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Pricing Equity-Indexed Annuities under a Stochastic Dividend Model

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Abstract: In this paper, we examine the valuations of equity-indexed annuities (EIAs) when their reference stocks distribute stochastic dividends. Due to the fact that stocks typically pay dividends at discrete times after the payment dates are announced, pricing EIAs with dividends is deemed to be practically significant. We directly model the discrete dividend payments using the jump diffusion process with regime switching, and then determine the dynamics of the stock price. The equivalent martingale measure of fair valuation in incomplete markets is determined by employing the Esscher transform. Finally, the pricing formulas of several of the most common EIAs in the market under the stochastic dividend model are obtained. Our model incorporates and extends the present literature on EIAs with accurate and effective valuation methods.

Keywords: equity-indexed annuities; stochastic dividend; announced in advance; Esscher transform

MSC: 91G20; 91G50; 62P05

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1. Introduction

Equity-linked annuities (ELAs) are a significant innovation in the insurance industry. ELAs provide policyholders with insurance protection and an investment return from the stock market. The most popular type of ELA is the equity-indexed annuity, which has enjoyed record industry sales in a comparatively short time. EIAs are essentially equity-linked contracts whose returns depend on the performance of an equity index, usually the S&P 500. EIAs also have a minimum guarantee clause to eliminate the downside risk for customers. Since EIA was first launched by Keyport Life Insurance Co in 1995, it has not only gained great popularity in the market, but also received extensive attention from scholars. Tiong [1] investigated the pricing of several common EIA product designs with the use of the Esscher transform. Lin and Tan [2] studied the valuations of some prevalent EIAs when the interest rates followed a mean-reverting diffusion process. Hainaut [3] discussed the effect of stock market volatility clustering on the valuation and risk of EIA. Shi and Zhang [4] utilized a Fourier transform approach to price path-dependent EIAs under a time-varying Lévy process.

The product designs of EIAs may differ depending on the companies who sell them. In this article, we discuss the pricing of point-to-point EIAs and annual reset EIAs, which are the most common in the market. Both of these products earn the realized income of the index (or other risk assets) with the prescribed participation in a certain period of time, but the way they calculate returns is different. As in the literature mentioned above, most research models the reference asset (such as stock) price directly and ignores dividend payments. In practice, however, dividends are paid at discrete intervals of time, always after the disclosure of the dates of the dividend payment. Therefore, the study of EIAs pricing under the dividend situation has important application value. As pointed out by [5], the dividend discounted model suggests that the reference stock price ought to equal the present value of all future discrete dividend payments. Under the pricing measure in [5], we can acquire the price ex-dividend for a stock that distributes dividend D_n at time $t_n > t$ under the real probability measure by

$$S_t = \mathbf{E}_t\left[\sum_{t_n>t} e^{-\tilde{r}(t_n-t)} D_n\right], \tag{1}$$

where $\tilde{r}$ denotes the expected return on stocks. Thus, from this perspective, the stock price is a derivative object. A more appropriate method is to directly model the discrete dividends so that the stock price can be obtained from the dividend model. Numerous works have also shown that it is suitable to study the derivatives pricing problem by modeling the discrete dividend payments. Korn and Rogers [5] first introduced the geometric Brownian motion (GBM) to model discrete dividend payments and got the option price in the Black–Scholes setting. Kruse and Muller [6] derived analytic expressions for the American option prices when the dividend process was governed by the GBM. Yan et al. [7] discussed the valuation problem of an EIA under the assumption that dividends were announced and paid at the same time. Shan et al. [8] extended the model in [5] and attained the pricing formulas of European call option.

Regime-switching models have become extremely popular in derivatives pricing during the last few years. Lin et al. [9] investigated the EIA valuation problems when the reference stock was driven by a regime-switching GBM. Qian et al. [10] assumed that the reference risky asset obeyed the stochastic process with regime switching and studied the pricing of EIAs with stochastic mortality risk under this model. Qian et al. [11] applied the local risk-minimization method to hedge EIAs when the stock followed a jump-diffusion process with regime switching. Indeed, the change in the state of the economy has a significant effect on the number of dividends distributed by the firm. Therefore, the regime-switching model can also be employed to describe this feature (see [12–14]). Inspired by the aforementioned works, we exploit the regime-switching jump diffusion process to model the discrete dividend payments of reference stock. However, the market in our model becomes incomplete due to the additional risk, which implies that there are many martingale pricing measures. To explore fair valuation, we take into account the method of Esscher transform (see, [15–17]) to find a martingale pricing measure.

In reality, when the underlying stock distributes a dividend, its amount is declared prior to the dividend payment time. Therefore, the research on the pricing of EIAs in this context is extremely valuable in the application of financial markets. This extends the current literature on EIAs pricing. The key contributions of this article are outlined below.

