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On Stability Criteria induced by the Resolvent Kernel for a Fractional Neutral Linear System with Distributed Delays

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Abstract: We consider an initial problem (IP) for a linear neutral system with distributed delays and derivatives in Caputo's sense of incommensurate order, with different kinds of initial functions. In the case when the initial functions are with bounded variation, it is proven that this IP has a unique solution. The Krasnoselskii's fixed point theorem, a very appropriate tool, is used to prove the existence of solutions in the case of the neutral systems. As a corollary of this result, we obtain the existence and uniqueness of a fundamental matrix for the homogeneous system. In the general case, without additional assumptions of boundedness type, it is established that the existence and uniqueness of a fundamental matrix lead existence and uniqueness of a resolvent kernel and vice versa. Furthermore, an explicit formula describing the relationship between the fundamental matrix and the resolvent kernel is proven in the general case too. On the base of the existence and uniqueness of a resolvent kernel, necessary and sufficient conditions for the stability of the zero solution of the homogeneous system are established. Finally, it is considered a well-known economics model to describe the dynamics of the wealth of nations and comment on the possibilities of application of the obtained results for the considered systems, which include as partial case the considered model.

Keywords: fractional derivatives; neutral fractional systems; distributed delay; resolvent kernel; fundamental matrix; stability

MSC: 34A08; 34A12

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1. Introduction

Fractional calculus is not a new mathematical tool, but in view of its application for modeling many real-world phenomena, it has attracted considerable attention in recent decades. Comprehensive information about the fractional calculus theory, fractional differential equations and its applications can be found in the monographs [1–3]. For a practical oriented exposition of this theme, we refer to [4,5].

As is mentioned in [6], a predictable process can be physically realized only if it is stable in some suitable natural sense. It must be noted that the ascertainment of this fact is obtained in general from practical experience. This fact explains why the investigation of the property stability of the models of processes is so important. The same is true concerning the existence of a fundamental matrix and the existence of an integral representations of their solutions, since they are the main tools for investigation of the stability properties. It is well-known, generally speaking, that for the delayed fractional differential equations, stability properties are more complicated to be studied in comparison with fractional differential equations and systems without delay.

An excellent historical overview of the works related to this theme up until 2011 can be obtained from the survey [7] and the references therein. Concerning the recent works related to this theme devoted to fractional equations and systems without delay we refer to [8,9]. The autonomous case for retarded (delayed) and neutral fractional equations and systems on this theme are considered in [10] for single constant delay, while [11,12] discusses distributed delay. Regarding distributed delay, systems have also been studied using Riemann–Liouville type derivatives. The singular case is treated in [13] and the case of distributed order fractional derivatives is considered in [14]. The integral representation for the neutral case with distributed delay is studied in [15] and some explicit conditions for stability of the same type systems in terms of logarithmic norm (Losinskii measure) are given in [16]. From the works devoted to the integral representation of the solutions for retarded fractional equations and systems in the nonautonomous case, we refer to [17], which treated a system with single delay, and for the case of distributed delay, we refer to [18]. The same problem in the neutral case is considered in [19] (single variable delay), while in [20] discusses the case of distributed delay. The important problem of the existence and uniqueness of a fundamental matrix and its smoothness is studied for the retarded systems in [21] and for neutral systems in [22]. Concerning the investigation of the different kinds of stability of the zero solution for linear retarded fractional systems, we note the remarkable work of [23], devoted to the case of multiple concentrated delays, [24], which studies the asymptotic stability of a system with distributed delays, and [25], devoted to the finite time stability. Different kinds of stability criteria for a linear system with multiple delays are also considered in [26], while [27] considers the neutral case with distributed delays and with the Riemann–Liouville type derivatives. The stability of the zero solution of nonlinear fractional systems were studied under different approaches. In the work [28], the asymptotic stability is studied via linearization, and in [29,30], for retarded and neutral systems with distributed delays, respectively, the preservation of the asymptotic stability of linear systems is studied under nonlinear perturbation. For numerical aspects, we refer to [31,32].

The present work is inspired by [33] and is mainly devoted of the solving of an open problem posed in this work, which investigated a neutral system with fractional derivatives in Caputo sense, of incommensurate orders belonging to the interval $(0, 1)$ and distributed delays. Our approach is based on a direct construction of a fundamental matrix under as minimal as possible restrictions, which leads to the solving of an auxiliary matrix IP for the studied system with discontinuous initial matrix-valued piecewise continuous or bounded variation on every compact interval in $\mathbb{R}$ initial functions. As an application of the obtained results, we obtain some analytical properties of the fundamental matrix. Note that the conditions and the obtained results are similar to those in the case of delayed systems with integer derivatives.

The paper is organized as follows. In Section 2, we recall the definitions of Riemann–Liouville and Caputo fractional derivatives; some with additional theorems and notations. Section 3 is devoted to the problem of the existence and uniqueness of the fundamental matrix of the homogeneous system. Section 4 is devoted to an application of the established results. We prove the existence and uniqueness of the corresponding resolvent kernel, without assumptions of uniform boundness in t. In addition, we also study the relationship between the resolvent kernel and the fundamental matrix and establish some analytical properties. In Section 5, a well-known economics model is considered, describing the dynamic of the wealth of nations, and some comments are made on the possibilities of the application of the obtained results for the considered systems, which include as partial case the considered model. In this section, some conclusions are also given.

2. Preliminaries and Problem Statement

To avoid misunderstandings, below are given the definitions of the Riemann–Liouville and Caputo fractional derivatives as well as their properties. Detailed information can be found in the monographs [1,2].

For $a \in \mathbb{R}$ and $\alpha \in (0,1)$ arbitrarily, for each $t > a$, the left-sided fractional integral operator and the left side Riemann–Liouville of order α are defined via $(D_{a+}^{-\alpha} f)(t) = \Gamma^{-1}(\alpha) \int\limits_a^t (t-s)^{\alpha-1} f(s) ds$ and ${}_{RL}D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(D_{a+}^{\alpha-1} f(t) \right)$, respectively.

The Caputo fractional derivative of same order by ${}_{C}D_{a+}^{\alpha} f(t) = {}_{RL}D_{a+}^{\alpha}[f(s) - f(a)](t)$, where $f \in L_1^{loc}(\mathbb{R}, \mathbb{R})$, $L_1^{loc}(\mathbb{R}, \mathbb{R})$ denote the real linear space of all locally Lebesgue integrable functions $f : \mathbb{R} \to \mathbb{R}$ and $BL_1^{loc}(\mathbb{R}, \mathbb{R}) \subset L_1^{loc}(\mathbb{R}, \mathbb{R})$ for the subspace of all locally bounded functions.

Below the following notations will be used: $\mathbb{R}_+ = (0, \infty)$, $\overline{\mathbb{R}}_+ = [0, \infty)$, $\mathbb{N}_0 = \bigcup \{0\}$, $k \in \langle n \rangle = \{1, 2, ..., n\}$, $n \in \mathbb{N}$, $i \in \langle m \rangle_0 \bigcup \{0\}$, $J_a = [a, \infty)$, $a \in \mathbb{R}$, $J_{a+b} = [a, a+b]$, $b \in \mathbb{R}_+$, $I, \Theta \in \mathbb{R}^{n \times n}$, which are the identity and zero matrix, respectively, and $\mathbf{0} \in \mathbb{R}^n$ is the zero vector-column.

For $W : J_a \times \mathbb{R} \to \mathbb{R}^{n \times n}$, $W(t, \theta) = \{w_{kj}(t, \theta)\}_{k,j=1}^n$, we denote $|W(t, \theta)| = \sum\limits_{k,j=1}^{n} |w_{kj}(t, \theta)|$, $(t, \theta) \in J_a \times \mathbb{R}$, $W(t, \theta) \in BV_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n \times n})$, which denote the functions $W(t, \theta)$, which, for every fixed $t \in J_a$, has bounded variation in θ on every compact interval $K \subset \mathbb{R}$ and denotes $\mathrm{Var}_{\theta \in K} W(t, \theta) = \{\mathrm{Var}_{\theta \in K} w_{kj}(t, \theta)\}_{k,j=1}^n$ and $|\mathrm{Var}_{\theta \in K} W(t, \cdot)| = \sum\limits_{k,j=1}^{n} \mathrm{Var}_{\theta \in K} w_{kj}(t, \cdot)$.
$Y(t) = (y^1(t), ..., y^n(t))^T : J_a \to \mathbb{R}^n$, $Y(t) \in BV_{loc}(J_a, \mathbb{R}^n)$ denote the functions with bounded variation in every compact interval $K \subset \mathbb{R}$ and

$$I_\beta(Y(t)) = \mathrm{diag}((y_1(t))^{\beta_1}, \ldots, (y_n(t))^{\beta_n}), \text{ where } \beta = (\beta_1, ..., \beta_n),\ \beta_k \in [-1, 1],\ k \in \langle n \rangle.$$

It is well known that for $z \in \mathbb{R}_+$, the gamma function has a minimum at $z_{min} \approx +1.46$, where it attains the value $\Gamma(z_{min}) \approx +0.8856$ (truncated).

With **BL**, we denote the Banach space of all vector functions $\Phi = (\phi_1, ..., \phi_n)^T : [-h, 0] \to \mathbb{R}^n$, which are bounded and Lebesgue measurable on the interval $[-h, 0]$ with norm $||\Phi|| = \sum\limits_{k \in \langle n \rangle} \sup\limits_{s \in [-h,0]} |\phi_k(s)| < \infty$ and S^Φ denotes the set of all jumps point of arbitrary $\Phi \in \mathbf{BL}$.
$\mathbf{PC} = PC([-h, 0], \mathbb{R}^n)$ denote the subspace of all right continuous for $t \in S^\Phi$ piecewise continuous functions, $\mathbf{PC}^* = \mathbf{PC} \cap BV([-h, 0], \mathbb{R}^n)$, $\mathbf{C} = C([-h, 0], \mathbb{R}^n)$, and all of them are endowed with the same sup norm.

