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Transcendence and the Expression of the Spectral Series of a Class of Higher Order Differential Operators

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Abstract: In this paper, a relationship between the spectral zeta series of a class of higher order self-adjoint differential operators on the unit circle and the integral of Green functions is established by Mercer's Theorem. Furthermore, the explicit expression and the transcendental nature of the spectral series are obtained by the integral representation. Finally, several applications in physics about differential operators' spectral theory, yellow some further works, and the transcendental nature of some zeta series are listed.

Keywords: self-adjoint differential operators; Mercer's theorem; spectral series; transcendentality; green functions

MSC: 34B24; 34B27; 11M06

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1. Introduction

Let $f(x)$ and $g(x)$ be polynomials in $\overline{\mathbb{Q}}[x]$ with $\deg f < \deg g$ so that $g(x)$ has no integral zeros. Murty and Weatherby [1] and Nesterenko [2] studied the infinite series

$$\sum_{k\in\mathbb{Z}} \frac{f(k)}{g(k)} \tag{1}$$

and related the transcendental nature of the sum to Schneider's conjecture and Gel'fond–Schneider's conjecture. In the case that $f(x) = 1$ and $g(x) = ax^2 + bx + c$, differentiating the series successively with respect to c, Murty and Weatherby [3], ([4], §6) deduced an explicit formula for

$$\sum_{k\in\mathbb{Z}} \frac{1}{(ak^2 + bk + c)^n}, \tag{2}$$

and proved that the sum is transcendental if $a,\ b,\ c\ \in \mathbb{Z}$ and $b^2 - 4ac < 0$.

Saradha and Tigdeman [5] proved that

$$\sum_{k=0}^{+\infty} \frac{(-1)^n(ak+b)}{(qk+s_1)(qk+s_2)},\ a,\ b,\ s_1,\ s_2 \in \mathbb{Z},\ s_1 \neq s_2,$$

with $|a| + |b| > 0$ and $-s_1/q,\ -s_2/q$ never being a non-negative integer, is transcendental except when $s_1 \equiv s_2 \pmod{q}$ and $a = 0$. Moreover, under the similar conditions, Saradha and Tigdeman ([5], Theorem 2) obtained that

$$\sum_{k=0}^{+\infty} \frac{(ak+b)}{(qk+s_1)(qk+s_2)(qk+s_3)}$$

is transcendental.

In [6], Weatherby proved that the sums

$$\sum_{k\in\mathbb{Z}} \frac{1}{(k^4-q^4)^{2n}},\ \sum_{k\in\mathbb{Z}} \frac{1}{(k^6-q^6)^{2n}},\ \sum_{k\in\mathbb{Z}} \frac{1}{(k^3\pm q^3)^{2n}},\ q\in\mathbb{Q}\backslash\mathbb{Z}, \tag{3}$$

are transcendental, and the series

$$\sum_{k\in\mathbb{Z},k\neq\pm 1} \frac{1}{k^n-1}, \tag{4}$$

is transcendental for $n = 3, 4, 6$.

In this paper, thanks to Mercer's Theorem (cf. ([7], §3.5.4), we use the following self-adjoint differential operator of order $m \geq 2$

$$T_m u := (-i)^m u^{(m)} + \alpha u = \lambda u \tag{5}$$

on the circle $\mathbb{S}^1$ to investigate the following spectral series:

$$\sum_{k=-\infty}^{+\infty} \frac{1}{[(2k\pi)^m+\alpha]^n}, \tag{6}$$

and give an explicit formula. When $m = 1$, for any $n \geq 1$, in the paper ([8], Theorem 2.4), the special values of the series

$$\sum_{k=-\infty}^{+\infty} \frac{1}{(2k\pi+\alpha)^n},$$

were studied, and an expression was obtained by the combined method. When $m = 2$, the series (6) is a special case of (2). For higher order $m \geq 3$, the series (3) are special cases of (6); however, the case (6) cannot include the series (4).

The self-adjoint differential operators (5) on $\mathbb{S}^1$ are equivalent to the boundary value problems

$$T_m u = (-i)^m u^{(m)} + \alpha u = \lambda u,\ \text{on } (0,1), \tag{7}$$

with the periodic boundary condition

$$u(0) = u(1), \cdots, u^{(m-1)}(0) = u^{(m-1)}(1),$$

where $\alpha \neq -(2k\pi)^m$, $k = 0, \pm 1, \pm 2, \cdots$. Its k-th eigenvalue is

$$\lambda_k^{(m)} = (2k\pi)^m + \alpha,\ k = \begin{cases} 0, \pm 1, \pm 2, \cdots, & \text{for odd } m; \\ 0, 1, 2, \cdots, & \text{for even } m. \end{cases}$$

In the case that m is even, the eigenvalues $\{\lambda_k^{(m)}\}$ of T_m have lower bounds and tend to infinity as $k \to \infty$,

$$-\infty < \lambda_0^{(m)} < \lambda_1^{(m)} \leq \lambda_2^{(m)} \leq \cdots \leq \lambda_k^{(m)} \to +\infty.$$

In the case that m is odd, the corresponding eigenvalues have neither upper nor lower bounds, and satisfy

$$-\infty \leftarrow \lambda_{-k}^{(m)} \leq \cdots \leq \lambda_{-2}^{(m)} \leq \lambda_{-1}^{(m)} < \lambda_0^{(m)} < \lambda_1^{(m)} \leq \lambda_2^{(m)} \leq \cdots \leq \lambda_k^{(m)} \to +\infty.$$

For any positive integer n, the k-th eigenvalue of T_m^n is $[\lambda_k^{(m)}]^n$. Then, Mercer's Theorem (cf. [7] §3.5.4, [9]) tells us that

$$\begin{aligned}\sum_{k=-\infty}^{+\infty} \frac{1}{[(2k\pi)^m+\alpha]^n} &= \sum_{k=-\infty}^{+\infty} \frac{1}{[\lambda_k^{(m)}]^n} \\ &= \int_0^1 \cdots \int_0^1 G(x_1,x_2)\cdots G(x_n,x_1)\mathrm{d}x_1\cdots \mathrm{d}x_n,\end{aligned} \tag{8}$$

where $G(\cdot,\cdot)$ is the Green function of problem (7).

In this paper, we use differential operators (7) to give an explicit formula for series (8), and study whether the sums of the series

$$\sum_{k=-\infty}^{\infty} \frac{1}{(2^m k^m - q^m)^n} \text{ and } \sum_{k=-\infty}^{\infty} \frac{1}{(2^m k^m + q^m)^n}$$

are transcendental numbers. This series is closely related to the Dirichlet series and L-functions (cf. [4,10,11] §16), which arise out of number theory and other considerations, see Soulé [12] and Ramakrishnan [13].

The rest of this paper is organized as follows: In Section 2, first, some preliminary work is given, including some properties of differential operator spectral theory, Green function, and Mercer's Theorem. Then, a relationship between spectral zeta series and the integral of Green function is established by using Mercer's Theorem. Moreover, the explicit expressions and transcendentality of the spectral series of second and third order differential operators on $\mathbb{S}^1$ are obtained. The main results are given in Section 3. In this section, using the same method in Section 2, we can obtain an integral representation of spectral series of higher order self-adjoint differential operators; see (8). Using the integral representation, we prove that the spectral series is a linear combination of $\{\pi,\cdots,\pi^n\}$. In the last section, we make a summary of the conclusion of this paper and give some applications in physics. Furthermore, according to these applications and the problems discussed in this paper, some possible further work related to the special value and transcendental nature of zeta series is listed.

2. The Second and Third Order Differential Operators

In this section, we consider the second and third order self-adjoint differential operators on a circle $\mathbb{S}^1$. Using Mercer's Theorem, we will calculate the sum of the spectral series from (8),

$$\sum_{k=-\infty}^{\infty} \frac{1}{(2^m k^m \pm q^m)^n},$$

where $m = 2,3$ and n is any positive integer.

2.1. The Second Order Case

In the second order case, the conclusions of the special value and transcendentality of series

$$\sum_{k=-\infty}^{+\infty} \frac{1}{[k^2+\alpha]^n}, \ \alpha \in \mathbb{Q},\ \alpha > 0,$$

are very complete (cf. [3,4,6]). In this subsection, we will consider the case $\alpha < 0$ and show the process of connecting the integral of Green function with the series by using Mercer's Theorem.

As $m = 2$, problem (7) becomes as (cf. [14], (1.2))

$$T_2 u = -u'' + \alpha u = \lambda u, \text{ on } (0,1), \tag{9}$$

with the boundary condition

$$u(0) = u(1),\ u'(0) = u'(1), \tag{10}$$

where $\alpha \neq -(2k\pi)^2$, $k = 0, 1, 2, \cdots$. Then, the k-th eigenvalue is

$$\lambda_k = (2k\pi)^2 + \alpha,\ k = 0, 1, 2, \cdots,$$

and the corresponding eigenfunctions are

$$\varphi_{\pm k}(x) = e^{\pm i2k\pi x} \text{ or } \cos(2k\pi x),\ \sin(2k\pi x).$$

Hence, for $k \geq 1$, the geometric multiplicity of eigenvalue λ_k is 2.

