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Mean Square Finite-Approximate Controllability of Semilinear Stochastic Differential Equations with Non-Lipschitz Coefficients

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Abstract: In this paper, we present a study on mean square approximate controllability and finite-dimensional mean exact controllability for the system governed by linear/semilinear infinite-dimensional stochastic evolution equations. We introduce a stochastic resolvent-like operator and, using this operator, we formulate a criterion for mean square finite-approximate controllability of linear stochastic evolution systems. A control is also found that provides finite-dimensional mean exact controllability in addition to the requirement of approximate mean square controllability. Under the assumption of approximate mean square controllability of the associated linear stochastic system, we obtain sufficient conditions for the mean square finite-approximate controllability of a semilinear stochastic systems with non-Lipschitz drift and diffusion coefficients using the Picard-type iterations. An application to stochastic heat conduction equations is considered.

Keywords: approximate controllability; mean square finite-approximate controllability; stochastic evolution systems

MSC: 93B05; 60H17; 93C25; 34K30; 34K35

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1. Introduction

Stochastic differential equations have been successfully used in recent years in many applied problems in physics, economics, electricity, mechanics, etc. Many real systems and biological procedures exhibit some form of dynamic action under random perturbation, with continuous and discrete properties. In the last few decades, controllability concepts (approximate/exact approximate/finite-approximate controllability and so on) for different types of stochastic semilinear evolutionary systems have been studied in many articles using various methods. We divide scientific articles devoted to stochastic controllability concepts into groups as follows.

- Linear stochastic systems: Approximate controllability notions for stochastic linear systems were studied in [1–10]. In [1,2], stochastic Ljapunov methods are used to give sufficient conditions for these types of stochastic observability and controllability. In [3,4], the authors study the controllability of linear dynamical systems in the presence of random perturbations. In [7], with the help of dual equations the duality between approximate controllability and observability is deduced. In [8,9], necessary and sufficient conditions, in terms of uniform and strong convergence of a certain sequence of operators involving the resolvent of the negative of the controllability operator, are formulated.
- Semilinear stochastic systems: Studies on the approximate controllability concepts of semilinear/nonlinear stochastic systems have progressed slowly as compared to linear stochastic systems, see [11–20]. There are several approaches: a resolvent approach applied together with fixed point methods, integral contractor, sequencing method

and the monotone technique. Several researchers—Sunahara et al. [11,12], Mahmudov [9], George [13], Sakthivel and Kim [14], Tand and Zhang [15], Mokkedem and Fu [21], Ain et al. [22], Anguraj and Ramkumar [23]—have used different methods to study approximate controllability for several stochastic evolution systems.

- Non-Lipschitz stochastic systems: Approximate controllability of non-Lipschitz stochastic systems was considered in Sing et al. [24], Ren et al. [25], Mahmudov et al. [26].
- Finite-approximate controllability: Simultaneous mean square approximate and finite-dimensional mean exact controllability, referred to as the finite-approximate mean square controllability of linear/semilinear stochastic systems in infinite-dimensional spaces, is studied in [10,27].

As far as we know, no attempts have been made to study the analogue of mean square finite-approximate controllability for linear stochastic evolution systems as well as for semilinear stochastic evolution systems with non-Lipschitz coefficients. In contrast, approximate controllability problems for the mean square finite-approximate controllability for linear/semilinear stochastic systems investigated in this manuscript have not been tackled in the existing literature. This study explores the mean square approximate controllability for linear/semilinear stochastic systems with non-Lipschitz drift and diffusion coefficients and fills this gap in the literature.

Therefore, motivated by the above discussions, we study the mean square finite-approximate controllability of the following stochastic differential equation:

$$\begin{aligned} d\mathfrak{z}(\tau) &= [\mathfrak{A}\mathfrak{z}(\tau) + \mathfrak{B}\mathfrak{u}(\tau) + \mathfrak{f}(\tau, \mathfrak{z}(\tau), \mathfrak{u}(\tau))]d\tau + \mathfrak{g}(\tau, \mathfrak{z}(\tau), \mathfrak{u}(\tau))dw(\tau), \\ \mathfrak{z}(0) &= \mathfrak{z}_0. \end{aligned} \tag{1}$$

Here, $\mathfrak{X}$ is a Hilbert space, $\mathfrak{z} : [0, T] \times \Omega \to \mathfrak{X}$ is the state process, $\mathfrak{u} : [0, T] \times \Omega \to \mathfrak{U}$ is the control process, $\mathfrak{U}$ is a Hilbert space, $\mathfrak{A}$ is an infinitesimal generator of C_0-semigroup, $\mathfrak{B} \in L(\mathfrak{U}, \mathfrak{X})$ is a linear continuous operator, $\mathfrak{f} : [0, T] \times \mathfrak{X} \times \mathfrak{U} \to \mathfrak{X}$, $\mathfrak{g} : [0, T] \times \mathfrak{X} \times \mathfrak{U} \to L_2^0$ are functions to be defined later.

We introduce mean square finite-approximate controllability for Equation (1).

Definition 1. *Let M be a subspace of $\mathfrak{X}$ with finite-dimension. $\pi : \mathfrak{X} \to M$ is the orthogonal projection operator. System (1) is said to be mean square approximately controllable if for a given $\mathfrak{z}_0 \in \mathfrak{X}$, $\mathfrak{z}_T \in L^2(\mathfrak{F}_T, \mathfrak{X})$ and $\varepsilon > 0$, there exists a control process $\mathfrak{u}_\varepsilon \in L^2_{\mathfrak{F}}(0, T; \mathfrak{U})$ such that the solution to (1) satisfies*

$$\mathbb{E}\|\mathfrak{z}(T; \mathfrak{u}_\varepsilon) - \mathfrak{z}_T\|^2 < \varepsilon^2.$$

Definition 2. *Let M be a subspace of $\mathfrak{X}$ with finite-dimension. $\pi : \mathfrak{X} \to M$ is the orthogonal projection operator. System (1) is said to be exact mean finite-dimensional controllable if for a given $\mathfrak{z}_0 \in \mathfrak{X}$, $\mathfrak{z}_T \in L^2(\mathfrak{F}_T, \mathfrak{X})$, there exists a control process $\mathfrak{u} \in L^2_{\mathfrak{F}}(0, T; \mathfrak{U}$ such that the solution to (1) satisfies*

$$\pi\mathbb{E}\mathfrak{z}(T; \mathfrak{u}_\varepsilon) = \pi\mathbb{E}\mathfrak{z}_T.$$

Definition 3 ([10])**.** *Let M be a subspace of $\mathfrak{X}$ with finite-dimension. $\pi : \mathfrak{X} \to M$ is the orthogonal projection operator. System (1) is said to be mean square finite-approximately controllable if for a given $\mathfrak{z}_0 \in \mathfrak{X}$, $\mathfrak{z}_T \in L^2(\mathfrak{F}_T, \mathfrak{X})$ and $\varepsilon > 0$, there exists a control process $\mathfrak{u}_\varepsilon \in L^2_{\mathfrak{F}}(0, T; \mathfrak{U})$ such that the solution to (1) satisfies*

$$\mathbb{E}\|\mathfrak{z}(T; \mathfrak{u}_\varepsilon) - \mathfrak{z}_T\|^2 < \varepsilon^2, \tag{2}$$

$$\pi\mathbb{E}\mathfrak{z}(T; \mathfrak{u}_\varepsilon) = \pi\mathbb{E}\mathfrak{z}_T. \tag{3}$$

Simultaneous exact mean finite-dimensional and approximate mean square controllability is referred to as mean square finite-approximate controllability. A control process $\mathfrak{u}_\varepsilon$ can be selected such that $\mathfrak{z}(T; \mathfrak{u}_\varepsilon)$ satisfies (2) and a finite number of constrains (3). It is clear that mean square finite-approximate controllability implies both exact mean finite-dimensional and approximate mean square controllability. However, the converse is not obvious.

The following are the main contributions of the paper.

(i) We introduce and study the simultaneous mean exact finite-dimensional and approximate mean square controllability (mean square finite-approximate controllability) concept for the linear/semilinear infinite-dimensional stochastic systems.
(ii) We prove that the finite-approximate mean square controllability of the stochastic linear system (4) is equivalent to the mean square approximate controllability of the system (4). We give an explicit analytical form of the control that provides finite-dimensional mean square controllability of the linear stochastic system (1) in terms of stochastic resolvent-like operators.
(iii) We present sufficient conditions for the mean square finite-dimensional controllability semilinear stochastic differential systems in infinite dimensional Hilbert spaces. We prove that mean square approximate controllability of the linear part of the stochastic system implies the mean square finite-approximate controllability of the semilinear stochastic differential equation with non-Lipschitz coefficients. Our results are new even for the semilinear stochastic differential equation with Lipschitz coefficients.

The following is how the rest of this paper is structured: In Section 2, we provide some fundamental notations and definitions, as well as some relevant assumptions. In Section 3, we show that for a linear stochastic evolution system (5) approximate mean square controllability on $[0, T]$ is equivalent to finite-approximate controllability in the mean square sense on $[0, T]$. Necessary and sufficient conditions are given for a finite-approximate mean square controllability concept of linear stochastic evolutionary systems in Hilbert spaces in terms of stochastic resolvent-like operators. In addition, we find an explicit form of the finitely approximating control in terms of the stochastic resolvent-like operator $\left(\varepsilon(I - \pi\mathbb{E}\{\cdot\}) + \Pi_0^T\right)^{-1}$. In Section 4, by applying the Picard approximation method, we establish sufficient conditions for the mean square finite-dimensional controllability of (1). Finally, to illustrate the theoretical findings, we provide numerical examples.

2. Preliminaries

We give notations and some preliminary results needed to present our principal results.

- For any pair $\mathfrak{X}_1$ and $\mathfrak{X}_2$ of separable real Hilbert spaces, we denote by $L(\mathfrak{X}_1, \mathfrak{X}_2)$ the space of bounded (continuous) linear operators from $\mathfrak{X}_1$ to $\mathfrak{X}_2$.
- $(\mathfrak{F}_\tau)_{\tau \geq 0}$ is a normal filtration, $(\Omega, \mathfrak{F}, (\mathfrak{F}_\tau)_{\tau\geq 0}, \mathfrak{P})$ is a probability space.
- $w(\tau)$ is a Wiener process on $(\Omega, \mathfrak{F}, \mathfrak{P})$. The covariance operator $Q \in L(K, K)$, with $\mathrm{tr}Q < \infty$ satisfies the following assumption: there exists a basis $\{e_k\}_{k\geq 1}$ in K, a bounded sequence of positive real numbers $\lambda_k \geq 0$ such that $Qe_k = \lambda_k e_k,\ k = 1, 2, \dots,$ and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that

$$\langle w(\tau), e\rangle = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\langle e_k, e\rangle \beta_k(\tau),\ e \in K,\ \tau \in [0, T],$$

  and $\mathfrak{F}_\tau = \mathfrak{F}_\tau^w$, where $\mathfrak{F}_\tau^w$ is the sigma algebra generated by $\{w(\theta) : 0 \leq \theta \leq \tau\}$
- K, $\mathfrak{X}$ and $\mathfrak{U}$ are separable Hilbert spaces.
- $L_2^0 = L_2\left(Q^{1/2}K; \mathfrak{X}\right)$ is the space of all Hilbert–Schmidt operators $\psi : Q^{1/2}K \to \mathfrak{X}$ with the inner product $\langle \psi, \phi\rangle_{L_2^0} = \sum_{k=1}^{\infty} (\psi Q\phi e_k, e_k)_K = \mathrm{tr}[\psi Q\phi]$.
- $L^2(\mathfrak{F}_T, \mathfrak{X})$ is the (Hilbert) space of all $\mathfrak{F}_T$-measurable square integrable functions $\mathfrak{f} : [0, T] \times \Omega \to \mathfrak{X}$.
- $L^2_{\mathfrak{F}}(0, T; \mathfrak{X})$ is the Hilbert space of all square integrable and $\mathfrak{F}$-adapted processes $\mathfrak{f} : [0, T] \times \Omega \to \mathfrak{X}$.
- $C\left(0, T; L^2(\mathfrak{F}, \mathfrak{X})\right)$ is the Banach space of continuous maps from $[0, T]$ into $L^2(\mathfrak{F}, \mathfrak{X})$ satisfying the condition $\sup\left\{\mathbb{E}\|\varphi(\tau)\|^2_{\mathfrak{X}} : \tau \in [0, T]\right\} < \infty$.