For $t > a$ consider the inhomogeneous neutral linear delayed system with and distributed delays in the form:

$$D_{a+}^{\alpha}(X(t) - \int\limits_{-h}^{0} [d_\theta V(t, \theta)] X(t+\theta)) = \int\limits_{-h}^{0} [d_\theta U(t, \theta)] X(t+\theta) + F(t), \tag{1}$$

and the corresponding homogeneous neutral linear system:

$$D_{a+}^{\alpha}(X(t) - \int\limits_{-h}^{0} [d_\theta V(t, \theta)] X(t+\theta)) = \int\limits_{-h}^{0} [d_\theta U(t, \theta)] X(t+\theta), \tag{2}$$

where the differential orders, $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_k \in (0,1)$, can be incommensurate, $U^i, V^l : J_a \times \mathbb{R} \to \mathbb{R}^{n \times n}$, $V(t, \theta) = \{v_{kj}(t, \theta)\}_{k,j=1}^n = \sum\limits_{l \in \langle r \rangle} V^l(t, \theta)$, $r \in \mathbb{N}$, $U(t, \theta) = \{u_{kj}(t, \theta)\}_{k,j=1}^n = \sum\limits_{i \in \langle m \rangle_0} U^i(t, \theta)$, $F : J_a \to \mathbb{R}^n$, $X(t) = (x_1(t), ..., x_n(t))^T$, $X_t(\theta) = (x_t^1(\theta), ..., x_t^n(\theta))^T$, $X_t(\theta) = X(t+\theta)$, $-h \le \theta \le 0$, $h \in \mathbb{R}_+$, $F \in BL_1^{loc}(J_a, \mathbb{R})$, $D_{a+}^{\alpha} X(t) = (D_{a+}^{\alpha_1} x_1(t), ..., D_{a+}^{\alpha_n} x_n(t))^T$, for every $t \in J_a$, where $D^{\alpha_k}_{a+} = {}_C D^{\alpha_k}_{a+}$ denotes the left side Caputo fractional derivative. For clarity, we rewrite the system (2) in more detail form:

$$D^{\alpha_k}_{a+}(x_k(t) - \sum_{l=1}^{r}(\sum_{j=1}^{n}\int_{-\tau_l}^{0} x_j(t+\theta)d_\theta v^l_{kj}(t,\theta))) = \sum_{i=0}^{m}(\sum_{j=1}^{n}\int_{-\sigma_i}^{0} x_j(t+\theta)d_\theta u^i_{kj}(t,\theta)),$$

where $\sigma_i, \tau_l \in \mathbb{R}_+$, $\tau = \max_{l\in\langle r\rangle}\tau_l$, $\sigma = \max_{i\in\langle m\rangle_0}\sigma_i$, $h = \max(\tau,\sigma)$ and for arbitrary $\Phi \in \mathbf{BL}$, we introduce the following initial condition:

$$X_t(\theta) = X(t+\theta) = \Phi(t-a+\theta) \text{ for } t+\theta \le a, \theta \in [-h,0]. \tag{3}$$

In our consideration below, we need the following auxiliary system:

$$\begin{aligned} X(t) &= C_{\Phi(0)} + \int_{-h}^{0}[d_\theta V(t,\theta)]X(t+\theta) \\ &\quad + I_{-1}(\Gamma(\alpha))\int_{a}^{t} I_{\alpha-1}(t-\tau)(\int_{-h}^{0}[d_\theta U(\tau,\theta)]X(\tau+\theta) + F(\tau))d\tau, \end{aligned} \tag{4}$$

where $C_{\Phi(0)} = \Phi(0) - \int_{-h}^{0}[d_\theta V^l(a,\theta)]\,\Phi(\theta)$.

Definition 1. *We say that for the kernels $U^i, V^l : J_a \times \mathbb{R} \to \mathbb{R}^{n\times n}$, the conditions* **(S)** *are fulfilled if for every $l \in \langle r\rangle, i \in \langle m\rangle_0$, the following conditions hold (see [6,33]):*

(S1) *The functions $(t,\theta) \to U^i(t,\theta)$, $(t,\theta) \to V^l(t,\theta)$ are measurable in $(t,\theta) \in J_a \times \mathbb{R}$ and normalized so that $U(t,\theta) = V(t,\theta) = 0$ for $\theta \ge 0$, $U^i(t,\theta) = U^i(t,-\sigma_i)$ for $\theta \le -\sigma_i$, $V^l(t,\theta) = V^l(t,-\tau_l)$ for $\theta \le -\tau_l$ and every $t \in J_a$. The kernels $U^i(t,\theta)$ and $V^l(t,\theta)$ are continuous from left in θ on $(-\sigma_i,0)$ and $(-\tau_l,0)$, respectively, for $t \in J_a$, and $U^i(t,\theta), V^l(t,\theta) \in BV_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ for every fixed $t \in J_a$.*

(S2) *For $t \in J_a$, the functions*

$$V^*(t) = \mathrm{Var}_{[-h,0]}V(t,\cdot),\ U^*(t) = \mathrm{Var}_{[-h,0]}U(t,\cdot) \in BL^{loc}_1(J_a,\mathbb{R}^{n\times n})$$

and the kernel $V(t,\theta)$ is uniformly nonatomic at zero (i.e. for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for each $t \in J_a$, we have $\mathrm{Var}_{[-\delta,0]}V(t,\cdot) < \varepsilon$ ([34]).

(S3) *For $t \in J_a$ and $\theta \in [-h,0]$, the Lebesgue decompositions of the kernels have the form: $U^i(t,\theta) = U^i_d(t,\theta) + U^i_c(t,\theta)$, $V^l(t,\theta) = V^l_d(t,\theta) + V^l_c(t,\theta)$, $U^i_c(t,\theta) = U^i_{ac}(t,\theta) + U^i_s(t,\theta)$, $V^l_c(t,\theta) = V^l_c(t,\theta) + V^l_s(t,\theta)$, where the indexes d, ac, s denoted the jump, the absolutely continuous and the singular part, respectively, in the Lebesgue decompositions and $U^i_c(t,\theta), V^l_c(t,\theta)$ are the continuous part of these decompositions. In addition, $U^i_d(t,\theta) = \{a^i_{kj}(t)H(\theta+\sigma_i(t))\}^n_{k,j=1}$, $V^l_d(t,\theta) = \{\bar{a}^l_{kj}(t)H(\theta+\tau_l(t))\}^n_{k,j=1}$, where $H(t)$ is the Heaviside function:*

$$A^i(t) = \{a^i_{kj}(t)\}^n_{k,j=1} \in BL^{loc}_1(J_a,\mathbb{R}^{n\times n}),\ \overline{A}^l(t) = \{\bar{a}^l_{kj}(t)\}^n_{k,j=1} \in C(J_a,\mathbb{R}^{n\times n}),$$

$\sigma_0(t) \equiv 0$, $\sigma_i(t) \in C(J_a,[0,\sigma_i])$, $\tau_l(t) \in C(J_a,[0,\tau_l])$ for every $t \in J_a$.

(S4) *For each $t^* \in J_a$, the relations:*

$$\lim_{t\to t^*}\int_{-\sigma}^{0}|U^i(t,\theta) - U^i(t^*,\theta)|d\theta = \lim_{t\to t^*}\int_{-\tau}^{0}|V^l(t,\theta) - V^l(t^*,\theta)|d\theta = 0$$

hold, and there exists $\gamma \in (0,\infty)$ such that the kernels $V^l_{ac}(t,\theta)$ and $V^l_s(t,\theta)$ are continuous in t when $t \in [a, a+\gamma]$, $\theta \in [-h,0]$ and $l \in \langle r\rangle$.

(S5) *The sets $S^l_\Phi = \{t \in [a,a+h] | t - \tau_l(t) - a \in S^\Phi\}$, $S^i_\Phi = \{t \in [a,a+h] | t - \sigma_i(t) - a \in S^\Phi\}$, for every $\Phi \in \mathbf{PC}$, do not have limit points.*

Definition 2. *The vector function $X(t) = (x_1(t), ..., x_n(t))^T$ is a solution of the initial problem (IP) (1), (3) in $J_{a+b}(J_a)$, $b \in \mathbb{R}_+$, if $X|_{[a,a+b]} \in C([a,a+b], \mathbb{R}^n)$, $(X|_{J_a} \in C(J_a,\mathbb{R}^n))$, $X(t)$ satisfies the system (1) for all $t \in (a,b]$ ($t \in (a,\infty)$) and the initial condition (3) for $t+\theta \le a$, $\theta \in [-h,0]$.*

Definition 3. *The vector function $X(t) = (x_1(t), ..., x_n(t))^T$ is a solution of the initial problem (IP) (4), where (3) in $J_{a+b}(J_a)$ if $X|_{[a,a+b]} \in C([a,a+b], \mathbb{R}^n)$, $(X|_{J_a} \in C(J_a,\mathbb{R}^n))$, $X(t)$ satisfies the system (4) for all $t \in (a,b]$ $(t \in (a,\infty))$ and the initial condition (3) for $t+\theta \le a$, $\theta \in [-h,0]$.*

In this paper, we assume that the condition **(S)** holds. For arbitrary fixed $s \in J_a$, we define for $(t,s) \in J_a \times \mathbb{R}$ two initial matrix functions:

$$\Phi_C(t,s) = \begin{cases} I, \ t = s \\ \Theta, \ \ t < s, \ (t,s) \in J_a \times \mathbb{R} \\ \Theta, \ s < a \end{cases}, \ \Phi_Q(t,s) = \begin{cases} I, \ -h \le s \le t \le a \\ \Theta, \ \ t < s \\ \Theta, \ s < -h \end{cases}$$

and introduce the auxiliary matrix initial problem:

$$D^\alpha_{a+}(W(t,s) - \int_{-h}^{0} [d_\theta V(t,\theta)]W(t+\theta,s)) = \int_{-h}^{0} [d_\theta U(t,\theta)]W(t+\theta,s), \ t \in (s,\infty); \tag{5}$$

$$W(t,s) = \Phi_C(t,s), \ t \in (-\infty,s]; \tag{6}$$

$$W(t,s) = \Phi_Q(t,s), \ t \in (-\infty,s]. \tag{7}$$

Definition 4. *For each $s \in J_a$, the matrix-valued function $C(\cdot,s) : \mathbb{R} \to \mathbb{R}^{n\times n}$, $C(t,s) = \{c_{kj}(t,s)\}_{k,j\in\langle n\rangle}$ is called a solution of the IP (5), (6) for $t \in (s,\infty)$, if $C(\cdot,s)$ is continuous in t on J_s, satisfying the matrix Equation (5) for $t \in (s,\infty)$, as well as the initial condition (6).*

Definition 5. *For each $s \in J_a$ the matrix-valued function $Q(\cdot,s) : \mathbb{R} \to \mathbb{R}^{n\times n}$, $Q(t,s) = \{q_{kj}(t,s)\}_{k,j\in\langle n\rangle}$ is called a solution of the IP (5), (7) for $t \in (a,\infty)$, if $Q(\cdot,s)$ is continuous in t on J_a, satisfying the matrix Equation (5) for $t \in (a,\infty)$, as well as the initial condition (7).*

The solution of the IP (5), (6) $C(t,s)$ is called the fundamental matrix of the system (2), and obviously, $C(t,a) = Q(t,a)$.

In our exposition below, we will need three theorems presented below, one of them in a slightly modified version.

