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New Results on Finite-Time Synchronization Control of Chaotic Memristor-Based Inertial Neural Networks with Time-Varying Delays

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Abstract: In this work, we are concerned with the finite-time synchronization (FTS) control issue of the drive and response delayed memristor-based inertial neural networks (MINNs). Firstly, a novel finite-time stability lemma is developed, which is different from the existing finite-time stability criteria and extends the previous results. Secondly, by constructing an appropriate Lyapunov function, designing effective delay-dependent feedback controllers and combining the finite-time control theory with a new non-reduced order method (NROD), several novel theoretical criteria to ensure the FTS for the studied MINNs are provided. In addition, the obtained theoretical results are established in a more general framework than the previous works and widen the application scope. Lastly, we illustrate the practicality and validity of the theoretical results via some numerical examples.

Keywords: novel finite-time stability theorems; generalized MINNs; mixed time-varying delays; new non-reduced order method

MSC: 93D40

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1. Introduction

Neural networks (NNs) have garnered considerable attention from researchers, due to being widely applied in numerous different fields, such as cryptography, model identification, and signal processing [1–7]. The essential issues of these applications are to research the dynamical behaviors of NNs. As a significant dynamical property, synchronization has attracted wide-scale attention in recent years [8–12]. However, in the existing papers, most of these published works are connected with infinite-time synchronization, such as exponential synchronization and asymptotical synchronization [13–15]. Considering that the lifespan of biologies and apparatus is limited, we always desire to obtain faster or even finite-time convergent speed in practice. Hence, the investigation of FTS is more meaningful [16–19]. In the meantime, compared with the infinite-time synchronization of NNs, FTS has a better convergence rate and exhibits several other desirable features.

It is well known that conventional NNs are usually presented by first-order differential equations. In 1986, Westervelt and Babcock established the NNs with an inertial item by adding the inductors into the NNs' model, which could be described by a second-order form [20]. Inertial neural networks (INNs) have evident engineering and biological backgrounds, and the dynamical properties of INNs are more complicated than conventional NNs [21–24]. Recently, some results on the dynamical behaviors of INNs, which include periodicity, stability, dissipativity and so on, have been presented [25–29]. With the application of INNs to various fields such as secure communication, signal processing and image encryption [30–32], some further studies of the INNs are essential and significant. Meanwhile, due to the inherent communication times between the neurons and the limited switching speed of amplifiers, the time delays unavoidably exist in the process of the implementation of INNs (see Figure 1a). However, it has been proved in many previous works that the existence of time delays may cause divergence, oscillation, and even instability of the system [33–35]. Hence, the dynamic analysis of INNs with mixed time-varying delays (MTVDs) is necessary for their successful applications.

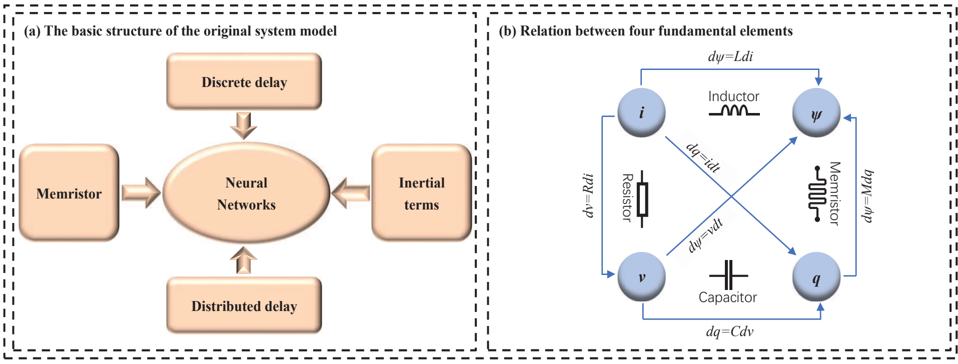

Figure 1. System constitution and four fundamental elements.

On the other hand, a memristor device, which is the fourth basic circuit element (see Figure 1b) successfully manufactured by HP Laboratory [36], is a perfect element to simulate the function of the neural synapse [37,38]. Adopting memristors as the synapses connection weights in the circuit of NNs has a lot of merits, including simple structure, nano-dimension, ease of integration and low power consumption [39,40]. Considering the advantages of memristors, memristor-based INNs have become a hot topic [41–44]. In [45], by using a hybrid feedback controller, novel results were acquired to insure the FTS of drive and response MINNs, and the authors deeply discuss the relationship between the estimated value of settling time and the parameter ξ_i in variable substitution. In [46], the authors investigated the FTS and fixed time synchronization of MINNs by using the Lyapunov stability theory and Filippov discontinuous theory. However, so far, all authors have studied the FTS of MINN by utilizing a variable substitution method to transform MINNs' system into a first-order form. There are hardly any articles studying the dynamical properties of MINNs by using NROD, and the existing results are just focused on infinite-time synchronization or stability. As mentioned in the papers [28,41], the variable transformation method for the discontinuous MINNs' system may be short of rigor in practice. Hence, researching MINNs themselves directly is a better method, instead of utilizing the reduced-order method.

Motivated by the aforementioned discussions, in this work, we will investigate the MINNs with MTVDs by using a new study method and acquire several new sufficient conditions to insure the FTS for such considered systems. The main contributions are highlighted below:

1. A novel finite-time stability criterion is derived (see Lemma 2), which is different from the existing finite-time stability criteria and extends the previous results.
2. By combining finite-time control theory with a new NROD, which can study MINNs themselves directly instead of using the variable substitution method, some new theoretical criteria to ensure the FTS for the studied MINNs are developed.

3. Taking the memristors, inertial items, and MTVDs into account, the acquired theoretical results are established in a more general framework than the previous results and widens the application scope.

The remainder of this paper is organized as follows. In Section 2, the INNs model, some useful definition, and lemmas are introduced. In Section 3, new criteria for IMNNs with MTVDs derived are given. In Section 4, illustrative examples are presented to validate the effectiveness of the theoretical results. Finally, Section 5 shows the conclusions.

Notations: Let $\mathbb{R}$ denote the space of real numbers. $C([t_0-\tau,t_0],\mathbb{R})$ denotes the set of all continuous functions from $[t_0-\tau,t_0]$ to $\mathbb{R}$. x^T denotes the transpose of a vector (or a matrix) x. $diag\{\cdots\}$ denotes a diagonal matrix. $(\cdot)_{n\times n}$ denotes a $n\times n$ matrix. $||\cdot||_1$ are the 1-norm of a vector (or a matrix).

2. Preliminaries

In this work, we consider the MINNs with MTVDs as follows:

$$\begin{aligned}\frac{d^2x_i(t)}{dt^2}=&-a_i\frac{dx_i(t)}{dt}-b_ix_i(t)+\sum_{j=1}^{n}c_{ij}(x_i(t))f_j(x_j(t))+\sum_{j=1}^{n}d_{ij}(x_i(t))\\&\times f_j(x_j(t-\tau(t)))+\sum_{j=1}^{n}e_{ij}(x_i(t))\int_{t-\ell(t)}^{t}f_j(x_j(s))ds+I_i(t)\end{aligned}\tag{1}$$