- $\mathfrak{Z}_\tau = \mathfrak{Z}(0,\tau;\mathfrak{X}) \subset C\big(0,\tau;L^2(\mathfrak{F},\mathfrak{X})\big)$ is a closed subspace consisting of measurable and $\mathfrak{F}$-adapted $\mathfrak{X}$-valued processes $\varphi \in C\big(0,\tau;L^2(\mathfrak{F},\mathfrak{X})\big)$ endowed with the norm $\|\varphi\|_\tau = \left(\sup\limits_{0\le s\le\tau} \mathbb{E}\|\varphi(s)\|^2_{\mathfrak{X}}\right)^{\frac{1}{2}}$.
- $S : \mathfrak{X} \to \mathfrak{X}$ is a C_0-semigroup generated by $\mathfrak{A} : D(\mathfrak{A}) \subset \mathfrak{X} \to \mathfrak{X}$ and $\mathfrak{B} \in L(\mathfrak{U},\mathfrak{X})$ such that
$$\sup_{0\le\tau\le T} \|S(\tau)\|_{L(\mathfrak{X})} = M_S \text{ and } \|\mathfrak{B}\|_{L(\mathfrak{U},\mathfrak{X})} = M_{\mathfrak{B}}.$$

To formulate and prove our main results, we require the following assumptions.

($\mathbf{H}_1$) $(\mathfrak{f},\mathfrak{g}) : [0,T] \times \mathfrak{X} \times \mathfrak{U} \to \mathfrak{X} \times L_2^0$ is a function that satisfies the following conditions:

(a) The function $\mathfrak{f}(\cdot,\mathfrak{z},\mathfrak{u}) : [0,T] \to \mathfrak{X}$ is measurable strongly for all $(\mathfrak{z},\mathfrak{u}) \in \mathfrak{X} \times \mathfrak{U}$ and the function $\mathfrak{f}(\tau,\cdot,\cdot) : \mathfrak{X} \times \mathfrak{U} \to \mathfrak{X}$ is continuous in $(\mathfrak{z},\mathfrak{u})$ for each $\tau \in [0,T]$;

(b) The function $\mathfrak{g}(\cdot,\mathfrak{z},\mathfrak{u}) : [0,T] \to L_2^0$ is measurable strongly for all $(\mathfrak{z},\mathfrak{u}) \in L_2^0 \times \mathfrak{U}$ and the function $\mathfrak{g}(\tau,\cdot,\cdot) : \mathfrak{X} \to L_2^0$ is continuous in $(\mathfrak{z},\mathfrak{u})$ for each $\tau \in [0,T]$;

(c) For any $(\mathfrak{z},\mathfrak{u}) \in L^2(\mathfrak{F}_T,\mathfrak{X}) \times L^2(\mathfrak{F}_T,\mathfrak{U})$ and $\tau \in [0,T]$, there exist non-decreasing functions $F_1, F_2 : [0,T] \times [0,\infty) \to [0,\infty)$ such that
$$\mathbb{E}\|\mathfrak{f}(\tau,\mathfrak{z},\mathfrak{u})\|^2 + \mathbb{E}\|\mathfrak{g}(\tau,\mathfrak{z},\mathfrak{u})\|^2_{L_2^0} \le F_1\Big(\tau,\mathbb{E}\|\mathfrak{z}\|^2\Big) + F_2\Big(\tau,\mathbb{E}\|\mathfrak{u}\|^2\Big).$$

($\mathbf{H}_2$) The functions $F(\tau,p)$ and $G(\tau,p)$ are continuous in p for each fixed $\tau \in [0,T]$ and locally integrable in τ for each fixed $p \in [0,\infty)$. Moreover, the integral equation
$$p(\tau) = p_0 + a\int_0^\tau (F_1(s,p(s)) + F_2(s,p(s)))ds$$
admits a solution for all $a > 0$ and $p_0 \ge 0$.

($\mathbf{H}_3$) There exist non-decreasing functions $H_1, H_2 : [0,T] \times [0,\infty) \to [0,\infty)$ such that for all $(\mathfrak{z}_1,\mathfrak{u}_1),(\mathfrak{z}_2,\mathfrak{u}_2) \in L^2(\mathfrak{F}_T,\mathfrak{X}) \times L^2(\mathfrak{F}_T,\mathfrak{U})$ and $\tau \in [0,T]$
$$\begin{aligned}&\mathbb{E}\|\mathfrak{f}(\tau,\mathfrak{z}_1,\mathfrak{u}_1) - \mathfrak{f}(\tau,\mathfrak{z}_2,\mathfrak{u}_2)\|^2 + \mathbb{E}\|\mathfrak{g}(\tau,\mathfrak{z}_1,\mathfrak{u}_1) - \mathfrak{g}(\tau,\mathfrak{z}_2,\mathfrak{u}_2)\|^2_{L_2^0}\\ &\le H_1\Big(\tau,\mathbb{E}\|\mathfrak{z}_1-\mathfrak{z}_2\|^2\Big) + H_2\Big(\tau,\mathbb{E}\|\mathfrak{u}_1-\mathfrak{u}_2\|^2\Big).\end{aligned}$$

($\mathbf{H}_4$) The functions $H_1(\tau,p)$, $H_2(\tau,p)$ are continuous in p for each fixed $\tau \in [0,T]$ and locally integrable with $H_1(\tau,0) = H_2(,0) = 0$. Moreover, if the inequality
$$r(\tau) \le b\int_0^\tau (H_1(s,r(s)) + H_2(s,r(s)))ds$$
is satisfied by a nonnegative continuous function $r()$ for $\tau \in [0,T]$ subject to $r(0) = 0$ for some $b > 0$, then $r(\tau) = 0$ for all $\tau \in [0,T]$.

(AC) The stochastic linear system
$$\mathfrak{z}(\tau) = \mathfrak{T}(\tau)\mathfrak{z}_0 + \int_0^\tau \mathfrak{T}(\tau-\theta)\mathfrak{B}\mathfrak{u}(\theta)d\theta + \int_0^\tau \mathfrak{T}(\tau-\theta)\sigma(\theta)dw(\theta) \tag{4}$$
is mean square approximately controllable on $[0,T]$. Here $\sigma \in \mathfrak{X}_2\big(0,T;L_2^0\big)$.

Remark 1. *(i) If $H_1(t,y) = H_2(t,y) = Cy$, $C > 0$, then the functions in the assumption (H_3) become the Lipschitz functions.*

(ii) *If H_1, H_2 are concave and for all $(\mathfrak{z}_1, \mathfrak{u}_1), (\mathfrak{z}_2, \mathfrak{u}_2) \in \mathfrak{X} \times \mathfrak{U}$*

$$\begin{aligned}&\|\mathfrak{f}(\tau, \mathfrak{z}_1, \mathfrak{u}_1) - \mathfrak{f}(\tau, \mathfrak{z}_2, \mathfrak{u}_2)\|^2 + \|\mathfrak{g}(\tau, \mathfrak{z}_1, \mathfrak{u}_1) - \mathfrak{g}(\tau, \mathfrak{z}_2, \mathfrak{u}_2)\|^2_{L_2^0} \\ &\leq H_1\Big(\tau, \|\mathfrak{z}_1 - \mathfrak{z}_2\|^2\Big) + H_2\Big(\tau, \|\mathfrak{u}_1 - \mathfrak{u}_2\|^2\Big),\end{aligned}$$

then the Jensen inequality implies (H_3).

(iii) *For some concrete examples, see [25].*

We present the following definition of mild solutions to (1).

Definition 4 ([28]). *Stochastic process $\mathfrak{z} \in \mathfrak{Z}_T$ is said to be a mild solution of (1) if for any $\mathfrak{u} \in L^2_{\mathfrak{F}}(0, T; \mathfrak{U})$ it satisfies the following stochastic integral equation*

$$\begin{aligned}\mathfrak{z}(\tau) = \mathfrak{T}(\tau)\mathfrak{z}_0 &+ \int_0^\tau \mathfrak{T}(\tau - \theta)[\mathfrak{B}\mathfrak{u}(\theta) + \mathfrak{f}(\theta, \mathfrak{z}(\theta), \mathfrak{u}(\theta))]d\theta \\ &+ \int_0^\tau \mathfrak{T}(\tau - \theta)\mathfrak{g}(\theta, \mathfrak{z}(\theta), \mathfrak{u}(\theta))dw(\theta).\end{aligned}$$

3. Linear Systems: Finite-Approximate Controllability

In this section, we study the mean square finite-approximate controllability of the stochastic linear evolution system:

$$\begin{cases} d\mathfrak{z}(\tau) = [\mathfrak{A}\mathfrak{z}(\tau) + \mathfrak{B}\mathfrak{u}(\tau)]d\tau + \sigma(\tau)dw(\tau), \\ \mathfrak{z}(0) = \mathfrak{z}_0 \in \mathfrak{X}. \end{cases} \tag{5}$$

The continuous linear operator $L_0^T : L^2_{\mathfrak{F}}(0, T; \mathfrak{U}) \to L^2(\mathfrak{F}_T, \mathfrak{X})$ defined by

$$L_0^T \mathfrak{u} := \int_0^T \mathfrak{T}(T - \theta)\mathfrak{B}\mathfrak{u}(\theta)d\theta$$

is called a controllability operator. Its adjoint is defined by

$$\left(L_0^T\right)^* \varphi := \mathfrak{B}^*\mathfrak{T}^*(T - \theta)\mathbb{E}\{\varphi \mid \mathfrak{F}_\theta\}, \;\; \varphi \in L^2(\mathfrak{F}_T, \mathfrak{X}).$$

The controllability Gramian operator is defined by

$$\begin{aligned}\Pi_0^T &:= L_0^T \left(L_0^T\right)^* \\ &= \int_0^T \mathfrak{T}(T - \theta)\mathfrak{B}\mathfrak{B}^*\mathfrak{T}^*(T - \theta)\mathbb{E}\{\cdot \mid \mathfrak{F}_\theta\}d\theta : L^2(\mathfrak{F}_T, \mathfrak{X}) \to L^2(\mathfrak{F}_T, \mathfrak{X}).\end{aligned}$$

The resolvent operator $\left(\varepsilon I + \Pi_0^T\right)^{-1}$ is known to be useful in studying the approximate/exact controllability properties of linear and semilinear deterministic/stochastic evolution systems, see [1,6]. In this regard, a new criterion for finite-approximation controllability of a linear stochastic evolutionary system (5) is formulated in terms of a resolvent-like operator $\left(\varepsilon(I - \pi\mathbb{E}\{\cdot\}) + \Pi_0^T\right)^{-1}$. We show that for a linear stochastic evolution system (5) approximate mean square controllability on $[0, T]$ is equivalent to finite-approximate controllability in the mean square sense on $[0, T]$. Necessary and sufficient conditions are given for a finite-approximate mean square controllability concept of linear stochastic evolutionary systems in Hilbert spaces in terms of stochastic resolvent-like operators. In addition, we find an explicit form of the finitely approximating control in terms of the stochastic resolvent-like operator $\left(\varepsilon(I - \pi\mathbb{E}\{\cdot\}) + \Pi_0^T\right)^{-1}$.

The following two types of operators:

- Operator $\Pi_0^T \in L\big(L^2(\mathfrak{F}_T, \mathfrak{X})\big)$ is said to be nonnegative if $\mathbb{E}\langle \Pi_0^T \varphi, \varphi\rangle \geq 0$ for all $\varphi \in L^2(\mathfrak{F}_T, \mathfrak{X})$.
- Operator $\Pi_0^T \in L\big(L^2(\mathfrak{F}_T, \mathfrak{X})\big)$ is said to be positive if $\mathbb{E}\langle \Pi_0^T \varphi, \varphi\rangle > 0$ for all $\varphi \in L^2(\mathfrak{F}_T, \mathfrak{X})$ with $\varphi \neq 0$.

Firstly, we present two properties on the resolvent operator $\big(\varepsilon I + \Pi_0^T\big)^{-1}$.