Theorem 1. *([35], page 17) Let the following conditions be fulfilled:*

(i) *The conditions* **(S)** *hold and $h \ge \max(\tau,\sigma)$ is arbitrary fixed number.*

(ii) *The functions $G \in C([a-h,\infty),\mathbb{R}^n)$.*

Then, the function $Y(t) = \int_{h}^{0} [d_\theta V(t,\theta]G(t+\theta)$ is continuous in t for $(t,\theta) \in J_a \times [-h,0]$.

Theorem 2. *([36] Krasnosel'skii's fixed point theorem) Let E be a Banach space with norm $\|\cdot\|$, M be a nonempty, closed and convex subset of E and for the maps $\mathbf{T}, \mathbf{S} : M \to E$, the following conditions hold:*

(i) $\mathbf{T}$ *is contraction with constant $\beta \in (0,1)$;*

(ii) $\mathbf{S}$ *is continuous and the set* $\mathbf{S}(M)$ *is contained in a compact set;*
(iii) $\mathbf{T}x + \mathbf{S}y \in M$ *for every* $x, y \in M$.

Then, there exist a $y \in M$ *with* $\mathbf{T}y + \mathbf{S}y = y$.

Theorem 3. *(Theorem 4 in [16]) Let the following conditions hold:*

(i) *The conditions* **(S)** *hold;*
(ii) $\sup\limits_{t\in J_a} Var_{\theta\in[-h,0]}|V(t,\theta)| = \mathbf{V} < 1$;
(iii) *For each solution of IP (4), (3) with arbitrary initial function* $\Phi \in \mathbf{C}$ *the following inequality holds:*

$$|Y(t)| \leq a(t) + g(t)\left|\int\limits_a^t |(I_{-1}(\Gamma(\alpha))I_{\alpha-1}t-\eta)||Y(\eta)|d\tau\right|$$

for $t \in [a, T)$, *where* $a(t) \in L_1^{loc}([a, T), \mathbb{R}_+)$ *and* $g(t) \in C([a, T), \mathbb{R}_+)$ *are nondecreasing,* $a < T \leq \infty$.

Then, the solution of IP (4), (3) is Mittag–Leffler (ML) bounded of order α_M *for* $t \in [a, T)$, *i.e.*

$$Y(t) \leq a(t)E_{\alpha_M}(g(t)\Gamma(\alpha_M)t^{\alpha_M}).$$

3. Main Results

The aim of this section is two-fold: First, with Theorem 4, we provide a positive answer of the open problem stated in [33]. Second, we study the existence and uniqueness of the solutions of the IP (4) and (3) in the cases when $\Phi \in \mathbf{PC}^*$.

The next definitions clear the possible interaction between the concentrated delays in the neutral part of the system, i.e. the low terminal a of the fractional derivatives, in the case when $\Phi \in \mathbf{PC}$ in both cases, i.e. $a \notin S^\Phi$ and $a \in S^\Phi$.

Definition 6. *[20] The low terminal* a *will be called a noncritical point (noncritical jump point) for some initial function* $\Phi \in \mathbf{BL}$ *relative to the delay* $\tau_l(t)$, $l \in \langle r\rangle$ *if the equality* $\tau_l(a) = 0$ *implies that there exists a constant* $\varepsilon \in (0, h]$ *(eventually depending from* τ_l*), such that* $t - a < \tau_l(t) \leq t$ *for* $t \in (a, a+\varepsilon]$.

Definition 7. *[20] The low terminal* a *for arbitrary function* $\Phi \in \mathbf{BL}$ *with* $a \notin S^\Phi$ *(*$a \in S^\Phi$*) will be called a critical point (critical jump point) relative to some delay* $\tau_l(t)$, $l \in \langle r\rangle$ *if the equality* $\tau_l(a) = 0$ *implies that there exists a constant* $\varepsilon \in (0, h]$ *(eventually depending from* τ_l*), such that* $t - a \geq \tau_l(t)$ *for* $t \in [a, a+\varepsilon]$.

It is simple to see that without loss of generality, we can renumber all delays $\tau_l(t)$, so that those for which $a \notin S^\Phi$ ($a \in S^\Phi$) is a noncritical or critical point (noncritical or critical jump point), to have the numbers $\tau_1(t), ..., \tau_q(t)$, $q \leq r$. In the next exposition, for convenience, we assume that this renumbering is made.

Theorem 4. *Let the following conditions hold:*

(i) *The conditions* **(S)** *hold;*
(ii) *The low terminal* a *is either a critical point (*$a \notin S^\Phi$*), or a critical jump point (*$a \in S^\Phi$*), relative the delays* $\tau_1(t), ..., \tau_q(t)$, $1 \leq q \leq r$.

Then, either $\sum\limits_{l\in\langle q\rangle}\left|\overline{A}^l(a)\right| < 1$ *or there exists* $\varepsilon \in (0, h]$ *such that* $V_d^l(t,\theta) \equiv \Theta$ *for* $t \in [a, a+\varepsilon]$, $\theta \in [-h, 0]$ *and* $l \in \langle q\rangle$.

Proof. Assume the contrary, that $\sum_{l\in\langle q\rangle} \left|\overline{A}^l(a)\right| \geq 1$. Then, there exists $\varepsilon_1 > 0$ such that $\sum_{l\in\langle q\rangle} \left|\overline{A}^l(t)\right| \geq \frac{1}{2}$ for $t \in [a, a+\varepsilon_1]$.

From condition **(S2)**, it follows that $\delta > 0$, such that for $t \in [a, a+\varepsilon_1]$, we have

$$\sum_{l\in\langle q\rangle} \left|\mathrm{Var}_{\theta\in[-\delta,0]} V_d^l(t,\theta)\right| \leq \frac{1}{4}.$$

Since $\left|V_d^l(t,\theta)\right| \leq \left|V_d^l(t,0)\right| + \left|Var_{\theta\in[-\delta,0]} V_d^l(t,\theta)\right| = \left|Var_{\theta\in[-\delta,0]} V_d^l(t,\theta)\right|$, we have $\sum_{l\in\langle q\rangle} \left|V_d^l(t,\theta)\right| \leq \sum_{l\in\langle q\rangle} \left|Var_{\theta\in[-\delta,0]} V_d^l(t,\theta)\right| \leq \frac{1}{4}$ for $t \in [a, a+\varepsilon_1]$ too.

For definiteness, we assume that the delays with critical points nave numbers from 1 to q and we will consider both cases: $(\mathbf{i})$ $a \in S^\Phi$, then a critical jump point for the delays $\tau_1(t), \dots, \tau_q(t)$, and $(\mathbf{ii})$ $a \notin S^\Phi$, which is a critical point for these delays. In both cases, two possibilities exist:

(a) There exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that $\tau_l(t) > 0$, for $t \in (a, a+\varepsilon_2]$ and $l \in \langle q\rangle$;
(b) There exists a monotone decreasing sequence $\{t_j\}_{j\in\mathbb{N}} \subset J_a$ with $\lim_{j\to\infty} t_j = a$ such that $\tau_l(t_j) = \tau_l(a) = 0$, $j \in \mathbb{N}$ for some $l \in \langle q\rangle$.

$(\mathbf{i})$ Consider first the case when $a \in S^\Phi$ is a critical jump point for some $l \in \langle q\rangle$. Then, in the case (b), the set S^Φ includes infinitely many points, and hence, we will have at least one accumulation point which contradicts condition **(S5)**.
For case (a), let $t^* \in (a, a+\varepsilon_2]$ be arbitrary with $\tau_l(t^*) \in (0, \delta)$. Then, for $\theta \in [-\tau_l(t^*), 0]$, we have:

$$\begin{aligned}\sum_{l\in\langle q\rangle} \left|V_d^l(t^*,\theta)\right| &\leq \sum_{l\in\langle q\rangle} \left|V_d^l(t^*,0)\right| + \left|Var_{\theta\in[-\tau_l(t^*),0]} V_d^l(t^*,\theta)\right| \\ &= \sum_{l\in\langle q\rangle} \left|Var_{\theta\in[-\tau_l(t^*),0]} V_d^l(t^*,\theta)\right| \leq \sum_{l\in\langle q\rangle} \left|Var_{\theta\in[-\delta,0]} V_d^l(t^*,\theta)\right| \leq \frac{1}{4}.\end{aligned} \tag{8}$$

From the other side for $\theta \in [-\tau_l(t^*), 0]$, we have:

$$\sum_{l\in\langle q\rangle} \left|V_d^l(t^*,\theta)\right| = \sum_{l\in\langle q\rangle} \left|\overline{A}^l(t^*)H(\theta+\tau_l(t^*))\right| = \sum_{l\in\langle q\rangle} \left|\overline{A}^l(t^*)\right| \geq \frac{1}{2},$$

which contradicts with (8).
$(\mathbf{ii})$ Let $a \notin S^\Phi$ be a critical point for the delays. It is simple to see that case (a) can be treated as case (a) in the former point $(\mathbf{i})$.

In case (b), we have two possibilities: the first is that for all $l \in \langle q\rangle$, there exists $\varepsilon > 0$, such that $\tau_l(t) \equiv 0$ for $t \in [a, a+\varepsilon]$. Then, we obtain that $H(\theta+\tau_l(t)) = H(\theta) = 0$ holds for $t \in [a, a+\varepsilon_2]$ and $l \in \langle q\rangle$. Then, we have:

$$\sum_{l\in\langle q\rangle} V_d^l(t,\theta) = \sum_{l\in\langle q\rangle} \overline{A}^l(t)H(\theta+\tau_l(t)) = \sum_{l\in\langle q\rangle} \overline{A}^l(t)H(\theta) = \Theta. \tag{9}$$

The second possibility is that there exists a monotone decreasing sequence $\{t_j\}_{j\in\mathbb{N}} \subset J_a$, with $\lim_{j\to\infty} t_j = a$, such that $\tau_l(t_j) = \tau(a) = 0$, $j \in \mathbb{N}$ for some $l \in \langle q\rangle$ and there exists $j_{n^0} \in \mathbb{N}$, such that for $j \geq j_{n^0}$, we have that $\delta > \tau_l(t_j^*) > 0$, where $j \geq j_{n^0}$ and $t_j^* = \max_{t\in[t_{j+1},t_j]} \tau_l(t)$.