$i\in\{1,2,\cdots,n\}$, where $x_i(t)$ represents the state of the ith neuron, and the second derivative of $x_i(t)$ is called an inertial term of the system (1). $a_i>0$ and $b_i>0$ are constants. The function $f_j(\cdot)$ represents the nonlinear activation function. $I_i(t)$ is the external input on the ith neuron. $\tau(t)$ and $\ell(t)$ are the discrete and distributed delay, respectively, and there exist constants τ and ℓ such that $0\le\tau(t)\le\tau$, $0\le\ell(t)\le\ell$. $c_{ij}(x_i(t))$, $d_{ij}(x_i(t))$ and $e_{ij}(x_i(t))$ are memristor-based connection weights, which are given by

$$c_{ij}(x_i(t))=\begin{cases}c'_{ij},|x_i(t)|\le T_i,\\c''_{ij},|x_i(t)|>T_i,\end{cases}\quad d_{ij}(x_i(t))=\begin{cases}d'_{ij},|x_i(t)|\le T_i,\\d''_{ij},|x_i(t)|>T_i,\end{cases}\quad e_{ij}(x_i(t))=\begin{cases}e'_{ij},|x_i(t)|\le T_i\\e''_{ij},|x_i(t)|>T_i\end{cases}$$

for $i,j=1,2,\cdots,n$, where $T_i>0$ represents the switching jumps, and $c'_{ij},c''_{ij},d'_{ij},d''_{ij},e'_{ij},e''_{ij}$, are known constants. Denote

$$\begin{aligned}&\bar{c}_{ij}=\max\{|c'_{ij}|,|c''_{ij}|\},\bar{d}_{ij}=\max\{|d'_{ij}|,|d''_{ij}|\},\bar{e}_{ij}=\max\{|e'_{ij}|,e''_{ij}|\},\hat{c}_{ij}=\max\{c'_{ij},c''_{ij}\},\\&\check{c}_{ij}=\min\{c'_{ij},c''_{ij}\},\hat{d}_{ij}=\max\{d'_{ij},d''_{ij}\},\check{d}_{ij}=\min\{d'_{ij},d''_{ij}\},\hat{e}_{ij}=\max\{e'_{ij},e''_{ij}\},\check{e}_{ij}=\min\{e'_{ij},e''_{ij}\}\end{aligned}$$

and

$$\begin{aligned}&\bar{C}=\left(\bar{c}_{ij}\right)_{n\times n},\bar{D}=\left(\bar{d}_{ij}\right)_{n\times n},\bar{E}=\left(\bar{e}_{ij}\right)_{n\times n},\hat{C}=\left(\hat{c}_{ij}\right)_{n\times n},\check{C}=\left(\check{c}_{ij}\right)_{n\times n},\\&\hat{D}=\left(\hat{d}_{ij}\right)_{n\times n},\check{D}=\left(\check{d}_{ij}\right)_{n\times n},\hat{E}=\left(\hat{e}_{ij}\right)_{n\times n},\check{E}=\left(\check{e}_{ij}\right)_{n\times n}.\end{aligned}$$

The initial value of system (1) is given by

$$x_i(s)=\phi_i(s),\frac{dx_i(s)}{ds}=\phi'_i(s),t_0\ge0,s\in[t_0-\tau,t_0]$$

where $\phi_i(s),\phi'_i(s)\in C([t_0-\tau,t_0],\mathbb{R})$, $i=1,2,\cdots,n$.

For the activation functions $f_j(\cdot)$, we conduct the following assumptions.

Assumption 1. *For $\forall x,y\in\mathbb{R}$, $x\ne y$, there exist constants $l_j>0$ such that the activation function $f_j:\mathbb{R}\to\mathbb{R}$ satisfies $|f_j(x)-f_j(y)|\le l_j|x-y|,i=1,2,\cdots,n,$, i.e., the function f_j satisfies the Lipschitz condition,*

Assumption 2. *For $\forall x \in \mathbb{R}$, there exist constants $M_j > 0$ such that the activation function $f_j : \mathbb{R} \to \mathbb{R}$ satisfies $|f_j(x)| \leq M_j, i = 1, 2, \cdots, n$,, i.e., the function f_j is bounded.*

Then, the response MINNs is given as follows:

$$\begin{aligned}\frac{d^2 y_i(t)}{dt^2} =& - a_i \frac{dy_i(t)}{dt} - b_i y_i(t) + \sum_{j=1}^{n} c_{ij}(y_i(t)) f_j(y_j(t)) + \sum_{j=1}^{n} d_{ij}(y_i(t)) \\ & \times f_j(y_j(t-\tau(t))) + \sum_{j=1}^{n} e_{ij}(y_i(t)) \int_{t-\ell(t)}^{t} f_j(y_j(s)) ds + I_i(t) + U_i(t) \end{aligned} \tag{2}$$

in which $y_i(t)$ is the state of the response MINNs and $U_i(t)$ is the properly designed controller; the memristor-based connection weights $c_{ij}(y_i(t))$, $d_{ij}(y_i(t))$, $e_{ij}(y_i(t))$ are defined as in (1). The initial value of system (2) is as follows:

$$y_i(s) = \psi_i(s), \frac{dy_i(s)}{ds} = \psi_i'(s), t_0 \geq 0, s \in [t_0 - \tau, t_0]$$

where $\psi_i(s), \psi_i'(s) \in C([t_0 - \tau, t_0], \mathbb{R})$, $i = 1, 2, \cdots, n$.

Denote synchronization error $r_i(t) = y_i(t) - x_i(t)$. From (1) and (2), we obtain the error systems as follows:

$$\begin{aligned}\frac{d^2 r_i(t)}{dt^2} =& - a_i \frac{dr_i(t)}{dt} - b_i r_i(t) + \sum_{j=1}^{n} c_{ij}(y_i(t)) g_j(r_j(t)) + \sum_{j=1}^{n} d_{ij}(y_i(t)) g_j(r_j(t-\tau(t))) \\ & + \sum_{j=1}^{n} e_{ij}(y_i(t)) \int_{t-\ell(t)}^{t} g_j(r_j(s)) ds + \sum_{j=1}^{n} [c_{ij}(y_i(t)) - c_{ij}(x_i(t))] f_j(x_j(t)) \\ & + \sum_{j=1}^{n} [d_{ij}(y_i(t)) - d_{ij}(x_i(t))] f_j(x_j(t-\tau(t))) \\ & + \sum_{j=1}^{n} [e_{ij}(y_i(t)) - e_{ij}(x_i(t))] \int_{t-\ell(t)}^{t} f_j(x_j(s)) ds + U_i(t) \end{aligned} \tag{3}$$

for $i = 1, 2, \cdots, n$, where $g_j(r_j(\cdot)) = f_j(y_j(\cdot)) - f_j(x_j(\cdot))$.

The synchronization error system (3) can be transformed into the vector form as

$$\begin{aligned} r''(t) =& - A r'(t) - B r(t) + C_y g(r(t)) + D_y g(r(t-\tau(t))) + E_y \int_{t-\ell(t)}^{t} g(r(s)) ds \\ & + (C_y - C_x) f(x(t)) + (D_y - D_x) f(x(t-\tau(t))) \\ & + (E_y - E_x) \int_{t-\ell(t)}^{t} f(x(s)) ds + U(t) \end{aligned} \tag{4}$$

where

$$\begin{aligned} & r(t) = \big(r_1(t), r_2(t), \cdots, r_n(t)\big)^T, g(r(\cdot)) = \big(g_1(r_1(\cdot)), g_2(r_2(\cdot)), \cdots, g_n(r_n(\cdot))\big)^T, \\ & f(x(\cdot)) = \big(f_1(x_1(\cdot)), f_2(x_2(\cdot)), \cdots, f_n(x_n(\cdot))\big)^T, U(t) = \big(U_1(t), U_2(t), \cdots, U_n(t)\big)^T. \end{aligned}$$

and

$$\begin{aligned} & A = \operatorname{diag}\{a_1, a_2, \cdots, a_n\}, B = \operatorname{diag}\{b_1, b_2, \cdots, b_n\}, C_y = \big(c_{ij}(y_i(t))\big)_{n \times n}, C_x = \big(c_{ij}(x_i(t))\big)_{n \times n}, \\ & D_y = \big(d_{ij}(y_i(t))\big)_{n \times n}, D_x = \big(d_{ij}(x_i(t))\big)_{n \times n}, E_y = \big(e_{ij}(y_i(t))\big)_{n \times n}, E_x = \big(e_{ij}(x_i(t))\big)_{n \times n}, \end{aligned}$$

The following definition and lemmas are given, which are helpful in proving the main results.

Definition 1. *[46] Drive and response MINNs (1) and (2) are said to be FTS, if for appropriate designed controller $U_i(t)$, there exists a constant $0 < T < +\infty$, such that $\lim_{t \to t_0+T} r(t) = 0$, and $r(t) \equiv 0$ for $\forall t \geq t_0 + T$, and T is called the settling time.*

Since FTS requires that the system trajectories converge to a Lyapunov equilibrium state in the finite time, the FTS is a stronger condition than the exponential or asymptotic synchronization. The following Lemma 1, which has been widely used in many previous works, gives a sufficient condition to ensure FTS.