(i) We replace the specific pricing measure in [5] with the general real probability measure and obtain the dynamic of the reference stock price under the martingale pricing measure.
(ii) We employ the Esscher transform to identify the unique martingale pricing measure in the incomplete setting under the stochastic dividend model.
(iii) We derive the pricing formulas for the point-to-point EIA and the annual reset EIA when the dividends are declared earlier than the payment time.

The remaining part of the article is organized below. Section 2 describes the dynamics of the dividend model and identifies the unique martingale pricing measure by applying the Escher transform. Section 3 presents the point-to-point EIA and annual reset EIA pricing under the stochastic dividend model. The last section contains conclusions and some potential works.

2. Literature Review

Here, we briefly review the literature on the valuation of EIAs. Much attention has been paid to the pricing of EIAs in the Black–Scholes framework, including studies by [1,18,19]. These works examined the application of option pricing theory and its techniques to EIAs valuation. To address the problem of volatile smiles, ref. [20] assumed that the underlying asset price followed a variance-gamma process and derived analytical solutions for EIAs with different product designs. Scholars also investigated the impact of alternative dynamics on the prices of EIAs. For example, refs. [3,10,21] examined the valuation of various EIAs in a jump diffusion setting. Refs. [2,22,23] studied the problems of pricing EIAs with stochastic interest rates and mortality risk, while [24,25] considered the EIAs pricing with stochastic volatility. Regime switching models are more popular in pricing EIAs in recent years. Ref. [9] discussed the valuation of EIAs when the market value dynamic of the underlying asset is driven by a regime switching GBM. Meanwhile, the Esscher transform was utilized to determine a pricing measure for fair valuation. Some examples of EIA pricing under regime-switching models include [26–28]. Unlike the Esscher transform, ref. [27] utilized the minimal martingale measure method to identify a pricing measure. Ref. [4] used a Fourier transform approach to price path-dependent EIAs under a time-varying Lévy process and utilized the minimal martingale measure method to identify a pricing measure.

3. The Modeling Assumptions

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\mathbf{P}$ is the real-world probability measure. We suppose that the economic states are modeled by a continuous-time Markov chain $\{\xi_t\}_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbf{P})$ with a state space $\mathscr{X}$. The state space of $\{\xi_t\}_{t\in[0,T]}$ is a finite set of unit vectors $(e_1, \dots e_N)$, where $e_i = (0, \dots 1, \dots 0) \in R^N$. Here, we take the usual practice of Elliott et al. [15] and give the following representation form of $\{\xi_t\}_{t\in[0,T]}$:

$$\xi_t = \xi_0 + \int_0^t Q\xi_s \mathrm{d}s + M_t, \tag{2}$$

where $\{M_t\}_{t\in[0,T]}$ is a martingale process with respect to the filtration generated by $\{\xi_t\}_{t\in[0,T]}$ and $Q = (q_{ij})_{i,j=1,\dots N}$ is the rate matrix of $\{\xi_t\}_{t\in[0,T]}$.

Assumes that the reference equity distributes dividends at equidistant times. Specifically, these dividends $D_1 = \mathrm{X}_{t_1}, \dots, D_n = \mathrm{X}_{t_n} \dots$ paid at times $0 < t_1 = h, \dots, t_n = nh \dots$, where $\{\mathrm{X}_t\}_{t\in[0,T]}$ is the dividend process described by the following jump diffusion process:

$$\frac{\mathrm{dX}_t}{\mathrm{X}_{t-}} = (\mu - k\lambda_t)\mathrm{d}t + \sigma_t \mathrm{d}\widetilde{W}_t + \left(\mathrm{e}^{Z_{t-}} - 1\right)\mathrm{d}N_t, \tag{3}$$

where $\mathrm{X}_0 = x_0$, $\mu < \tilde{r}$, $\{\widetilde{W}_t\}_{t\in[0,T]}$ is a standard $\mathbf{P}$-Brownian motion ($\mathbf{P}$-BM), $\{\lambda_t\}_{t\in[0,T]}$ is the intensity of the Poisson process $\{N_t\}_{t\in[0,T]}$ and $\{\sigma_t\}_{t\in[0,T]}$ is the volatility of $\{\mathrm{X}_t\}_{t\in[0,T]}$. If the jump arrives at time t, the jump amplitude is controlled by $Z_t \sim N(\mu_1, \sigma_1^2)$, where $\sigma_1 > 0$. Thus, the percentage change in the dividend process is $k = \mathrm{e}^{\mu_1 + \frac{1}{2}\sigma_1^2} - 1$. Write $(\mathcal{F}_t^S)_{t\in[0,T]}$ and $\left(\mathcal{F}_t^{\xi}\right)_{t\in[0,T]}$ for the $\mathbf{P}$-augmentations of the filtration generated by $\{S_t\}_{t\in[0,T]}$ and $\{\xi_t\}_{t\in[0,T]}$, respectively. Assume that for every t, $\{Z_t\}$ are i.i.d., and $\{Z_t\}_{t\in[0,T]}$, $\{N_t\}_{t\in[0,T]}$, $\{\xi_t\}_{t\in[0,T]}$ and $\{\widetilde{W}_t\}_{t\in[0,T]}$ are independent of each other. The parameters σ_t, λ_t depend on the chain $\{\xi_t\}_{t\in[0,T]}$, as follows:

$$\sigma_t := \langle \sigma, \xi_t \rangle, \quad \lambda_t := \langle \lambda, \xi_t \rangle,$$

where $\sigma, \lambda \in R_+^N$, and $\langle \cdot, \cdot \rangle$ represents the inner product operation. Moreover, we can get the same relation as [5]:

$$\mathbf{E}\mathrm{X}_t/\mathrm{X}_0 = e^{\mu t}\mathbf{E}\left[\exp\left\{\int_0^t \sigma_s \mathrm{d}\widetilde{W}_s - \frac{1}{2}\int_0^t \sigma_s^2 \mathrm{d}s + \int_0^t Z_{s-}\mathrm{d}N_s - \int_0^t k\lambda_s \mathrm{d}s\right\}\right] = e^{\mu t}.$$

Assuming that dividends are announced and paid at the same time, the underlying share price can be shown as

$$S_t = S_0 \exp\left\{\int_0^t \left(\tilde{r} - k\lambda_s - \frac{1}{2}\sigma_s^2\right)\mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\widetilde{W}_s + \int_0^t Z_{s-}\mathrm{d}N_s\right\}, \tag{4}$$

for $t \in ((m-1)h, mh)$, where $S_0 = X_0 e^{-(\tilde{r}-\mu)mh} / \left(1 - e^{-(\tilde{r}-\mu)h}\right)$. Particularly, we have

$$S_{mh} = S_{mh-} - X_{mh} = S_{mh-} e^{-(\tilde{r}-\mu)h}. \tag{5}$$

This relational equation indicates that S_{mh} is determined by S_{mh-}. That is, the absolute amount of dividend payments is stochastic, but not their relative amount.

As mentioned earlier, our market model is incomplete, and thus, lots of martingale pricing measures exist. Next, we employ the Esscher transform in [9,10] to find a martingale pricing measure for fair valuation. Define $Y_T = \log \frac{S_t}{S_0}$. Then, we divide Y_T into a diffusive process Y_T^1 plus a pure jump process Y_T^2, where

$$Y_T^1 = \int_0^t \left(\tilde{r} - k\lambda_s - \frac{1}{2}\sigma_s^2\right) \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\widetilde{W}_s, \quad Y_T^2 = \int_0^t Z_{s-} \mathrm{d}N_s.$$

For each $t \in [0, T]$, we write $\mathcal{G}_t = \mathcal{F}_t^S \vee \mathcal{F}_T^\xi$. Then, the Esscher transform $\mathbf{Q} \sim \mathbf{P}$ on $\mathcal{G}_t$ with respect to the regime-switching parameters $(\theta_t^1)_{t\in[0,T]}$ and $(\theta_t^2)_{t\in[0,T]}$ is given by

$$\begin{aligned}
\Lambda_t = \left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{G}_t} &= \frac{\exp\left(\int_0^t \theta_s^1 \mathrm{d}Y_s^1 + \int_0^t \theta_{s-}^2 \mathrm{d}Y_s^2\right)}{E\left[\exp\left(\int_0^t \theta_s^1 \mathrm{d}Y_s^1 + \int_0^t \theta_{s-}^2 \mathrm{d}Y_s^2\right) \mid \mathcal{F}_t^\xi\right]} \\
&= \exp\left(\int_0^t \theta_{s-}^2 Z_{s-} \mathrm{d}N_s - \int_0^t \lambda_s \left(e^{\theta_s^2 \mu_1 + \frac{1}{2}\left(\theta_s^2 \sigma_1\right)^2} - 1\right) \mathrm{d}s\right) \\
&\quad \cdot \exp\left(\int_0^t \theta_s^1 \sigma_s \mathrm{d}\widetilde{W}_s - \frac{1}{2}\int_0^t \left(\theta_s^1 \sigma_s\right)^2 \mathrm{d}s\right),
\end{aligned} \tag{6}$$

where $\theta_t^m = \langle \theta^m, \xi_t \rangle$, $\theta^m = \left(\theta_1^m, \theta_2^m, \ldots, \theta_n^m\right) \in R^n$, $m = 1, 2$. It is not difficult to verify that the Radon–Nikodym derivative process Λ_t is an exponential martingale.