Then, for $j \geq j_{n^0}$ and $\theta \in [-\tau_l(t_j^*), 0]$, we have:

$$\sum_{l\in\langle q\rangle} \left|V_d^l(t_j^*,\theta)\right| = \sum_{l\in\langle q\rangle} \left|\overline{A}^l(t_j^*)H(\theta+\tau_l(t_j^*))\right| = \sum_{l\in\langle q\rangle} \left|\overline{A}^l(t^*)\right| \geq \frac{1}{2}. \tag{10}$$

From the other side, as in (8) for $\theta \in [-\tau_l(t_j^*), 0]$, we obtain:

$$\sum_{l \in \langle q \rangle} \left| V_d^l(t_j^*, \theta) \right| \leq \sum_{l \in \langle q \rangle} \left| Var_{\theta \in [-\tau_l(t^*), 0]} V_d^l(t_j^*, \theta) \right| \leq \sum_{l \in \langle q \rangle} \left| Var_{\theta \in [-\delta, 0]} V_d^l(t_j^*, \theta) \right| \leq \frac{1}{4},$$

which contradicts with (10). □

Remark 1. *Both Definitions 6 and 7 lead to the following essentially question: are there only these two possibilities or do others exist? In fact, in both definitions, we suppose that either there exists* $\varepsilon \in (0, h]$ *such that the equation* $t - \tau_l(t) = a$ *has no roots for each* $t \in (a, a+\varepsilon]$ *and* $l \in \langle r \rangle$*. Then, for* $t \in (a, a+\varepsilon]$ *and* $l \in \langle r \rangle$*, we have that either* $t - \tau_l(t) < a$ *(noncritical case), or* $t - \tau_l(t) \geq a$ *(critical case). The answer is that in the case when* $a \in S^\Phi$*, from the condition* (**S5**)*, it follows that there are no other possibilities. Indeed, if we assume that for every* $\varepsilon \in (0, h]$*, the equation* $t - \tau_l(t) = a$ *has at least one root for some* $l \in \langle r \rangle$*, then the set* $\bigcup_{l \in \langle r \rangle} S_\Phi^l$ *will have an accumulation point, which contradicts with condition* (**S5**)*. In the case when* $a \notin S^\Phi$*, it is possible that the case* (**ii**) *(b) can appear in addition. Thus, from Theorem 4, it follows that the Condition 3 in Theorem 2 in [20] is unnecessary.*

Let $J_L = \{t_0, ..., t_L\},\ L \in \mathbb{N}$, with $J_L \subset (a-h, a]$ being an arbitrary finite particle of $[a-h, a]$ and defining the real linear subspaces of **PC** and **PC*** as follows:

$$E(J_L) = \{\Phi : [a-h, a] \to \mathbb{R}^n |\ \Phi \in \mathbf{PC},\ S^\Phi \subseteq J_L\},$$
$$E^*(J_L) = \{\Phi : [a-h, a] \to \mathbb{R}^n |\ \Phi \in \mathbf{PC}^*,\ S^\Phi \subseteq J_L\}.$$

It is clear that since $\mathbf{PC}^* \subset \mathbf{PC}$, then $E^*(J_L)$ is a subspace of $E(J_L)$.

Following the approach introduced in [18], we can define the real linear space:

$$E = \{G : [a-h, \infty) \to \mathbb{R}^n |\ \ G|_{J_a} \in C(J_a, \mathbb{R}^n),\ G(t) = \Phi(t),\ t \in [a-h, a], \Phi \in E(J_L)\},$$

and its linear subspace:

$$\begin{aligned} E^* = &\{\overline{G} : [a-h, \infty), \to \mathbb{R}^n |\ \ \overline{G}|_{J_a} \in C(J_a, \mathbb{R}^n) \cap BV_{loc}(J_a, \mathbb{R}^n), \\ &\overline{G}(t) = \Phi(t),\ t \in [a-h, a], \Phi \in E^*(J_L)\}. \end{aligned}$$

For arbitrary fixed $M \in (0, h]$, the real linear space is as follows:

$$E_M = \{G : [a-h, a+M] \to \mathbb{R}^n |\ \ G = G|_{[a-h, a+M]},\ G \in E\},$$

endowed with the norm $\|G\|_s = \sup_{t \in [a-h, a+M]} |G(t)|$, which is the real Banach space and considers the linear subspace of E_M:

$$E_M^* = \{G : [a-h, a+M] \to \mathbb{R}^n |\ \ G = \overline{G}|_{[a-h, a+M]},\ \overline{G} \in E^*\}.$$

The subspace E_M^* is endowed with the following norm:

$$\begin{aligned} \|G\|_{Var} &= |G(a-h)| + \underset{s \in [a-h, a+M]}{Var} |G(s)| = |G(a-h)| + \underset{s \in [a-h, a]}{Var} |\Phi(s)| + \underset{s \in [a, a+M]}{Var} |G(s)| \\ &= |G(a-h)| + \sum_{k=1}^{n} \underset{s \in [a-h, a]}{Var} |\phi_k(s)| + \sum_{k=1}^{n} \underset{s \in [a, a+M}{Var} |g_k(s)|, \end{aligned}$$

which is a Banach space concerning the norm $\|\cdot\|_{Var}$ for every $M \in (0, h]$ (see [25]). The important question which needs an answer is as follows: is the space E_M^* endowed with the norm $\|\cdot\|_s$ a closed subspace of E_M? It must be noted that the following lemma that answers the question above is a generalization of Lemma 1 in [26] in the case of Banach spaces.

Lemma 1. *For arbitrary fixed $M \in (0, h]$, the space E_M^* endowed with the norm $\|\cdot\|_s$ is a closed subspace of E_M.*

Proof. The proof of the statement uses the idea of the proof of Lemma 1 in [26], and in that way, we will mainly be sketching the differences.

Let $M \in (0, h]$ be an arbitrary fixed number and let $\{G^q(t) = (g_1^q(t), ..., g_n^q(t, s))^T\}_{l=1}^{\infty} \subset E_M^*$ be an arbitrary Cauchy sequence under the norm $\|\cdot\|_s$. That means that for each $\varepsilon > 0$, there exists a number $q_0(\varepsilon) \in \mathbb{N}$, such that for every $p \in \mathbb{N}$, we have $\|G^{q_0} - G^{q_0+p}\|_s < \varepsilon$, and since E_M is a Banach space, there exists $G^0(t, s) = (g_1^0(t), ..., g_n^0(t))^T \in E_M$, such that $\lim\limits_{p \to \infty} \|G^0 - G^{q_0+p}\|_s = 0$.

Thus, there exists a number $q_1(\varepsilon) \geq q_0(\varepsilon)$, such that for every $p \in \mathbb{N}$, we have $\|\Phi^0 - \Phi^{q_1+p}\|_s < \varepsilon$, where $\Phi^0 = G^0|_{[a-h,a]}$, $\Phi^q = G^q|_{[a-h,a]}$, and hence, $\Phi^0(t)$ has finitely many jumps and $S^{\Phi^0} \subseteq J_N$.

For an arbitrary partition $P_N = \{t_0, ..., t_N\},\ N \in \mathbb{N}$, of the interval $[a-h, a+M]$, $t_0 = a - h,\ t_N = a + M$, we have:

$$\sum_{i=0}^{N-1} |g_k^q(t_{i+1}) - g_k^q(t_i)| \leq V_k^q, \tag{11}$$

for every $q \in \mathbb{N}$ and arbitrary fixed $k \in \langle n \rangle$, where $V^q = \left| Var_{t \in [a-h, s+M]} G^q(t) \right| = \sum\limits_{k=1}^{n} Var_{t \in [a-h, s+M]} g_k^q(\cdot, s) = \sum\limits_{k=1}^{n} V_k^q$. Then, there exists a number $q_2(\varepsilon) \geq q_1(\varepsilon)$, such that $\|G^{q_2} - G^{q_2+p}\|_s < \varepsilon$ for each $p \in \mathbb{N}$ and the inequality $|g_k^{q_2}(t_i) - g_k^{q_2+p}(t_i)| < \frac{\varepsilon}{2N}$ holds for $t_j \in [a-h, a+M]$. Thus, for $j \in \langle N-1 \rangle_0$, the inequalities:

$$\begin{aligned} g_k^{q_2}(t_{j+1}) - \frac{\varepsilon}{2N} &< g_k^{q_2+p}(t_{j+1}, s) < g_k^{q_2}(t_{j+1}) + \frac{\varepsilon}{2N}, \\ -g_k^{q_2}(t_j) - \frac{\varepsilon}{2N} &< -g_k^{q_2+p}(t_j) < -g_k^{q_2}(t_j) + \frac{\varepsilon}{2N} \end{aligned} \tag{12}$$

holds too.

For each $p \in \mathbb{N}$ and $j \in \langle N-1 \rangle_0$, by adding the inequalities (11), we obtain:

$$|g_k^{q_2+p}(t_{j+1}) - g_k^{q_2+p}(t_j)| < |g_k^{q_2}(t_{j+1}) - g_k^{q_2}(t_j)| + \frac{\varepsilon}{N}. \tag{13}$$

From the inequalities (11) and (13) for each $p \in \mathbb{N}$, the following estimation is obtained:

$$\sum_{j=0}^{N-1} |g_k^{q_2+p}(t_{j+1}) - g_k^{q_2+p}(t_j)| < \sum_{j=0}^{N-1} |g_k^{q_2}(t_{j+1}) - g_k^{q_2}(t_j)| + \varepsilon < V_k^{q_2} + \varepsilon.$$

Therefore, the sequence $\{V^q\}_{q \in \mathbb{N}}$ is bounded from above and lets us denote $V = \sup\limits_{q \in \mathbb{N}} \{V^q\}_{q \in \mathbb{N}}$.

We will prove that $G^0(t)$ has bounded the variation on $[a-h, a+M]$.

Since $\lim\limits_{p \to \infty} \|G^0 - G^{q_0+p}\|_s = 0$, there exists a number $q_3(\varepsilon) \geq q_2(\varepsilon)$, such that for each $p \in \mathbb{N}$, we have that $\|G^0 - G^{q_2+p}\|_s < \varepsilon$, and for $j \in \langle N-1 \rangle_0$, the following inequalities $|g_k^{q_3(\varepsilon)+p}(t_j) - g_k^0(t_j)| < \frac{\varepsilon}{2N}$ hold. Then, as above, we can prove that:

$$\sum_{j=0}^{N-1} |g_k^0(t_{j+1}) - g_k^0(t_j)| < \sum_{j=0}^{N-1} |g_k^{q_3(\varepsilon)+p}(t_{j+1}) - g_k^{q_3(\varepsilon)+p}(t_j)| + \varepsilon \leq V + \varepsilon$$

hence, $G^0 \in E_M^*$. $\square$

The statement of the next theorem is essentially based off of the statement of Theorem 4, and in this sense, it is a generalization of Theorem 2 in [20], i.e. the statement is still true under weaker conditions.