Lemma 1. *Assume that $\Pi_0^T : L^2(\mathfrak{F}_T, \mathfrak{X}) \to L^2(\mathfrak{F}_T, \mathfrak{X})$ is a linear positive operator. Then*

(a) *For any $\varepsilon > 0$, we have $\mathbb{E}\Big\|\varepsilon\big(\varepsilon I + \Pi_0^T\big)^{-1}\pi\mathbb{E}\{\cdot\}\Big\|^2 < 1$.*

(b) $\mathbb{E}\Big\|\varepsilon\big(\varepsilon I + \Pi_0^T\big)^{-1}\pi\mathbb{E}\{\cdot\}\Big\|^2$ *is continuous in ε and*

$$\gamma = \max_{0\leq\varepsilon\leq 1} \mathbb{E}\left\|\varepsilon\left(\varepsilon I + \Pi_0^T\right)^{-1}\pi\mathbb{E}\{\cdot\}\right\|^2 < 1.$$

Proof. It is clear that $\big(\varepsilon I + \Pi_0^T\big)^{-1}\pi\mathbb{E}\{\cdot\}$ maps $L^2(\mathfrak{F}_T, \mathfrak{X})$ into a finite-dimensional subspace of $\mathfrak{X}$ and

$$\left\|\varepsilon\left(\varepsilon I + \Pi_0^T\right)^{-1}\pi\mathbb{E}\{\cdot\}\right\| \leq 1.$$

To show that $\Big\|\varepsilon\big(\varepsilon I + \Pi_0^T\big)^{-1}\pi\mathbb{E}\{\cdot\}\Big\| < 1$, in contrast, suppose that there exists a sequence $\Big\{h_n \in L^2(\mathfrak{F}_T, \mathfrak{X}) : \mathbb{E}\|h_n\|^2 = 1\Big\}$ such that

$$\varepsilon\left(\varepsilon I + \Pi_0^T\right)^{-1}\pi\mathbb{E}\{h_n\} =: z_n, \quad \|z_n\| \to 1 \text{ as } n \to \infty. \tag{6}$$

It follows from Equation (6) that $\{z_n\}$ is a sequence of finite-dimensional vectors and

$$\varepsilon\pi\mathbb{E}\{h_n\} = \varepsilon z_n + \Pi_0^T z_n \text{ and } z_n \to z_0 \text{ strongly in } \mathfrak{X}. \tag{7}$$

$$\begin{aligned}
\langle \pi\mathbb{E}\{h_n\}, z_n\rangle &= \langle z_n, z_n\rangle + \frac{1}{\varepsilon}\left\langle \Pi_0^T z_n, z_n\right\rangle, \\
\|z_n\|^2 < \langle z_n, z_n\rangle &+ \frac{1}{\varepsilon}\left\langle \Pi_0^T z_n, z_n\right\rangle = \langle \pi\mathbb{E}\{h_n\}, z_n\rangle \leq \|\pi\mathbb{E}\{h_n\}\|\|z_n\| \leq \|z_n\|.
\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{gathered}
1 \leq 1 + \frac{1}{\varepsilon}\left\langle \Pi_0^T z_0, z_0\right\rangle \leq 1, \\
\left\langle \Pi_0^T z_0, z_0\right\rangle = 0 \Longrightarrow z_0 = 0.
\end{gathered}$$

Now, from Equation (7), it follows that $\|z_n\| \to 0$ as $n \to \infty$, which is a contradiction. The lemma is proved. □

The next lemma establishes a connection between the stochastic resolvent operator $\big(\varepsilon I + \Pi_0^T\big)^{-1}$ and the stochastic resolvent-like operator $\big(\varepsilon(I - \pi\mathbb{E}\{\cdot\}) + \Pi_0^T\big)^{-1}$.

Lemma 2. *If $\Pi_0^T : L^2(\mathfrak{F}_T, \mathfrak{X}) \to L^2(\mathfrak{F}_T, \mathfrak{X})$ is a non-negative linear operator, then the operator $\varepsilon(I - \pi\mathbb{E}\{\cdot\}) + \Pi_0^T : L^2(\mathfrak{F}_T, \mathfrak{X}) \to L^2(\mathfrak{F}_T, \mathfrak{X})$ is invertible and*

$$\mathbb{E}\left\|\left(\varepsilon(I - \pi\mathbb{E}\{\cdot\}) + \Pi_0^T\right)^{-1} h\right\|^2 \leq \frac{1}{\min(\varepsilon, \delta)}\mathbb{E}\|h\|^2, \quad h \in L^2(\mathfrak{F}_T, \mathfrak{X}), \tag{8}$$

where $\delta = \min\{\langle \pi\mathbb{E}\{\cdot\}\Pi_0^T \pi\mathbb{E}\{\varphi\}, \varphi\rangle : \|\pi\mathbb{E}\{\varphi\}\| = 1\}$. *Moreover, if* $\Pi_0^T : L^2(\mathfrak{F}_T, \mathfrak{X}) \to L^2(\mathfrak{F}_T, \mathfrak{X})$ *is a linear positive operator then*

$$\left(\varepsilon(I - \pi\mathbb{E}\{\cdot\}) + \Pi_0^T\right)^{-1} = \left(I - \varepsilon\left(\varepsilon I + \Pi_0^T\right)^{-1} \pi\mathbb{E}\{\cdot\}\right)^{-1} \left(\varepsilon I + \Pi_0^T\right)^{-1}. \tag{9}$$

Proof. We write $\varepsilon(I - \pi) + \Pi_0^T$ as follows.

$$\varepsilon(I - \pi\mathbb{E}\{\cdot\}) + \Pi_0^T = \varepsilon(I - \pi\mathbb{E}\{\cdot\}) + (I - \pi\mathbb{E}\{\cdot\})\Pi_0^T + \pi\mathbb{E}\{\cdot\}\Pi_0^T.$$

It is clear that

$$\begin{aligned}
&\mathbb{E}\left\langle \varepsilon(I - \pi\mathbb{E}\{\cdot\}) + \Pi_0^T \varphi, \varphi \right\rangle \\
&= \mathbb{E}\left\langle \left(\varepsilon(I - \pi\mathbb{E}\{\cdot\}) + (I - \pi\mathbb{E}\{\cdot\})\Pi_0^T\right)\varphi, \varphi \right\rangle + \left\langle \pi\mathbb{E}\{\cdot\}\Pi_0^T \varphi, \varphi \right\rangle \\
&\geq \begin{cases} \mathbb{E}\langle \pi\mathbb{E}\{\cdot\}\Pi_0^T \pi\mathbb{E}\{\cdot\}\varphi, \varphi\rangle, & \varphi \in M, \\ \mathbb{E}\langle (\varepsilon(I - \pi\mathbb{E}\{\cdot\}) + (I - \pi\mathbb{E}\{\cdot\})\Pi_0^T (I - \pi\mathbb{E}\{\cdot\}))\varphi, \varphi\rangle, & \varphi \in \mathfrak{X} \ominus M \end{cases} \\
&\geq \min(\varepsilon, \delta)\mathbb{E}\|\varphi\|^2.
\end{aligned}$$

It follows that $\varepsilon(I - \pi\mathbb{E}\{\cdot\}) + \Pi_0^T$ is invertible and inequality (8) is satisfied. If $\Pi_0^T : L^2(\mathfrak{F}_T, \mathfrak{X}) \to L^2(\mathfrak{F}_T, \mathfrak{X})$ is a positive linear operator then by Lemma 1, $\left(I - \varepsilon(\varepsilon I + \Pi_0^T)^{-1} \pi\mathbb{E}\{\cdot\}\right)^{-1}$ exists. On the other hand, since $(\varepsilon I + \Pi_0^T)$ is invertible and

$$\varepsilon(I - \pi\mathbb{E}\{\cdot\}) + \Pi_0^T = \left(\varepsilon I + \Pi_0^T\right)\left(I - \varepsilon\left(\varepsilon I + \Pi_0^T\right)^{-1} \pi\mathbb{E}\{\cdot\}\right),$$

the operator $\varepsilon(I - \pi\mathbb{E}\{\cdot\}) + \Pi_0^T$ is boundedly invertible and (9) is satisfied. □

Next, we present new criteria for the mean square finite-approximate controllability of linear stochastic system (5).

Theorem 1. *The following assertions are equivalent:*

(i) *Linear stochastic system (5) is mean square approximately controllable on* $[0, T]$*;*

(ii) Π_0^T *is positive;*

(iii) *For any* $h \in L^2(\mathfrak{F}_T, \mathfrak{X})$*, we have* $\mathbb{E}\left\|\varepsilon\left(\varepsilon + \Pi_0^T\right)^{-1} h\right\|^2 \to 0$ *as* $\varepsilon \to 0^+$*;*

(iv) *For any* $h \in L^2(\mathfrak{F}_T, \mathfrak{X})$*, we have* $\mathbb{E}\left\|\varepsilon\left(\varepsilon(I - \pi\mathbb{E}\{\cdot\}) + \Pi_0^T\right)^{-1} h\right\|^2 \to 0$ *as* $\varepsilon \to 0^+$*;*

(v) *Linear stochastic system (5) is mean square finite-approximately controllable on* $[0, T]$*.*

Proof. We show that (i)$\Longleftrightarrow$(ii). By definition, system (5) is approximately controllable if $\operatorname{Im} L_0^T$ is dense in $L^2(\mathfrak{F}_T, \mathfrak{X})$. Then, we know that

$$\left(\ker\left(L_0^T\right)^*\right)^\perp = \overline{\operatorname{Im} L_0^T}.$$

Moreover

$$\mathbb{E}\left\langle \Pi_0^T h, h \right\rangle = \mathbb{E}\left\|\left(L_0^T\right)^* h\right\|^2, \ h \in L^2(\mathfrak{F}_T, \mathfrak{X}).$$

It follows that

$$\Pi_0^T > 0 \Longleftrightarrow \ker\left(L_0^T\right)^* = 0 \Longleftrightarrow \overline{\operatorname{Im} L_0^T} = \left(\ker\left(L_0^T\right)^*\right)^\perp = \left(\ker\left(L_0^T\right)^*\right)^\perp.$$

We show that (ii)$\Longleftrightarrow$(iii).

Suppose (iii) fails. Then, for some $h \in L^2(\mathfrak{F}_T, \mathfrak{X})$, we have

$$\lim_{\varepsilon \to 0^+} \varepsilon \mathbb{E} \left\| \left(\varepsilon I + \Pi_0^T \right)^{-1} h \right\|^2 = z \neq 0.$$

Set $z_\varepsilon = \varepsilon \big(\varepsilon I + \Pi_0^T \big)^{-1} h$. Then, $\big(\varepsilon I + \Pi_0^T \big) z_\varepsilon = \varepsilon h$, and taking the limit of both sides, we obtain

$$\Pi_0^T z = 0 \Longrightarrow \mathbb{E} \left\langle \Pi_0^T z, z \right\rangle = 0$$

for nonzero z, which contradicts the positivity of Π_0^T.

Now, assuming that (ii) fails, for some nonzero $z \in L^2(\mathfrak{F}_T, \mathfrak{X})$, we have

$$\mathbb{E} \left\langle \Pi_0^T z, z \right\rangle = \mathbb{E} \left\| \left(L_0^T \right)^* z \right\|^2 = 0 \Longrightarrow L_0^T \left(L_0^T \right)^* z = \Pi_0^T z = 0.$$

It follows that

$$\begin{aligned} \left(\varepsilon I + \Pi_0^T \right) z = \varepsilon z &\Longrightarrow z = \varepsilon \left(\varepsilon I + \Pi_0^T \right)^{-1} z \\ &\Longrightarrow \lim_{\varepsilon \to 0^+} \varepsilon \mathbb{E} \left\| \left(\varepsilon I + \Pi_0^T \right)^{-1} z \right\|^2 = z \neq 0, \end{aligned}$$

which leads to a contradiction.

To prove the implication (iii)$\Longrightarrow$(iv), suppose that

$$\lim_{\varepsilon \to 0^+} \mathbb{E} \left\| \left(\varepsilon I + \Gamma_0^T \right)^{-1} h \right\|^2 = 0, \ \ h \in L^2(\mathfrak{F}_T, \mathfrak{X}).$$

It follows from (9) that for any $h \in L^2(\mathfrak{F}_T, \mathfrak{X})$

$$\begin{aligned} &\sqrt{\mathbb{E} \left\| \varepsilon \big(\varepsilon (I - \pi \mathbb{E}\{\cdot\}) + \Pi_0^T \big)^{-1} h \right\|^2} \\ &\leq \left\| \left(I - \varepsilon \left(\varepsilon I + \Pi_0^T \right)^{-1} \pi \mathbb{E}\{\cdot\} \right)^{-1} \right\| \sqrt{\mathbb{E} \left\| \varepsilon \big(\varepsilon I + \Pi_0^T \big)^{-1} h \right\|^2} \\ &\leq \frac{1}{1 - \left\| \varepsilon \big(\varepsilon I + \Pi_0^T \big)^{-1} \pi \mathbb{E}\{\cdot\} \right\|} \sqrt{\mathbb{E} \left\| \varepsilon \big(\varepsilon I + \Pi_0^T \big)^{-1} h \right\|^2}. \end{aligned} \tag{10}$$

On the other hand, from

$$\begin{aligned} &\varepsilon_1 \left(\varepsilon_1 I + \Pi_0^T \right)^{-1} \pi \mathbb{E}\{\cdot\} - \varepsilon \left(\varepsilon I + \Pi_0^T \right)^{-1} \pi \mathbb{E}\{\cdot\} \\ &= \varepsilon_1 \left(\varepsilon_1 I + \Pi_0^T \right)^{-1} \left(I + \varepsilon^{-1} \Pi_0^T - I - \varepsilon_1^{-1} \Pi_0^T \right) \varepsilon \left(\varepsilon I + \Pi_0^T \right)^{-1} \pi \mathbb{E}\{\cdot\} \\ &= \varepsilon_1 \left(\varepsilon_1 I + \Pi_0^T \right)^{-1} \left(\varepsilon^{-1} \Pi_0^T - \varepsilon_1^{-1} \Pi_0^T \right) \varepsilon \left(\varepsilon I + \Pi_0^T \right)^{-1} \pi \mathbb{E}\{\cdot\} \\ &= \left(\varepsilon_1 I + \Pi_0^T \right)^{-1} \left(\varepsilon_1 \Pi_0^T - \varepsilon \Pi_0^T \right) \left(\varepsilon I + \Pi_0^T \right)^{-1} \pi \mathbb{E}\{\cdot\} \\ &= \left(\varepsilon_1 I + \Pi_0^T \right)^{-1} (\varepsilon_1 - \varepsilon) \Pi_0^T \left(\varepsilon I + \Pi_0^T \right)^{-1} \pi \mathbb{E}\{\cdot\}, \end{aligned}$$

it follows that $\varepsilon \big(\varepsilon I + \Pi_0^T \big)^{-1} \pi \mathbb{E}\{\cdot\}$ is continuous in ε. Indeed,

$$\left\| \varepsilon_1 \left(\varepsilon_1 I + \Pi_0^T \right)^{-1} \pi \mathbb{E}\{\cdot\} - \varepsilon \left(\varepsilon I + \Pi_0^T \right)^{-1} \pi \mathbb{E}\{\cdot\} \right\| \leq \frac{|\varepsilon_1 - \varepsilon|}{\varepsilon_1} \to 0 \ \text{ as } \ \varepsilon_1 \to \varepsilon.$$

By (10), the continuity of $\varepsilon\big(\varepsilon I+\Pi_0^T\big)^{-1}\pi\mathbb{E}\{\cdot\}$ and Lemma 1, we have

$$\gamma=\max_{0\le\varepsilon\le1}\left\|\varepsilon\Big(\varepsilon I+\Pi_0^T\Big)^{-1}\pi\mathbb{E}\{\cdot\}\right\|<1,$$
$$\sqrt{\mathbb{E}\left\|\varepsilon\big(\varepsilon(I-\pi\mathbb{E}\{\cdot\})+\Pi_0^T\big)^{-1}h\right\|^2}\le\frac{1}{1-\gamma}\sqrt{\mathbb{E}\left\|\varepsilon\big(\varepsilon I+\Pi_0^T\big)^{-1}h\right\|^2}.$$

Thus, $\varepsilon\big(\varepsilon(I-\pi\mathbb{E}\{\cdot\})+\Pi_0^T\big)^{-1}$ converges to zero as $\varepsilon\to0^+$ in the strong operator topology.