Since we assume that $\alpha \neq -(2k\pi)^2$, $k = 0, 1, \cdots$. We know that 0 is not the eigenvalue of T_2. Hence, T_2^{-1} exists and is a bounded linear operator on $L^2[0, 1]$, and the Green function $G(s, t) = G(t, s)$ of (9) at $\lambda = 0$ is defined as that for any fixed $s \in [0, 1]$, the function $G(s, t)$ satisfies the boundary condition (10) and for any $f \in L^2[0, 1]$,

$$(T_2^{-1} f)(s) =: \int_0^1 G(s, t) f(t) \mathrm{d}t,\ s \in [0, 1]. \tag{11}$$

Here,

$$L^2[0, 1] := \{u \text{ is measurable} : \int_0^1 |u|^2 \mathrm{d}x < \infty\}.$$

The definition (11) is equivalent to for any fixed $s \in [0, 1]$,

$$T_2 G(s, t) = \delta_s(t),$$

where $\delta_s(t)$ is the Delta function at s (cf. [15]). By the definition, we can obtain that the Green function of (9) at 0 is

$$G(s, t) = \frac{1}{2\sqrt{\alpha}} \begin{cases} \frac{e^{\sqrt{\alpha}(t-s)}}{e^{\sqrt{\alpha}}-1} - \frac{e^{-\sqrt{\alpha}(t-s)}}{e^{-\sqrt{\alpha}}-1}, & 0 \leq s \leq t \leq 1; \\ \frac{e^{\sqrt{\alpha}(t-s+1)}}{e^{\sqrt{\alpha}}-1} - \frac{e^{-\sqrt{\alpha}(t-s+1)}}{e^{-\sqrt{\alpha}}-1}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{12}$$

Mercer's Theorem (cf. [7], §3.5.4) tells us that

$$\frac{1}{\alpha} + 2\sum_{k=1}^{\infty} \frac{1}{(2k\pi)^2 + \alpha} = \int_0^1 G(t, t) \mathrm{d}t = \frac{1}{2\sqrt{\alpha}} \left(\frac{e^{\sqrt{\alpha}} + 1}{e^{\sqrt{\alpha}} - 1} \right). \tag{13}$$

In fact, we have

$$G(s, t) = \sum_{k=-\infty}^{+\infty} \frac{\varphi_k(s)\overline{\varphi_k(t)}}{\lambda_k} = \sum_{k=-\infty}^{+\infty} \frac{\varphi_k(s)\overline{\varphi_k(t)}}{(2k\pi)^2 + \alpha} = \sum_{k=-\infty}^{+\infty} \frac{\eta^k(s, t)}{(2k\pi)^2 + \alpha},$$

where $\varphi_k(x) = e^{i2k\pi x}$ and $\eta(s, t) := e^{i2\pi(s-t)}$. Then, the identification (13) follows from (12).

Taking $\sqrt{\alpha} = i\theta$, i.e., $\alpha = -\theta^2$, we have

$$\frac{1}{\sqrt{\alpha}} \left(\frac{e^{\sqrt{\alpha}} + 1}{e^{\sqrt{\alpha}} - 1} \right) = -\frac{1}{\theta} \cot\left(\frac{\theta}{2}\right).$$

Set $\theta = q\pi$, $\alpha = -q^2\pi^2$. Then,

$$\frac{-1}{q^2} + 2\sum_{k=1}^{\infty} \frac{1}{4k^2 - q^2} = \pi^2 \int_0^1 G(t, t) \mathrm{d}t = -\frac{\pi}{2q} \cot\left(\frac{q}{2}\pi\right).$$

Hence, for any q, such that $q\tan\left(\frac{q}{2}\pi\right) \in \mathbb{Q}$, we have that

$$\frac{-1}{q^2} + 2\sum_{k=1}^{\infty} \frac{1}{4k^2 - q^2}$$

is a rational multiple of π. Moreover, for any $q \in \mathbb{Q}$, $\cot\left(\frac{q}{2}\pi\right)$ is an algebraic number.

Similarly, as $\alpha = q^2\pi^2$, we can obtain that

$$\frac{1}{q^2} + 2\sum_{k=1}^{\infty} \frac{1}{4k^2 + q^2} = \pi^2 \frac{1}{2\sqrt{\alpha}} \left(\frac{e^{\sqrt{\alpha}} + 1}{e^{\sqrt{\alpha}} - 1} \right) = \frac{\pi}{2q} \coth\left(\frac{q}{2}\pi\right).$$

In summary, we have the following lemma.

Lemma 1. *For any $q \in \mathbb{Q} \backslash \{\pm 2k,\ k = 0, 1, \cdots\}$, we have*

$$\frac{-1}{q^2} + 2\sum_{k=1}^{\infty} \frac{1}{4k^2 - q^2} = -\frac{\pi}{2q}\cot\left(\frac{q}{2}\pi\right) = -\frac{\pi}{2q}\frac{1+\cos(q\pi)}{\sin(q\pi)} =: c_1\pi,$$

as $q \neq 2k$ for any $k \in \mathbb{Z}$, and

$$\frac{1}{q^2} + 2\sum_{k=1}^{\infty} \frac{1}{4k^2 + q^2} = \frac{\pi}{2q}\coth\left(\frac{q}{2}\pi\right) = \frac{\pi}{2q}\left(\frac{e^{q\pi}+1}{e^{q\pi}-1}\right) =: \tilde{c}_1\pi,$$

where $c_1 \in \mathbb{Q}(\sin(q\pi), \cos(q\pi)) \subset \overline{\mathbb{Q}}$ and $\tilde{c}_1 \in \mathbb{Q}(e^{q\pi})$.

In fact, using the Fourier series of $\cos(z\,x)$,

$$\cos(z\,x) = \frac{2\,z}{\pi}\sin(\pi\,z)\left(\frac{1}{2\,z^2} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\,x)}{z^2 - n^2}\right),$$

the following identities of $\cot(z\,\pi)$ and $\coth(z\,\pi)$ can be proved (cf. [16] §8.5)

$$\cot(z\,\pi) = \frac{2}{\pi}\left(\frac{1}{2\,z} + \sum_{k=1}^{\infty} \frac{z}{z^2 - k^2}\right),$$
$$\coth(z\,\pi) = \frac{2}{\pi}\left(\frac{1}{2\,z} + \sum_{k=1}^{\infty} \frac{z}{z^2 + k^2}\right).$$

These two identities can also express Lemma 2. In the following, we consider the case $m \geq 2$.

Firstly, we calculate integral $\int_0^1\int_0^1 |G(s,t)|^2 \mathrm{d}s\mathrm{d}t$. Suppose $\alpha = -q^2\pi^2$. Then, the Green function can be rewritten as

$$G(s,t) = \frac{1}{2iq\pi}\begin{cases} \frac{e^{iq\pi(t-s)}}{e^{iq\pi}-1} + \frac{e^{iq\pi(s-t)}}{1-e^{-iq\pi}}, & 0 \leq s \leq t \leq 1; \\ \frac{e^{iq\pi(s-t)}}{e^{iq\pi}-1} + \frac{e^{iq\pi(t-s)}}{1-e^{-iq\pi}}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{14}$$

Hence,

$$\begin{aligned}
&4q^2\pi^2|e^{iq\pi}-1|^2\int_0^1\int_0^1|G(s,t)|^2\mathrm{d}s\mathrm{d}t\\
&=\int_0^1\mathrm{d}t\left\{\int_0^t[2+2\cos(2s-2t+1)q\pi]\mathrm{d}s+\int_t^1[2+2\cos(2s-2t-1)q\pi]\mathrm{d}s\right\}\\
&=\int_0^1\left\{2+\frac{1}{q\pi}\sin[(2s-2t+1)q\pi]\Big|_0^t+\frac{1}{q\pi}\sin[(2s-2t-1)q\pi]\Big|_t^1\right\}\mathrm{d}t\\
&=\int_0^1\left\{2+2\frac{1}{q\pi}\sin[q\pi]\right\}\mathrm{d}t\\
&=2\left[1+\frac{1}{q\pi}\sin(q\pi)\right].
\end{aligned}$$

Then, Mercer's Theorem tells us that

$$\frac{1}{q^4}+2\sum_{k=1}^{\infty}\frac{1}{(4k^2-q^2)^2}=\pi^4\int_0^1\int_0^1|G(s,t)|^2\mathrm{d}s\mathrm{d}t=\frac{\pi^2}{4q^2[1-\cos(q\pi)]}\left[1+\frac{1}{q\pi}\sin(q\pi)\right].$$