Theorem 5. *Let the conditions* **(S)** *hold.*

Then, for arbitrary finite partition $J_L = \{t_0, ..., t_L\},\ L \in \mathbb{N}$ *of* $[a-h, a]$ *(* $J_L \subset (a-h, a]$*) and each initial function* $\Phi \in E^*(J_L)$*, there exists* $M_* \in \mathbb{R}_+$ *such that the IP* (4), (3) *has at least one solution* $X(t) = (x_1(t), ..., x_n(t))^T$ *in the sense of Definition 3 with an interval of existence* $[a, a+M_*]$.

Proof. Let $J_L = \{t_0, ..., t_L\},\ L \in \mathbb{N}$ be the arbitrary finite partition of $[a-h, a]$, with $J_L \subset (a-h, a]$, and $\Phi^* \in E^*(J_L)$ be an arbitrary fixed function, introducing the subset $E_M^{\Phi_*} = \{G_{\Phi_*} \in E_M^* |\ \ G(t) = \Phi_*(t), t \in [a-h, a]\}$, $E_M^{\Phi_*} \subset E_M^*$. The set $E_M^{\Phi_*}$ is nonempty, convex and closed (in virtue of the Lemma 1) subset of E_M^* for arbitrary $M \in (0, h]$, concerning the norm $\|G\|_s = \sup\limits_{t \in [a-h, a+M]} |G(t)|$, $G \in E_M^*$.

Define the operator $\mathbf{T}$ for arbitrary $t \in (a, a+M]$ and $G \in E_M^*$ as follows:

$$
\begin{aligned}
&(\mathbf{T}G)(t) = C_{\Phi(0)} + \int\limits_{-h}^{0} [d_\theta V(t,\theta)]G(t+\theta);\\
&(\mathbf{T}G)(t) = \Phi(t-a),\ t \in [a-h, a],\ \Phi \in \mathbf{PC}^*;\\
&\lim_{t \to a+M-0} (\mathbf{T}G)(t) = (\mathbf{T}G)(a+M).
\end{aligned} \tag{14}
$$

For arbitrary $M \in (0, h]$ and $G \in E_M^*$, define the operator $\mathbf{S}$ via the following equalities:

$$
\begin{aligned}
&(\mathbf{S}G)(t) = I_{-1}(\Gamma(\alpha)) \int\limits_a^t I_{\alpha-1}(t-\eta)(\int\limits_{-h}^{0} [d_\theta U(\eta,\theta)]G(\eta+\theta) + F(\eta))d\tau,\ t \in (a, a+M];\\
&(\mathbf{S}G)(t) = \mathbf{0},\ t \in [a-h, a]
\end{aligned} \tag{15}
$$

and for shortness, we rewrite the system (4) in the form $G(t) = (\mathbf{T}G)(t) + (\mathbf{S}G)(t)$.

As first, we will prove that $\mathbf{T}E_M^{\Phi_*} \subseteq E_M^{\Phi_*}$ for arbitrary fixed function $\Phi^* \in E^*(J_L)$, and all $M \in (0, h]$ are small enough.

Let $G \in E_M^{\Phi_*}$ and $M \in (0, \tau]$ be arbitrary. Then, we have:

$$
(\mathbf{T}G)(t) = C_{\Phi_*(0)} + \int\limits_{-h}^{0} [d_\theta V_d(t,\theta)]G_{\Phi_*}(t+\theta) + \int\limits_{-h}^{0} [d_\theta V_c(t,\theta)]G_{\Phi_*}(t+\theta). \tag{16}
$$

Let $\gamma \in \mathbb{R}_+$ be the constant existing according to condition **(S4)**, and hence, we have $V_c(t,\theta)$ and the continuous functions for $(t,\theta) \in [a, a+\gamma] \times [-h, 0]$.

Then, by integrating, by parts, the third addend in the right side of (16), we have:

$$\begin{aligned}
&\int_{-h}^{0}[d_\theta V_c(t,\theta)]G(t+\theta) = -V_c(t,-h)G(t-h) + \int_{-h}^{0}[d_\theta G(t+\theta)]V_c(t,\theta) \\
&= -V_c(t,-h)\Phi_*(t-a-h) + \int_{-h}^{0}[d_\theta G(t+\theta)]V_c(t,\theta) \\
&= -V_c(t,-h)\Phi_*(t-a-h) + \int_{-h}^{a-t}[d_\theta \Phi_*(t-a+\theta)]V_c(t,\theta) \\
&+ \int_{a-t}^{0}[d_\theta G(t+\theta)]V_c(t,\theta).
\end{aligned} \tag{17}$$

Since $\Phi_*(t-a+\theta)$ is right continuous at $\theta=-h$, then from condition **(S4)**, it follows that there exists $M_1 \in (0, \min(M,\gamma))$ such that the first addend in the right side of (17) are continuous functions for $t \in [a, a+M_1]$. For the second and third addends, we have that $V_c(t,\theta)$ is continuous in $[a,a+M_1]\times[-h,0]$, while the functions $\Phi_*(t-a+\theta)$ and $G(t+\theta)$ have bounded variation for $t+\theta \in [a-h,a]$ and $t+\theta \in [a,a+M_1]$, respectively. Then, for $(t,\theta) \in [a,a+M_1]\times[-h,0]$, and hence, for $t+\theta \in [a-h,a+M_1]$, the second and third addends in the right side of (17), according to Theorem 1, are continuous functions in t for $t \in [a,a+M_1]$. Thus, the third addend in the right side of (16) is also a continuous function for $t \in [a,a+M_1]$. Below, for definiteness, we will assume that $1 \le q_1 \le q \le r$, where the critical points are numbered from 1 to q_1 the noncritical points are numbered from q_1+1 to q. Then, for the second addend in the right side of (16) we have:

$$\begin{aligned}
&\sum_{l\in\langle q\rangle}\int_{-h}^{0}[d_\theta V_d^l(t,\theta)]G_{\Phi_*}(t+\theta)) = \sum_{l\in\langle q_1\rangle}A^l(t)G_{\Phi_*}(t-\tau_l(t)) \\
&+\sum_{l\in\langle q\rangle\backslash\langle q_1\rangle}A^l(t)\Phi_*(t-a-\tau_l(t)) + \sum_{l\in\langle r\rangle\backslash\langle q\rangle}A^l(t)\Phi_*(t-a-\tau_l(t)),
\end{aligned} \tag{18}$$

where $\tau_l(a) > 0$ for $l \in \langle r\rangle\backslash\langle q\rangle$. From Definition 7, it follows that there exists a constant $\varepsilon_1 \in (0,h]$, such that $t-\tau_l(t) \ge a$ for $t \in [a,a+\varepsilon_1]$ and $l \in \langle q_1\rangle$. Since $\tau_l(a) > 0$ for $l \in \langle r\rangle\backslash\langle q\rangle$ and taking into account Lemma 3 in [25] and Definition 7, we conclude that there exists a constant $\varepsilon_2 \in (0,\varepsilon_1]$ such that $t-\tau_l(t) < a$ for $t \in (a,a+\varepsilon_2]$ and $\Phi_*(t-a-\tau_l(t))$ is a continuous function in $t \in (a,a+\varepsilon_2]$ and $l \in \langle r\rangle\backslash\langle q_1\rangle$. Thus, the left side of (18) is a continuous function for $t \in (a,a+\varepsilon_2]$ (only right continuous at $t=a$), and hence, $(\mathbf{T}G_{\Phi_*})(t)$ for all $M \in (0,M_2]$, $M_2=\min(M_1,\varepsilon_2)$ is also continuous function. From (11), it follows that $\lim\limits_{t\to a+0}(\mathbf{T}G_{\Phi_*})(t) = \Phi_*(0)$ and $\lim\limits_{t\to a+M-0}(\mathbf{T}G_{\Phi_*})(t) = (\mathbf{T}G_{\Phi_*})(a+M)$.

Let $J_N = \{t_0,\ldots,t_N\}$, $N \in \mathbb{N}$ be an arbitrary particle of $[a,a+M]$, $M \in (0,M_2]$, and, then we have:

$$\begin{aligned}
&\sum_{z\in\langle N-1\rangle_0}\left|\int_{-h}^{0}G(t_{z+1}+\theta)d_\theta V(t_{z+1},\theta) - \int_{-h}^{0}G(t_z+\theta)d_\theta V(t_z,\theta)\right| \\
&\le \left|Var_{s\in[a-h,a+M]}G(s)\right| \sup_{t\in[a,a+M]}\left|Var_{\theta\in[-h,0]}V(t,\theta)\right|,
\end{aligned}$$

where $C > 0$ is a constant not depending from N. Thus, $\mathbf{T}E_M^{\Phi_*} \subseteq E_M^{\Phi_*}$, and hence, $\mathbf{T}E_M^* \subseteq E_M^*$.

To verify that condition 1 of Theorem 2 holds, we will prove that for arbitrary fixed function $\Phi_* \in \mathbf{PC}^*$ and all $M \in (0,h]$ are small enough so that the operator $\mathbf{T}$ is a contraction in $E_M^{\Phi_*}$.

Let $G, \widetilde{G} \in E_M^{\Phi_*}$ be arbitrary, and then for all $t \in [a, a+M_2]$, $M \in (0, M_2]$, we have:

$$\begin{aligned}\left\|\mathbf{T}G - \mathbf{T}\overline{G}\right\|_s &= \sup_{t\in[a-h,a+M]}\left|\mathbf{T}G(t) - \mathbf{T}\widetilde{G}(t)\right| \\ &\leq \sup_{t\in[a-h,a+M]}\left|\int_{-h}^{0}[d_\theta V_d(t,\theta)](G(t+\theta) - \widetilde{G}(t+\theta))\right| \\ &+ \sup_{t\in[a-h,a+M]}\left|\int_{-h}^{0}[d_\theta V_c(t,\theta)](G(t+\theta) - \widetilde{G}(t+\theta))\right|.\end{aligned} \tag{19}$$

Applying Lemma 1, we have $\sum_{l\in\langle q_1\rangle}\left|\overline{A}^l(a)\right| = \beta < 1$, and there exists $M_3 \in (0, M_2]$ such that for $t \in [a, a+M_3]$, we have $\sum_{l\in\langle q_1\rangle}\left|\overline{A}^l(t)\right| \leq \beta + \frac{(1-\beta)}{8}$. Condition **(S2)** implies that there exists $\delta > 0$, such that for $\theta \in [-\delta, 0]$ and $t \in [a, a+M_4]$, $M_4 = \min(M_3, \delta)$ we have $\sup_{t\in[a,a+M]} Var_{\theta\in[-\delta,0]}V(t,\theta) = \frac{(1-\beta)}{8}$ and

$$\Phi_*(t - a - \tau_l(t)) = G(t + \tau_l(t)) = \widetilde{G}(t + \tau_l(t)) \tag{20}$$

when $l \in \langle r\rangle\backslash\langle q_1\rangle$.