To prove the equivalence (iii)$\Longleftrightarrow$(v), we take any $\varepsilon>0$, $h\in L^2(\mathfrak{F}_T,\mathfrak{X})$, and consider the functional $J_\varepsilon(\cdot,h):L^2(\mathfrak{F}_T,\mathfrak{X})\to R$ defined as follows :

$$J_\varepsilon(\varphi,h)=\frac{1}{2}\int_0^T\mathbb{E}\|\mathfrak{B}^*\mathfrak{T}^*(T-\theta)\mathbb{E}\{\varphi\mid\mathfrak{F}_\theta\}\|^2d\theta$$
$$+\frac{\varepsilon}{2}\mathbb{E}\langle(I-\pi\mathbb{E}\{\cdot\})\varphi,\varphi\rangle-\mathbb{E}\langle\varphi,h-\mathfrak{T}(T)\mathfrak{z}_0\rangle.$$

Suppose that (iii) ($\Leftrightarrow$(ii)) is satisfied. It is obvious that $J_\varepsilon(\cdot,h)$ is Gateaux differentiable and $J'_\varepsilon(\varphi,h)=\Pi_0^T\varphi+\varepsilon(I-\pi\mathbb{E}\{\cdot\})\varphi-h+\mathfrak{T}(T)\mathfrak{z}_0$ is strictly monotonic. The positivity of Π_0^T implies that the functional $J_\varepsilon(\cdot,h)$ is strictly convex. Thus, $J_\varepsilon(\cdot,h)$ has a unique minimum and can be calculated as follows:

$$\Pi_0^T\varphi+\varepsilon(I-\pi\mathbb{E}\{\cdot\})\varphi-h+\mathfrak{T}(T)\mathfrak{z}_0=0,$$
$$\varphi_{\min}=-\Big(\varepsilon(I-\pi\mathbb{E}\{\cdot\})+\Pi_0^T\Big)^{-1}(\mathfrak{T}(T)\mathfrak{z}_0-h).$$

For the control $\mathfrak{u}_\varepsilon(\theta)=\mathfrak{B}^*\mathfrak{T}^*(T-\theta)\mathbb{E}\{\varphi_{\min}\mid\mathfrak{F}_\theta\}$

$$\begin{aligned}\mathfrak{z}_\varepsilon(T)-h&=\mathfrak{T}(T)\mathfrak{z}_0+\int_0^T\mathfrak{T}(T-\theta)\mathfrak{B}\mathfrak{u}(\theta)d\theta-h\\&=\mathfrak{T}(T)\mathfrak{z}_0-h-\Pi_0^T\Big(\varepsilon(I-\pi\mathbb{E}\{\cdot\})+\Pi_0^T\Big)^{-1}(\mathfrak{T}(T)\mathfrak{z}_0-h)\\&=\mathfrak{T}(T)\mathfrak{z}_0-h-\Big(\Pi_0^T+\varepsilon(I-\pi\mathbb{E}\{\cdot\})-\varepsilon(I-\pi\mathbb{E}\{\cdot\})\Big)\\&\times\Big(\varepsilon(I-\pi\mathbb{E}\{\cdot\})+\Pi_0^T\Big)^{-1}(\mathfrak{T}(T)\mathfrak{z}_0-h)\\&=\varepsilon(I-\pi\mathbb{E}\{\cdot\})\Big(\varepsilon(I-\pi\mathbb{E}\{\cdot\})+\Pi_0^T\Big)^{-1}(\mathfrak{T}(T)\mathfrak{z}_0-h).\end{aligned}\tag{11}$$

Since (iii)$\Rightarrow$(iv), we have

$$\lim_{\varepsilon\to0^+}\mathbb{E}\|\mathfrak{z}_\varepsilon(T)-h\|^2=\lim_{\varepsilon\to0^+}\varepsilon\mathbb{E}\left\|(I-\pi\mathbb{E}\{\cdot\})\Big(\varepsilon(I-\pi\mathbb{E}\{\cdot\})+\Pi_0^T\Big)^{-1}(\mathfrak{T}(T)\mathfrak{z}_0-h)\right\|^2=0,$$
$$\pi\mathbb{E}\{\mathfrak{z}_\varepsilon(T)-h\}=0,$$

That is, system (5) is finite-approximately mean square controllable. Thus, (iii) implies (v). The implication (v) $\Rightarrow$ (iii) is obvious, since mean square finite-approximate controllability implies the mean square approximate controllability. (iv)$\Rightarrow$(v) follows from equality (11). Thus, we have

$$(i)\Leftrightarrow(ii)\Leftrightarrow(iii)\Leftrightarrow(iv)\Rightarrow(v)\Rightarrow(i).$$

□

Theorem 2. *The (deterministic) system*

$$\begin{cases}\mathfrak{z}'(\tau)=\mathfrak{A}\mathfrak{z}(\tau)+\mathfrak{B}\mathfrak{u}(\tau),\ \ \mathfrak{u}\in L^2(0,T;\mathfrak{U})\\ \mathfrak{z}(0)=\mathfrak{z}_0\in\mathfrak{X}.\end{cases}\tag{12}$$

is approximately controllable on every $[r, T]$, $0 \leq r < T$, *if and only if the linear stochastic system (5) is (mean square) approximate controllable on* $[0, T]$.

Proof. Suppose that the deterministic system (12) is approximately controllable on every $[r, T]$. Then, it is known that

$$W_r^T := \int_r^T \mathfrak{T}(T-\theta)\mathfrak{B}\mathfrak{B}^*\mathfrak{T}^*(T-\theta)d\theta, \quad 0 \leq r < T,$$

is positive. On the other hand, by the martingale representation theorem, for any $h \in L^2(\mathfrak{F}_T, \mathfrak{X})$, there exists a stochastic process $\psi \in L^2_{\mathfrak{F}}(0, T; L_2^0)$ such that

$$\mathbb{E}\{h \mid \mathfrak{F}_\tau\} = \mathbb{E}h + \int_0^\tau \psi(\theta)dw(\theta);$$

see, for example, [9]. Using the above representation, we can write Π_0^T in terms of matrix W_r^T :

$$\begin{aligned}
\Pi_0^T h &= \int_0^T \mathfrak{T}(T-\theta)\mathfrak{B}\mathfrak{B}^*\mathfrak{T}^*(T-\theta)\mathbb{E}\{h \mid \mathfrak{F}_\theta\}d\theta \\
&= \int_0^T \mathfrak{T}(T-\theta)\mathfrak{B}\mathfrak{B}^*\mathfrak{T}^*(T-\theta)\left(\mathbb{E}h + \int_0^\theta \psi(r)dw(r)\right)d\theta \\
&= W_0^T\mathbb{E}h + \int_0^T \int_r^T \mathfrak{T}(T-\theta)\mathfrak{B}\mathfrak{B}^*\mathfrak{T}^*(T-\theta)d\theta\psi(r)dw(r) \\
&= W_0^T\mathbb{E}h + \int_0^T W_r^T\psi(r)dw(r).
\end{aligned}$$

Therefore, for any nonzero $h \in L^2(\mathfrak{F}_T, \mathfrak{X})$

$$\mathbb{E}\left\langle \Pi_0^T h, h\right\rangle = \left\langle W_0^T\mathbb{E}h, \mathbb{E}h\right\rangle + \mathbb{E}\int_0^T \left\langle W_r^T\psi(r), \psi(r)\right\rangle dr > 0.$$

Thus, the positivity of the operator $\Pi_0^T = L_0^T\left(L_0^T\right)^* : L^2(\mathfrak{F}_T, \mathfrak{X}) \to L^2(\mathfrak{F}_T, \mathfrak{X})$ is equivalent to the positivity of W_r^T, $0 \leq r < T$. Therefore, by Theorem 1, the stochastic linear system (5) is approximately mean square controllable on $[0, T]$ if and only if the deterministic counterpart (12) is approximately controllable on any $[r, T]$, $0 \leq r < T$. □

4. Semilinear Systems: Mean Square Finite-Approximate Controllability

The proof of the main result of this section is based on the Picard approximation method. To apply the Picard method, for any $\varepsilon > 0$ we introduce the non-linear operator $\mathfrak{C}_\varepsilon : \mathfrak{Z}_T \times \mathfrak{U}_T \to \mathfrak{Z}_T \times \mathfrak{U}_T$ which is defined as follows

$$\mathfrak{C}_\varepsilon(\mathfrak{z}, \mathfrak{u}) = \left(\mathfrak{C}_\varepsilon^1, \mathfrak{C}_\varepsilon^2\right)(\mathfrak{z}, \mathfrak{u}) = (z, v),$$

where

$$\begin{aligned}
z(\tau) = \mathfrak{C}_\varepsilon^1(\mathfrak{z}, \mathfrak{u}) &= \mathfrak{T}(\tau)\mathfrak{z}_0 + \int_0^\tau \mathfrak{T}(-\theta)[\mathfrak{B}v(\theta) + \mathfrak{f}(\theta, \mathfrak{z}(\theta), \mathfrak{u}(\theta))]d\theta \\
&+ \int_0^\tau \mathfrak{T}(\tau-\theta)\mathfrak{g}(\theta, \mathfrak{z}(\theta), \mathfrak{u}(\theta))dw(\theta),
\end{aligned}$$

$$
\begin{aligned}
v(\tau) = \mathfrak{C}_\varepsilon^2(\mathfrak{z}, \mathfrak{u}) &= \mathfrak{B}^*\mathfrak{T}^*(T-\tau)\Big(\varepsilon(I-\pi)+\Gamma_0^T\Big)^{-1}(\mathbb{E}h - \mathfrak{T}(T)\mathfrak{z}_0) \\
&- \mathfrak{B}^*\mathfrak{T}^*(T-\tau)\int_0^\tau \Big(\varepsilon(I-\pi)+\Gamma_\theta^T\Big)^{-1}\mathfrak{T}(T-\theta)\mathfrak{f}(\theta,\mathfrak{z}(\theta),\mathfrak{u}(\theta))d\theta \\
&- \mathfrak{B}^*\mathfrak{T}^*(T-\tau)\int_0^\tau \Big(\varepsilon(I-\pi)+\Gamma_\theta^T\Big)^{-1}[\mathfrak{T}(T-\theta)\mathfrak{g}(\theta,\mathfrak{z}(\theta),\mathfrak{u}(\theta)) - \varphi(\theta)]dw(\theta),
\end{aligned}
$$

and $\varphi \in L^2_{\mathfrak{F}}(0,T;L_2^0)$ comes from the representation

$$
h = \mathbb{E}h + \int_0^T \varphi(\theta)dw(\theta)
$$

of $h \in L^2(\mathfrak{F}_T, \mathfrak{X})$.