Similarly, for $\alpha=q^2\pi^2$, the Green function is

$$G(s,t)=\frac{1}{2q\pi}\begin{cases}\frac{e^{q\pi(t-s)}}{e^{q\pi}-1}-\frac{e^{-q\pi(t-s)}}{e^{-q\pi}-1}, & 0\le s\le t\le 1;\\ \frac{e^{q\pi(t-s+1)}}{e^{q\pi}-1}-\frac{e^{-q\pi(t-s+1)}}{e^{-q\pi}-1}, & 0\le t\le s\le 1.\end{cases}\tag{15}$$

Then,

$$\begin{aligned}
&\int_0^1\int_0^1|G(s,t)|^2\mathrm{d}s\mathrm{d}t\\
&=\frac{1}{4q^2\pi^2}\int_0^1\mathrm{d}t\left\{\int_0^t\left[\frac{e^{q\pi(t-s)}}{e^{q\pi}-1}-\frac{e^{-q\pi(t-s)}}{e^{-q\pi}-1}\right]^2\mathrm{d}s+\int_t^1\left[\frac{e^{q\pi(t-s+1)}}{e^{q\pi}-1}-\frac{e^{-q\pi(t-s+1)}}{e^{-q\pi}-1}\right]^2\mathrm{d}s\right\}
\end{aligned}$$

Set $\tau:=t-s$, as $0\le s\le t\le 1$ and $\tau:=t-s+1$, as $0\le t\le s\le 1$; then,

$$\begin{aligned}
\pi^4\int_0^1\int_0^1|G(s,t)|^2\mathrm{d}s\mathrm{d}t&=\frac{\pi^2}{4q^2}\int_0^1\mathrm{d}t\int_0^1\left[\frac{e^{q\pi\tau}}{e^{q\pi}-1}-\frac{e^{-q\pi\tau}}{e^{-q\pi}-1}\right]^2\mathrm{d}\tau\\
&=\frac{\pi^2}{4q^2(e^{q\pi}-1)^2}\int_0^1\left[e^{q\pi\tau}+e^{q\pi(1-\tau)}\right]^2\mathrm{d}\tau\\
&=\frac{\pi^2}{4q^2(e^{q\pi}-1)^2}\left[2e^{q\pi}+\frac{e^{2q\pi}-1}{q\pi}\right].
\end{aligned}\tag{16}$$

Again Mercer's Theorem tells us that

$$\frac{1}{q^4}+2\sum_{k=1}^{\infty}\frac{1}{(4k^2+q^2)^2}=\pi^4\int_0^1\int_0^1|G(s,t)|^2\mathrm{d}s\mathrm{d}t=\frac{\pi^2}{4q^2(e^{q\pi}-1)^2}\left[2e^{q\pi}+\frac{e^{2q\pi}-1}{q\pi}\right].$$

Therefore, we obtain the following conclusions.

Lemma 2. *[cf. [3] Theorem 4, [6] Theorem 3.2 (ii)] For any $q\in\mathbb{Q}$, we have*

$$\frac{1}{q^4}+2\sum_{k=1}^{\infty}\frac{1}{(4k^2-q^2)^2}=\frac{\pi^2}{4q^2[1-\cos(q\pi)]}\left[1+\frac{1}{q\pi}\sin(q\pi)\right]=c_2\pi^2+c_1\pi,\tag{17}$$

as $q \neq 2k$ *for any* $k \in \mathbb{Z}$, *and*

$$\frac{1}{q^4} + 2\sum_{k=1}^{\infty} \frac{1}{(4k^2+q^2)^2} = \frac{\pi^2}{4q^2(e^{q\pi}-1)^2}\left[2e^{q\pi} + \frac{e^{2q\pi}-1}{q\pi}\right] = \tilde{c}_2\pi^2 + \tilde{c}_1\pi, \tag{18}$$

where $c_i \in \mathbb{Q}(\sin(q\pi), \cos(q\pi)) \subset \overline{\mathbb{Q}}$, *and* $\tilde{c}_i \in \mathbb{Q}(e^{q\pi})$, $i = 1, 2$.

Now, we consider the property of the coefficient of

$$\pi^n \int_0^1 \cdots \int_0^1 G(x_1, x_2) \cdots G(x_n, x_1) \mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{19}$$

First, from the expression of Green function (12), we know that the coefficient of Green function of operator T_2^2, i.e.,

$$\pi^2 \int_0^1 G(s, \tau) G(\tau, t) \mathrm{d}\tau,$$

must be $c_1 + c_0/\pi$. Then, by induction, the coefficient of

$$\pi^n \int_0^1 \cdots \int_0^1 G(x_1, x_2) \cdots G(x_{n-1}, x_n) G(x_n, x_1) \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1} \tag{20}$$

must be $c_n + c_{n-1}/\pi \cdots + c_1/\pi^{n-1}$. Applying the substitution in (16), we set

$$\tau_{n-1} := \begin{cases} x_n - x_{n-1}, & 0 \leq x_{n-1} \leq x_n \leq 1; \\ x_n - x_{n-1} + 1, & 0 \leq x_n \leq x_{n-1} \leq 1; \end{cases}$$

and

$$\tau_1 := \begin{cases} x_n - x_1, & 0 \leq x_1 \leq x_n \leq 1; \\ x_n - x_1 + 1, & 0 \leq x_n \leq x_1 \leq 1. \end{cases}$$

Then, $G(x_{n-1}, x_n) = G(\tau_{n-1}),\ G(x_n, x_1) = G(\tau_1)$. Hence, the integral (20) is independent of variable x_n, and the coefficients of integral (19) are also $c_n + c_{n-1}/\pi \cdots + c_1/\pi^{n-1}$, which are the same as the coefficients of integral (20).

Moreover, similar to the above two lemmas, for any $q \in \mathbb{Q} \backslash \{\pm 2k,\ k = 0, 1, \cdots\}$, the coefficients

$$c_i \in \mathbb{Q}(\sin(q\pi), \cos(q\pi)),\ \alpha = -q^2\pi^2,\ i = 1, \cdots, n,$$

and

$$c_i \in \mathbb{Q}(e^{q\pi}),\ \alpha = q^2\pi^2,\ i = 1, \cdots, n.$$

Therefore, we obtain a more general conclusion which includes Lemmas 2 and 2 as special cases.

Theorem 1. *For any* $q \in \mathbb{Q}$, *and any positive integer* n, *we have that*

$$\frac{1}{(-q^2)^n} + 2\sum_{k=1}^{\infty} \frac{1}{(4k^2-q^2)^n} = c_n\pi^n + \cdots + c_1\pi,\ q \neq 2k,\ k \in \mathbb{Z},$$

and

$$\frac{1}{q^{2n}} + 2\sum_{k=1}^{\infty} \frac{1}{(4k^2+q^2)^n} = \tilde{c}_n\pi^n + \cdots + \tilde{c}_1\pi,$$

where $c_i \in \mathbb{Q}(\sin(q\pi), \cos(q\pi)) \subset \overline{\mathbb{Q}}$, *and* $\tilde{c}_i \in \mathbb{Q}(e^{q\pi})$, $i = 0, 1, \cdots, n$.

Murty and Weatherby [1], ([4] §6), ([3] Theorem 4) or ([6] Theorem 3.2 (i)) obtained the expression

$$\sum_{k\in\mathbb{Z}}\frac{1}{k^2+c}=\frac{\pi(e^{2\pi\sqrt{c}}+1)}{\sqrt{c}(e^{2\pi\sqrt{c}}-1)},\ \text{for } c\geq 0.$$

Using the derivative of the series to c, the transcendentality of sum

$$\sum_{k\in\mathbb{Z}}\frac{1}{(k^2+q^2)^n},\ q\in\mathbb{Q}\backslash\{0\},$$

can be derived (cf. [6] Theorem 3.2 (ii)). In Theorem 1, a calculation method of series $\sum_{k\in\mathbb{Z}}\frac{1}{(k^2\pm q^2)^n}$ is obtained. Furthermore, by this calculation formula, we can find the transcendentality of the series.

2.2. The Third Order Case

In the following, we consider Problem (7) in the case $m=3$.

$$T_3u=iu'''+\alpha u=\lambda u,\ \text{on } (0,1), \tag{21}$$

with the boundary condition

$$u(0)=u(1),\ u'(0)=u'(1),\ u''(0)=u''(1),$$

where $\alpha\neq-(2k\pi)^3$, $k=0,\pm1,\pm2,\cdots$. Then, the k-th eigenvalue is $\lambda_k=(2k\pi)^3+\alpha$, $k=0,\pm1,\pm2,\cdots$, the corresponding eigenfunction is $\varphi_k(x)=e^{i2k\pi x}$. Hence, for any integer k, the geometric multiplicity of eigenvalue λ_k is simple.