According to fundamental theorem on asset pricing in refs. [29,30], the absence of arbitrage is equivalent to fulfilling the so-called martingale condition: there exists an equivalent martingale measure, such that the discounted stock price is a martingale under that measure. Let r denote the risk-free rate, as in refs. [31]; the martingale condition becomes

$$S_0 = \mathbf{E}_{\mathbf{Q}}\left[e^{-rt} S_t \mid \mathcal{G}_0\right], \tag{7}$$

To further identify the parameters θ_t^1 and θ_t^2, we adopt the view in refs. [31,32] that the jump risk is diversifiable and not priced, which implies that $\theta_t^2 = 0$. Then, we solve the martingale condition by the following theorem.

Theorem 1. *When the dividend payments of the reference asset follow a stochastic process, the martingale condition in (7) is valid if and only if*

$$\theta_t^1 = \frac{r - \tilde{r}}{\sigma_t^2}. \tag{8}$$

Proof. By means of Bayes' rule, we can obtain

$$\begin{aligned}
\mathbf{E}_{\mathbf{Q}}\left[e^{-rt} S_t \mid \mathcal{G}_0\right] =& \frac{\mathbf{E}\left[e^{-rt} S_t \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} \mid \mathcal{G}_0\right]}{\mathbf{E}\left[\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} \mid \mathcal{G}_0\right]} \\
=& S_0 \mathbf{E}\Bigg[\exp\left(\int_0^t \sigma_s \left(1 + \theta_s^1\right) \mathrm{d}W_s - \frac{1}{2}\int_0^t \sigma_s^2 \left(1 + \theta_s^1\right)^2 \mathrm{d}s - \int_0^t \left(1 - e^{\mu_1 + \frac{1}{2}\sigma_1^2}\right) \lambda_s \mathrm{d}s\right. \\
&\left. + \int_0^t Z_{s-} \mathrm{d}N_s + \int_0^t \left(e^{\mu_1 + \frac{1}{2}\sigma_1^2} - 1\right) \mathrm{d}s + \int_0^t \left(\tilde{r} - r - k\lambda_s - \theta_s^1 \sigma_s^2\right) \mathrm{d}s \mid \mathcal{G}_0\right)\Bigg] \\
=& S_0 \exp\left(\int_0^t \left(\tilde{r} - r - k\lambda_s + \theta_s^1 \sigma_s^2 + k\lambda_s\right) \mathrm{d}s\right).
\end{aligned}$$

Hence, the martingale condition (7) is valid if and only if $\theta_t^1 = \frac{r-\tilde{r}}{\sigma_t^2}$. Vice versa. This proves the theorem. □

Substituting the regime-switching parameter, Λ_t can be rewritten as

$$\Lambda_t = \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} \mid \mathcal{G}_t = \exp\left(\int_0^t \frac{r-\tilde{r}}{\sigma_s} \mathrm{d}\widetilde{W}_s - \frac{1}{2} \int_0^t \left(\frac{r-\tilde{r}}{\sigma_s} \right)^2 \mathrm{d}s \right). \tag{9}$$

Applying the Girsanov's theorem, $W_t = \widetilde{W}_t + \int_0^t \frac{r-\tilde{r}}{\sigma_s} \mathrm{d}s$ is a standard $\mathbf{Q}$-BM. So, the price of reference stock under the martingale pricing measure $\mathbf{Q}$ is given by

$$\frac{\mathrm{d}S_t}{S_{t-}} = (r - k\lambda_t)\mathrm{d}t + \sigma_t \mathrm{d}W_t + \left(e^{Z_{t-}} - 1 \right) \mathrm{d}N_t. \tag{10}$$

Indeed, the expected return of the stock under the equivalent martingale measure equals the risk-free rate, so the assumption that $\mu < r$ in [5] still holds.

