\mathbf{y} \in L$, $c^{(1)}_{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{y}} + c^{(2)}_{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{y}} + c^{(3)}_{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{y}} = 0$ since $x^{\mathbf{y}}_{(1)} + x^{\mathbf{y}}_{(2)} + x^{\mathbf{y}}_{(3)} = 0$. Hence proved. □

Proof of Theorem 2. According to the definition of $c^{(i)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}}$ in (7) for $i = 1, 2, 3$, we have

$$
\begin{aligned}
c^{(1)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} &= \frac{1}{3^{3s}} \sum_{x \in \mathcal{F}} x^{\mathbf{u}} x^{\mathbf{v}} x^{\mathbf{y}}_{(1)} = \frac{1}{3^{3s}} \sum_{x \in \mathcal{F}} x^{\mathbf{u}} x^{\mathbf{v}} x^{\bar{\mathbf{y}}} = \frac{1}{3^{3s}} \sum_{x \in \mathcal{F}} x^{(\mathbf{u}+\mathbf{v}+\bar{\mathbf{y}})_{[3]}} = \frac{1}{3^{2s}} b_{(\mathbf{u}+\mathbf{v}+\bar{\mathbf{y}})_{[3]}}, \\
c^{(2)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} &= \frac{1}{3^{3s}} \sum_{x \in \mathcal{F}} x^{\mathbf{u}} x^{\mathbf{v}}_{(4)} x^{\mathbf{y}}_{(2)}, c^{(3)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} = \frac{1}{3^{3s}} \sum_{x \in \mathcal{F}} x^{\mathbf{u}} x^{\mathbf{v}}_{(5)} x^{\mathbf{y}}_{(3)}.
\end{aligned}
$$

From Lemma 3, it is obvious that the conclusions are true. Hence proved. □

Proof of Theorem 3. From (6), for $i = 1, 2, 3$, we have

$$
\begin{aligned}
f_{\mathcal{B}_i}(z) &= \sum_{\mathbf{u}\in L} \sum_{\mathbf{v}\in L} \sum_{\mathbf{y}\in L} c^{(i)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} z^{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} \\
&= \sum_{\mathbf{u}\in L} \sum_{\mathbf{v}\in L,(A,B)\in H} \sum_{\mathbf{y}\in L,(M,N)\in H} c^{(i)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} z^{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} \\
&\quad + \sum_{\mathbf{u}\in L} \sum_{\mathbf{v}\in L,(A,B)\in J} \sum_{\mathbf{y}\in L,(M,N)\in K} c^{(i)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} z^{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} \\
&\quad + \sum_{\mathbf{u}\in L} \sum_{\mathbf{v}\in L,(A,B)\in K} \sum_{\mathbf{y}\in L,(M,N)\in J} c^{(i)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} z^{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}}
\end{aligned}
$$

$$+\sum_{\mathbf{u}\in L}\sum_{\mathbf{v}\in L,(A,B)\in J}\sum_{\mathbf{y}\in L,(M,N)\in J} c^{(i)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} z^{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}}$$
$$+\sum_{\mathbf{u}\in L}\sum_{\mathbf{v}\in L,(A,B)\in K}\sum_{\mathbf{y}\in L,(M,N)\in K} c^{(i)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} z^{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}}$$
$$+\sum_{\mathbf{u}\in L}\sum_{\mathbf{v}\in L,(A,B)\in H}\sum_{\mathbf{y}\in L,(M,N)\in J} c^{(i)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} z^{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}}$$
$$+\sum_{\mathbf{u}\in L}\sum_{\mathbf{v}\in L,(A,B)\in H}\sum_{\mathbf{y}\in L,(M,N)\in K} c^{(i)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} z^{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}}$$
$$+\sum_{\mathbf{u}\in L}\sum_{\mathbf{v}\in L,(A,B)\in J}\sum_{\mathbf{y}\in L,(M,N)\in H} c^{(i)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} z^{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}}$$
$$+\sum_{\mathbf{u}\in L}\sum_{\mathbf{v}\in L,(A,B)\in K}\sum_{\mathbf{y}\in L,(M,N)\in H} c^{(i)}_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}} z^{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}}.$$

By Lemma 1, $f_{T(\mathcal{F})}(z) = f_{\mathcal{B}_1}(z) + f_{\mathcal{B}_2}(z) + f_{\mathcal{B}_3}(z)$, therefore, it is obvious that the conclusion is true according to Theorem 2. Hence proved. □