Set $\alpha=-q^3\pi^3$. Then, the Green function of (21) at 0 is

$$G(s,t)=\frac{i}{3q^2\pi^2}\begin{cases}\dfrac{\exp\left(-(\frac{\sqrt{3}}{2}+\frac{1}{2}i)q\pi(t-s)\right)}{(\frac{1}{2}+\frac{\sqrt{3}}{2}i)\left\{1-\exp\left(-(\frac{\sqrt{3}}{2}+\frac{1}{2}i)q\pi\right)\right\}}+\dfrac{\exp\left((\frac{\sqrt{3}}{2}-\frac{1}{2}i)q\pi(t-s)\right)}{(\frac{1}{2}-\frac{\sqrt{3}}{2}i)\left\{1-\exp\left((\frac{\sqrt{3}}{2}-\frac{1}{2}i)q\pi\right)\right\}}+\dfrac{\exp(iq\pi(t-s))}{-\{1-\exp(iq\pi)\}}, \\ \qquad 0\leq s\leq t\leq 1; \\ \dfrac{\exp\left(-(\frac{\sqrt{3}}{2}+\frac{1}{2}i)q\pi(t-s+1)\right)}{(\frac{1}{2}+\frac{\sqrt{3}}{2}i)\left\{1-\exp\left(-(\frac{\sqrt{3}}{2}+\frac{1}{2}i)q\pi\right)\right\}}+\dfrac{\exp\left((\frac{\sqrt{3}}{2}-\frac{1}{2}i)q\pi(t-s+1)\right)}{(\frac{1}{2}-\frac{\sqrt{3}}{2}i)\left\{1-\exp\left((\frac{\sqrt{3}}{2}-\frac{1}{2}i)q\pi\right)\right\}}+\dfrac{\exp(iq\pi(t-s+1))}{-\{1-\exp(iq\pi)\}}, \\ \qquad 0\leq t\leq s\leq 1.\end{cases}$$

Set

$$z_1=\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)q\pi,\ z_2=-\bar{z}_1=-\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)q\pi,\ \text{and } z_3=iq\pi.$$

Then,

$$G(s,t)=\frac{1}{3q\pi}\begin{cases}\frac{\exp(z_2(t-s))}{z_1(1-\exp(z_2))}+\frac{\exp(z_1(t-s))}{z_2(1-\exp(z_1))}+\frac{\exp(z_3(t-s))}{z_3(1-\exp(z_3))}, & 0\leq s\leq t\leq 1; \\ \frac{\exp(z_2(t-s+1))}{z_1[1-\exp(z_2)]}+\frac{\exp(z_1(t-s+1))}{z_2[1-\exp(z_1)]}+\frac{\exp(z_3(t-s+1))}{z_3[1-\exp(z_3)]}, & 0\leq t\leq s\leq 1.\end{cases} \tag{22}$$

In particular, we have that, for any $t\in[0,1]$,

$$\begin{aligned}G(t,t)&=\frac{-1}{3q^3\pi^3}\sum_{j=1}^{3}\frac{z_j}{1-\exp(z_j)}\\&=\frac{1}{6q^2\pi^2}\left[\frac{\sqrt{3}\sinh(\frac{\sqrt{3}}{2}q\pi)+\sin(\frac{1}{2}q\pi)}{\cosh(\frac{\sqrt{3}}{2}q\pi)-\cos(\frac{1}{2}q\pi)}+\cot(\frac{1}{2}q\pi)\right].\end{aligned} \tag{23}$$

As $q=1$, for any $t\in[0,1]$,

$$G(t,t)=\frac{1}{3\pi^2}\left[\frac{\sqrt{3}}{2}\tanh(\frac{\sqrt{3}}{2}\pi)+\frac{1}{2}\cosh^{-1}(\frac{\sqrt{3}}{2}\pi)\right],$$

and Mercer's Theorem tells us that

$$\sum_{k=-\infty}^{\infty}\frac{1}{(2k)^3-1}=\frac{\pi}{3}\left[\frac{\sqrt{3}}{2}\tanh(\frac{\sqrt{3}}{2}\pi)+\frac{1}{2}\cosh^{-1}(\frac{\sqrt{3}}{2}\pi)\right]=:c_1\pi, \tag{24}$$

where

$$c_1\in\mathbb{Q}\left(\frac{\sqrt{3}}{2};\exp(\frac{\sqrt{3}}{2}q\pi)\right)\subset\overline{\mathbb{Q}}\left(\exp(\frac{\sqrt{3}}{2}q\pi)\right).$$

In the following lemma, we obtain the more general case.

Lemma 3. *For any $q\in\mathbb{Q}\backslash\{\pm 2k,\ k=0,1,\cdots\}$, we have*

$$\sum_{k=-\infty}^{\infty}\frac{1}{(2k)^3-q^3}=\frac{\pi}{6q^2}\left[\frac{\sqrt{3}\sinh(\frac{\sqrt{3}}{2}q\pi)+\sin(\frac{1}{2}q\pi)}{\cosh(\frac{\sqrt{3}}{2}q\pi)-\cos(\frac{1}{2}q\pi)}+\cot(\frac{1}{2}q\pi)\right]=:c_1\pi+c_0, \tag{25}$$

where

$$c_i\in\mathbb{Q}\left(\frac{\sqrt{3}}{2},\sin(\frac{1}{2}q\pi),\cos(\frac{1}{2}q\pi);\exp(\frac{\sqrt{3}}{2}q\pi)\right)\subset\overline{\mathbb{Q}}\left(\exp(\frac{\sqrt{3}}{2}q\pi)\right),\ i=0,1.$$

Weatherby ([6], Theorem 3.2 (ii)) proved that the sums

$$\sum_{k\in\mathbb{Z}}\frac{1}{k^3+q^3},\ q\in\mathbb{Q}\backslash\mathbb{Z},$$

are transcendental; however, the explicit expression is not obtained in the paper.

Now, we calculate the integral $\int_0^1\int_0^1|G(s,t)|^2\mathrm{d}s\mathrm{d}t$. We note that

$$\int_0^1\int_t^1|G(s,t)|^2\mathrm{d}s\mathrm{d}t=\int_0^1\int_0^s|G(s,t)|^2\mathrm{d}t\mathrm{d}s=\int_0^1\int_0^t|G(s,t)|^2\mathrm{d}s\mathrm{d}t.$$

Hence,

$$\int_0^1\int_0^1|G(s,t)|^2\mathrm{d}s\mathrm{d}t=2\int_0^1\int_0^t|G(s,t)|^2\mathrm{d}s\mathrm{d}t.$$

Recall the Green function, we obtain that, for any $0\le s\le t\le 1$,

$$9q^2\pi^2|G(s,t)|^2=\left|\frac{\exp(z_2(t-s))}{z_1[1-\exp(z_2)]}+\frac{\exp(z_1(t-s))}{z_2[1-\exp(z_1)]}+\frac{\exp(z_3(t-s))}{z_3[1-\exp(z_3)]}\right|^2,$$

where $z_1=\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)q\pi,\ z_2=-\bar{z}_1=-\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)q\pi,$ and $z_3=iq\pi$.

First, we calculate three square terms.

$$\begin{aligned}&\int_0^1\int_0^t\left|\frac{\exp(z_2(t-s))}{z_1[1-\exp(z_2)]}\right|^2\mathrm{d}s\mathrm{d}t\\&=\frac{1}{q^2\pi^2|1-\exp(z_2)|^2}\int_0^1\int_0^t\exp(\sqrt{3}q\pi(s-t))\mathrm{d}s\mathrm{d}t\\&=\frac{1}{q^2\pi^2|1-\exp(z_2)|^2}\left[\frac{1}{\sqrt{3}q\pi}-\frac{1}{3q^2\pi^2}+\frac{\exp(-\sqrt{3}q\pi)}{3q^2\pi^2}\right],\end{aligned}$$

where
$$|1-\exp(z_2)|^2 = 2\exp\left(-\frac{\sqrt{3}}{2}q\pi\right)\left[\cosh(\frac{\sqrt{3}}{2}q\pi) - \cos(\frac{1}{2}q\pi)\right].$$

$$\begin{aligned}
&\int_0^1\int_0^t\left|\frac{\exp(z_1(t-s))}{z_2[1-\exp(z_1)]}\right|^2 \mathrm{d}s\mathrm{d}t \\
&= \frac{1}{q^2\pi^2|1-\exp(z_1)|^2}\int_0^1\int_0^t \exp(\sqrt{3}q\pi(t-s))\mathrm{d}s\mathrm{d}t \\
&= \frac{1}{q^2\pi^2|1-\exp(z_1)|^2}\left[-\frac{1}{\sqrt{3}q\pi} - \frac{1}{3q^2\pi^2} + \frac{\exp(\sqrt{3}q\pi)}{3q^2\pi^2}\right],
\end{aligned}$$

where
$$|1-\exp(z_1)|^2 = 2\exp\left(\frac{\sqrt{3}}{2}q\pi\right)\left[\cosh(\frac{\sqrt{3}}{2}q\pi) - \cos(\frac{1}{2}q\pi)\right].$$