Remark 1. *Under the martingale pricing measure* $\mathbf{Q}$, *when the discrete dividend payments of reference asset follow a jump-diffusion process with regime switching, the discounted stock price process* $\{e^{-rt}S_t\}$ *is a martingale between the adjacent dividend payment time.*

In practice, dividends of underlying stocks are always paid after the dates of their payment are disclosed. To solve this, we adopt the view in [5], supposing that the dividend distributed at time mh is declared at time $(m-1+\rho)h$ and equals $X_{(m+1-\rho)h}$, where $\rho \in (0,1)$. From the declaration of a dividend to the date of dividend payment, the underlying stock price S_t incorporates a deterministic part, reflecting the present value of the next already known dividend. The stock price at this time is considered the sum of the ex-dividend price and the evolution of the present value of the next dividend payment. So, the dividends for this time interval are referred to as cum dividends. When we interpret the announcement of the dividend as the payment of its present value at the announcement time, then with $D_{(m-1+\rho)h} \mathrm{X}_{mh} e^{-r(1-\rho)h}$, we utilize the prior work to derive the ex-dividend price of underlying stock as

$$S_t^{ex} = \frac{\mathrm{X}_t e^{-(r-\mu)\{(m+\rho)h-t\}}}{1 - e^{-(r-\mu)h}}, \tag{11}$$

and the cum-dividend price is given by

$$S_t^{\mathrm{cum}} = S_t^{ex} + \mathrm{X}_{(m-1+\rho)h} e^{r(t-(m-1+\rho)h)}, \tag{12}$$

for $t \in ((m-1+\rho)h, mh)$. For $t \in ((m-1)h, (m-1+\rho)h)$, the stock price as follows

$$S_t = \frac{\mathrm{X}_t e^{-(r-\mu)\{(m-1+\rho)h-t\}}}{1 - e^{-(r-\mu)h}}. \tag{13}$$

In particular, we can get the following relational equation, similar to (5),

$$S_{mh} = S_{mh-} - \mathrm{X}mh = S_{mh-} e^{-(r-\mu)\rho h}. \tag{14}$$

In particular, $\rho = 1$ indicates that the dividends are declared and paid at the same time.

From (6), the martingale pricing measure gained through the Esscher transform is also applicable to the case where the dividends are declared earlier than the payment time. The reference stock price in this case is then given by the below theorem.

Theorem 2. *When the dividend payments of the reference asset follow a jump-diffusion process with regime switching, the price of discounted stock $\{e^{-rt}S_t\}$ is a martingale between the adjacent dividend announcement and dividend payment time.*

Proof. If $t \in ((m-1)h, (m+\rho)h)$, let $S_1 = \frac{X_0 e^{-(r-\mu)(m-1+\rho)h}}{1-e^{-(r-\mu)h}}$, then the price of discounted stock is given by

$$e^{-rt}S_t = S_1 \exp\left\{ \int_0^t (-k\lambda_s - \frac{1}{2}\sigma_s^2)\mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s + \int_0^t Z_{s-}\mathrm{d}N_s \right\}. \quad (15)$$

If $t \in ((m-1+\rho)h, mh)$, we write $S_2 = \frac{X_0 e^{-(r-\mu)(m+\rho)h}}{1-e^{-(r-\mu)h}}$; the price of discounted stock can be described by

$$e^{-rt}S_t = S_2 \exp\left\{ \int_0^t (-k\lambda_s - \frac{1}{2}\sigma_s^2)\mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s + \int_0^t Z_{s-}\mathrm{d}N_s \right\} + X_{(m-1+\rho)h} e^{-r(m-1+\rho)h}. \quad (16)$$

With the definition of exponential martingale, we can quickly determine that the price of discounted stock (15) and (16) are **Q**- martingales. This completes the proof. □

4. Pricing EIAs

In this section, we introduce two common EIA designs and pricing formulas when the dividends are declared earlier than the payment time. In order to make our model more suitable for the actual market, we also take the mortality risk of the policyholder into consideration.

4.1. Pricing the Point-to-Point Design EIA

Suppose that the initial premium be 1, thus the contingent claim $C_{pp}(t)$ can be expressed as

$$C_{pp}(t) = \max\{\min(e^{\alpha \Upsilon_t}, e^{\beta t}), e^{\gamma t}\}, \quad (17)$$

where α is the participation rate, which is usually less than or equal to 1, β is the upper limit rate that indicates the maximum annualized rate that can be credited, and $\gamma(< \beta)$ is the guaranteed minimum return during the entire contract period. As in [9], we denote $\kappa(x)$ as the future lifetime of the policyholder at age x. We suppose that $\kappa(x)$, S_t and ξ_t are independent of each other. We also assume that T is an integer, and if the insured survives after maturity time, the policy will pay $C_{pp}(T)$; if the insured died in $(t-1, t], t \leq T$, the insurer will pay $C_{pp}(t)$. For ease of expression, the following standard actuarial notation will be used: ${}_tp_x = P(\kappa(x) > t)$ represents the probability that a x-year-old human lives to age $x+t$, and ${}_tq_x = 1 - {}_tp_x$.