Lemma 1. *[47] Suppose there exist a continuous positive-definite function $V : D \to \mathbb{R}^+$, and an open neighborhood $\chi \subset D$ of the origin, such that*

$$V'(x(t)) \leq -\alpha V^{\eta}(x(t)), \forall x(t) \in \chi \setminus \{0\}.$$

In which $\alpha > 0$, $\eta \in (0,1)$ are constant. Then, the origin of the system is FTS, and the settling time T^1 satisfies

$$T^1 \leq t_0 + \frac{V^{1-\eta}(x(t_0))}{\alpha(1-\eta)}.$$

The term $V(x(t))$ will inevitably appear in the derivative of the Lyapunov function, which is ignored to estimate the settling time in the Lemma 1. Thus, there is room for further research to reduce the conservatism of the FTS conditions. In the following, we will derive a new lemma that takes into account the useful term $V(x(t))$.

Lemma 2. *Assume there exist a positive definite continuous function $V : D \to \mathbb{R}^+$, a continuous differentiable function $\varphi : \mathbb{R} \to \mathbb{R}^+$, constants $k > 0$, and an open neighborhood $\chi \subset D$ of the origin, such that for $\forall t \geq t_0$,*

$$V'(x(t)) \leq -kV(x(t)) - \varphi(t), \forall x(t) \in \chi \setminus \{0\} \tag{5}$$

$$\varphi(t) > 0, \varphi'(t) \leq 0 \tag{6}$$

Then, if $t \geq t_0 + \frac{1}{k} \ln \frac{kV(x(t_0)) + \varphi(t_0)}{\varphi(t)}$ holds, the origin of system is FTS.

Proof. Please see Appendix A for the detailed proof of Lemma 2. □

Remark 1. *The novel FTS lemma is different from the existing finite-time stability criteria and extends the previous results. Compared with the Lemma 1, the novel Lemma 2 fully considers the information in the derivative of the Lyapunov function. Note that the settling time is not explicit; it needs to be evaluated for the specific $\varphi(t)$ function. Next, we will provide explicit estimations of the settling time for some specific functions satisfying the condition (6).*

Lemma 3. *Under the conditions in Lemma 2, and $\varphi(t) = \alpha V^{\eta}(x(t))$, where $\alpha > 0, \eta \in (0,1)$ are constant. Then, the origin of the system is FTS, and the settling-time T^2 satisfies*

$$T^2 \leq t_0 + \frac{\ln\left(\frac{k}{\alpha} V^{1-\eta}(x(t_0)) + 1\right)}{k(1-\eta)}.$$

Remark 2. *It is not hard to see that $\varphi(t) = \alpha V^{\eta}(x(t)) > 0$ and $\varphi'(t) = \eta\alpha V^{\eta-1}(x(t))V'(x(t)) < 0$; thus, the condition (6) is satisfied. By the condition (5), one has $V(t) \leq V(t_0)e^{-k(t-t_0)}$. Thus, we have $\ln(\varphi(t)) = \ln(\alpha V^{\eta}(x(t))) \leq \ln(\alpha) + \eta[\ln(V(t_0)) - k(t-t_0)]$. Hence, when $t \geq t_0 + \frac{\ln\left(1+\frac{k}{\alpha}V^{1-\eta}(x(t_0))\right)}{k(1-\eta)}$,, one has $V(x(t)) \equiv 0$. Namely, the system can achieve FTS and the settling time is bound by $T^2 \leq t_0 + \frac{\ln\left(1+\frac{k}{\alpha}V^{1-\eta}(x(t_0))\right)}{k(1-\eta)}$. Obviously, $\frac{\ln\left(1+\frac{k}{\alpha}V^{1-\eta}(x(t_0))\right)}{k(1-\eta)} \Big/ \frac{V^{1-\eta}(x(t_0))}{\alpha(1-\eta)} < 1$,* *namely, $T^2 < T^1$. Therefore, Lemma 3 can provide a tighter settling time estimation than Lemma 1. In Lemma 2 of [48], a similar result has been given.*

Lemma 4. *Under the conditions in Lemma 2, and $\varphi(t) = \theta$, where $\theta > 0$ are constant. Then, the origin of the system is FTS, and the settling-time T^3 satisfies*

$$T^3 \leq t_0 + \frac{\ln\left(\frac{k}{\theta}V(x(t_0)) + 1\right)}{k}.$$

Remark 3. *Obviously, $\varphi(t) = \theta > 0$ and $\varphi'(t) = 0$. Based on Lemma 2, one has $T^3 \leq t_0 + \frac{\ln\left(\frac{k}{\theta}V(x(t_0))+1\right)}{k}$. Compared with the Lemmas 1 and 3, the derivative of the Lyapunov function in Lemma 4 does not need to have exponential terms, and the settling time is more concise and easy to test. In addition, the FTS theorem in [49] requires that the Lyapunov function satisfies $V'(x(t)) \leq -\theta$, and the settling time is $T = t_0 + \frac{V(t_0)}{\theta}$. Obviously, if setting $k = 0$, then Lemma 4 is reduced to the FTS lemma in [49]; thus, Lemma 4 is a strengthened result. Moreover, $\frac{1}{k}\ln\frac{kV(t_0)+\theta}{\theta} < \frac{V(t_0)}{\theta}$,, i.e., Lemma 4 can provide a tighter estimate value of settling time than FTS lemma in [49].*

Lemma 5. *Under the conditions in Lemma 2, and $\varphi(t) = \varrho e^{-\xi(t-t_0)}$, where $\varrho > 0, 0 < \xi < k$ are constant. Then, the origin of the system is FTS, and the settling time T^4 is bounded by*

$$T^4 \leq t_0 + \frac{\ln\left(\frac{k}{\varrho}V(x(t_0)) + 1\right)}{k - \xi}.$$

Remark 4. *It is easy to know that $\varphi(t) = \varrho e^{-\xi(t-t_0)} > 0$ and $\varphi'(t) = -\xi\varrho e^{-\xi(t-t_0)} < 0$; thus, the condition (6) is satisfied. We have $\ln(\varphi(t)) = \ln(\varrho) - \xi(t - t_0)$. Therefore, when $t \geq t_0 + \frac{\ln\left(\frac{k}{\varrho}V(x(t_0))+1\right)}{k-\xi}$,, one has $V(x(t)) \equiv 0$. Namely, the system can realize FTS. Moreover, the settling time is bounded by $T^4 \leq t_0 + \frac{\ln\left(\frac{k}{\varrho}V(x(t_0))+1\right)}{k-\xi}$.*

Lemma 6. *(see [46]) Setting $x_1, x_2, \cdots, x_n \geq 0, 0 < p \leq 1$,; then, the following inequality holds:*

$$\sum_{i=1}^{n} x_i^p \geq \left(\sum_{i=1}^{n} x_i\right)^p.$$

3. Main Results

On the basis of the Lyapunov functions' approach and finite-time control theory, we will directly study the FTS from MINNs themselves instead of utilizing the reduced-order method.