Lemma 3. *Under assumptions (H_1)-(H_4), the operator $\mathfrak{C}_\varepsilon$ is well defined and there exist positive numbers $k_1(\varepsilon)$, $k_2(\varepsilon)$ such that for $(\mathfrak{z}, \mathfrak{u}) \in \mathfrak{Z}_T \times \mathfrak{U}_T$ then*

$$
\begin{aligned}
&\|\mathfrak{C}_\varepsilon(\mathfrak{z},\mathfrak{u})\|_\tau^2 \\
&\leq k_1(\varepsilon) + k_2(\varepsilon)\int_0^\tau F_1\left(\theta, \sup_{0\leq r\leq\theta}\mathbb{E}\|\mathfrak{z}(r)\|^2\right)d\theta \\
&+ k_2(\varepsilon)\int_0^\tau F_2\left(\theta, \sup_{0\leq r\leq\theta}\mathbb{E}\|\mathfrak{u}(r)\|^2\right)d\theta.
\end{aligned}
$$

Proof. Firstly, we estimate $\sup_{0\leq\theta\leq\tau}\mathbb{E}\|v(\theta)\|^2$ as follows.

$$
\begin{aligned}
&\mathbb{E}\|v(\tau)\|^2 = \mathbb{E}\|v(\tau,\mathfrak{z},\mathfrak{u})\|^2 \\
&\leq 4\mathbb{E}\left\|\mathfrak{B}^*\mathfrak{T}^*(T-\tau)\Big(\varepsilon(I-\pi)+\Gamma_0^T\Big)^{-1}(\mathbb{E}h - \mathfrak{T}(T)\mathfrak{z}_0)\right\|^2 \\
&+ 4\mathbb{E}\left\|\mathfrak{B}^*\mathfrak{T}^*(T-\tau)\int_0^\tau\Big(\varepsilon(I-\pi)+\Gamma_\theta^T\Big)^{-1}\mathfrak{T}(T-\theta)\mathfrak{f}(\theta,\mathfrak{z}(\theta),\mathfrak{u}(\theta))d\theta\right\|^2 \\
&+ 4\mathbb{E}\left\|\mathfrak{B}^*\mathfrak{T}^*(T-\tau)\int_0^\tau\Big(\varepsilon(I-\pi)+\Gamma_\theta^T\Big)^{-1}\mathfrak{T}(T-\theta)\mathfrak{g}(\theta,\mathfrak{z}(\theta),\mathfrak{u}(\theta))dw(\theta)\right\|^2 \\
&+ 4\mathbb{E}\left\|\mathfrak{B}^*\mathfrak{T}^*(T-\tau)\int_0^\tau\Big(\varepsilon(I-\pi)+\Gamma_\theta^T\Big)^{-1}\mathfrak{T}(T-\theta)\varphi(\theta)dw(\theta)\right\|^2 \\
&\leq \frac{4}{\varepsilon^2}M_S^2M_{\mathfrak{B}}^2\mathbb{E}\|(\mathbb{E}h - \mathfrak{T}(T)\mathfrak{z}_0)\|^2 + \frac{4}{\varepsilon^2}M_S^4M_{\mathfrak{B}}^2\mathbb{E}\int_0^\tau\|\varphi(\theta)\|^2d\theta \\
&+ \frac{4}{\varepsilon^2}M_S^4M_{\mathfrak{B}}^2T\mathbb{E}\int_0^\tau\|\mathfrak{f}(\theta,\mathfrak{z}(\theta),\mathfrak{u}(\theta))\|^2d\theta \\
&+ \frac{4}{\varepsilon^2}M_S^4M_{\mathfrak{B}}^2\mathbb{E}\int_0^\tau\|\mathfrak{g}(\theta,\mathfrak{z}(\theta),\mathfrak{u}(\theta))\|^2d\theta. \qquad (13)
\end{aligned}
$$

Next, we estimate $\sup_{0\leq\theta\leq\tau}\mathbb{E}\|z(\theta)\|^2$:

$$
\begin{aligned}
&\mathbb{E}\|z(\tau)\|^2 \\
&\leq 4M_S^2\|\mathfrak{z}_0\|^2 + 4M_S^2M_{\mathfrak{B}}^2T\mathbb{E}\int_0^\tau\|v(\theta)\|^2d\theta \\
&+ 4M_S^2M_{\mathfrak{B}}^2T\mathbb{E}\int_0^\tau\|\mathfrak{f}(\theta,\mathfrak{z}(\theta),\mathfrak{u}(\theta))\|^2d\theta \\
&+ 4M_S^2M_{\mathfrak{B}}^2\mathbb{E}\int_0^\tau\|\mathfrak{g}(\theta,\mathfrak{z}(\theta),\mathfrak{u}(\theta))\|_{L_2^0}^2d\theta \qquad (14)
\end{aligned}
$$

Inserting inequality (14) into Equation (13), we obtain

$$
\begin{aligned}
&\|\mathfrak{C}_\varepsilon(\mathfrak{z},\mathfrak{u})\|_\tau^2 = \sup_{0\le\theta\le\tau}\mathbb{E}\|z(\theta)\|^2 + \sup_{0\le\theta\le\tau}\mathbb{E}\|v(\theta)\|^2\\
&\le 4M_S^2\|\mathfrak{z}_0\|^2\\
&+4M_S^2M_{\mathfrak{B}}^2T\mathbb{E}\int_0^\tau\|\mathfrak{f}(\theta,\mathfrak{z}(\theta),\mathfrak{u}(\theta))\|^2d\theta+4M_S^2M_{\mathfrak{B}}^2\mathbb{E}\int_0^\tau\|\mathfrak{g}(\theta,\mathfrak{z}(\theta),\mathfrak{u}(\theta))\|_{L_2^0}^2d\theta\\
&+\frac{4}{\varepsilon^2}M_S^2M_{\mathfrak{B}}^2\mathbb{E}\|(\mathbb{E}h-\mathfrak{T}(T)\mathfrak{z}_0)\|^2+\frac{4}{\varepsilon^2}M_S^4M_{\mathfrak{B}}^2\mathbb{E}\int_0^\tau\|\varphi(\theta)\|^2d\theta\\
&+\frac{4}{\varepsilon^2}M_S^4M_{\mathfrak{B}}^2T\mathbb{E}\int_0^\tau\|\mathfrak{f}(\theta,\mathfrak{z}(\theta),\mathfrak{u}(\theta))\|^2d\theta+\frac{4}{\varepsilon^2}M_S^4M_{\mathfrak{B}}^2\mathbb{E}\int_0^\tau\|\mathfrak{g}(\theta,\mathfrak{z}(\theta),\mathfrak{u}(\theta))\|^2d\theta
\end{aligned}
$$

$$
\begin{aligned}
&+4M_S^2M_{\mathfrak{B}}^2T\mathbb{E}\int_0^\tau\left(\frac{4}{\varepsilon^2}M_S^2M_{\mathfrak{B}}^2\mathbb{E}\|(\mathbb{E}h-\mathfrak{T}(T)\mathfrak{z}_0)\|^2+\frac{4}{\varepsilon^2}M_S^4M_{\mathfrak{B}}^2\mathbb{E}\int_0^r\|\varphi(\theta)\|^2d\theta\right)dr\\
&+4M_S^2M_{\mathfrak{B}}^2T\mathbb{E}\int_0^\tau\left(\frac{4}{\varepsilon^2}M_S^4M_{\mathfrak{B}}^2T\mathbb{E}\int_0^r\|\mathfrak{f}(\theta,\mathfrak{z}(\theta),\mathfrak{u}(\theta))\|^2d\theta\right)dr\\
&+4M_S^2M_{\mathfrak{B}}^2T\mathbb{E}\int_0^\tau\left(\frac{4}{\varepsilon^2}M_S^4M_{\mathfrak{B}}^2\mathbb{E}\int_0^r\|\mathfrak{g}(\theta,\mathfrak{z}(\theta),\mathfrak{u}(\theta))\|_{L_2^0}^2d\theta\right)dr\\
&\le k_1(\varepsilon)+k_2(\varepsilon)\int_0^\tau F_1\left(\theta,\sup_{0\le r\le\theta}\mathbb{E}\|\mathfrak{z}(r)\|^2\right)d\theta+k_2(\varepsilon)\int_0^\tau F_2\left(\theta,\sup_{0\le r\le\theta}\mathbb{E}\|\mathfrak{u}(r)\|^2\right)d\theta,
\end{aligned}
$$

where

$$
\begin{aligned}
k_1(\varepsilon) :=\ & 4M_S^2\|\mathfrak{z}_0\|^2+\frac{4}{\varepsilon^2}M_S^2M_{\mathfrak{B}}^2\mathbb{E}\|(\mathbb{E}h-\mathfrak{T}(T)\mathfrak{z}_0)\|^2+\frac{4}{\varepsilon^2}M_S^4M_{\mathfrak{B}}^2\mathbb{E}\int_0^\tau\|\varphi(\theta)\|^2d\theta\\
&+4M_S^2M_{\mathfrak{B}}^2T\mathbb{E}\int_0^\tau\left(\frac{4}{\varepsilon^2}M_S^2M_{\mathfrak{B}}^2\mathbb{E}\|(\mathbb{E}h-\mathfrak{T}(T)\mathfrak{z}_0)\|^2+\frac{4}{\varepsilon^2}M_S^4M_{\mathfrak{B}}^2\mathbb{E}\int_0^r\|\varphi(\theta)\|^2d\theta\right)dr,
\end{aligned}
$$

$$
k_2(\varepsilon) := \left(4M_S^2M_{\mathfrak{B}}^2+\frac{4}{\varepsilon^2}M_S^4M_{\mathfrak{B}}^2+\frac{16}{\varepsilon^2}M_S^6M_{\mathfrak{B}}^4\right)\max(T,1).
$$

□

Lemma 4. *Under assumptions* (H_1)–(H_4)*, the operator* $\mathfrak{C}_\varepsilon$ *is well defined and there exist positive numbers* $L(\varepsilon), k_1(\varepsilon), k_2(\varepsilon)$ *such that for* $(\mathfrak{z}_1,\mathfrak{u}_1), (\mathfrak{z}_2,\mathfrak{u}_2)\in\mathfrak{Z}_T\times\mathfrak{U}_T$*, then*

$$
\begin{aligned}
&\|\mathfrak{C}_\varepsilon(\mathfrak{z}_1,\mathfrak{u}_1)-\mathfrak{C}_\varepsilon(\mathfrak{z}_2,\mathfrak{u}_2)\|_\tau^2\\
&\le L(\varepsilon)\int_0^\tau H_1\left(\theta,\sup_{0\le r\le\theta}\mathbb{E}\|\mathfrak{z}_1(r)-\mathfrak{z}_2(r)\|^2\right)d\theta\\
&+L(\varepsilon)\int_0^\tau H_2\left(\theta,\sup_{0\le r\le\theta}\mathbb{E}\|\mathfrak{u}_1(r)-\mathfrak{u}_2(r)\|^2\right)d\theta,
\end{aligned}
$$

Proof. It is clear that

$$
\begin{aligned}
&\|\mathfrak{C}_\varepsilon(\mathfrak{z}_1,\mathfrak{u}_1)-\mathfrak{C}_\varepsilon(\mathfrak{z}_2,\mathfrak{u}_2)\|_\tau^2\\
&=\sup_{0\le\theta\le\tau}\mathbb{E}\|z_1(\theta)-z_2(\theta)\|^2+\sup_{0\le\theta\le\tau}\mathbb{E}\|v_1(\theta)-v_2(\theta)\|^2
\end{aligned}
$$

Firstly, we estimate $\sup_{0\le\theta\le\tau}\mathbb{E}\|v_1(\theta)-v_2(\theta)\|^2$ as follows.