Hence, from (18) for the first addend in the right side of (19) when $t \in [a, a+M_4]$, it follows that:

$$\sup_{t\in[a-h,a+M]}\left|\int_{-h}^{0}[d_\theta V_d(t,\theta)](G(t+\theta) - \widetilde{G}(t+\theta))\right| \leq (\beta + \frac{1-\beta}{8})\left\|G - \widetilde{G}\right\|_s. \tag{21}$$

For the second addend in the right side of (19), taking into account that $t + \theta \leq a$ and $l \in \langle r\rangle$ (23) hold, we obtain:

$$\begin{aligned}&\sup_{t\in[a-h,a+M]}\left|\int_{-h}^{0}[d_\theta V_c(t,\theta)](G(t+\theta) - \widetilde{G}(t+\theta))\right| \\ &\leq \sup_{t\in[a-h,a+M]}\left|\int_{-\delta}^{0}[d_\theta V_c(t,\theta)](G(t+\theta) - \widetilde{G}(t+\theta))\right| \\ &+ \sup_{t\in[a-h,a+M]}\left|\int_{-h}^{-\delta}[d_\theta V_c(t,\theta)](G(t+\theta) - \widetilde{G}(t+\theta))\right| \\ &\leq \sup_{t\in[a,a+M]} Var_{\theta\in[-\delta,0]}V(t,\theta)\left\|G - \widetilde{G}\right\|_s = \frac{(1-\beta)}{8}\left\|G - \widetilde{G}\right\|_s.\end{aligned} \tag{22}$$

Thus, from (24) and (25), it follows that the operator $\mathbf{T}$ is a contraction in $E_M^{\Phi_*}$, since $\frac{(1+3\beta)}{4} < 1$.

To verify that condition 2 of Theorem 2 holds, we must prove that $\mathbf{S}$ is continuous and the set $\mathbf{S}(E_M^{\Phi_*})$ is a relative compact set.

For arbitrary $G \in E_M^*$ from condition **(S2)** , it follows that:

$$\left| \int_{-h}^{0} [d_\theta U(\tau,\theta)]G(\tau+\theta) \right| \leq U^* \|G\|_M, U^* = \sup_{t\in[a,a+M]} U^*(t)$$

hence, the integral on the left side is at least locally bounded by the Lebesgue integrable function in t for $t \in [a, a+M],\ M \in (0,h]$.

Estimating the following integral:

$$\begin{aligned} \int_a^t |I_{-1}(\Gamma(\alpha))I_{\alpha-1}(t-\eta)|d\eta &= \sum_{k\in\langle n\rangle} \Gamma^{-1}(\alpha_k) \int_a^t (t-\eta)^{\alpha_k-1} d\eta \\ &= \sum_{k\in\langle n\rangle} \frac{(t-a)^{\alpha_k}}{\Gamma(1+\alpha_k)} \leq (t-a)^{\alpha_{min}} n\Gamma^{-1}(z_{\min}). \end{aligned} \tag{23}$$

Taking into account (23) for every $M \in (0, M_5]$ and $t \in [a, a+M],\ M_5 = \min(1, M_4)$, we have that:

$$\begin{aligned} \|(\mathbf{S}G)(t)\|_M &= \sup_{t\in[a-h,a+M]} \left| \int_a^t I_{-1}(\Gamma(\alpha))I_{\alpha-1}(t-\eta)(\int_{-h}^{0} [d_\theta U(\eta,\theta)]G(\eta+\theta) + F(\eta))d\tau \right| \\ &\leq (U^*\|G\|_M + F_M) \sup_{t\in[a-h,a+M]} \int_a^t |I_{-1}(\Gamma(\alpha))I_{\alpha-1}(t-\eta)|d\tau \\ &\leq (t-a)^{\alpha_{min}} n\Gamma^{-1}(z_{\min})(U^*\|G\|_M + F_M), \end{aligned} \tag{24}$$

where $F_M = \sup\limits_{t\in[a,a+M]} |F(t)|$. Then, from (24), it follows that $\lim\limits_{t\to a+0} (\mathbf{S}G)(t) = \mathbf{0}$, and we can conclude that $\mathbf{S}E_M^* \subseteq E_M^*$.

Let $G^1, G^2 \in E_M^*$ be arbitrary and $t \in [a, a+M],\ M \in (0,h]$. Then, using (24), we obtain:

$$\begin{aligned} &\left\| (\mathbf{S}G^1)(t) - (\mathbf{S}G^2)(t) \right\|_M \\ &\leq \sup_{t\in[a,a+M} \left| I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\eta)(\int_{-h}^{0} [d_\theta U(\eta,\theta)]G^1(\eta+\theta) - G^2(\eta+\theta))d\tau \right| \\ &\leq \sup_{t\in[a,a+M} \int_a^t |I_{-1}(\Gamma(\alpha)(I_{\alpha-1}(t-\eta)| \left| \int_{-h}^{0} [d_\theta U(\eta,\theta)]G^1(\eta+\theta) - G^2(\eta+\theta))d\tau \right| d\tau \\ &\leq U^* \left\| G^1 - G^2 \right\|_M \sup_{t\in[a,a+M} \int_a^t |I_{-1}(\Gamma(\alpha)(I_{\alpha-1}(t-\eta)|d\tau \\ &\leq M^{\alpha_{min}} U^* n\Gamma^{-1}(z_{\min}) \left\| G^1 - G^2 \right\|_M \end{aligned} \tag{25}$$

hence, from (25), it follows that the map $\mathbf{S} : E_M^* \to E_M^*$ is continuous.

Let $R \geq \|\Phi_*\|$ be an arbitrary fixed number, denoted by:

$$B(G^*, R) = \{G \in E_M^* |\ \|G - G^*\|_M \leq R\},\ \text{where } G^*(t) = \begin{cases} \Phi_*(t-a), & t \in [a-h, a]; \\ \Phi_*(0), & t \in [a, a+M]. \end{cases}$$

From (24) for arbitrary $G_M \in B(G^*, R)$, we have the following estimation:

$$\|(\mathbf{S}G)(t)\|_M \leq M^{\alpha_{min}} n\Gamma^{-1}(z_{\min})(U^*R + F_M)$$

hence, the set $\mathbf{S}(B(G^*, R))$ is uniformly bounded, and thus, $\mathbf{S}$ maps every bounded subset of E^*_M in a uniformly bounded subset of E^*_M.

To apply Theorem 2, we must prove that the set $\mathbf{S}(B(G^*, R))$ is relatively compact, and according to Arzela–Ascoli's theorem, it is enough to prove that the set $\mathbf{S}(B(G^*, R))$ is equicontinuous.

Let $\varepsilon > 0$, $t_1, t_2 \in [a, a+M]$, $M \in (0, M_5]$ be arbitrary, and for definiteness, assume that $t_1 < t_2$. Then, for every $G \in B(G^*, R)$, when $|t_2 - t_1| < \delta$, $\delta < \left(\frac{\varepsilon}{C^*(n+1)}\right)^{\frac{1}{\alpha_{min}}}$, $C^* = n\Gamma^{-1}(z_{\min})(U^*R + F_M)$, we have the following estimation:

$$
\begin{aligned}
&|(\mathbf{S}G)(t_2) - (\mathbf{S}G)(t_1)| \leq \left| I_{-1}(\Gamma(\alpha)) \int_{t_1}^{t_2} I_{\alpha-1}(t_2-\eta)(\int_{-h}^{0} [d_\theta U(\eta,\theta)]G(\eta+\theta) + F(\eta))d\tau \right| \\
&+ \left| I_{-1}(\Gamma(\alpha)) \int_{a}^{t_1} (I_{\alpha-1}(t_2-\eta) - I_{\alpha-1}(t_1-\eta))(\int_{-h}^{0} [d_\theta U(\eta,\theta)]G(\eta+\theta))d\tau \right| \\
&\leq (t_2-t_1)^{\alpha_{min}} n\Gamma^{-1}(z_{\min})(U^*R+F_M) + U^*R \int_{a}^{t_1} |I_{-1}(\Gamma(\alpha)(I_{\alpha-1}(t_2-\eta) - I_{\alpha-1}(t_1-\eta))|d\tau \\
&\leq (t_2-t_1)^{\alpha_{min}} n\Gamma^{-1}(z_{\min})(U^*R+F_M) + U^*R \sum_{k\in\langle n\rangle} \Gamma^{-1}(\alpha_k) \int_{a}^{t_1} \left|(t_2-\eta)^{\alpha_k-1} - (t_1-\eta)^{\alpha_k-1}\right| d\eta \\
&\leq C^*(t_2-t_1)^{\alpha_{min}} + U^*R \sum_{k\in\langle n\rangle} \Gamma^{-1}(\alpha_k) \int_{a}^{t_1} ((t_1-\eta)^{\alpha_k-1} - (t_2-\eta)^{\alpha_k-1})d\eta \\
&\leq C^*(t_2-t_1)^{\alpha_{min}} + U^*R\Gamma^{-1}(z_{\min}) \sum_{k\in\langle n\rangle} ((t_1-a)^{\alpha_k} - (t_2-a)^{\alpha_k} + (t_2-t_1)^{\alpha_k}) \\
&\leq C^*(t_2-t_1)^{\alpha_{min}} + C^* \sum_{k\in\langle n\rangle} (\alpha_k(t_1-t_2)(\xi-a)^{\alpha_k-1} + (t_2-t_1)^{\alpha_k}) \\
&\leq C^*(t_2-t_1)^{\alpha_{min}} + C^* \sum_{k\in\langle n\rangle} (t_2-t_1)^{\alpha_k}(1 - \alpha_k(t_2-t_1)^{1-\alpha_k}(t_2-a)^{-1}) \\
&\leq C^*(t_2-t_1)^{\alpha_{min}}(1 + \sum_{k\in\langle n\rangle} (1 - \alpha_k(t_2-t_1)^{1-\alpha_k}(t_2-a)^{-1}) \leq C^*(t_2-t_1)^{\alpha_{min}}(n+1) < \varepsilon.
\end{aligned}
$$

Thus, we proved that the set $\mathbf{S}(B(G^*, R))$ is equicontinuous, and hence, the map $\mathbf{S} : E^*_M \to E^*_M$ is compact.