Proof of Theorem 4. If $\mathcal{F}$ is a design of resolution III, then the length of the shortest word in $f_{\mathcal{F}}(x)$ is 3. It means that there exists at least a $\alpha \in L$ such that $b_\alpha \neq 0$ and $\alpha_{(1)} + \alpha_{(2)} = 3$, for any $\alpha \in L$, if $\alpha_{(1)} + \alpha_{(2)} < 3$, $b_\alpha = 0$. Therefore, for $\mathbf{u}, \mathbf{v}, \mathbf{y} \in L$, if $[(\mathbf{u}+\mathbf{v}+\bar{\mathbf{y}})_{[3]}]_{(1)} + [(\mathbf{u}+\mathbf{v}+\bar{\mathbf{y}})_{[3]}]_{(2)} < 3$, $b_{(\mathbf{u}+\mathbf{v}+\bar{\mathbf{y}})_{[3]}} = 0$. In other words, if $b_{(\mathbf{u}+\mathbf{v}+\bar{\mathbf{y}})_{[3]}} \neq 0$, $[(\mathbf{u}+\mathbf{v}+\bar{\mathbf{y}})_{[3]}]_{(1)} + [(\mathbf{u}+\mathbf{v}+\bar{\mathbf{y}})_{[3]}]_{(2)} \geq 3$. It is to be noted that $(\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y})_{(1)} + (\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y})_{(2)} \geq [(\mathbf{u}+\mathbf{v}+\bar{\mathbf{y}})_{[3]}]_{(1)} + [(\mathbf{u}+\mathbf{v}+\bar{\mathbf{y}})_{[3]}]_{(2)}$. Therefore, $(\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y})_{(1)} + (\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y})_{(2)} \geq 3$. This means that the length of the word $z_{\mathbf{u}\oplus\mathbf{v}\oplus\mathbf{y}}$ is at least 3. On the other hand, if $\mathbf{v} = \mathbf{y} = \mathbf{0}$, the words in $\sum_{\mathbf{u}\in L} b_{\mathbf{u}} z^{\mathbf{u}\oplus\mathbf{0}\oplus\mathbf{0}}$ are just the words of $f_{\mathcal{F}}(x)$, and the length of the shortest word in $f_{\mathcal{F}}(x)$ is 3. Thus the length of the words in $f_{T(\mathcal{F})}(x)$ are at least 3, and there exists at least a word with length 3. It means that the resolution of $T(\mathcal{F})$ is III.

Moreover, if $\mathcal{F}$ is regular $|b_\alpha/b_{\mathbf{0}}| = 1$ for any $\alpha \in L$ and $b_\alpha \neq 0$. The coefficients in $f_{T(\mathcal{F})}(x)$ are $3b_\alpha/3^{2s}$, where $\alpha = (\mathbf{u}+\mathbf{v}+\mathbf{y})_{[3]}$. Therefore, $T(\mathcal{F})$ also is regular since $|\frac{3b_\alpha}{3^{2s}}/\frac{3n}{3^{3s}}| = |b_\alpha/b_{\mathbf{0}}| = 1$ for $\alpha \in L$ and $b_\alpha \neq 0$. □

Proof of Theorem 5. From Theorem 3, one can easily obtain the indicator function of $T_1(\mathcal{F})$ as follows

$$f_{T_1(\mathcal{F})}(z) = \frac{1}{3^{s-1}}\left[\sum_{\mathbf{v}\in L,(A,B)\in H}\sum_{\mathbf{y}\in L,(M,N)\in H} b_{(\mathbf{v}+\bar{\mathbf{y}})_{[3]}} z^{\mathbf{v}\oplus\mathbf{y}} + \sum_{\mathbf{v}\in L,(A,B)\in J}\sum_{\mathbf{y}\in L,(M,N)\in K} b_{(\mathbf{v}+\bar{\mathbf{y}})_{[3]}} z^{\mathbf{v}\oplus\mathbf{y}} + \sum_{\mathbf{v}\in L,(A,B)\in K}\sum_{\mathbf{y}\in L,(M,N)\in J} b_{(\mathbf{v}+\bar{\mathbf{y}})_{[3]}} z^{\mathbf{v}\oplus\mathbf{y}}\right].$$

If the resolution of $\mathcal{F}$ is III (or IV), there exists some $\alpha_0 \in L$ such that $\alpha_{0(1)} + \alpha_{0(2)} = 3$ (or 4) and $b_{\alpha_0} \neq 0$, and $b_\alpha = 0$ for any $\alpha \in L$ and $\alpha_{(1)} + \alpha_{(2)} < 3$ (or 4). If $b_\alpha \neq 0$ for any $\alpha \in L$, there exists $\mathbf{v}, \mathbf{y}$ such that $\mathbf{v} + \bar{\mathbf{y}} = \alpha$ and $b_{\mathbf{v}+\bar{\mathbf{y}}} \neq 0$. Hence, there exists word(s) with length 3 (or 4), and there is no word with a length smaller than 3 (or 4). That is, the resolution of $T_1(\mathcal{F})$ is III (or IV). Moreover, if $\mathcal{F}$ is regular, $T_1(\mathcal{F})$ also is regular since it is a projection of $T(\mathcal{F})$. Similarly, the indicator functions of $T_2(\mathcal{F})$ and $T_3(\mathcal{F})$ can also be obtained from Theorem 3. According to Theorem 1, the conclusions are true for $T_2(\mathcal{F})$ and $T_3(\mathcal{F})$. Hence proved. □

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