$$
\begin{aligned}
&\mathbb{E}\|v_1(\tau)-v_2(\tau)\|^2=\mathbb{E}\|v(\tau,\mathfrak{z}_1,\mathfrak{u}_1)-v(\tau,\mathfrak{z}_2,\mathfrak{u}_2)\|^2\\
&\leq 2\mathbb{E}\left\|\mathfrak{B}^*\mathfrak{T}^*(T-\tau)\int_0^\tau\Big(\varepsilon(I-\pi)+\Gamma_\theta^T\Big)^{-1}\mathfrak{T}(T-\theta)[\mathfrak{f}(\theta,\mathfrak{z}_1(\theta),\mathfrak{u}_1(\theta))-\mathfrak{f}(\theta,\mathfrak{z}_2(\theta),\mathfrak{u}_2(\theta))]d\theta\right\|^2\\
&+2\mathbb{E}\left\|\mathfrak{B}^*\mathfrak{T}^*(T-\tau)\int_0^\tau\Big(\varepsilon(I-\pi)+\Gamma_\theta^T\Big)^{-1}\mathfrak{T}(T-\theta)[\mathfrak{g}(\theta,\mathfrak{z}_1(\theta),\mathfrak{u}_1(\theta))-\mathfrak{g}(\theta,\mathfrak{z}_2(\theta),\mathfrak{u}_2(\theta))]dw(\theta)\right\|^2\\
&\leq\frac{1}{\varepsilon^2}2M_S^2M_{\mathfrak{B}}^2\left\{\mathbb{E}\left\|\int_0^\tau\mathfrak{T}(T-\theta)[\mathfrak{f}(\theta,\mathfrak{z}_1(\theta),\mathfrak{u}_1(\theta))-\mathfrak{f}(\theta,\mathfrak{z}_2(\theta),\mathfrak{u}_2(\theta))]d\theta\right\|^2\right.\\
&\left.+\mathbb{E}\left\|\int_0^\tau\mathfrak{T}(T-\theta)[\mathfrak{g}(\theta,\mathfrak{z}_1(\theta),\mathfrak{u}_1(\theta))-\mathfrak{g}(\theta,\mathfrak{z}_2(\theta),\mathfrak{u}_2(\theta))]dw(\theta)\right\|^2\right\}\\
&\leq\frac{1}{\varepsilon^2}2M_S^2M_{\mathfrak{B}}^2\left\{\mathbb{E}\left\|\int_0^\tau\mathfrak{T}(T-\theta)[\mathfrak{f}(\theta,\mathfrak{z}_1(\theta),\mathfrak{u}_1(\theta))-\mathfrak{f}(\theta,\mathfrak{z}_2(\theta),\mathfrak{u}_2(\theta))]d\theta\right\|^2\right.\\
&\left.+\mathbb{E}\int_0^\tau\|\mathfrak{T}(T-\theta)\|^2\|\mathfrak{g}(\theta,\mathfrak{z}_1(\theta),\mathfrak{u}_1(\theta))-\mathfrak{g}(\theta,\mathfrak{z}_2(\theta),\mathfrak{u}_2(\theta))\|_{L_2^0}^2d\theta\right\}\\
&\leq\frac{1}{\varepsilon^2}2M_S^2M_{\mathfrak{B}}^2\left\{M_S^2T\mathbb{E}\int_0^\tau\|\mathfrak{f}(\theta,\mathfrak{z}_1(\theta),\mathfrak{u}_1(\theta))-\mathfrak{f}(\theta,\mathfrak{z}_2(\theta),\mathfrak{u}_2(\theta))\|^2d\theta\right.\\
&\left.+M_S^2\mathbb{E}\int_0^\tau\|\mathfrak{g}(\theta,\mathfrak{z}_1(\theta),\mathfrak{u}_1(\theta))-\mathfrak{g}(\theta,\mathfrak{z}_2(\theta),\mathfrak{u}_2(\theta))\|_{L_2^0}^2d\theta\right\}.
\end{aligned}\tag{15}
$$

Using assumption (H_4), we obtain

$$
\mathbb{E}\|v(\tau,\mathfrak{z}_1,\mathfrak{u}_1)-v(\tau,\mathfrak{z}_2,\mathfrak{u}_2)\|^2\leq c\int_0^\tau H_1\Big(\tau,\mathbb{E}\|\mathfrak{z}_1(\theta)-\mathfrak{z}_2(\theta)\|^2\Big)d\theta+c\int_0^\tau H_2\Big(\tau,\mathbb{E}\|\mathfrak{u}_1(\theta)-\mathfrak{u}_2(\theta)\|^2\Big)d\theta,
$$

where

$$
c=\frac{1}{\varepsilon^2}2M_S^2M_{\mathfrak{B}}^2\Big[M_S^2T+M_S^2\Big].
$$

Next, we estimate $\sup_{0\leq\theta\leq\tau}\mathbb{E}\|z_1(\theta)-z_2(\theta)\|^2$:

$$
\begin{aligned}
&\mathbb{E}\|z_1(\tau)-z_2(\tau)\|^2\\
&\leq 3M_\theta^2M_{\mathfrak{B}}^2T\mathbb{E}\int_0\|v_1(\theta)-v_2(\theta)\|^2d\theta\\
&+3M_S^2T\mathbb{E}\int_0^\tau\|\mathfrak{f}(\theta,\mathfrak{z}_1(\theta),\mathfrak{u}_1(\theta))-\mathfrak{f}(\theta,\mathfrak{z}_2(\theta),\mathfrak{u}_2(\theta))\|^2d\theta\\
&+3M_S^2\mathbb{E}\int_0^\tau\|\mathfrak{g}(\theta,\mathfrak{z}_1(\theta),\mathfrak{u}_1(\theta))-\mathfrak{g}(\theta,\mathfrak{z}_2(\theta),\mathfrak{u}_2(\theta))\|_{L_2^0}^2d\theta
\end{aligned}\tag{16}
$$

Combining inequalities (15) and (16), we obtain

$$
\begin{aligned}
&\|\mathfrak{C}_\varepsilon(\mathfrak{z}_1,\mathfrak{u}_1)-\mathfrak{C}_\varepsilon(\mathfrak{z}_2,\mathfrak{u}_2)\|_\tau^2\\
&\le 3M_S^2T\mathbb{E}\int_0^\tau\|\mathfrak{f}(\theta,\mathfrak{z}_1(\theta),\mathfrak{u}_1(\theta))-\mathfrak{f}(\theta,\mathfrak{z}_2(\theta),\mathfrak{u}_2(\theta))\|^2d\theta\\
&+3M_S^2\mathbb{E}\int_0^\tau\|\mathfrak{g}(\theta,\mathfrak{z}_1(\theta),\mathfrak{u}_1(\theta))-\mathfrak{g}(\theta,\mathfrak{z}_2(\theta),\mathfrak{u}_2(\theta))\|_{L_2^0}^2d\theta\\
&+\frac{1}{\varepsilon^2}6M_S^6M_{\mathfrak{B}}^4T^2\mathbb{E}\int_0^\tau\|\mathfrak{f}(\theta,\mathfrak{z}_1(\theta),\mathfrak{u}_1(\theta))-\mathfrak{f}(\theta,\mathfrak{z}_2(\theta),\mathfrak{u}_2(\theta))\|^2d\theta\\
&+\frac{1}{\varepsilon^2}6M_S^6M_{\mathfrak{B}}^4\mathbb{E}\int_0^\tau\|\mathfrak{g}(\theta,\mathfrak{z}_1(\theta),\mathfrak{u}_1(\theta))-\mathfrak{g}(\theta,\mathfrak{z}_2(\theta),\mathfrak{u}_2(\theta))\|_{L_2^0}^2d\theta\\
&\le L(\varepsilon)\int_0^\tau H_1\left(\theta,\sup_{0\le r\le\theta}\mathbb{E}\|\mathfrak{z}_1(r)-\mathfrak{z}_2(r)\|^2\right)d\theta\\
&+L(\varepsilon)\int_0^\tau H_2\left(\theta,\sup_{0\le r\le\theta}\mathbb{E}\|\mathfrak{z}_1(r)-\mathfrak{z}_2(r)\|^2\right)d\theta.
\end{aligned}
$$

□

Lemma 5. *Under assumptions (H_1)-(H_4), the sequence $\{(\mathfrak{z}_n,\mathfrak{u}_n):n\ge 0\}$ is bounded in $\mathfrak{Z}_T\times\mathfrak{U}_T$.*

Proof. By Lemma 3, for any $n\ge 0$, we have

$$
\begin{aligned}
\|(\mathfrak{z}_{n+1},\mathfrak{u}_{n+1})\|_\tau^2&=\|\mathfrak{C}_\varepsilon(\mathfrak{z}_n,\mathfrak{u}_n)\|_\tau^2\\
&\le k_1(\varepsilon)+k_2(\varepsilon)\int_0^\tau F_1\left(\theta,\sup_{0\le r\le\theta}\mathbb{E}\|\mathfrak{z}_n(r)\|^2\right)d\theta\\
&+k_2(\varepsilon)\int_0^\tau F_2\left(\theta,\sup_{0\le r\le\theta}\mathbb{E}\|\mathfrak{u}_n(r)\|^2\right)d\theta.
\end{aligned}
$$

where k_1,k_2 are constants independent of n. Let $p(\tau)$ be a global solution of the equation

$$p(\tau)=k_0+k_2\int_0^\tau(F_1(\theta,p(\theta))+F_2(\theta,p(\theta)))d\theta$$

with an initial condition $k_0>\max\left(k_1,\|\mathfrak{z}_0\|_T^2\right)$. We will prove by mathematical induction that

$$\|(\mathfrak{z}_n,\mathfrak{u}_n)\|_\tau^2\le p(\tau),\quad \tau\in[0,T]. \tag{17}$$

For $n=0$ inequality (17) holds by definition of p. Suppose that

$$\|(\mathfrak{z}_m,\mathfrak{u}_m)\|_\tau^2\le p(\tau),\quad \tau\in[0,T],\ m\ge 0. \tag{18}$$

Then, by inequality (18) we obtain that

$$
\begin{aligned}
p(\tau)-\|(\mathfrak{z}_m,\mathfrak{u}_m)\|_\tau^2&\ge k_2\int_0^\tau\left(F_1(\theta,p(\theta))-F\left(\theta,\|\mathfrak{z}_m\|_\theta^2\right)\right)d\theta\\
&+k_2\int_0^\tau\left(F_2(\theta,p(\theta))-F_2\left(\theta,\|\mathfrak{z}_m\|_\theta^2\right)\right)d\theta\ge 0.
\end{aligned}
$$

It follows that $\{(\mathfrak{z}_n,\mathfrak{u}_n):n\ge 0\}$ is bounded in $\mathfrak{Z}_T\times\mathfrak{U}_T$:

$$\sup_{n\ge 0}\|(\mathfrak{z}_n,\mathfrak{u}_n)\|_T^2\le p(T). \tag{19}$$

□

Lemma 6. *Under assumptions* (H_1)*-*(H_4)*, the sequence* $\{(\mathfrak{z}_n, \mathfrak{u}_n) : n \geq 0\}$ *is a Cauchy sequence in* $\mathfrak{Z}_T \times \mathfrak{U}_T$.

Proof. Define

$$p_n(\tau) = \sup_{m \geq n} \|\mathfrak{z}_m - \mathfrak{z}_n\|_\tau^2,$$
$$q_n(\tau) = \sup_{m \geq n} \|\mathfrak{u}_m - \mathfrak{u}_n\|_\tau^2,$$
$$r_n(\tau) = \sup_{m \geq n} \|(\mathfrak{z}_m, \mathfrak{u}_m) - (\mathfrak{z}_n, \mathfrak{u}_n)\|_\tau^2$$

The functions p_n, q_n, r_n are well defined, uniformly bounded and evidently nondecreasing. Then, there exist nondecreasing functions $p(\tau), q(\tau), r(\tau)$ such that

$$\lim_{n\to\infty} p_n(\tau) = p(\tau), \quad \lim_{n\to\infty} q_n(\tau) = q(\tau), \quad \lim_{n\to\infty} r_n(\tau) = r(\tau).$$

By Lemma 4, we obtain that

$$\begin{aligned}
&\|\mathfrak{C}_\varepsilon(\mathfrak{z}_m, \mathfrak{u}_m) - \mathfrak{C}_\varepsilon(\mathfrak{z}_n, \mathfrak{u}_n)\|_\tau^2 \\
&\leq L(\varepsilon) \int_0^\tau H_1\left(\theta, \sup_{0\leq r\leq\theta} \mathbb{E}\|\mathfrak{z}_{m-1}(r) - \mathfrak{z}_{n-1}(r)\|^2\right) d\theta \\
&+ L(\varepsilon) \int_0^\tau H_2\left(\theta, \sup_{0\leq r\leq\theta} \mathbb{E}\|\mathfrak{u}_{m-1}(r) - \mathfrak{u}_{n-1}(r)\|^2\right) d\theta
\end{aligned}$$

from which in turn it follows that

$$\begin{aligned}
r(\tau) &\leq r_n(\tau) = p_n(\tau) + q_n(\tau) \\
&\leq L(\varepsilon) \int_0^\tau H_1\left(\theta, \sup_{0\leq r\leq\theta} \mathbb{E}\|\mathfrak{z}_{m-1}(r) - \mathfrak{z}_{n-1}(r)\|^2\right) d\theta \\
&+ L(\varepsilon) \int_0^\tau H_2\left(\theta, \sup_{0\leq r\leq\theta} \mathbb{E}\|\mathfrak{u}_{m-1}(r) - \mathfrak{u}_{n-1}(r)\|^2\right) d\theta \\
&= L(\varepsilon) \int_0^\tau H_1(\theta, p_{n-1}(\theta)) d\theta + L(\varepsilon) \int_0^\tau \mathfrak{X}_2(\theta, q_{n-1}(\theta)) d\theta.
\end{aligned}$$

By the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
r(\tau) &\leq p(\tau) + q(\tau) \\
&\leq L(\varepsilon) \int_0^\tau H_1(\theta, p(\theta)) d\theta + L(\varepsilon) \int_0^\tau H_2(\theta, q(\theta)) d\theta.
\end{aligned}$$

If follows that

$$\begin{gathered}
p(\tau) + q(\tau) \\
\leq L(\varepsilon) \int_0^\tau H_1(\theta, p(\theta) + q(\theta)) d\theta \\
+ L(\varepsilon) \int_0^\tau H_2(\theta, p(\theta) + q(\theta)) d\theta, \\
p(0) + q(0) = 0.
\end{gathered}$$

By the Bihari inequality, it follows that $p(\tau) + q(\tau) = 0$. However,

$$\begin{aligned}\|(\mathfrak{z}_m, \mathfrak{u}_m) - (\mathfrak{z}_n, \mathfrak{u}_n)\|_\tau^2 &\leq p_n(T) + q_n(T) \\ &\to p(T) + q(T) = 0.\end{aligned}$$

Therefore

$$\lim_{n,m\to\infty} \|(\mathfrak{z}_m, \mathfrak{u}_m) - (\mathfrak{z}_n, \mathfrak{u}_n)\|_\tau^2 = 0.$$

□

Theorem 3. *Under assumptions* (H_1)–(H_4)*, the operator* $\mathfrak{C}_\varepsilon$ *has a unique fixed point in* $\mathfrak{Z}_T \times \mathfrak{U}_T$.