We define the delay-dependent feedback controller $U(t)$, as follows:

$$\begin{aligned} U(t) = &- sgn(r'(t))\Big[\Lambda\bar{r}'(t) + \Theta\bar{r}(t) + \bar{D}L\bar{r}(t-\tau(t)) + \beta + \bar{E}L\int_{t-\ell(t)}^{t}\bar{r}(s)ds \\ &+ \xi(\bar{r}(t))^{\mu} + \xi(\bar{r}'(t))^{\mu}\Big] \end{aligned} \tag{7}$$

in which $sgn(r'(t)) = \mathrm{diag}\big(\mathrm{sign}(r_1'(t)), \mathrm{sign}(r_2'(t)), \cdots, \mathrm{sign}(r_n'(t))\big), \bar{r}'(t) = \big(|r_1'(t)|, |r_2'(t)|, \cdots, |r_n'(t)|\big)^T, \bar{r}(\cdot) = \big(|r_1(\cdot)|, |r_2(\cdot)|, \cdots, |r_n(\cdot)|\big)^T, L = \mathrm{diag}\{l_1, l_2, \cdots, l_n\}.\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}, \Theta = \mathrm{diag}\{\theta_1, \theta_2, \cdots, \theta_n\}, \beta = (\beta_1, \beta_2, \cdots, \beta_n)^T, (\bar{r}(t))^{\mu} = \big(|r_1(t)|^{\mu}, |r_2(t)|^{\mu}, \cdots, |r_n(t)|^{\mu}\big)^T, (\bar{r}'(t))^{\mu} = \big(|r_1'(t)|^{\mu}, |r_2'(t)|^{\mu}, \cdots, |r_n'(t)|^{\mu}\big)^T$, and $\lambda_i, \theta_i, \beta_i > 0 (i = 1, 2, \cdots, n)$, $\xi > 0, 0 < \mu < 1$ are constants,

Theorem 1. *Suppose Assumptions 1 and 2 hold. If $\lambda_i, \theta_i (i = 1, 2, \cdots, n)$ and β satisfy*

$$\lambda_i > 1 - a_i, \quad \theta_i > b_i + l_i \sum_{j=1}^{n} \bar{c}_{ji},$$
$$||\beta||_1 > ||(\hat{C} - \check{C})M||_1 + ||(\hat{D} - \check{D})M||_1 + \ell||(\hat{E} - \check{E})M||_1,$$

then the drive-response system (1) and (2) can be synchronized in a finite time under the controller (7). Moreover, the settling time

$$T \leq t_0 + \frac{\ln\left(1 + \frac{a}{\xi}\left(2V(t_0)\right)^{\frac{1-\mu}{2}}\right)}{a(1-\mu)}. \tag{8}$$

where $a = \min\{\lambda, \theta\}$, and $\lambda = \min\limits_{1 \leq i \leq n} \{\lambda_i + a_i - 1\}$, $\theta = \min\limits_{1 \leq i \leq n} \left\{\theta_i - b_i - l_i \sum_{j=1}^{n} \bar{c}_{ji}\right\}$.

Proof. Please see Appendix B for a detailed proof of Theorem 1. □

Theorem 2. *Suppose Assumptions 1 and 2 hold. If $\lambda_i, \theta_i (i = 1, 2, \cdots, n)$ and β satisfy*

$$\lambda_i > 1 - a_i, \quad \theta_i > b_i + l_i \sum_{j=1}^{n} \bar{c}_{ji},$$
$$||\beta||_1 > ||(\hat{C} - \check{C})M||_1 + ||(\hat{D} - \check{D})M||_1 + \ell||(\hat{E} - \check{E})M||_1,$$

then the drive-response MINNs (1) and (2) are synchronized in a finite time under the controller (7). Moreover, the settling time

$$T \leq t_0 + \frac{\left(V(t_0)\right)^{1-\mu}}{\xi(1-\mu)}. \tag{9}$$

Proof. Please see Appendix C for a detailed proof of Theorem 2. □

Remark 5. *Obviously,*

$$\frac{\ln\left(1 + \frac{a}{\xi}\left(V(t_0)\right)^{1-\mu}\right)}{a(1-\mu)} \Bigg/ \frac{\left(V(t_0)\right)^{1-\mu}}{\xi(1-\mu)} = \frac{\ln\left(1 + \frac{a}{\xi}\left(V(t_0)\right)^{1-\mu}\right)}{\frac{a}{\xi}\left(V(t_0)\right)^{1-\mu}} < 1$$

Therefore, $\frac{\ln\left(1+\frac{a}{\xi}\left(V(t_0)\right)^{1-\mu}\right)}{a(1-\mu)} < \frac{\left(V(t_0)\right)^{1-\mu}}{\xi(1-\mu)}$, i.e., Theorem 1 can give a tighter estimate value of settling time than Theorem 2. Theorem 1 is a strengthened result. However, Theorem 2 has fewer parameters than Theorem 1 and is easier to test. Therefore, we can adopt the Theorem 1 or the Theorem 2 in actual applications.

Next, we propose the following delay-dependent feedback control scheme:

$$U(t) = -\mathrm{sgn}(r'(t))\left[\Lambda\bar{r}'(t) + \Theta\bar{r}(t) + \bar{D}L\bar{r}(t - \tau(t)) + \bar{E}L\int_{t-\ell(t)}^{t} \bar{r}(s)ds + \beta\right]. \tag{10}$$

Theorem 3. *Suppose Assumptions 1 and 2 hold. If $\lambda_i, \theta_i (i = 1, 2, \cdots, n)$ and β satisfy*

$$\lambda_i > 1 - a_i, \quad \theta_i > b_i + l_i \sum_{j=1}^{n} \bar{c}_{ji},$$
$$||\beta||_1 > ||(\hat{C} - \check{C})M||_1 + ||(\hat{D} - \check{D})M||_1 + \ell||(\hat{E} - \check{E})M||_1,$$

then the drive-response MINNs (1) and (2) can achieve FTS with the controller (10). Moreover, the settling time

$$T \leq t_0 + \frac{1}{a} \ln \frac{aV(t_0) + \beta^*}{\beta^*}. \tag{11}$$

Proof. Please see Appendix D for a detailed proof of Theorem 3. □

Remark 6. *Compared with Theorem 1 (or Theorem 2), we can observe that the controller of Theorem 3 does not need to have an exponential term, and the drive-response MINNs (1) and (2) can also achieve FTS, which can effectively simplify the controller. In addition, the controller (10) is easier to implement and operate in the actual application. In addition, the settling time in Theorem 3 is more concise and easier to test than Theorem 2 (or Theorem 3).*

Moreover, we propose the following delay-dependent feedback control scheme:

$$U(t) = -\text{sgn}(r'(t))\left[\Lambda\bar{r}'(t) + \Theta\bar{r}(t) + \bar{D}L\bar{r}(t-\tau(t)) + \bar{E}L\int_{t-\ell(t)}^{t} \bar{r}(s)ds + \beta + \Delta e^{-\mu(t-t_0)}\right]. \tag{12}$$

where $\Delta = (\delta_1, \delta_2, \cdots, \delta_n)^T$ and $\delta_i > 0(i = 1, 2, \cdots, n)$, $\mu > 0$ are constants.

Theorem 4. *Suppose Assumptions 1 and 2 hold. If* $\lambda_i, \theta_i(i = 1, 2, \cdots, n)$ *and* β *satisfy*

$$\lambda_i > 1 - a_i, \quad \theta_i > b_i + l_i \sum_{j=1}^{n} \bar{c}_{ji}, \quad \mu < a,$$
$$||\beta||_1 > ||(\hat{C} - \check{C})M||_1 + ||(\hat{D} - \check{D})M||_1 + \ell||(\hat{E} - \check{E})M||_1,$$

then, the drive-response MINNs (1) and (2) can achieve FTS with the controller (12). Moreover, the settling time

$$T \leq t_0 + \frac{1}{a-\mu} \ln \frac{aV(t_0) + \omega}{\omega}. \tag{13}$$

Proof. Please see Appendix E for a detailed proof of Theorem 4. □

Remark 7. *Up to now, in several earlier papers [44–46], which discussed the FTS of MINNs, the theoretical criteria have been acquired basically by utilizing the reduced-order method. As is known to all, the variable transformation method for the discontinuous MINNs system may be short of rigor in practice. In this work, we directly study the MINNs themselves without using the reduced-order method, and several novel sufficient conditions to insure the FTS for the considered MINNs are developed.*

Remark 8. *Unlike the INNs without memristor [15,17,18], the MNNs without inertial term [8,13] and the INNs with discrete or constant delays [15,17,18,44–46], this work takes into account the memristors, inertial items and MTVDs. Our considered MINNs is more general. Hence, the results in this work are acquired in a more general framework than the previous results, which widens the application scope.*