Theorem 3. *Under the martingale pricing measure* **Q**, *when the dividend payments of the reference asset obey a jump-diffusion process with regime switching, the time-zero price of T-annual maturity point-to-point EIA $P_{pp}(S_0, T; \xi)$ is as follows*
(i) When $T \in ((m-1)h, (m+\rho)h)$,

$$P_{PP}(S_0, T; \xi) = \mathbf{E_Q}\left[e^{-rT} \sum_{n=1}^{\infty} \frac{e^{-\int_0^T \lambda_s \mathrm{d}s} \left(\int_0^T \lambda_s \mathrm{d}s\right)^n}{n!} \Psi(0, T) \right] \times {}_Tp_x$$
$$+ \sum_{t=1}^{T} \mathbf{E_Q}\left[e^{-rt} \sum_{n=1}^{\infty} \frac{e^{-\int_0^t \lambda_s \mathrm{d}s} \left(\int_0^t \lambda_s \mathrm{d}s\right)^n}{n!} \Psi(0, t) \right] \times {}_{t-1}p_x q_{x+t-1}, \quad (18)$$

where

$$\Psi(0,t) = e^{\gamma t}\Phi(d_1) + e^{\beta t}\Phi(d_2) + e^{\alpha\left(rt - \int_0^t \frac{1}{2}\sigma_s^2 \mathrm{d}s - \int_0^t k\lambda_s \mathrm{d}s + n\mu_1\right) + \frac{\alpha^2}{2}\left(\int_0^t \sigma_s^2 \mathrm{d}s - n\sigma_1^2\right)}[\Phi(d_3) - \Phi(d_4)],$$

$$d_1 = \frac{\frac{\gamma t}{\alpha} - \int_0^t \left(r - \frac{1}{2}\sigma_s^2 - k\lambda_s\right)\mathrm{d}s - n\mu_1}{\sqrt{\int_0^t \sigma_s^2 \mathrm{d}s + n\sigma_1^2}},$$

$$d_2 = \frac{\frac{-\beta t}{\alpha} + \int_0^t \left(r - \frac{1}{2}\sigma_s^2 - k\lambda_s\right)\mathrm{d}s + n\mu_1}{\sqrt{\int_0^t \sigma_s^2 \mathrm{d}s + n\sigma_1^2}},$$

$$d_3 = \frac{\frac{\beta t}{\alpha} - \int_0^t \left(r - \frac{1}{2}\sigma_s^2 - k\lambda_s\right)\mathrm{d}s - n\mu_1 - \alpha\left(\int_0^t \sigma_s^2 \mathrm{d}s + n\sigma_1^2\right)}{\sqrt{\int_0^t \sigma_s^2 \mathrm{d}s + n\sigma_1^2}},$$

$$d_4 = \frac{\frac{\gamma t}{\alpha} - \int_0^t \left(r - \frac{1}{2}\sigma_s^2 - k\lambda_s\right)\mathrm{d}s - n\mu_1 - \alpha\left(\int_0^t \sigma_s^2 \mathrm{d}s + n\sigma_1^2\right)}{\sqrt{\int_0^t \sigma_s^2 \mathrm{d}s + n\sigma_1^2}}.$$

(ii) When $T \in ((m-1+\rho)h, mh)$,

$$\begin{aligned} P_{PP}(S_0, T; \xi) = {} & \mathbf{E_Q}\left[e^{-rT}\sum_{n=1}^{\infty}\frac{e^{-\int_0^T \lambda_s \mathrm{d}s}\left(\int_0^T \lambda_s \mathrm{d}s\right)^n}{n!}\widetilde{\Psi}(0,T)\right] \times {}_T p_x \\ & + \sum_{t=1}^{T}\mathbf{E_Q}\left[e^{-rt}\sum_{n=1}^{\infty}\frac{e^{-\int_0^t \lambda_s \mathrm{d}s}\left(\int_0^t \lambda_s \mathrm{d}s\right)^n}{n!}\widetilde{\Psi}(0,t)\right] \times {}_{t-1}p_x q_{x+t-1}, \end{aligned} \tag{19}$$

where

$$\widetilde{\Psi}(0,t) = e^{\gamma t}\Phi(d_1) + e^{\beta t}\Phi(d_2) + e^{\alpha\left(rt - \int_0^t \frac{1}{2}\sigma_s^2 \mathrm{d}s - \int_0^t k\lambda_s \mathrm{d}s - (r-\mu)h + n\mu_1\right) + \frac{\alpha^2}{2}\left(\int_0^t \sigma_s^2 \mathrm{d}s - n\sigma_1^2\right)}[\Phi(d_3) - \Phi(d_4)],$$