Proof. By Lemma 6, $(\mathfrak{z}_n, \mathfrak{u}_n)$ is a Cauchy sequence in $\mathfrak{Z}_T \times \mathfrak{U}_T$. The completeness of $\mathfrak{Z}_T \times \mathfrak{U}_T$ implies the existence of a process $(\mathfrak{z}, \mathfrak{u}) \in \mathfrak{Z}_T \times \mathfrak{U}_T$ such that

$$\lim_{n\to\infty} \|(\mathfrak{z}, \mathfrak{u}) - (\mathfrak{z}_n, \mathfrak{u}_n)\|_T^2 = 0.$$

Taking the limit

$$(\mathfrak{z}, \mathfrak{u}) = \lim_{n\to\infty} (\mathfrak{z}_{n+1}, \mathfrak{u}_{n+1}) = \lim_{n\to\infty} \mathfrak{C}_\varepsilon(\mathfrak{z}_n, \mathfrak{u}_n) = \mathfrak{C}_\varepsilon(\mathfrak{z}, \mathfrak{u}),$$

we see that $(\mathfrak{z}, \mathfrak{u})$ is a fixed point of $\mathfrak{C}_\varepsilon$.

Further, if $(\mathfrak{z}_1, \mathfrak{u}_1), (\mathfrak{z}_2, \mathfrak{u}_2) \in \mathfrak{Z}_T \times \mathfrak{U}_T$ are two fixed points of $\mathfrak{C}_\varepsilon$, then Lemma 4 would imply that

$$\begin{aligned}&\|\mathfrak{C}_\varepsilon(\mathfrak{z}_1, \mathfrak{u}_1) - \mathfrak{C}_\varepsilon(\mathfrak{z}_2, \mathfrak{u}_2)\|_\tau^2 \\ &\leq L(\varepsilon) \int_0^\tau H_1\left(\theta, \sup_{0\leq r\leq\theta} \mathbb{E}\|\mathfrak{z}_1(r) - \mathfrak{z}_2(r)\|^2\right) d\theta \\ &+ L(\varepsilon) \int_0^\tau H_2\left(\theta, \sup_{0\leq r\leq\theta} \mathbb{E}\|\mathfrak{u}_1(r) - \mathfrak{u}_2(r)\|^2\right) d\theta.\end{aligned}$$

So, as in the proof of Lemma 6, we obtain that

$$\|\mathfrak{C}_\varepsilon(\mathfrak{z}_1, \mathfrak{u}_1) - \mathfrak{C}_\varepsilon(\mathfrak{z}_2, \mathfrak{u}_2)\|_T^2 = 0.$$

Therefore, $(\mathfrak{z}_1, \mathfrak{u}_1) = (\mathfrak{z}_2, \mathfrak{u}_2)$ and $\mathfrak{C}_\varepsilon$ has a unique fixed point in $\mathfrak{Z}_T \times \mathfrak{U}_T$. □

Theorem 4. *Let assumptions* (H_1)–(H_4) *and (AC) hold. Assume that the operator* $\mathfrak{T}(\tau)$, $\tau > 0$ *is compact and analytic. Moreover, suppose the functions* $\mathfrak{f}$ *and* $\mathfrak{g}$ *are uniformly bounded. Then, system (1) is mean square finite-approximately controllable on* $[0, T]$.

Proof. Let $(\mathfrak{z}^\varepsilon, \mathfrak{u}^\varepsilon)$ be a fixed point of $\mathfrak{C}_\varepsilon$ in $\mathfrak{Z}_T \times \mathfrak{U}_T$. Then

$$\begin{aligned}&\mathfrak{z}^\varepsilon(T) - h = \\ &\quad -\varepsilon(I-\pi)\left(\varepsilon(I-\pi) + \Gamma_0^T\right)^{-1}(\mathbb{E}h - \mathfrak{T}(T)\mathfrak{z}_0) \\ &\quad + \varepsilon(I-\pi)\int_0^T \left(\varepsilon(I-\pi) + \Gamma_\theta^T\right)^{-1} \mathfrak{T}(T-\theta)\mathfrak{f}(\theta, \mathfrak{z}^\varepsilon(\theta), \mathfrak{u}^\varepsilon(\theta))d\theta \\ &\quad + \varepsilon(I-\pi)\int_0^T \left(\varepsilon(I-\pi) + \Gamma_\theta^T\right)^{-1} [\mathfrak{T}(T-\theta)\mathfrak{g}(\theta, \mathfrak{z}^\varepsilon(\theta), \mathfrak{u}^\varepsilon(\theta)) - \varphi(\theta)]dw(\theta). \quad (20)\end{aligned}$$

Since the functions $\mathfrak{f}$ and $\mathfrak{g}$ are uniformly bounded, there exists a constant $L > 0$ such that

$$\|\mathfrak{f}(\theta, \mathfrak{z}^\varepsilon(\theta), \mathfrak{u}^\varepsilon(\theta))\| + \|\mathfrak{g}(\theta, \mathfrak{z}^\varepsilon(\theta), \mathfrak{u}^\varepsilon(\theta))\|_{L_2^0} \leq L.$$

Then, there exists a subsequence still denoted by $\{\mathfrak{f}(\theta,\mathfrak{z}^{\varepsilon}(\theta),\mathfrak{u}^{\varepsilon}(\theta)),\mathfrak{g}(\theta,\mathfrak{z}^{\varepsilon}(\theta),\mathfrak{u}^{\varepsilon}(\theta))\}$ which converges weakly to say $(\mathfrak{f},\mathfrak{g}) \in \mathfrak{X} \times L_2^0$. Now, due to compactness, $\mathfrak{T}(\theta),\theta > 0$, it follows that

$$\mathfrak{T}(T-\theta)\mathfrak{f}(\theta,\mathfrak{z}^{\varepsilon}(\theta),\mathfrak{u}^{\varepsilon}(\theta)) \to \mathfrak{T}(T-\theta)\mathfrak{f}(\theta),$$
$$\mathfrak{T}(T-\theta)\mathfrak{g}(\theta,\mathfrak{z}^{\varepsilon}(\theta),\mathfrak{u}^{\varepsilon}(\theta)) \to \mathfrak{T}(T-\theta)\mathfrak{g}(\theta),$$

in $[0,T] \times \Omega$. From equation (20), we have

$$\begin{aligned}
&\mathbb{E}\|\mathfrak{z}^{\varepsilon}(T)-h\|^2 \\
&\leq 6\mathbb{E}\left\|\varepsilon(I-\pi)\left(\varepsilon(I-\pi)+\Gamma_0^T\right)^{-1}(\mathbb{E}h-\mathfrak{T}(T)\mathfrak{z}_0)\right\|^2 \\
&+6\mathbb{E}\int_0^T\left\|\varepsilon(I-\pi)\left(\varepsilon(I-\pi)+\Gamma_0^T\right)^{-1}\varphi(\theta)\right\|_{L_2^0}^2 d\theta \\
&+6\mathbb{E}\left(\int_0^T\left\|\varepsilon(I-\pi)\left(\varepsilon(I-\pi)+\Gamma_\theta^T\right)^{-1}\right\|\|\mathfrak{T}(T-\theta)[\mathfrak{f}(\theta,\mathfrak{z}^{\varepsilon}(\theta),\mathfrak{u}^{\varepsilon}(\theta))-\mathfrak{f}(\theta)]\|d\theta\right)^2 \\
&+6\mathbb{E}\left(\int_0^T\left\|\varepsilon(I-\pi)\left(\varepsilon(I-\pi)+\Gamma_\theta^T\right)^{-1}\right\|\|\mathfrak{f}(\theta)\|d\theta\right)^2 \\
&+6\mathbb{E}\int_0^T\left\|\varepsilon(I-\pi)\left(\varepsilon(I-\pi)+\Gamma_\theta^T\right)^{-1}\right\|^2\|\mathfrak{T}(T-\theta)[\mathfrak{g}(\theta,\mathfrak{z}^{\varepsilon}(\theta),\mathfrak{u}^{\varepsilon}(\theta))-\mathfrak{g}(\theta)]\|_{L_2^0}^2 d\theta \\
&+6\mathbb{E}\int_0^T\left\|\varepsilon(I-\pi)\left(\varepsilon(I-\pi)+\Gamma_\theta^T\right)^{-1}\right\|^2\|\mathfrak{g}(\theta)\|_{L_2^0}^2 d\theta.
\end{aligned}$$

On the other hand, $\varepsilon(I-\pi)\left(\varepsilon(I-\pi)+\Gamma_\theta^T\right)^{-1} \to 0$ strongly as $\varepsilon \to 0^+$ and $\left\|\varepsilon(I-\pi)\left(\varepsilon(I-\pi)+\Gamma_\theta^T\right)^{-1}\right\|^2 \leq 1$. Therefore, by the Lebesgue dominated convergence theorem, we can easily obtain that $\mathbb{E}\|\mathfrak{z}^{\varepsilon}(T)-h\|^2 \to 0$ as $\varepsilon \to 0^+$. This implies the approximate controllability in the mean square of system (1). Mean exact finite-dimensional controllability follows from Equation (20):

$$\pi(\mathbb{E}\mathfrak{z}^{\varepsilon}(T)-\mathbb{E}h)=0.$$

□

5. Applications

Example 1. We consider a system governed by the semilinear heat equation with lumped control

$$\begin{cases} \frac{\partial}{\partial\tau}y(\tau,\theta) = \dfrac{\partial^2 y(\tau,\theta)}{\partial\theta^2} + \chi_{(\alpha_1,\alpha_2)}(\theta)u(\tau), \\ y(\tau,0) = y(\tau,\pi) = 0, \quad 0<\tau<T, \\ y(0,\theta) = y_0(\theta), \quad 0 \leq \theta \leq \pi, \end{cases} \tag{21}$$

where $\chi_{(\alpha_1,\alpha_2)}(\theta)$ is the characteristic function of $(\alpha_1,\alpha_2) \subset (0,\pi)$. Let $\mathfrak{X} = L^2[0,\pi]$, $\mathfrak{U} = R$ and $\mathfrak{A} = d^2/d\theta^2$ with $D(\mathfrak{A}) = H_0^1[0,\pi] \cap H^2[0,\pi]$. We define the bounded linear operator $B: R \to L^2[0,\pi]$ by $(Bu)(\tau) = \chi_{(\alpha_1,\alpha_2)}(\theta)u()$ and the nonlinear operator f is assumed to be bounded.

Set $M = L_K^2[0,\pi] := \left\{\varphi : \varphi(\theta) = \sum_{i=1}^{K} \alpha_i e_i(\theta),\ \alpha_i \in R\right\}$ and denote by π the operator of the orthogonal projection $L^2[0,\pi]$ onto $L_K^2[0,\pi]$. $\mathfrak{A}$ generates a compact analytic semigroup $\mathfrak{T}(\tau)$ which is defined as follows.

$$\mathfrak{T}(\tau)h = \sum_{n=1}^{\infty} \exp\left(-n^2\pi^2\tau\right)\langle h, e_n\rangle e_n,$$
$$L_s^T u = \int_s^T \mathfrak{T}(T-\tau)(Bu)(\tau)d\tau = \sum_{n=1}^{\infty}\int_s^T \exp(-\lambda_n(T-\tau))\left\langle \chi_{(\alpha_1,\alpha_2)}(\theta), e_n\right\rangle u(\tau)d\tau e_n,$$
$$\left(L_s^T\right)^* h = \sum_{n=1}^{\infty}\int_s^T \exp(-\lambda_n(T-\tau))\left\langle \chi_{(\alpha_1,\alpha_2)}(\theta), e_n\right\rangle\langle h, e_n\rangle d\tau,$$
$$\Gamma_s^T h = L_s^T\left(L_s^T\right)^* h = \sum_{n=1}^{\infty}\int_s^T \exp(-2\lambda_n(T-\tau)^{\alpha})d\tau\left\langle \chi_{(\alpha_1,\alpha_2)}(\theta), e_n\right\rangle^2\langle h, e_n\rangle e_n,$$

where $e_n(\theta) = \sqrt{\frac{2}{\pi}}\sin(n\theta)$, $n = 1, 2, \ldots$ is a complete orthonormal set of eigen vectors of $\mathfrak{A}$. Subsequently, we attain

$$\begin{aligned}\left(\varepsilon(I-\pi_M) + \Gamma_s^T\right)^{-1} g &= \sum_{n=1}^{\infty} \frac{1}{\left(\varepsilon(I-\pi_M) + \int_s^T \exp(-2\lambda_n(T-\tau))d\left\langle \chi_{(\alpha_1,\alpha_2)}(\theta), e_n\right\rangle^2\right)}\langle g, e_n\rangle e_n \\ &= \sum_{n=1}^{K} \frac{1}{\int_s^T \exp(-2\lambda_n(T-))d\tau\left\langle \chi_{(\alpha_1,\alpha_2)}(\theta), e_n\right\rangle^2}\langle g, e_n\rangle e_n \\ &+ \sum_{n=K+1}^{\infty} \frac{1}{\left(\varepsilon + \int_s^T \exp(-2\lambda_n(T-\tau))d\tau\left\langle \chi_{(\alpha_1,\alpha_2)}(\theta), e_n\right\rangle^2\right)}\langle g, e_n\rangle e_n\end{aligned}$$

It is clear that $\varepsilon\left(\varepsilon(I-\pi_M) + \Gamma_0^T\right)^{-1} g \to 0$ as $\varepsilon \to 0^+$ if $\left\langle \chi_{(\alpha_1,\alpha_2)}(\theta), e_n\right\rangle = \int_{\alpha_1}^{\alpha_2} \sqrt{2}\sin(n\pi\theta)d\theta = -\frac{\sqrt{2}}{n\pi}\cos(n\pi\theta)\mid_{\alpha_1}^{\alpha_2} \neq 0$, which holds whenever $\alpha_1 \pm \alpha_2$ is an irrational number.