Therefore, we have

$$\begin{aligned}
&\int_0^1\int_0^t\left|\frac{\exp(z_2(t-s))}{z_1[1-\exp(z_2)]}\right|^2 + \left|\frac{\exp(z_1(t-s))}{z_2[1-\exp(z_1)]}\right|^2 \mathrm{d}s\mathrm{d}t \\
&= \frac{1}{\sqrt{3}q^3\pi^3\left[\cosh(\frac{\sqrt{3}}{2}q\pi) - \cos(\frac{1}{2}q\pi)\right]}\sinh\left(\frac{\sqrt{3}}{2}q\pi\right).
\end{aligned} \tag{26}$$

The last square term is

$$\int_0^1\int_0^t\left|\frac{\exp(z_3(t-s))}{z_3[1-\exp(z_3)]}\right|^2 \mathrm{d}s\mathrm{d}t = \frac{1/2}{q^2\pi^2|1-\exp(z_3)|^2} = \frac{1}{4q^2\pi^2[1-\cos(q\pi)]}. \tag{27}$$

In the next, we will calculate the three cross terms:

$$2\int_0^1\int_0^t \frac{\exp(z_2(t-s))}{z_1[1-\exp(z_2)]}\frac{\exp(\bar{z}_1(t-s))}{\bar{z}_2[1-\exp(\bar{z}_1)]}\mathrm{d}s\mathrm{d}t = \frac{1}{z_1\bar{z}_2[1-\exp(z_2)][1-\exp(\bar{z}_1)]},$$

then we can obtain the real part of the integral

$$\begin{aligned}
&2\Re\int_0^1\int_0^t \frac{\exp(z_2(t-s))}{z_1[1-\exp(z_2)]}\frac{\exp(\bar{z}_1(t-s))}{\bar{z}_2[1-\exp(\bar{z}_1)]}\mathrm{d}s\mathrm{d}t \\
&= \frac{-1+\cosh(\frac{\sqrt{3}}{2}q\pi)\cos(\frac{1}{2}q\pi) + \sqrt{3}\sinh(\frac{\sqrt{3}}{2}q\pi)\sin(\frac{1}{2}q\pi)}{q^2\pi^2|1-\exp(z_2)|^2|1-\exp(\bar{z}_1)|^2} \\
&= \frac{-1+\cosh(\frac{\sqrt{3}}{2}q\pi)\cos(\frac{1}{2}q\pi) + \sqrt{3}\sinh(\frac{\sqrt{3}}{2}q\pi)\sin(\frac{1}{2}q\pi)}{4q^2\pi^2\left[\cosh(\frac{\sqrt{3}}{2}q\pi) - \cos(\frac{1}{2}q\pi)\right]^2}.
\end{aligned} \tag{28}$$

We will calculate the remaining two cross terms:

$$\begin{aligned}
&2\int_0^1\int_0^t \frac{\exp(z_2(t-s))}{z_1[1-\exp(z_2)]}\frac{\exp(\bar{z}_3(t-s))}{\bar{z}_3[1-\exp(\bar{z}_3)]}\mathrm{d}s\mathrm{d}t \\
&= \frac{2}{z_1\bar{z}_3[1-\exp(z_2)][1-\exp(\bar{z}_3)]}\int_0^1\int_0^t \exp((z_2+\bar{z}_3)(t-s))\mathrm{d}s\mathrm{d}t \\
&= \frac{2}{z_1\bar{z}_3[1-\exp(z_2)][1-\exp(\bar{z}_3)]}\frac{\exp(z_2+\bar{z}_3)-1-(z_2+\bar{z}_3)}{(z_2+\bar{z}_3)^2} \\
&= \frac{2}{3q^4\pi^4}\frac{\exp(z_2+\bar{z}_3)-1-(z_2+\bar{z}_3)}{[1-\exp(z_2)][1-\exp(\bar{z}_3)]} \\
&= \frac{2}{3q^4\pi^4}\left\{1-\frac{1}{1-\exp(z_2)}-\frac{1}{1-\exp(\bar{z}_3)}-\frac{z_2+\bar{z}_3}{[1-\exp(z_2)][1-\exp(\bar{z}_3)]}\right\},
\end{aligned}$$

$$\Re\frac{1}{1-\exp(\bar{z}_3)} = \frac{1-\cos(q\pi)}{|1-\exp(-iq\pi)|^2} = \frac{1}{2},$$

$$\Re\frac{1}{1-\exp(z_2)} = \frac{1}{2}\frac{\exp(\frac{\sqrt{3}}{2}q\pi)-\cos(\frac{1}{2}q\pi)}{\cosh(\frac{\sqrt{3}}{2}q\pi)-\cos(\frac{1}{2}q\pi)},$$

and

$$\begin{aligned}
&\Re\frac{z_2+\bar{z}_3}{[1-\exp(z_2)][1-\exp(\bar{z}_3)]} \\
=&\frac{-\sqrt{3}q\pi\left[\frac{1}{2}\exp(\frac{\sqrt{3}}{2}q\pi)-\exp(\frac{\sqrt{3}}{2}q\pi)\cos(q\pi+\frac{\pi}{3})-\cos(\frac{1}{2}q\pi+\frac{\pi}{3})+\cos(\frac{3}{2}q\pi+\frac{\pi}{3})\right]}{4[1-\cos(q\pi)]\left[\cosh(\frac{\sqrt{3}}{2}q\pi)-\cos(\frac{1}{2}q\pi)\right]}.
\end{aligned}$$

Similarly, we can obtain that

$$\begin{aligned}
&2\int_0^1\int_0^t \frac{\exp(z_1(t-s))}{z_2[1-\exp(z_1)]}\frac{\exp(\bar{z}_3(t-s))}{\bar{z}_3[1-\exp(\bar{z}_3)]}\mathrm{d}s\mathrm{d}t \\
=&\frac{2}{3q^4\pi^4}\left\{1-\frac{1}{1-\exp(z_1)}-\frac{1}{1-\exp(\bar{z}_3)}-\frac{z_1+\bar{z}_3}{[1-\exp(z_1)][1-\exp(\bar{z}_3)]}\right\}.
\end{aligned}$$

Note that

$$\Re\frac{1}{1-\exp(z_1)} = \frac{1}{2}\frac{\exp(-\frac{\sqrt{3}}{2}q\pi)-\cos(\frac{1}{2}q\pi)}{\cosh(\frac{\sqrt{3}}{2}q\pi)-\cos(\frac{1}{2}q\pi)},$$

and

$$\begin{aligned}
&\Re\frac{z_1+\bar{z}_3}{[1-\exp(z_1)][1-\exp(\bar{z}_3)]} \\
=&\frac{\sqrt{3}q\pi\left[\frac{1}{2}\exp(-\frac{\sqrt{3}}{2}q\pi)-\exp(-\frac{\sqrt{3}}{2}q\pi)\cos(q\pi-\frac{\pi}{3})-\cos(\frac{1}{2}q\pi-\frac{\pi}{3})+\cos(\frac{3}{2}q\pi-\frac{\pi}{3})\right]}{4[1-\cos(q\pi)]\left[\cosh(\frac{\sqrt{3}}{2}q\pi)-\cos(\frac{1}{2}q\pi)\right]}.
\end{aligned}$$

Summing the above two cross terms respectively, we obtain

$$\Re\frac{1}{1-\exp(z_1)}+\Re\frac{1}{1-\exp(z_1)} = \frac{1}{2},$$

and

$$\begin{aligned}
&\Re\frac{z_2+\bar{z}_3}{[1-\exp(z_2)][1-\exp(\bar{z}_3)]}+\Re\frac{z_1+\bar{z}_3}{[1-\exp(z_1)][1-\exp(\bar{z}_3)]}\\
=&\sqrt{3}q\pi\frac{[\cos(q\pi)-1]\sinh(\frac{\sqrt{3}}{2}q\pi)+\sqrt{3}\Big[\sin(\frac{3}{2}q\pi)-\sin(\frac{1}{2}q\pi)\Big]-\sqrt{3}\sin(q\pi)\cosh(\frac{\sqrt{3}}{2}q\pi)}{4[1-\cos(q\pi)]\Big[\cosh(\frac{\sqrt{3}}{2}q\pi)-\cos(\frac{1}{2}q\pi)\Big]}\\
=&\sqrt{3}q\pi\frac{[\cos(q\pi)-1]\Big[\sinh(\frac{\sqrt{3}}{2}q\pi)+\sqrt{3}\sin(\frac{1}{2}q\pi)\Big]+\sqrt{3}\sin(q\pi)\Big[\cos(\frac{1}{2}q\pi)-\cosh(\frac{\sqrt{3}}{2}q\pi)\Big]}{4[1-\cos(q\pi)]\Big[\cosh(\frac{\sqrt{3}}{2}q\pi)-\cos(\frac{1}{2}q\pi)\Big]}\\
=&\frac{-\sqrt{3}q\pi\Big[\sinh(\frac{\sqrt{3}}{2}q\pi)+\sqrt{3}\sin(\frac{1}{2}q\pi)\Big]}{4\Big[\cosh(\frac{\sqrt{3}}{2}q\pi)-\cos(\frac{1}{2}q\pi)\Big]}-\frac{3q\pi\sin(q\pi)}{4[1-\cos(q\pi)]}.
\end{aligned}$$