$$d_1 = \frac{\frac{\gamma t}{\alpha} - \int_0^t \left(r - \frac{1}{2}\sigma_s^2 - k\lambda_s\right)\mathrm{d}s + (r-\mu)h - n\mu_1}{\sqrt{\int_0^t \sigma_s^2 \mathrm{d}s + n\sigma_1^2}},$$

$$d_2 = \frac{\frac{-\beta t}{\alpha} + \int_0^t \left(r - \frac{1}{2}\sigma_s^2 - k\lambda_s\right)\mathrm{d}s - (r-\mu)h + n\mu_1}{\sqrt{\int_0^t \sigma_s^2 \mathrm{d}s + n\sigma_1^2}},$$

$$d_3 = \frac{\frac{\beta t}{\alpha} - \int_0^t \left(r - \frac{1}{2}\sigma_s^2 - k\lambda_s\right)\mathrm{d}s + (r-\mu)h - n\mu_1 - \alpha\left(\int_0^t \sigma_s^2 \mathrm{d}s + n\sigma_1^2\right)}{\sqrt{\int_0^t \sigma_s^2 \mathrm{d}s + n\sigma_1^2}},$$

$$d_4 = \frac{\frac{\gamma t}{\alpha} - \int_0^t \left(r - \frac{1}{2}\sigma_s^2 - k\lambda_s\right)\mathrm{d}s + (r-\mu)h - n\mu_1 - \alpha\left(\int_0^t \sigma_s^2 \mathrm{d}s + n\sigma_1^2\right)}{\sqrt{\int_0^t \sigma_s^2 \mathrm{d}s + n\sigma_1^2}}.$$

Proof. For (i), when $T \in ((m-1)h, (m+\rho)h)$,

$$Y_T = \log\frac{S_T}{S_1} = \int_0^T (r - k\lambda_s - \frac{1}{2}\sigma_s^2)\mathrm{d}s + \int_0^T \sigma_s \mathrm{d}W_s + \int_0^T Z_{s-}\mathrm{d}N_s. \tag{20}$$

The time-zero price of this policy under the equivalent martingale measure $\mathbf{Q}$ is

$$
\begin{aligned}
P_{PP}(S_0, T; \xi) &= \mathbf{E_Q}\left[e^{-rT} C_{PP}(T) I(\kappa(x) > T)\right] + \sum_{t=1}^{T} \mathbf{E_Q}\left[e^{-rT} C_{PP}(T) I(t-1 < \kappa(x) \le t)\right] \\
&= \mathbf{E_Q}\left[e^{-rT} C_{PP}(T)\right] \times {}_T p_x + \sum_{t=1}^{T} \mathbf{E_Q}\left[e^{-rt} C_{PP}(t)\right] \times {}_{t-1} p_x q_{x+t-1}.
\end{aligned} \tag{21}
$$

Substituting for (20), we have

$$
\begin{aligned}
\mathbf{E_Q}[e^{-rT} C_{PP}(T)] &= \mathbf{E_Q}\left[\mathbf{E_Q}\left[e^{-rT} C_{PP}(T) \mid \mathcal{G}_0\right]\right] \\
&= \mathbf{E_Q}\left[e^{-rT} \mathbf{E_Q}\left[e^{\gamma T} I(Y_T \le \frac{\gamma T}{\alpha}) \mid \mathcal{G}_0\right]\right] + \mathbf{E_Q}\left[e^{-rT} \mathbf{E_Q}\left[e^{\beta T} I(Y_T > \frac{\beta T}{\alpha}) \mid \mathcal{G}_0\right]\right] \\
&\quad + \mathbf{E_Q}\left[e^{-rT} \mathbf{E_Q}\left[e^{\alpha Y_T} I(\gamma T < \alpha Y_T \le \beta T) \mid \mathcal{G}_0\right]\right] \\
&= \mathbf{E_Q}\left[e^{-rT} e^{\gamma T} \sum_{n=1}^{\infty} \frac{e^{-\int_0^T \lambda s \mathrm{d}s} \left(\int_0^T \lambda s \mathrm{d}s\right)^n}{n!} \Phi(d_1)\right] \\
&\quad + \mathbf{E_Q}\left[e^{-rT} e^{\beta T} \sum_{n=1}^{\infty} \frac{e^{-\int_0^T \lambda s \mathrm{d}s} \left(\int_0^T \lambda s \mathrm{d}s\right)^n}{n!} \Phi(d_2)\right] \\
&\quad + \mathbf{E_Q}\Bigg[e^{-rT} e^{\alpha\left(\int_0^T (r - \frac{1}{2}\sigma_s^2 - k\lambda_s)\mathrm{d}s + n\mu_1\right) + \frac{\alpha^2}{2}\left(\int_0^T \sigma_s^2 \mathrm{d}s - n\sigma_1^2\right)} \\
&\quad \cdot \sum_{n=1}^{\infty} \frac{e^{-\int_0^T \lambda s \mathrm{d}s} \left(\int_0^T \lambda s \mathrm{d}s\right)^n}{n!} [\Phi(d_3) - \Phi(d_4)]\Bigg].
\end{aligned} \tag{22}
$$