4. Illustrative Example

Example 1. *Consider the two-neuron drive MINNs with MTVDs, as follows:*

$$\begin{aligned}\frac{d^2x_i(t)}{dt^2} =& -a_i\frac{dx_i(t)}{dt} - b_ix_i(t) + \sum_{j=1}^{2} c_{ij}(x_i(t))f_j(x_j(t)) + \sum_{j=1}^{2} d_{ij}(x_i(t)) \\ &\times f_j(x_j(t-\tau(t))) + \sum_{j=1}^{2} e_{ij}(x_i(t))\int_{t-\ell(t)}^{t} f_j(x_j(s))ds + I_i(t) \end{aligned} \tag{14}$$

$i = 1, 2$, where $a_1 = 2.52$, $a_2 = 3.14$, $b_1 = 3.39$, $b_2 = 3.78$, and the activation functions $f_1(\cdot) = f_2(\cdot) = tanh(\cdot)$, $\tau(t) = \ell(t) = \frac{e^t}{1+e^t}$, $I_1(t) = 5sin(t)$, $I_2(t) = 5cos(t)$. We can clearly observe that the functions f_1 and f_2 satisfy Assumption 1 and 2 with $l_1 = l_2 = 1$, $M_1 = M_2 = 1$, and $0 \le \tau(t), \ell(t) \le 1$. The memristor-based connection weights take the following forms:

$$c_{11}(x_1(t)) = \begin{cases} 5.21, |x_1(t)| \le T_1, \\ -3.48, |x_1(t)| > T_1, \end{cases} \quad c_{12}(x_1(t)) = \begin{cases} 4.25, |x_1(t)| \le T_1, \\ -5.37, |x_1(t)| > T_1, \end{cases}$$
$$c_{21}(x_2(t)) = \begin{cases} 3.37, |x_2(t)| \le T_2, \\ 2.62, |x_2(t)| > T_2, \end{cases} \quad c_{22}(x_2(t)) = \begin{cases} -4.63, |x_2(t)| \le T_2, \\ 3.77, |x_2(t)| > T_2, \end{cases}$$
$$d_{11}(x_1(t)) = \begin{cases} 2.59, |x_1(t)| \le T_1, \\ -5.47, |x_1(t)| > T_1, \end{cases} \quad d_{12}(x_1(t)) = \begin{cases} -3.56, |x_1(t)| \le T_1, \\ -4.28, |x_1(t)| > T_1, \end{cases}$$
$$d_{21}(x_2(t)) = \begin{cases} 2.07, |x_2(t)| \le T_2, \\ 3.69, |x_2(t)| > T_2, \end{cases} \quad d_{22}(x_2(t)) = \begin{cases} -4.37, |x_2(t)| \le T_2, \\ 4.92, |x_2(t)| > T_2, \end{cases}$$
$$e_{11}(x_1(t)) = \begin{cases} -4.36, |x_1(t)| \le T_1, \\ 2.55, |x_1(t)| > T_1, \end{cases} \quad e_{12}(x_1(t)) = \begin{cases} 2.33, |x_1(t)| \le T_1, \\ 5.47, |x_1(t)| > T_1, \end{cases}$$
$$e_{21}(x_2(t)) = \begin{cases} -3.78, |x_2(t)| \le T_2, \\ 5.64, |x_2(t)| > T_2, \end{cases} \quad e_{22}(x_2(t)) = \begin{cases} 4.39, |x_2(t)| \le T_2, \\ 5.24, |x_2(t)| > T_2. \end{cases}$$

where $T_1 = 3$, $T_2 = 10$. The initial value of system (14) are $x_1(0) = -3.8$, $x_1'(0) = 7.8$, $x_2(0) = -2.5$, $x_2'(0) = 22.3$. The corresponding response MINNs as follows:

$$\begin{aligned}\frac{d^2y_i(t)}{dt^2} =& -a_i\frac{dy_i(t)}{dt} - b_iy_i(t) + \sum_{j=1}^{2} c_{ij}(y_i(t))f_j(y_j(t)) + \sum_{j=1}^{2} d_{ij}(y_i(t)) \\ &\times f_j(y_j(t-\tau(t))) + \sum_{j=1}^{2} e_{ij}(y_i(t))\int_{t-\ell(t)}^{t} f_j(y_j(s))ds + I_i(t) + U_i(t) \end{aligned} \tag{15}$$

where $i = 1, 2$, the initial value of system (15) are $y_1(0) = -2.6$, $y_1'(0) = -8.35$, $y_2(0) = 2.3$, $y_2'(0) = -24.4$, and the controllers $U_i(t)$ are defined in (7). The drive and response MINNs (14) and (15) and their synchronization error system without controller are given in Figures 2 and 3, which imply that drive-response MINNs will not achieve synchronization as time increases.

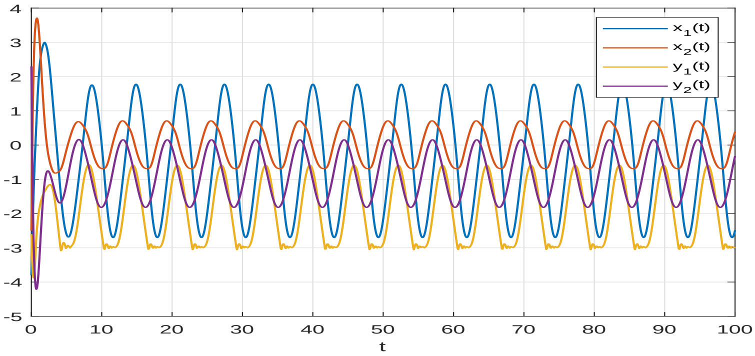

Figure 2. The drive-response MINNs without the controller.

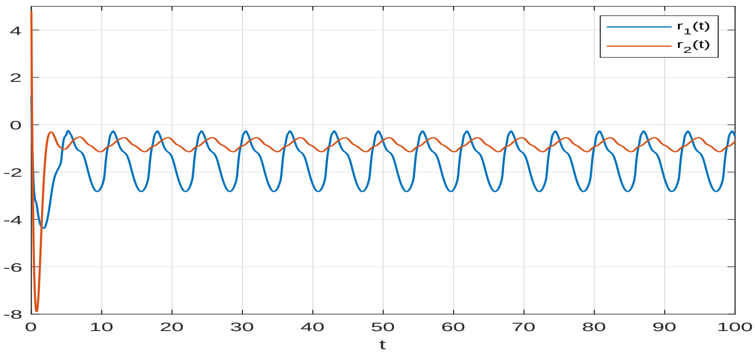

Figure 3. The synchronization error system without the controller.

Obviously, we can obtain that $1 - a_1 = -1.52, 1 - a_2 = -2.14$, $b_1 + l_1 \sum_{j=1}^{2} \bar{c}_{j1} = 11.97$, $b_2 + l_2 \sum_{j=1}^{2} \bar{c}_{j2} = 13.78$, and $\left|\left|(\hat{C} - \check{C})M\right|\right|_1 = 32.70$, $\left|\left|(\hat{D} - \check{D})M\right|\right|_1 = 30.95$, $\ell\left|\left|(\hat{E} - \check{E})M\right|\right|_1 = 33.76$, so we have $\left|\left|\beta\right|\right|_1 > 97.41$. Choose $\lambda_1 = \lambda_2 = 1$, $\theta_1 = \theta_2 = 15$, $\beta_1 = \beta_2 = 50$, $\xi = 1.5$ and $\mu = 0.8$, then the condition in Theorem 1 or Theorem 2 is satisfied. According to Theorem 1 or Theorem 2, the drive MINNs (14) and the response MINNS (15) can achieve FTS with the controller (14), and the estimated value of settling time $T \leq 7.77$ or $T \leq 4.36$. Figure 4 shows the synchronization error trajectories of drive-response MINNs (14) and (15) under the controller (7). The drive-response MINNs are successfully synchronized in the finite time. Therefore, the correctness of Theorems 1 and 2 is certified.