Hence, we can obtain that

$$\begin{aligned}
&2\Re\int_0^1\int_0^t\frac{\exp(z_2(t-s))}{z_1[1-\exp(z_2)]}\frac{\exp(\bar{z}_3(t-s))}{\bar{z}_3[1-\exp(\bar{z}_3)]}+\frac{\exp(z_1(t-s))}{z_2[1-\exp(z_1)]}\frac{\exp(\bar{z}_3(t-s))}{\bar{z}_3[1-\exp(\bar{z}_3)]}\mathrm{d}s\mathrm{d}t\\
=&\frac{2}{3q^4\pi^4}\left\{\frac{\sqrt{3}q\pi\Big[\sinh(\frac{\sqrt{3}}{2}q\pi)+\sqrt{3}\sin(\frac{1}{2}q\pi)\Big]}{4\Big[\cosh(\frac{\sqrt{3}}{2}q\pi)-\cos(\frac{1}{2}q\pi)\Big]}+\frac{3q\pi\sin(q\pi)}{4[1-\cos(q\pi)]}\right\}\\
=&\frac{1}{2\sqrt{3}q^3\pi^3}\frac{\sinh(\frac{\sqrt{3}}{2}q\pi)+\sqrt{3}\sin(\frac{1}{2}q\pi)}{\cosh(\frac{\sqrt{3}}{2}q\pi)-\cos(\frac{1}{2}q\pi)}+\frac{1}{2q^3\pi^3}\frac{\sin(q\pi)}{1-\cos(q\pi)}.
\end{aligned}\tag{29}$$

Using Mercer's Theorem again, we have

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}\frac{1}{(2^3k^3-q^3)^2}=\pi^6\int_0^1\int_0^1|G(s,t)|^2\mathrm{d}s\mathrm{d}t=2\pi^6\int_0^1\int_0^t|G(s,t)|^2\mathrm{d}s\mathrm{d}t\\
=&\frac{2\pi^6}{9q^2\pi^2}\int_0^1\int_0^t\left|\frac{\exp(z_2(t-s))}{z_1[1-\exp(z_2)]}+\frac{\exp(z_1(t-s))}{z_2[1-\exp(z_1)]}+\frac{\exp(z_3(t-s))}{z_3[1-\exp(z_3)]}\right|^2\mathrm{d}s\mathrm{d}t.
\end{aligned}$$

Then, add up Equations (26), (27), (28) and (29), we can obtain that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}\frac{1}{(2^3k^3-q^3)^2}=\frac{2\pi}{9\sqrt{3}q^5}\frac{\sinh(\frac{\sqrt{3}}{2}q\pi)}{\cosh(\frac{\sqrt{3}}{2}q\pi)-\cos(\frac{1}{2}q\pi)}+\frac{\pi^2}{18q^4}\frac{1}{1-\cos(q\pi)}\\
&+\frac{\pi^2}{18q^4}\frac{-1+\cosh(\frac{\sqrt{3}}{2}q\pi)\cos(\frac{1}{2}q\pi)+\sqrt{3}\sinh(\frac{\sqrt{3}}{2}q\pi)\sin(\frac{1}{2}q\pi)}{\Big[\cosh\Big(\frac{\sqrt{3}}{2}q\pi\Big)-\cos\Big(\frac{1}{2}q\pi\Big)\Big]^2}\\
&+\frac{\pi}{9\sqrt{3}q^5}\frac{\sinh(\frac{\sqrt{3}}{2}q\pi)+\sqrt{3}\sin(\frac{1}{2}q\pi)}{\cosh(\frac{\sqrt{3}}{2}q\pi)-\cos(\frac{1}{2}q\pi)}+\frac{\pi^2}{9q^4}\frac{\sin(q\pi)}{1-\cos(q\pi)}\\
=&\frac{\pi^2}{18q^4}\frac{1+2\sin(q\pi)}{1-\cos(q\pi)}+\frac{\pi^2}{18q^4}\frac{-1+\cosh(\frac{\sqrt{3}}{2}q\pi)\cos(\frac{1}{2}q\pi)+\sqrt{3}\sinh(\frac{\sqrt{3}}{2}q\pi)\sin(\frac{1}{2}q\pi)}{\Big[\cosh\Big(\frac{\sqrt{3}}{2}q\pi\Big)-\cos\Big(\frac{1}{2}q\pi\Big)\Big]^2}\\
&+\frac{\pi}{9q^5}\frac{\sqrt{3}\sinh(\frac{\sqrt{3}}{2}q\pi)+\sin(\frac{1}{2}q\pi)}{\cosh(\frac{\sqrt{3}}{2}q\pi)-\cos(\frac{1}{2}q\pi)}=:c_2\pi^2+c_1\pi.
\end{aligned}\tag{30}$$

Supposing that $q\in\mathbb{Q}$, then $c_1,\ c_2$ in (30) satisfy that

$$c_1,c_2\in\mathbb{Q}\left(\frac{\sqrt{3}}{2},\sin(\frac{1}{2}q\pi),\cos(\frac{1}{2}q\pi);\exp(\frac{\sqrt{3}}{2}q\pi)\right)\subset\overline{\mathbb{Q}}\left(\exp(\frac{\sqrt{3}}{2}q\pi)\right).$$

As $q = 1$, we have that

$$\sum_{k=-\infty}^{\infty} \frac{1}{(2^3k^3-1)^2} = \frac{\pi^2}{18}\left[\frac{1}{2} + \sqrt{3}\tanh(\frac{\sqrt{3}}{2}\pi)\cosh^{-1}(\frac{\sqrt{3}}{2}\pi) - \cosh^{-2}(\frac{\sqrt{3}}{2}\pi)\right] + \frac{\pi}{9}\left[\sqrt{3}\tanh(\frac{\sqrt{3}}{2}\pi) + \cosh^{-1}(\frac{\sqrt{3}}{2}\pi)\right] =: c_2\pi^2 + c_1\pi, \tag{31}$$

where

$$c_1, c_2 \in \mathbb{Q}\left(\frac{\sqrt{3}}{2}; \exp(\frac{\sqrt{3}}{2}\pi)\right) \subset \overline{\mathbb{Q}}\left(\exp(\frac{\sqrt{3}}{2}\pi)\right).$$

For the more general case, similar to Theorem 1, we can obtain that

Theorem 2. *For any $q \in \mathbb{Q}\backslash\{\pm 2k,\ k = 0, 1, \cdots\}$, and any $n \geq 1$, we have*

$$\sum_{k=-\infty}^{\infty} \frac{1}{(2^3k^3-q^3)^n} = c_n\pi^n + \cdots + c_1\pi,$$

where

$$c_i \in \mathbb{Q}\left(\frac{\sqrt{3}}{2}, \sin(\frac{1}{2}q\pi), \cos(\frac{1}{2}q\pi); \exp(\frac{\sqrt{3}}{2}q\pi)\right) \subset \overline{\mathbb{Q}}\left(\exp(\frac{\sqrt{3}}{2}q\pi)\right),\ i = 1, \cdots, n.$$

In ([6], Theorem 3.3), Weatherby proved that the sum

$$\sum_{k\in\mathbb{Z}} \frac{1}{(k^3-q^3)^{2n}},\ q \in \mathbb{Q}\backslash\mathbb{Z},$$

is transcendental and the calculation formula is not obtained in this paper. In Theorem 2, a calculation method of series $\sum_{k\in\mathbb{Z}} \frac{1}{(k^3-q^3)^n}$ is obtained. Furthermore, by this calculation formula, we can judge the transcendentality of the series.