Let $G \in E^*_M$ and $\widetilde{G} \in E^{\Phi_*}_M$ be arbitrary. Then, $t \in [a, a+M]$, $M \in (0, M_*]$, $M_* = M_5$, where the function $\mathbf{T}\widetilde{G}(t) + \mathbf{S}G(t)$ is continuous, $\lim\limits_{t\to a+0}(\mathbf{T}\widetilde{G}(t) + \mathbf{S}G(t)) = \Phi_*(0)$, $\mathbf{T}\widetilde{G}(t) + \mathbf{S}G(t) = \Phi_*(t-a)$, $t \in [a-h, a]$, $\lim\limits_{t\to a+M-0}(\mathbf{T}\widetilde{G}(t) + \mathbf{S}G(t)) = (\mathbf{T}\widetilde{G})(a+M) + \mathbf{S}G(a+M)$, and hence, $\mathbf{T}\widetilde{G}(t) + \mathbf{S}G(t) \in E^{\Phi_*}_M$, i.e. the condition 3 of Theorem 2 holds. Thus, the system (4) has at least one fixed point in $E^{\Phi_*}_M$. □

Theorem 6. *Let the conditions* **(S)** *hold.*

Then, for each $\Phi_* \in \mathbf{PC}^*$, there exists $M_* \in \mathbb{R}_+$, such that the IP (4) and (3) has a unique solution $X(t) = (x_1(t), ..., x_n(t))^T$ in the interval $[a, a+M_*]$.

Proof. Let $\Phi_* \in \mathbf{PC}^*$ be arbitrary. Then, according to Theorem 5, the IP (4) and (3) has at least one solution $X(t)$ in the interval $M_* \in \mathbb{R}_+$, and we can assume that there exist two different solutions, i.e. X_1 and X_2, of the IP (4) and (3) in the same interval.

Then, the function $Y(t) = \sup_{s\in[a,a+t]} |X_2(s) - X_1(s)| > \mathbf{0}$ for $t \in [a, a+M_*]$, which is a continuous solution of the IP (4) and (3), with $F(t) \equiv \mathbf{0}$ and $\Phi(t) \equiv \mathbf{0}$.

Then, from (4) and (3), it follows that the following inequality holds:

$$|Y(t)| \le |(\mathbf{T}Y)(t)| + |\mathbf{S}Y(t)| \le \beta|Y(t)| + |\mathbf{S}Y(t)|,\ \beta \in (0,1)$$

hence:

$$|Y(t)| \le |\mathbf{S}Y(t)| \le \frac{U^*}{1-\beta}\left|\int_a^t |(I_{-1}(\Gamma(\alpha))I_{\alpha-1}t-\eta))||Y(\eta)|d\tau\right|. \tag{26}$$

Since in (26), we have that $a(t) \equiv 0$,; then, from Theorem 3, it follows that $Y(t) \equiv \mathbf{0}$, which contradicts our assumption. Thus, the IP (4) and (3) has a unique solution in the interval $[a, a+M_*]$. □

Corollary 1. *Let the conditions* **(S)** *hold.*

Then, for each $\Phi_* \in \mathbf{PC}^*$, the IP (4) and (3) has a unique solution $X(t) = (x_1(t), ..., x_n(t))^T$ with the interval of existence J_a and $X(t) \in BV_{loc}(J_a, \mathbb{R}^n)$.

Proof. From Theorems 3 and 4 in [19], it follows that the IP (4) and (3) has a unique solution $X(t)$ in the sense of Definition 3 with the interval of existence J_a. Then, the statement of the theorem follows from Lemma 1. □

Theorem 7. *Let the conditions* **(S)** *hold.*

Then, the following statements hold:

(i) For each fixed $s \in J_a$, the matrix IP (5), (6) has a unique solution $C(\cdot, s) : \mathbb{R} \to \mathbb{R}^{n\times n}$ with interval of existence J_s, and $C(t,s) = \{c_{kj}(t,s)\}_{k,j\in\langle n\rangle} \in BV_{loc}(\mathbb{R}, \mathbb{R}^n)$;

(ii) For each $s \in [a-h, a]$, the matrix IP (5), (7) has a unique solution $Q(\cdot, s) : \mathbb{R} \to \mathbb{R}^{n\times n}$ with the interval of existence J_a, and $Q(t,s) = \{q_{kj}(t,s)\}_{k,j\in\langle n\rangle} \in BV_{loc}(\mathbb{R}, \mathbb{R}^n)$.

Proof. (i) Let $j \in \langle n\rangle$ and $s \in J_a$ be arbitrary fixed numbers, define the initial functions $\Phi_*(t,s) = \Phi^j_C(t,s)$, where $\Phi^j_C(t,s)$ is the j-th column of the matrix function $\Phi_C(t,s)$, and consider the IP (2) and (3).

Then, according to Theorems 5 and 6, the IP (4) and (3) has unique solution $C^j(t,s) = (c_{1j}(t,s), ..., c_{nj}(t,s))^T$, where (2) and (3) have a unique solution $C^j(t,s) = (c_{1j}(t,s), \ldots, c_{nj}(t,s))^T$, with Definition 2, with the interval of existence J_s and from Corollary 1, it follows that $C^j(\cdot, s) \in BV_{loc}(\mathbb{R}, \mathbb{R}^n)$.

This matrix $C(t,s) = (C^1(t,s), ..., C^n(t,s))$ is the unique fundamental matrix for the system (2).

Case (ii) can be treated in an analogical way. □

4. Applications

In this section, for the application of the obtained results concerning the fundamental matrix of the system (2), we establish that the problem of the existence of a unique resolvent kernel $R(t,s) \in \mathbf{SVB}^\infty_{loc}(J \times \mathbb{R}, \mathbb{R}^{n\times n})$ corresponding to the kernel $K(t,s) \in \mathbf{SBV}^\infty_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ defined via (28), is equivalent to the problem of the existence of a unique fundamental matrix $C(t,s) \in BV_{loc}(\mathbb{R}, \mathbb{R}^n)$ of the system (2). Some stability results are also established.

Definition 8. *([33]) The function $K(t,s) : J_a \times J_a \to \mathbb{R}^{n\times n}$ is called Stieltjes-Volterra type $\mathbf{B}^\infty$ kernel on $J_a \times J_a$ ($K \in \mathbf{SVB}^\infty(J_a \times J_a, \mathbb{R}^{n\times n})$), if the following conditions* **(K)** *hold:*

(K1) *The function $(t,s) \to K(t,s)$ is measurable in t for each fixed s, right continuous in s on (a,t) and $K(t,s) = \Theta$ for $s > t$;*

(K2) *$K(t,s)$ is bounded, and the total variation in s of $K(t,s)$ for every fixed t is uniformly bounded in s on J too.*

With $K \in \mathbf{SVB}^{\infty}_{loc}(J \times \mathbb{R}, \mathbb{R}^{n\times n})$,, we denote the set of kernels that restrictions to an arbitrary compact subset J, $J \subset J_a$ belong to $\mathbf{B}^{\infty}$.

Definition 9. *A kernel $R \in \mathbf{SVB}^{\infty}_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ ($R \in \mathbf{SVB}^{\infty}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$) is called a Stieltjes-Volterra resolvent of type $\mathbf{B}^{\infty}$, corresponding to a kernel $K \in \mathbf{SVB}^{\infty}_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ ($K \in \mathbf{SVB}^{\infty}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$) if for $s \in \mathbb{R}$, $t \in J \cap \{t \geq s\}$, ($t \in J_a \cap \{t \geq s\}$), it satisfies the following system:*

$$R(t,s) = -K(t,s) + \int_s^t d_\eta[K(t,\eta)]R(\eta,s) = -K(t,s) + \int_s^t d_\eta[R(t,\eta)]K(\eta,s), \tag{27}$$

where the integrals in (27) are understood in the sense of Lebesgue-Stieltjes and $J \subset J_a$ is an arbitrary compact subset.

According to Lemma 1 in [27] for arbitrary $\Phi \in \mathbf{C}$, the solution $X(t)$ of the IP (4), (3) satisfies for $t \in J_a$ the Volterra-Stieltjes equation $X(t) = \int_s^t [d_s K(t,s)]X(s) + f(t)$, where the function $f(t)$ is given (i.e. it depends only on the kernels U and V, as well as on the initial functions $\Phi(t)$ and $F(t)$) and the kernel $K(t,s)$ has the following form:

$$K(t,s) = V(t, s-t) + I_{-1}(\Gamma(\alpha) \int_s^t (U(\tau, s-\tau) I_{\alpha-1}(t-\tau) d\tau. \tag{28}$$

The condition **(S)** implies that $K(t,s) \in \mathbf{SBV}^{\infty}_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ (see [27]) when the kernel $K(t,s)$, defined with (28), and for arbitrary fixed $s \in J_a$, we introduce the matrix functions $R(t,s)$ via the following relation:

$$R(t,s) = C(t,s) - H(t,s), \tag{29}$$

where $H(t,s) = I$ for $t \geq s$ and $H(t,s) = \Theta$ when $t < s$.

The next theorem is a generalization of Theorem 2 in [27] for the case when kernel $K(t,s)$ is defined via (28) and satisfies only the conditions ($\mathbf{S}$), but possibly does not satisfy the conditions ($\mathbf{K}$) and solves the open problem stated in the same work. Practically, we establish that the problem of the existence of a unique resolvent kernel $R(t,s) \in \mathbf{SVB}^{\infty}_{loc}(J \times \mathbb{R}, \mathbb{R}^{n\times n})$ corresponding to the kernel $K(t,s) \in \mathbf{SBV}^{\infty}_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ defined via (28) is equivalent to the problem of the existence of a unique fundamental matrix $C(t,s) \in BV_{loc}(\mathbb{R}, \mathbb{R}^n)$ of the system (2).

Theorem 8. *Let the following conditions hold:*

(i) *The conditions ($\mathbf{S}$) are fulfilled;*
(ii) *The kernel $K(t,s)$ have the form (28).*

Then, the relation (29) holds if and only if when $C(t,s) \in BV_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ is the unique fundamental matrix of (2) and the function $R(t,s) \in \mathbf{SVB}^{\infty}_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ defined via (29) is the unique solution of the resolvent Equation (27) corresponding to the kernel $K(t,s) \in \mathbf{SBV}^{\infty}_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ defined with (28).