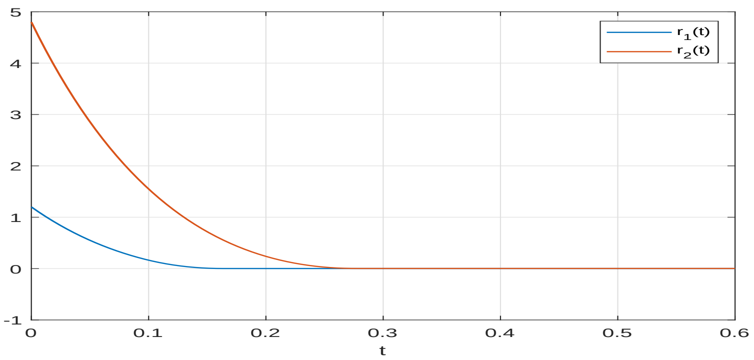

Figure 4. The synchronization error system with the controller.

Example 2. *Consider the two-neuron drive MINNs with MTVDs, as follows:*

$$\begin{aligned}\frac{d^2x_i(t)}{dt^2} =& - a_i\frac{dx_i(t)}{dt} - b_ix_i(t) + \sum_{j=1}^{2} c_{ij}(x_i(t))f_j(x_j(t)) + \sum_{j=1}^{2} d_{ij}(x_i(t)) \\ &\times f_j(x_j(t-\tau(t))) + \sum_{j=1}^{2} e_{ij}(x_i(t)) \int_{t-\ell(t)}^{t} f_j(x_j(s))ds + I_i(t) \end{aligned} \tag{16}$$

$i = 1, 2$, *where* $a_1 = 1.32$, $a_2 = 1.43$, $b_1 = 1.57$, $b_2 = 1.33$, *and the activation functions* $f_1(\cdot) = f_2(\cdot) = tanh(\cdot)$, $\tau(t) = \ell(t) = \frac{e^t}{1+e^t}$, $I_1(t) = sin(t)$, $I_2(t) = cos(t)$. *We can clearly observe that the functions* f_1 *and* f_2 *satisfy Assumption 1 and 2 with* $l_1 = l_2 = 1$, $M_1 = M_2 = 1$, *and* $0 \leq \tau(t), \ell(t) \leq 1$. *The memristor-based connection weights take the following forms:*

$$c_{11}(x_1(t)) = \begin{cases} 1.15, |x_1(t)| \leq T_1, \\ -1.34, |x_1(t)| > T_1, \end{cases} \quad c_{12}(x_1(t)) = \begin{cases} 1.44, |x_1(t)| \leq T_1, \\ 1.53, |x_1(t)| > T_1, \end{cases}$$

$$c_{21}(x_2(t)) = \begin{cases} -1.83, |x_2(t)| \leq T_2, \\ 1.47, |x_2(t)| > T_2, \end{cases} \quad c_{22}(x_2(t)) = \begin{cases} 1.28, |x_2(t)| \leq T_2, \\ -1.66, |x_2(t)| > T_2, \end{cases}$$

$$d_{11}(x_1(t)) = \begin{cases} -1.73, |x_1(t)| \leq T_1, \\ -1.48, |x_1(t)| > T_1, \end{cases} \quad d_{12}(x_1(t)) = \begin{cases} 1.27, |x_1(t)| \leq T_1, \\ -1.19, |x_1(t)| > T_1, \end{cases}$$

$$d_{21}(x_2(t)) = \begin{cases} -1.63, |x_2(t)| \leq T_2, \\ -1.38, |x_2(t)| > T_2, \end{cases} \quad d_{22}(x_2(t)) = \begin{cases} -1.55, |x_2(t)| \leq T_2, \\ 1.82, |x_2(t)| > T_2, \end{cases}$$

$$e_{11}(x_1(t)) = \begin{cases} -1.14, |x_1(t)| \leq T_1, \\ 1.52, |x_1(t)| > T_1, \end{cases} \quad e_{12}(x_1(t)) = \begin{cases} 1.57, |x_1(t)| \leq T_1, \\ 1.46, |x_1(t)| > T_1, \end{cases}$$

$$e_{21}(x_2(t)) = \begin{cases} -1.35, |x_2(t)| \leq T_2, \\ 1.86, |x_2(t)| > T_2, \end{cases} \quad e_{22}(x_2(t)) = \begin{cases} 1.74, |x_2(t)| \leq T_2, \\ -1.58, |x_2(t)| > T_2. \end{cases}$$

where $T_1 = 1$, $T_2 = 1$. *The initial value of system (16) are* $x_1(0) = 0.8$, $x_1'(0) = -4.8$, $x_2(0) = -1.5$, $x_2'(0) = 1.8$. *The corresponding response MINNs are as follows:*

$$\begin{aligned}\frac{d^2y_i(t)}{dt^2} =& - a_i\frac{dy_i(t)}{dt} - b_iy_i(t) + \sum_{j=1}^{2} c_{ij}(y_i(t))f_j(y_j(t)) + \sum_{j=1}^{2} d_{ij}(y_i(t)) \\ &\times f_j(y_j(t-\tau(t))) + \sum_{j=1}^{2} e_{ij}(y_i(t)) \int_{t-\ell(t)}^{t} f_j(y_j(s))ds + I_i(t) + U_i(t) \end{aligned} \tag{17}$$

where $i = 1, 2$, *the initial value of system (17) are* $y_1(0) = -0.6$, $y_1'(0) = 3.65$, $y_2(0) = -1.3$, $y_2'(0) = -1.3$, *and the controllers* $U_i(t)$ *are defined in (10). When the controllers* $U_i(t) = 0$, *the drive and response MINNs (16) and (17) and its synchronization error system are given in Figures 5 and 6, which imply that drive-response MINNs will not achieve synchronization as time increases.*

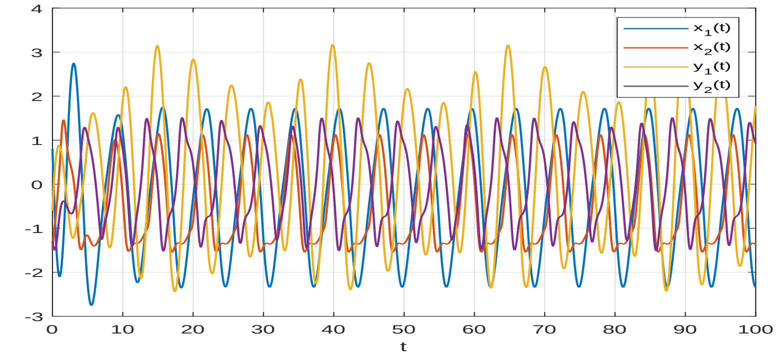

Figure 5. The drive-response MINNs without the controller.

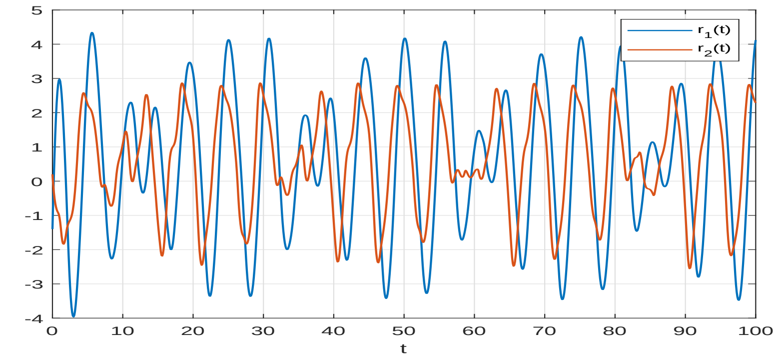

Figure 6. The synchronization error system without the controller.

According to Theorem 3, we can obtain that $1 - a_1 = -0.32, 1 - a_2 = -0.43$, $b_1 + l_1 \sum_{j=1}^{2} \bar{c}_{j1} = 4.74$, $b_2 + l_2 \sum_{j=1}^{2} \bar{c}_{j2} = 4.52$, and $||(\hat{C} - \check{C})M||_1 = 8.82$, $||(\hat{D} - \check{D})M||_1 = 6.33$, $\ell||(\hat{E} - \check{E})M||_1 = 9.30$,; thus, we have $||\beta||_1 > 24.45$. Then, choose $\lambda_1 = \lambda_2 = 1$, $\theta_1 = \theta_2 = 5$, and $\beta_1 = 10, \beta_2 = 16$,; the error systems with the controller are shown in Figure 7,and the settling time $T \leq 4.48$. It is easy to observe from Figure 7 that the drive-response MINNs (16) and (17) are successfully synchronization in the finite time with the controller (10); thus, the practicality and validity of Theorem 3 are shown.