If $\alpha_1 \pm \alpha_2$ is an irrational number, then the linear determinisitic system (21) is finite-approximately controllable on every $[s,T]$, $0 \le s < T$. By Theorem 2, the following linear stochastic system is mean square finite-approximately controllable on $[0,T]$.

$$\begin{cases} \frac{\partial}{\partial\tau}y(\tau,\theta) = \dfrac{\partial^2 y(\tau,\theta)}{\partial\theta^2} + \chi_{(\alpha_1,\alpha_2)}(\theta)u(\tau) + \sigma()dw(\tau), \\ y(\tau,0) = y(\tau,\pi) = 0, \quad 0 < \tau < T, \\ y(0,\theta) = y_0(\theta), \quad 0 \le \theta \le \pi, \end{cases}$$

where $w(\tau)$ denotes a standard real valued Wiener process, $\sigma \in \chi_2(0,T;R)$.

Example 2. Consider the following stochastic partial differential equation:

$$\begin{cases} \mathfrak{z}_\tau(\tau,\theta) = \mathfrak{z}_{\theta\theta}(\tau,\theta) + \mathfrak{B}\mu(\tau,\theta) + K_1(\tau,\mathfrak{z}(\tau,\theta)) + K_2(\tau,\mathfrak{z}(\tau,\theta))dw(\tau), \quad (\tau,\theta) \in [0,T]\times[0,\pi], \\ \mathfrak{z}(\tau,0) = \mathfrak{z}(\tau,\pi) = 0,\ 0 \le \tau \le 1, \quad \mathfrak{z}(0,\theta) = \mathfrak{z}_0(\theta),\ 0 \le \theta \le \pi, \end{cases} \tag{22}$$

where $w(\tau)$ denotes a standard real valued Wiener process on $(\Omega, \mathfrak{F}, \{\mathfrak{F}_\tau\}, P)$ and $\mathfrak{z}_0 \in L^2(0,\pi)$; $\mu : [0,T]\times(0,\pi) \to (0,T)$ is continuous in τ; $K_1, K_2 : R \times R \to R$ are continuous. Let $\mathfrak{X} = \mathfrak{U} = L^2_{\mathfrak{F}}(0,T)$ and define the operator $\mathfrak{A} = d^2/d\theta^2$ with $D(\mathfrak{A}) = H_0^1[0,\pi] \cap H^2[0,\pi]$. Then, $\mathfrak{A}$ generates a compact analytic semigroup $\mathfrak{T}(\tau)$ which is defined as follows

$$\mathfrak{T}(\tau)z = \sum_{n=1}^{\infty} e^{-n^2\tau}(z, e_n)e_n, \quad z \in \mathfrak{X},$$

where $e_n(\theta) = \sqrt{\frac{2}{\pi}} \sin(n\theta)$, $n = 1, 2, \ldots$ is a complete orthonormal set of eigen vectors of $\mathfrak{A}$. From these expressions, it follows that $\{T(\tau), \tau > 0\}$ is a uniformly bounded compact analytic semigroup.

Define an infinite-dimensional control space $\mathfrak{U}$ by $\mathfrak{U} = \{\mathfrak{u} : \mathfrak{u} = \sum_{n=2}^{\infty} \mathfrak{u}_n e_n$, such that $\sum_{n=2}^{\infty} \mathbf{E}\mathfrak{u}_n^2 < \infty\}$ endowed with the norm $\|\mathfrak{u}\|_{\mathfrak{U}} = (\sum_{n=2}^{\infty} \mathbf{E}\mathfrak{u}_n^2)^{1/2}$. Next, define a continuous linear mapping $\mathfrak{B}$ from $\mathfrak{U}$ into $\mathfrak{X}$ as follows

$$\mathfrak{B}\mathfrak{u}(\tau, \theta) = 2\mathfrak{u}_2(\tau)e_1(\theta) + \sum_{n=2}^{\infty} \mathfrak{u}_n(\tau)e_n(\theta) \text{ for } \mathfrak{u} = \sum_{n=2}^{\infty} \mathfrak{u}_n e_n \in \mathfrak{U}.$$

Let $\mathfrak{z}(\tau)(\theta) = \mathfrak{z}(\tau, \theta)$ and define the bounded linear operator $\mathfrak{B} : \mathfrak{U} \to \mathfrak{X}$ by $(\mathfrak{B}\mathfrak{u})(\tau)(\mathfrak{z}) = \mu(\tau, \mathfrak{z}), 0 \leq \mathfrak{z} \leq \pi$, $\mathfrak{f}(\tau, \mathfrak{z})(\cdot) = K_1(\tau, \mathfrak{z}(\cdot))$ and $\mathfrak{g}(\tau, \mathfrak{z})(\cdot) = K_2(\tau, \mathfrak{z}(\cdot))$.

The linear deterministic system that corresponds to (22) is approximately controllable on every $[s, T]$, $0 \leq s < T$ and all conditions of Theorem 4 are satisfied. Hence, by Theorem 4 the stochastic differential system (22) is finite-approximately controllable on $[0, 1]$.

6. Conclusions

The main aim of this work was to present:

- Necessary and sufficient conditions for finite-approximate mean square controllability of linear stochastic evolution systems in infinite-dimensional separable Hilbert spaces in terms of stochastic resolvent-like operators $\left(\varepsilon(I - \pi\mathbb{E}\{\cdot\}) + \Pi_0^T\right)^{-1}$. Moreover, we found an explicit analytical form of the contollability control which, in addition to the mean square approximate controllability property, ensures finite-dimensional mean exact controllability.
- The Picard approximation method to show a mean square finite-approximate controllability of a semilinear stochastic evolution system under non-Lipschitz conditions satisfied by the nonlinear drift and diffusion coefficients depending on control.

One can assume that the results of this work apply to a class of problems determined by various types of first order and second order fractional (impulsive) stochastic evolution systems, such as Caputo SDEs, Riemann–Liouville-type SDEs, Hadamard-type SDEs, Sobolev-type fractional SDEs and so on.

On the other hand, many real-world systems can sometimes experience different types of stochastic perturbations. For example, Poisson jumps are now used to describe various types of real-world systems. In the future, the same approach could be used for different types of systems with different stochastic perturbations.

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References

1. Sunahara, Y.; Aihara, S.; Kishino, K. On the stochastic observability and controllability for nonlinear systems. *Int. J. Control* **1975**, *22*, 65–82. [CrossRef]
2. Klamka, J.; Socha, L. Some remarks about stochastic controllability. *IEEE Trans. Autom. Control* **1977**, *22*, 880–881. [CrossRef]
3. Dubov, M.A.; Mordukhovich, B.S. Theory of controllability of linear stochastic systems. *Differ. Equ.* **1978**, *14*, 1609–1612.

4. Dubov, M.A.; Morduchovich, B.S. On controllability of infinite-dimensional linear stochastic systems. In Proceedings of the 2nd IFAC Symposium, Stochastic Control, Vilnius, Lithuania, 19–23 May; (IFAC Symposia Series); Pergamon Press: Oxford, UK; New York, NY, USA, 1987; Volume 2, pp. 307–310.
5. Chen, H.F. On stochastic observability and controllability. *Automatica* **1980**, *16*, 179–190. [CrossRef]
6. Zabczyk, J. Controllability of stochastic linear systems. Syst. *Control Lett.* **1981**, *1*, 25–31. [CrossRef]
7. Goreac, D. Approximate controllability for linear stochastic differential equations in infinite-dimensions. *Appl. Math. Optim.* **2009**, *60*, 105–132. [CrossRef]
8. Bashirov, A.E.; Mahmudov, N.I. On concepts of controllability for deterministic and stochastic systems. *SIAM J. Control Optim.* **1999**, *37*, 1808–1821. [CrossRef]
9. Mahmudov, N.I. Approximate controllability of semilinear deterministic and stochastic evolution equation in abstract spaces. *SIAM J. Control Optim.* **2003**, *42*, 1604–1622. [CrossRef]
10. Mahmudov, N.I. Mean Exact Finite-Approximate Contollability of Linear Stochastic Equations in Hilbert Spaces. *Filomat* **2022**, *36*, 2043–2050. [CrossRef]
11. Sunahara, Y.; Kishino, K.; Aihara, S. On stochastic observability of nonlinear discrete-time dynamical systems. *Int. J. Control* **1974**, *19*, 719–732. [CrossRef]
12. Sunahara, Y.; Kabeuchi, T.; Asada, Y.; Aihara, S.; Kishino, K. On stochastic controllability for nonlinear systems. *IEEE Trans. Autom. Control* **1974**, *19*, 49–54. [CrossRef]
13. George, R. K. Approximate controllability of semilinear systems using integral contractors. *Numer. Funct. Anal. Optim.* **1995**, *16*, 127–138. [CrossRef]
14. Sakthivel, R.; Kim, J.H. On controllability of nonlinear stochastic systems. *Rep. Math. Phys.* **2006**, *58*, 433–443. [CrossRef]
15. Tang, S.J.; Zhang, X. Null controllability for forward and backward stochastic parabolic equations. *SIAM J. Control Optim.* **2009**, *48*, 2191–2216. [CrossRef]
16. Liu, F.; Peng, S.G. On controllability for stochastic control systems when the cofficient is time-varying. *J. Syst. Sci. Complex.* **2010**, *23*, 270–278. [CrossRef]
17. Liu, Q. Some results on the controllability of forward stochastic heat equations with control on the drift. *J. Funct. Anal.* **2011**, *260*, 832–851. [CrossRef]
18. Sathya, R.; Balach, R.K. Controllability of neutral impulsive Ito type stochastic integrodifferential systems. *Vietnam J. Math.* **2013**, *41*, 59–80. [CrossRef]
19. Ning, H.; Qing, G. Approximate controllability of nonlinear stochastic partial differential systems with infinite delay. *Adv. Differ. Equ.* **2015**, *85*, 1–25. [CrossRef]
20. Gao, P.; Chen, M.; Li, Y. Observability estimates and null controllability for forward and backward linear stochastic Kuramoto-Sivashinsky equations. *SIAM J. Control Optim.* **2015**, *53*, 475–500. [CrossRef]
21. Mokkedem, F.Z.; Fu, X. Approximate controllability for a semilinear stochastic evolution systems with infinite delay in *Lp* space. *Appl. Math. Optim.* **2017**, *75*, 253–283. [CrossRef]
22. Ain, Q.T.; Nadeem, M.; Akgül, A.; De la Sen, M. Controllability of Impulsive Neutral Fractional Stochastic Systems. *Symmetry* **2022**, *14*, 2612. [CrossRef]
23. Anguraj, A.; Ramkumar, K. Approximate controllability of semilinear stochastic integrodifferential system with nonlocal conditions. *Fractal Fract.* **2018**, *29*, 29. [CrossRef]
24. Singh, Vikram; Chaudhary, Renu; Kumar Som, Lalit Approximate controllability of stochastic differential system with non-Lipschitz conditions. *Stoch. Anal. Appl.* **2022**, *40*, 505–519.
25. Ren Y, Dai H, Sakthivel R. () Approximate controllability of stochastic differential systems driven by a Lévy process. *Int. Control.* **2013**, *86*, 1158-1164. [CrossRef]
26. Mahmudov, N.I.; Zorlu, S. Controllability of non-linear stochastic systems. *Int. J. Control* **2003**, *76*, 95–104. [CrossRef]
27. Mahmudov, N.I.: Finite-approximate controllability of semilinear fractional stochastic integro-differential equations. *Chaos Solitons Fractals* **2020**, *139*, 110277 [CrossRef]
28. Da Prato, G.; Zabczyk, J. *Stochastic Equations in Infinite Dimensions*; Cambridge University Press: London, UK, 1992.