3. The Higher Order Self-Adjoint Differential Operators

Now, let us recall Problem (7) for any self-adjoint differential operators of order m on $\mathbb{S}^1$. For any positive integer m,

$$T_m u = (-i)^m u^{(m)} + \alpha u = \lambda u,\ \text{on } (0,1),$$

with the boundary condition

$$u(0) = u(1), \cdots, u^{(m-1)}(0) = u^{(m-1)}(1),$$

where $\alpha \neq -(2k\pi)^m$, $k = 0, \pm 1, \pm 2, \cdots$. Then, the eigenvalues of T_m are

$$\{\lambda_k = (2k\pi)^m + \alpha,\ k = 0, \pm 1, \pm 2, \cdots\},$$

and the corresponding eigenfunctions are

$$\varphi_k(x) = \begin{cases} e^{\pm i2k\pi x} \text{ or } \{\cos(2k\pi x),\ \sin(2k\pi x)\}, & \text{as m is even;} \\ e^{-i2k\pi x}, & \text{as m is odd.} \end{cases}$$

Hence, for an even m, for any $k \geq 1$, the geometric multiplicity of eigenvalue λ_k is 2, and only in the case $k = 0$ is the geometric multiplicity of eigenvalue λ_0 simple. For an odd m, for any k, the geometric multiplicity of λ_k is simple. Set $\alpha = \pm q^m \pi^m$, and using Mercer's Theorem again, we have

$$\pi^{nm} \int_0^1 \cdots \int_0^1 G(x_1, x_2) \cdots G(x_n, x_1) \mathrm{d}x_1 \cdots \mathrm{d}x_n$$
$$= \begin{cases} \frac{(\pm 1)^n}{q^{nm}} + 2\sum_{k=1}^{\infty} \frac{1}{(2^m k^m \pm q^m)^n}, & \text{if } m \text{ is even;} \\ \sum_{k=-\infty}^{\infty} \frac{1}{(2^m k^m \pm q^m)^n}, & \text{if } m \text{ is odd,} \end{cases}$$

where $G(s,t)$ is the Green function of T_m at 0 point. Note that, if m is even, we also have

$$\frac{(\pm 1)^n}{q^{nm}} + 2\sum_{k=1}^{\infty} \frac{1}{(2^m k^m \pm q^m)^n} = \sum_{k=-\infty}^{\infty} \frac{1}{(2^m k^m \pm q^m)^n}.$$

Let $\{\xi_j,\ j = 1, \cdots, m\}$ be m distinct roots in $\mathbb{C}$ of the algebraic equation

$$(-i)^m \xi^m + \alpha = 0.$$

Then, the Green function $G(s,t)$ is in the following form:

$$G(s,t) = \frac{i^m}{m} \begin{cases} \sum_{j=1}^{m} \frac{\exp(-\xi_j(t-s))}{[\exp(-\xi_j)-1]\xi_j^{m-1}}, & 0 \le s \le t \le 1; \\ \sum_{j=1}^{m} \frac{\exp(-\xi_j(t-s+1))}{[\exp(-\xi_j)-1]\xi_j^{m-1}}, & 0 \le t \le s \le 1. \end{cases} \tag{32}$$

Supposing $\alpha = -(q\pi)^m$, then, for any $j = 1, \cdots, m$, we have $\xi_j = iq\pi \exp(\frac{j}{m} 2\pi i)$, and

$$-\xi_j = \sin\left(\frac{j}{m}2\pi\right)q\pi - i\cos\left(\frac{j}{m}2\pi\right)q\pi,$$
$$\exp(-\xi_j) = \exp\left(\sin(\frac{j}{m}2\pi)q\pi\right)\left\{\cos\left(\cos(\frac{j}{m}2\pi)q\pi\right) - i\sin\left(\cos(\frac{j}{m}2\pi)q\pi\right)\right\}.$$

Note that, if m is odd, we have

$$\sin\left(\frac{(m-1)/2}{m}2\pi\right) = \sin\left(\frac{1}{m}\pi\right) \text{ and } \cos\left(\frac{(m-1)/2}{m}2\pi\right) = -\cos\left(\frac{1}{m}\pi\right).$$

Hence, for any $q \in \mathbb{Q}$, all the coefficients of Green function $G(s,t)$ satisfy

$$\xi_j,\ \exp(\xi_j) \in \mathbb{K}_m,\ j = 1, \cdots, m,$$

where $\mathbb{K}_m$ is a field, and is defined as

$$\mathbb{K}_m := \mathbb{Q}\left(\cos(\frac{1}{m}2\pi), \sin(\frac{1}{m}2\pi),\ \{\exp(\sin(\frac{j}{m}2\pi)q\pi), \cos(\cos(\frac{j}{m}2\pi)q\pi), \sin(\cos(\frac{j}{m}2\pi)q\pi), j = 1, \cdots, m\}\right), \text{ for any even } m,$$

and

$$\mathbb{K}_m := \mathbb{Q}\left(\cos(\frac{1}{m}\pi), \sin(\frac{1}{m}\pi),\ \{\exp(\sin(\frac{j}{m}2\pi)q\pi), \cos(\cos(\frac{j}{m}2\pi)q\pi), \sin(\cos(\frac{j}{m}2\pi)q\pi), j = 1, \cdots, m\}\right), \text{ for any odd } m.$$

Since

$$\cos\left(\frac{1}{m}2\pi\right), \sin\left(\frac{1}{m}2\pi\right), \cos\left(\frac{1}{m}\pi\right), \sin\left(\frac{1}{m}\pi\right) \in \overline{\mathbb{Q}},$$

we have that, for any m,

$$\mathbb{K}_m \subset \overline{\mathbb{Q}}\Big(\exp(\sin(\frac{j}{m}2\pi)q\pi), \cos(\cos(\frac{j}{m}2\pi)q\pi), \sin(\cos(\frac{j}{m}2\pi)q\pi), j=1,\cdots,m\Big).$$

If $\alpha=(q\pi)^m$, then for any $j=1,\cdots,m$, we have $\xi_j = iq\pi\exp(i\frac{2j-1}{m}\pi)$, and

$$-\xi_j = \sin\Big(\frac{2j-1}{m}\pi\Big)q\pi - i\cos\Big(\frac{2j-1}{m}\pi\Big)q\pi,$$
$$\exp(-\xi_j) = \exp\Big(\sin(\frac{2j-1}{m}\pi)q\pi\Big)\Big\{\cos(\cos(\frac{2j-1}{m}\pi)q\pi) - i\sin(\cos(\frac{2j-1}{m}\pi)q\pi)\Big\}.$$

Hence, for any $q\in\mathbb{Q}$, all the coefficients of Green function $G(s,t)$ satisfy

$$\xi_j, \exp(\xi_j) \in \widetilde{\mathbb{K}}_m,\ j=1,\cdots,m,$$

where, for any m, $\widetilde{\mathbb{K}}_m$ is defined as

$$\begin{aligned}\widetilde{\mathbb{K}}_m :=& \mathbb{Q}\Big(\cos(\frac{1}{m}\pi), \sin(\frac{1}{m}\pi),\\ &\{\exp(\sin(\frac{2j-1}{m}\pi)q\pi), \cos(\cos(\frac{2j-1}{m}\pi)q\pi), \sin(\cos(\frac{2j-1}{m}\pi)q\pi), j=1,\cdots,m\}\Big)\\ \subset& \overline{\mathbb{Q}}\Big(\exp(\sin(\frac{2j-1}{m}\pi)q\pi), \cos(\cos(\frac{2j-1}{m}\pi)q\pi), \sin(\cos(\frac{2j-1}{m}\pi)q\pi), j=1,\cdots,m\Big).\end{aligned}$$

Note that, for an odd m,

$$\sin\Big(\frac{2j-1}{m}\pi\Big) = \sin\Big(\frac{(m+1)/2-j}{m}2\pi\Big),\ \cos\Big(\frac{2j-1}{m}\pi\Big) = -\cos\Big(\frac{(m+1)/2-j}{m}2\pi\Big),$$

we have $\mathbb{K}_m = \widetilde{\mathbb{K}}_m$. This is consistent with the following facts:

$$\sum_{k=-\infty}^{\infty}\frac{1}{(2^m k^m - q^m)^n} = \sum_{k=-\infty}^{\infty}\frac{1}{[2^m(-k)^m - q^m]^n} = (-1)^n\sum_{k=-\infty}^{\infty}\frac{1}{(2^m k^m + q^m)^n},$$

for an odd m.

With these preparations, similar to Theorems 1 and 2, we have the following theorem.

Theorem 3. *For any $q\in\mathbb{Q}$, and any integer $m\geq 2$, $n\geq 1$, we have*

$$\sum_{k=-\infty}^{\infty}\frac{1}{(2^m k^m - q^m)^n} = c_n\pi^n + \cdots + c_1\pi,\ q^m \neq 2^m k^m,\ k\in\mathbb{Z},$$

and

$$\sum_{k=-\infty}^{\infty}\frac{1}{(2^m k^m + q^m)^n} = \tilde{c}_n\pi^n + \cdots + \tilde{c}_1\pi,\ q^m \neq -2^m k^m,\ k\in\mathbb{Z},$$

where $c_i\in\mathbb{K}_m$, $\tilde{c}_i\in\widetilde{\mathbb{K}}_m$, $i=1,\cdots,n$.