Proof. Sufficiency: Let $C(t,s) \in BV_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ is the unique fundamental matrix of (2), existing according to Theorem 7 and $K(t,s)$, with the form (28). Define the matrix functions $R(t,s)$ via (29), and then, for $s \in \mathbb{R}$, $t \in J \cap \{t \geq s\}$, ($t \in J_a \cap \{t \geq s\}$), we have that:

$$C(t,s) = I + \int_s^t [d_\eta K(t,\eta)]C(\eta,s) = H(t,s) + \int_s^t [d_\eta K(t,\eta)]C(\eta,s)$$

hence, we obtain:

$$\begin{aligned} R(t,s) = C(t,s) - H(t,s) &= \int_s^t [d_\eta K(t,\eta)]C(\eta,s) = \int_s^t [d_\eta K(t,\eta)](C(\eta,s) + H(t,s) - H(t,s)) \\ &= \int_s^t [d_\eta K(t,\eta)]H(t,s) + \int_s^t [d_\eta K(t,\eta)](C(\eta,s) - H(t,s)) = -K(t,s) + \int_s^t [d_\eta K(t,\eta)]R(t,s). \end{aligned} \tag{30}$$

Thus, (30) for $s \in \mathbb{R}, t \in J \cap \{t \geq s\}, (t \in J_a \cap \{t \geq s\})$ implies that $R(t,s)$ is the unique solution of the resolvent Equation (27). Obviously, since $C(s,s) = I$ and $H(t,s) = \Theta$ for $t < s$, then $R(t,s) = \Theta$ for $t \leq s$. From the conditions **(S)** and since $C(t,s) \in BV_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n \times n})$ and (29), it follows that $R(t,s) \in \mathbf{SVB}^{\infty}_{loc}(J \times \mathbb{R}, \mathbb{R}^{n \times n})$, and hence, $R(t,s)$ is the unique resolvent kernel corresponding to the kernel $K \in \mathbf{SVB}^{\infty}_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n \times n})$ defined via (28). The necessity can be proved in a reverse way. □

Corollary 2. *Let the following conditions hold:*

(i) *The conditions* (**S**) *are fulfilled;*
(ii) *The kernel* $K(t,s)$ *have the form* (28).

Then, we have the resolvent kernel $R(t,s) \in BV_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n \times n})$.

Proof. Since according Theorem 8, there exists a unique fundamental matrix $C(t,s) \in BV_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n \times n})$, then the statement follows from (29) and Theorem 8. □

Definition 10. *[6] The zero solution of* (2) *is called:*

(a) *Stable for a given* $a \in \mathbb{R}$ *if for any* $\epsilon > 0$ *there is a* $\delta = \delta(\varepsilon, a) > 0$*, such that* $|X(t;a;\Phi)| \leq \varepsilon$ *for any initial function* $\Phi \in PC$ *with* $\|\Phi\| < \delta$ *and* $t \in J_a$*. In the opposite case, the solution is called unstable.*
(b) *Uniformly stable if for any* $\epsilon > 0$*, there is a* $\delta(\epsilon) > 0$*, such that* $|X(t;a;\Phi)| \leq \varepsilon$ *for any initial function* Φ *with* $\|\Phi\| < \delta$ *and* $t \in J_a$*.*
(c) *Locally asymptotically stable (LAS) if for a given* $a \in \mathbb{R}$*, if it is stable and there is a* $\Delta = \Delta(a) > 0$*, such that* $\lim\limits_{t \to \infty} |X(t;a;\Phi)| = 0$ *for any initial function* Φ *with* $\|\Phi\| < \Delta$*. The set* $\Omega(a) \subset \mathbf{PC}$ *of all initial functions* Φ *for which* $\lim\limits_{t \to \infty} |X(t;\bar{t};\Phi)| = 0$ *is called the attraction domain of the zero solution for initial time* a*. The zero solution is said to be uniformly LAS, if* $\Delta > 0$ *is independent from the initial time.*
(d) *Globally asymptotically stable (GAS) if it is uniformly stable and* $\lim\limits_{t \to \infty} |X(t;a;\Phi)| = 0$ *for any initial function* Φ*.*

The next result gives a simple but useful necessary condition for the asymptotic stability of the zero solution of the system (2).

Theorem 9. *Let the following conditions hold:*

(i) *The conditions* (**S**) *are fulfilled;*
(ii) *The kernel* $K(t,s)$ *have the form* (28);
(iii) *The zero solution of system* (2) *is stable.*

Then, the corresponding resolvent kernel $R(t,s) \in BV_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n \times n})$*, for each fixed* $s \in \mathbb{R}$*, is bounded in* t*, i.e.* $\sup\limits_{t \in J_a} |R(t,s)| < \infty$*.*

Proof. Let $C(t,s) \in BV_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n \times n})$ be the unique fundamental matrix of (2) existing according Theorem 7 and the kernel $K(t,s)$ is defined via (28). Then, according to Theorem 8, there exists the corresponding resolvent kernel $R(t,s)$, which satisfies (29) for $s \in \mathbb{R}$, $t \in J_a$, and hence, $R(t,s) \in BV_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n \times n})$. Condition 2 of the theorem implies that for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that for arbitrary initial function $\Phi \in PC$ with $\|\Phi\| < \delta$, the corresponding solution $X(t;a;\Phi)$ of the IP (2), (3) satisfies the inequality $|X(t;a;\Phi)| \leq \frac{\varepsilon}{n}$ for $t \in J_a$.

Let $s \in \mathbb{R}$ be an arbitrary fixed number. Then, obviously, if $s \notin J_a$, or $s \in J_a$ and $s > t$, the statement of the theorem holds. Consider $t \in J_a \cap \{t \geq s\}$ and define the matrix $C_\delta(t,s) = \frac{\delta}{2}C(t,s) = (\frac{\delta}{2}C^1(t,s), \dots, \frac{\delta}{2}C^n(t,s))$. Then, we obtain that $|C_\delta(t,s)| \leq \varepsilon$ for $t \in J_a \cap \{t \geq s\}$, and hence, from (29), it follows that:

$$|R(t,s)| \leq |C(t,s)| + 1 \leq \frac{2}{\delta}|C_\delta(t,s)| \leq 1 + \frac{2\varepsilon}{\delta},$$

which completes the proof. □

Theorem 10. *Let the following conditions hold:*

(i) *The conditions* $(\mathbf{S})$ *are fulfilled;*
(ii) *The kernel* $K(t,s)$ *has the form* (28)*;*
(iii) *The corresponding resolvent kernel is uniformly bounded, i.e.* $\sup\limits_{(t,s)\in J_a\times\mathbb{R}} |R(t,s)| < \infty$ *and* $\sup\limits_{s\in[a-h,a]} |Q(t,s)| \leq \sup\limits_{s\in\mathbb{R}} |R(t,s)| < \infty$.

The zero solution of system (2) *is stable.*

Proof. Condition 3 implies that there exists a constant $Q > 0$ such that:

$$\sup_{(t,s)\in J_a\times[a-h,a]} |Q(t,s)| \leq Q$$

and for arbitrary initial function $\Phi \in \mathbf{PC}^*$, the solution $X(t;a;\Phi)$ of the IP (2) and (3) has the following integral representation (see [15,17]):

$$X(t;a;\Phi) = \int_{a-h}^{a} Q(t,s)d\widetilde{\Phi}(s), \tag{31}$$

where $\widetilde{\Phi}(a-h) = 0$ and $\widetilde{\Phi}(s) \equiv \Phi(s),\ s \in (a-h,a]$.

From condition 3 and (31), it follows that:

$$\begin{aligned}\sup_{t\in J_a}|X(t;a;\Phi)| &\leq \sup_{(t,s)\in J_a\times[a-h,a]} |Q(t,s)| \left|Var_{s\in[a-h,a]}\widetilde{\Phi}(s)\right| \\ &\leq Q\left|Var_{s\in[a-h,a]}\Phi(s)\right| \leq Q\|\Phi\|_{Var}.\end{aligned} \tag{32}$$

Let $\varepsilon > 0$ be arbitrary and let $\delta \in (0, \frac{\varepsilon}{Q})$ be an arbitrary fixed number. Then, since $\|\Phi\|_{Var} \geq \|\Phi\|_s$ from (32), it follows that for arbitrary initial function $\Phi \in \mathbf{PC}^*$ with $\|\Phi\|_{\mathrm{Var}} < \delta$, we have that $\sup\limits_{t\in J_a}|X(t;a;\Phi)| \leq \varepsilon$, and hence, the zero solution of system (2) is stable. □

5. Comments and Conclusions

As a motivation of our mathematical consideration, we note some possibilities of application of the studied systems as economics models.

As far we know, firstly, in the remarkable book [37], it was argued persuasively that delay differential equations are more suitable than ordinary differential equations alone or difference equations alone for an adequate treatment of dynamic economic phenomena. It is well known that there are at least two ways that time delays emerge in the dynamics of economic variables: there is some time lag between when the time economic decisions are made and the time the decisions bear fruit (see Chukwu [38]). There is a second "hidden" way, the way of rational expectation, see Fair [39] and Taylor [40] (inclusive expectation of bankruptcy [41]). In the second way, one assumes that expected future values of a variable are functions of the current and the past values of all relevant variables. In the monograph [42], the following model is introduced, describing the dynamic of the wealth of nations:

$$\frac{dX}{dt}\left(X(t)-\lambda\int_0^t \overline{A}(t-s)X(s)ds\right)=A_0X(t)+\lambda\int_0^t A(t-s)X(s)ds+F(t),$$

under the initial condition $X(t)=\Phi(t), t\in([h,0]$, where $\overline{A}(t),\ A(t)\in C^1(\overline{\mathbb{R}}_+,\mathbb{R}^{n\times n})$, $A_0\in\mathbb{R}^{n\times n}$, $n\in\mathbb{N}$, $X(t),F(t)\in C(\overline{\mathbb{R}}_+,\mathbb{R}^n)$ and $\Phi(t)\in C\in([-h,0],\mathbb{R}^n)$. The function $F(t)$ summarize the government and the private controls and some structural factors. The proposed model was derived from familiar economic principles and was used to study the dynamics of six important economic factors: national income, interest rate, employment, value of capital stock, prices, and cumulative balance of payment. The function $\Phi(t)$ describes the past (historical data) of $X(t)$, as well as which data have an impact on the dynamics of the state of the economy. Note that for the considered model, the conditions (S) hold, and then the stability criteria proved in the present work can be used to study the stability properties of the model.

As a first result for the considered IP for a linear neutral system with distributed delays and derivatives in Caputo's sense of incommensurate order, the existence of a unique solution is proven in the case when the initial functions are with a bounded variation. As corollary, we obtain the existence and uniqueness of a fundamental matrix for the homogeneous system, which has a bounded variation on each compact subinterval of J_a. Second, without any additional assumptions of the boundedness type, it is established that the existence and uniqueness of a fundamental matrix lead to the existence and uniqueness of a resolvent kernel and vice versa. The explicit formula describing the relationship between the fundamental matrix and the resolvent kernel is proven in the general case too. Furthermore, on the base of the existence and uniqueness of a resolvent kernel, a necessary condition as well as a sufficient condition for the stability of the zero solution of the homogeneous system are established. Finally, a well-known economics model is considered, describing the dynamics of the wealth of nations, and we comment on the possibilities of the application of the obtained results for the considered systems, which include as a partial case the considered model. Note that the validation of our conclusion follows from the results proved in Sections 3 and 4.

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