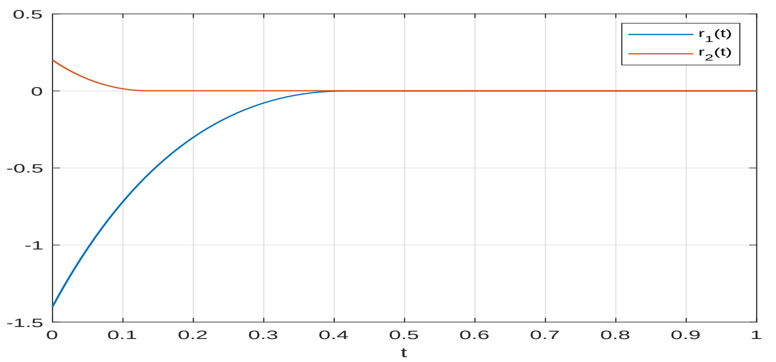

Figure 7. The synchronization error system with the controller.

5. Conclusions

In this work, novel sufficient conditions to guarantee the FTS for the studied MINNs have been given. Moreover, the novel finite-time stability criterion proposed in this paper is completely different from the existing ones, and it enriches the analytical tools for studying FTS. In addition, we directly investigated the FTS from the MINNs themselves without utilizing the variable substitution method widely used in the previous literature, which does not change the order of the second-order MINNs. Finally, several numerical examples have also been provided to demonstrate the validity of the novel research results. New useful study methods and theoretical results of this paper have widened the existing results, and this can be extended to many dynamical systems, such as nonlinear impulsive systems, and coupled neural networks, which are our possible future research topics.

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Appendix A

Proof of Lemma 2. Multiplying both sides of (5) with e^{kt} has

$$\frac{d[V(x(t))e^{kt}]}{dt} \leq -\varphi(t)e^{kt}. \tag{A1}$$

Integrating (A1) over $[t_0, t]$ yields

$$V(t, x(t))e^{kt} \leq V(t_0, x(t_0))e^{kt_0} - \int_{t_0}^{t} \varphi(s)e^{ks}ds \tag{A2}$$

On the basis of Integration by parts, one has

$$\begin{aligned}\int_{t_0}^{t} \varphi(s)e^{ks}ds &= \int_{t_0}^{t} \left(\frac{1}{k}\varphi(s)e^{ks}\right)' ds - \int_{t_0}^{t} \frac{1}{k}\varphi'(s)e^{ks}ds \\ &= \frac{1}{k}\varphi(t)e^{kt} - \frac{1}{k}\varphi(t_0)e^{kt_0} - \int_{t_0}^{t} \frac{1}{k}\varphi'(s)e^{ks}ds \end{aligned} \tag{A3}$$

By the conditions (6), substituting (A3) into (A2), we have

$$0 \leq V(x(t) \leq \left(V(x(t_0)) + \frac{\varphi(t_0)}{k}\right)e^{k(t_0-t)} - \frac{\varphi(t)}{k}. \tag{A4}$$

When $t \geq t_0 + \frac{1}{k}\ln\frac{kV(x(t_0))+\varphi(t_0)}{\varphi(t)}$ holds, we have $V(x(t)) \equiv 0$. Then, the proof is completed. □

Appendix B

Proof of Theorem 1. We adopt the following Lyapunov functions:

$$V(t) = ||r(t)||_1 + ||r'(t)||_1 = (\mathrm{sign}(r(t)))^T r(t) + (\mathrm{sign}(r'(t)))^T r'(t).$$

where

$$\begin{aligned}\mathrm{sign}(r(t)) &= \big(\mathrm{sign}(r_1(t)), \mathrm{sign}(r_2(t)), \cdots, \mathrm{sign}(r_n(t))\big)^T,\\ \mathrm{sign}(r'(t)) &= \big(\mathrm{sign}(r_1'(t)), \mathrm{sign}(r_2'(t)), \cdots, \mathrm{sign}(r_n'(t))\big)^T.\end{aligned}$$

By the system (3), we have

$$\begin{aligned}V'(t) =& \big(\mathrm{sign}(r(t))\big)^T r'(t) + \big(\mathrm{sign}(r'(t))\big)^T \Big[- Ar'(t) - Br(t) + C_y g(r(t)) \\ & + D_y g(r_j(t-\tau(t))) + E_y \int_{t-\ell(t)}^{t} g(r(s))ds + (C_y - C_x) f(x(t)) \\ & + (D_y - D_x) f(x(t-\tau(t))) + (E_y - E_x) \int_{t-\ell(t)}^{t} f(x(s))ds + U(t)\Big]. \end{aligned} \tag{A5}$$

Obviously, $\big(\mathrm{sign}(r(t))\big)^T r'(t) = \sum_{i=1}^n \mathrm{sign}(r_i(t)) r_i'(t) \le ||r'(t)||_1$, $\big(\mathrm{sign}(r'(t))\big)^T [-Ar'(t)] = -\sum_{i=1}^n a_i |r_i'(t)| = -||Ar'(t)||_1$, $\big(\mathrm{sign}(r'(t))\big)^T [-Br(t)] = -\sum_{i=1}^n \mathrm{sign}(r_i'(t)) b_i r_i(t) \le ||Br(t)||_1$. By Assumption 1, we can derive that

$$\begin{aligned}\big(\mathrm{sign}(r'(t))\big)^T C_y g(r(t)) =& \sum_{i=1}^n \sum_{j=1}^n \mathrm{sign}(r_i'(t)) c_{ij}(y_i(t)) g_j(r_j(t)) \le \sum_{i=1}^n \sum_{j=1}^n |c_{ij}(y_i(t))| \cdot |g_j(r_j(t))| \\ \le& \sum_{i=1}^n \sum_{j=1}^n \bar{c}_{ij} l_j |r_j(t)| = ||\bar{C} L r(t)||_1. \end{aligned} \tag{A6}$$

Similar to (A6), we obtain that $\big(\mathrm{sign}(r'(t))\big)^T D_y g(r_j(t-\tau(t))) \le ||\bar{D} L r(t-\tau(t))||_1$, $\big(\mathrm{sign}(r'(t))\big)^T E_y \int_{t-\ell(t)}^{t} g(r(s))ds \le ||\bar{E} L \int_{t-\ell(t)}^{t} \bar{r}(t)ds||_1$. By Assumption 2, we acquire

$$\begin{aligned}\big(\mathrm{sign}(r'(t))\big)^T (C_y - C_x) f(x(t)) =& \sum_{i=1}^n \sum_{j=1}^n \mathrm{sign}(r_i'(t)) [c_{ij}(y_i(t)) - c_{ij}(x_i(t))] f_j(x_j(t)) \\ \le& \sum_{i=1}^n \sum_{j=1}^n |c_{ij}(y_i(t)) - c_{ij}(x_i(t))| \cdot |f_j(x_j(t))| \\ \le& \sum_{i=1}^n \sum_{j=1}^n (\hat{c}_{ij} - \check{c}_{ij}) M_j = ||(\hat{C} - \check{C}) M||_1. \end{aligned} \tag{A7}$$

where $M = \big(M_1, M_2, \cdots, M_n\big)^T$. Similarly, we have $\big(\mathrm{sign}(r'(t))\big)^T (D_y - D_x) f(x(t-\tau(t))) \le ||(\hat{D} - \check{D}) M||_1$, $\big(\mathrm{sign}(r'(t))\big)^T (E_y - E_x) \int_{t-\ell(t)}^{t} f(x(s))ds \le \ell ||(\hat{E} - \check{E}) M||_1$. It is easy to derive that

$$\begin{aligned}\big(\mathrm{sign}(r'(t))\big)^T U(t) =& - ||\Lambda r'(t)||_1 - ||\Theta r(t)||_1 - ||\bar{D} L r(t-\tau(t))||_1 \\ & - ||\bar{E} L \int_{t-\ell(t)}^{t} \bar{r}(t)ds||_1 - ||\beta||_1 - \xi \sum_{i=1}^n |r_i(t)|^\mu - \xi \sum_{i=1}^n |r_i'(t)|^\mu. \end{aligned} \tag{A8}$$