In the case $m\geq 4$, Weatherby ([6], Theorem 3.2 (vi)) and ([6], Theorem 3.3) proved that the sums

$$\sum_{k\in\mathbb{Z}}\frac{1}{(k^4-q^4)^{2n}},\ \text{and}\ \sum_{k\in\mathbb{Z}}\frac{1}{(k^6-q^6)^{2n}},\ q\in\mathbb{Q}\backslash\mathbb{Z},$$

are transcendental. These sums are special cases of Theorem 3, and a calculation method of series $\sum_{k\in\mathbb{Z}}\frac{1}{(k^m-q^m)^n}$ is obtained in Theorem 3.

In Theorem 3, for $m = 2$, it is easy to verify that

$$\mathbb{K}_2 = \mathbb{Q}(\sin(q\pi), \cos(q\pi)) \subset \overline{\mathbb{Q}}.$$

For $m = 3$, since $\sin(\frac{1}{3}2\pi) = -\sin(\frac{2}{3}2\pi) = \frac{\sqrt{3}}{2}$ and $\cos(\frac{1}{3}2\pi) = \cos(\frac{2}{3}2\pi) = -\frac{1}{2}$, we have

$$\widetilde{\mathbb{K}}_3 = \mathbb{K}_3 = \mathbb{Q}\left(\frac{\sqrt{3}}{2}, \sin(\frac{1}{2}q\pi), \cos(\frac{1}{2}q\pi); \exp(\frac{\sqrt{3}}{2}q\pi)\right) \subset \overline{\mathbb{Q}}\left(\exp(\frac{\sqrt{3}}{2}q\pi)\right).$$

Therefore, Theorem 3 generalizes Theorems 1 and 2.

4. Conclusions and Further Work

In this section, we make a summary of the conclusions of this paper and discuss several applications in physics and possible further work.

4.1. Conclusions

In this paper, we consider the self-adjoint differential operator with order $m \geq 2$

$$T_m u = (-i)^m u^{(m)} + \alpha u = \lambda u \tag{33}$$

on the circle $\mathbb{S}^1$, where $\alpha \neq -(2k\pi)^m$, $k = 0, \pm 1, \pm 2, \cdots$. Its k-th eigenvalue is

$$\lambda_k^{(m)} = (2k\pi)^m + \alpha,\ k = \begin{cases} 0, \pm 1, \pm 2, \cdots, & \text{for odd } m; \\ 0, 1, 2, \cdots, & \text{for even } m. \end{cases}$$

For any positive integer n, the k-th eigenvalue of T_m^n is $[\lambda_k^{(m)}]^n$.
Then, Mercer's Theorem tells us that the spectral zeta function of T_m^n satisfies

$$\zeta_m(n) := \sum_{k=-\infty}^{+\infty} \frac{1}{[(2k\pi)^m + \alpha]^n} = \int_0^1 \cdots \int_0^1 G(x_1, x_2) \cdots G(x_n, x_1) \mathrm{d}x_1 \cdots \mathrm{d}x_n, \tag{34}$$

where $G(\cdot, \cdot)$ is the Green function of differential operator (33). The spectral zeta function is closely related to the Dirichlet series, Bernoulli number and L-functions (cf. [4,10,11] §16), which arise out of number theory and other considerations; see Soulé [12] and Ramakrishnan [13].

The formula (34) gives an integral representation of $\zeta_m(n)$. Using this integral representation, we can obtain the main conclusions of the paper. See Theorem 3. For any $q \in \mathbb{Q}$, and any integer $m \geq 2$, $n \geq 1$, we have

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k^m - q^m)^n} = c_n \pi^n + \cdots + c_1 \pi,\ q^m \neq k^m,\ k \in \mathbb{Z},$$

and

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k^m + q^m)^n} = \tilde{c}_n \pi^n + \cdots + \tilde{c}_1 \pi,\ q^m \neq -k^m,\ k \in \mathbb{Z},$$

where $c_i \in \mathbb{K}_m$, $\tilde{c}_i \in \widetilde{\mathbb{K}}_m$, $i = 1, \cdots, n$. The special value and transcendental nature of the sums are related to Schneider's conjecture and Gel'fond–Schneider's conjecture. See Murty and Weatherby [1,3,4], Nesterenko [2] and Saradha and Tigdeman [5].

4.2. Application and Further Work

In this subsection, firstly, some applications about eigenvalues, eigenfunctions, Green functions, and spectral series of self-adjoint operators in physics are given. Then, according to these applications and the problems discussed in this paper, some possible future work is listed.

Consider the self-adjoint differential operator in Section 2.1,

$$Tu = -u'' + \alpha u = \lambda u, \text{ on } \mathbb{R}. \tag{35}$$

This is called the stationary Schrödinger operator (cf. Carmona and LaCroix [17]), which was proposed by physicist Schrödinger in 1926. It describes the stable state of microscopic particles, which is a basic assumption of quantum mechanics. The Schrödinger operator is widely used in atomic physics, nuclear physics and solid state physics. The results of solving a series of problems such as atoms, molecules, nuclei, solids, etc. are in good agreement with the reality.

The eigenvalue λ and the corresponding eigenfunction ψ of (35) represent the energy of microscopic particles and the probability of their occurrence somewhere in space, resp.. Furthermore, ψ is also called a state function, which is normalized according to the requirement that $\int_{\mathbb{R}} |\psi|^2 = 1$. The position of the particle is then determined not as a definite point; instead, its probable location is given by the rules of quantum mechanics as follows: the probability that the particle is located in the interval (a, b) is $\int_a^b |\psi|^2$.

Riemann zeta function is a class of spectral function of Schrödinger operator,

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Physicists found that the distribution of $\zeta(s)$ zeros as energy levels is breathtakingly similar to those of a quantum system's, cf. Schumayer and Hutchinson [18]. This has inspired physicists to associate a dynamic system with the spectral zeta function. Hence, the examination of the spectral zeta function can help to understand physics and quantum mechanics.

In 1958, the physicist P. W. Anderson found that, if impurities were added to the conductor, the electrons would be scattered by these impurities during transmission, and the multiple scattering waves would interfere with each other, resulting in the stopping of the movement of the electrons, the disappearance of the conductivity of the metal and the appearance of the nature of insulator. The phenomenon from conducting state to insulating state caused by doping is called Anderson localization.

Anderson discussed the change of the eigenfunctions of Schrödinger operator by using Green's function method. The exponential decay of the Green's fucntion is defined as

$$G(\lambda, x, y) \le e^{-\gamma|x-y|},$$

where $\gamma > 0$ and $|x - y| > \delta > 0$. In fact, the estimation of the decay of the Green's function plays a key role in the exponential localization of eigenfunctions of the Schrödinger operator. For the lattice Schrödinger operator, Bourgain [19] studied the estimations of the Green's function, and thus obtained Anderson localization.

In the following, some possible further work related to the eigenvalues and the spectral zeta series $\zeta_m(n)$ are listed.

In [6], Weatherby proved that the series

$$\sum_{k\in\mathbb{Z}, k\neq\pm1} \frac{1}{k^m - 1}, \tag{36}$$

is transcendental for $m = 3, 4, 6$. However, if the Mercer's Theorem and Green function are used, it is required that the series

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k^m - c)^n}$$

satisfies $c \neq k^m$ for any $k \in \mathbb{Z}$. Hence, the series cannot include (36) and how to calculate and study the transcendentality of series

$$\sum_{k\in\mathbb{Z}, k\neq\pm 1} \frac{1}{(k^m-1)^n}, \ m \geq 2, \ n \geq 2$$

is a problem.

Recently, many papers [8], Wainger [20], Meiners and Vertman [21] study the special values of spectral zeta functions on the discrete tori. Recall that the spectral zeta function associated with the Cayley graph $\mathbb{Z}/N\mathbb{Z}$ is (cf. [22])

$$\sum_{m=1}^{N-1} \frac{1}{\sin^{2s}(\frac{m\pi}{N})} \in \mathbb{Q}.$$

The question is how to study the the spectral zeta function associated with the more general Cayley graph.

Murty and Weatherby [1] and Nesterenko [2] studied the transcendental nature of the infinite series

$$\sum_{k\in\mathbb{Z}} \frac{f(k)}{g(k)}, \tag{37}$$

where $f(x)$ and $g(x)$ are polynomials in $\overline{\mathbb{Q}}[x]$ with $\deg f < \deg g$ so that $g(x)$ has no integral zeros. Using the connection between series and differential operator spectral zeta function, this paper can only study the special case $f(x) = 1$ and $g(x) = (x^m + c)^n$. For more general cases, the method in this paper is difficult to implement. Therefore, it is a further problem to find a suitable research method for the research on infinite series (37).

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