Therefore,

$$\begin{aligned}
V'(t) \le & -\left(||\Lambda r'(t)||_1 + ||Ar'(t)||_1 - ||r'(t)||_1\right) - \left(||\Theta r(t)||_1 - ||Br(t)||_1\right. \\
& \left. - ||\bar{C}Lr(t)||_1\right) - \left(||\beta||_1 - ||(\hat{C} - \check{C})M||_1 - ||(\hat{D} - \check{D})M||_1\right. \\
& \left. - \ell||(\hat{E} - \check{E})M||_1\right) - \xi \sum_{i=1}^{n} |r_i(t)|^{\mu} - \xi \sum_{i=1}^{n} |r_i'(t)|^{\mu} \\
\le & -\lambda \sum_{i=1}^{n} |r_i'(t)| - \theta \sum_{i=1}^{n} |r_i(t)| - \beta^* - \xi \sum_{i=1}^{n} |r_i(t)|^{\mu} - \xi \sum_{i=1}^{n} |r_i'(t)|^{\mu} \\
\le & -aV(t) - \xi \sum_{i=1}^{n} |r_i(t)|^{\mu} - \xi \sum_{i=1}^{n} |r_i'(t)|^{\mu}. \qquad \text{(A9)}
\end{aligned}$$

where $\lambda = \min_{1\le i\le n}\{\lambda_i + a_i - 1\} > 0$, $\theta = \min_{1\le i\le n}\left\{\theta_i - b_i - l_i \sum_{j=1}^{n} \bar{c}_{ji}\right\} > 0$, $\beta^* = ||\beta||_1 - \left(||(\hat{C} - \check{C})M||_1 + ||(\hat{D} - \check{D})M||_1 + \ell||(\hat{E} - \check{E})M||_1\right) > 0$, and $a = \min\{\lambda, \theta\}$

According to Lemma 6, one has

$$\begin{aligned}
V'(t) \le & -aV(t) - \xi\left(\sum_{i=1}^{n} |r_i(t)|\right)^{\mu} - \xi\left(\sum_{i=1}^{n} |r_i'(t)|\right)^{\mu} \\
= & -aV(t) - \xi\left[\left(||r(t)||_1\right)^{\mu} + \left(||r'(t)||_1\right)^{\mu}\right] \\
\le & -aV(t) - \xi\left(V(t)\right)^{\mu}. \qquad \text{(A10)}
\end{aligned}$$

Based on Lemma 3, the drive MINNs (1) and response MINNs (2) can realize FTS. What is more, the settling time

$$T \le t_0 + \frac{\ln\left(1 + \frac{a}{\xi}\left(V(t_0)\right)^{1-\mu}\right)}{a(1-\mu)}.$$

This proof is completed. □

Appendix C

Proof of Theorem 2. The proofs are similar to the Theorem 1. From (A10), we have $V'(t) \le -\xi\left(V(t)\right)^{\mu}$. Based on Lemma 1, the drive MINNs (1) and response MINNs (2) can realize FTS. Moreover, the settling time $T \le t_0 + \frac{\left(V(t_0)\right)^{1-\mu}}{\xi(1-\mu)}$. This proof is completed. □

Appendix D

Proof of Theorem 3. We construct a Lyapunov function, as follows:

$$V(t) = ||r(t)||_1 + ||r'(t)||_1 = \left(\text{sign}(r(t))\right)^T r(t) + \left(\text{sign}(r'(t))\right)^T r'(t).$$

By the system (3), we have

$$\begin{aligned}
V'(t) = & \left(\text{sign}(r(t))\right)^T r'(t) + \left(\text{sign}(r'(t))\right)^T \Bigg[- Ar'(t) - Br(t) + C_y g(r(t)) \\
& + D_y g(r_j(t - \tau(t))) + E_y \int_{t-\ell(t)}^{t} g(r(s))ds + (C_y - C_x) f(x(t)) \\
& + (D_y - D_x) f(x(t - \tau(t))) + (E_y - E_x) \int_{t-\ell(t)}^{t} f(x(s))ds + U(t)\Bigg]. \qquad \text{(A11)}
\end{aligned}$$

Similarly, we can derive that

$$\begin{aligned} V'(t) \leq &-(||Ar'(t)||_1 - ||r'(t)||_1) + (||Br(t)||_1 + ||\bar{C}Lr(t)||_1) \\ &+ ||\bar{D}Lr(t-\tau(t))||_1 + ||\bar{E}L\int_{t-\ell(t)}^{t} \bar{r}(t)ds||_1 + (||(\hat{C}-\check{C})M||_1 \\ &+ ||(\hat{D}+\check{D})M||_1 + \ell||(\hat{E}-\check{E})M||_1) + (\mathrm{sign}(r'(t)))^T U(t). \end{aligned} \tag{A12}$$

It is easy to deduce that

$$\begin{aligned} (\mathrm{sign}(r'(t)))^T U(t) = &-||\Lambda r'(t)||_1 - ||\Theta r(t)||_1 - ||\bar{D}Lr(t-\tau(t))||_1 \\ &- ||\bar{E}L\int_{t-\ell(t)}^{t} \bar{r}(t)ds||_1 - ||\beta||_1. \end{aligned} \tag{A13}$$

Therefore,

$$\begin{aligned} V'(t) \leq &-(||\Lambda r'(t)||_1 + ||Ar'(t)||_1 - ||r'(t)||_1) - (||\Theta r(t)||_1 - ||Br(t)||_1 - ||\bar{C}Lr(t)||_1) \\ &-(||\beta||_1 - ||(\hat{C}-\check{C})M||_1 - ||(\hat{D}-\check{D})M||_1 - \ell||(\hat{E}-\check{E})M||_1) \\ = &-\sum_{i=1}^{n}\left(\theta_i - b_i - l_i\sum_{j=1}^{n}\bar{c}_{ji}\right)|r_i(t)| - \beta^* - \sum_{i=1}^{n}(\lambda_i + a_i - 1)|r_i'(t)| \\ \leq &-\lambda\sum_{i=1}^{n}|r_i'(t)| - \theta\sum_{i=1}^{n}|r_i(t)| - \beta^* \leq -aV(t) - \beta^* \end{aligned} \tag{A14}$$

By Lemma 4, the FTS between the drive MINNs (1) and the response MINN (3) under the controller (10) is realized. Moreover, the settling time is

$$T \leq t_0 + \frac{1}{a}\ln\frac{aV(t_0)+\beta^*}{\beta^*}.$$

The proof is completed. □

Appendix E

Proof of Theorem 4. We construct a Lyapunov function, as follows:

$$V(t) = ||r(t)||_1 + ||r'(t)||_1 = (\mathrm{sign}(r(t)))^T r(t) + (\mathrm{sign}(r'(t)))^T r'(t).$$

Similar to the proof of Theorem 3, we can derive that

$$\begin{aligned} V'(t) \leq &-(||\Lambda r'(t)||_1 + ||Ar'(t)||_1 - ||r'(t)||_1) - (||\Theta r(t)||_1 \\ &- ||Br(t)||_1 - ||\bar{C}Lr(t)||_1) - (||\beta||_1 - ||(\hat{C}-\check{C})M||_1 \\ &- ||(\hat{D}-\check{D})M||_1 - \ell||(\hat{E}-\check{E})M||_1) - ||\Delta e^{-\mu(t-t_0)}||_1 \\ \leq &-aV(t) - \omega e^{-\mu(t-t_0)} \end{aligned} \tag{A15}$$

where $\omega = \sum_{i=1}^{n}\delta_i$.

By Lemma 5, the FTS between the drive MINNs (1) and the response MINN (3) under the controller (12) is realized. In addition, the settling time is

$$T \leq t_0 + \frac{1}{a-\mu}\ln\frac{aV(t_0)+\omega}{\omega}.$$

The proof is completed. □

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