Then, (18) comes immediately from (21) and (22). For (ii), when $T \in ((m-1+\rho)h, mh)$,

$$
\begin{aligned}
Y_T &= \log \frac{S_T - X_{(m-1+\rho)h} e^{-r[(m-1+\rho)h - T]}}{S_1} \\
&= \int_0^T (r - k\lambda_s - \frac{1}{2}\sigma_s^2)\mathrm{d}s - (r - \mu)h + \int_0^T \sigma_s \mathrm{d}W_s + \int_0^T Z_{s-} \mathrm{d}N_s.
\end{aligned} \tag{23}
$$

According to the proof of (i), it is not difficult to get (19). Thus, the theorem's proof is finished. $\square$

Let J_i indicates the occupation time of $\{\xi_t\}_{t \in [0,T]}$ in state e_i over the time horizon $[0, T]$. Then,

$$
\lambda_T^* = \int_0^T \lambda_s \mathrm{d}s = \sum_{i=1}^{N} r_i J_i, \tag{24}
$$

$$
U_T = \int_0^T \sigma_s^2 \mathrm{d}s = \sum_{i=1}^{N} \sigma_i^2 J_i. \tag{25}
$$

Write $J = (J_1, J_2, \dots, J_N)$ for the vector of occupation times. Let B denote a diagonal matrix consisting of the elements in the vector $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_N)$ as its diagonal. For any ζ, the characteristic function of J under $\mathbf{Q}$ is

$$
E[\exp(i\langle \zeta, J \rangle) \mid \xi_0] = \langle \exp[(Q + iB)T]\xi_0, \mathbf{1} \rangle, \tag{26}
$$

where $i = \sqrt{-1}, \mathbf{1} = (1, 1, \dots, 1) \in R^N$. Let $\phi(J_1, J_2, \dots, J_N)$ denotes the joint probability distribution for the occupation times $J = (J_1, J_2, \dots, J_N)$. Note that $\phi(J_1, J_2, \dots, J_N)$ can be completely determined by the characteristic function.

For (i), when $T \in ((m-1)h, (m+\rho)h)$, the time-zero price of the point-to-point design EIA (18) becomes

$$\begin{aligned} P_{PP}(S_0, T; \xi) = &\int_{[0,T]^N} V(S_0, \lambda_T^*, U_T, \xi_0)\phi(J_1, J_2, \ldots, J_N)dJ_1 dJ_2 \ldots dJ_N \times {}_T p_x \\ &+ \sum_{t=1}^{T} \int_{[0,T]^N} V(S_0, \lambda_t^*, U_t, \xi_0)\phi(J_1, J_2, \ldots, J_N)dJ_1 dJ_2 \ldots dJ_N \times {}_{t-1}p_x q_{x+t-1}, \end{aligned} \tag{27}$$

where

$$V(S_0, \lambda_T^*, U_T; \xi_0) = e^{-rT} \sum_{n=1}^{\infty} \frac{e^{-\lambda_T^*}(\lambda_T^*)^n}{n!} \Psi(0, T), \tag{28}$$

$$\Psi(0,T) = e^{\gamma T}\Phi(d_1) + e^{\beta T}\Phi(d_2) + e^{\alpha\left(rT - \frac{1}{2}U_T - k\lambda_T^* + n\mu_1\right) + \frac{\alpha^2}{2}\left(U_T - n\sigma_1^2\right)}[\Phi(d_3) - \Phi(d_4)],$$

$$\begin{aligned} d_1 &= \frac{\frac{\gamma T}{\alpha} - \left(rT - \frac{1}{2}U_T - k\lambda_T^*\right) - n\mu_1}{\sqrt{U_T + n\sigma_1^2}}, \\ d_2 &= \frac{\frac{-\beta T}{\alpha} + rT - \frac{1}{2}U_T - k\lambda_T^* + n\mu_1}{\sqrt{U_T + n\sigma_1^2}}, \\ d_3 &= \frac{\frac{\beta T}{\alpha} - rT + \frac{1}{2}U_T + k\lambda_T^* - n\mu_1 - \alpha\left(U_T + n\sigma_1^2\right)}{\sqrt{U_T + n\sigma_1^2}}, \\ d_4 &= \frac{\frac{\gamma T}{\alpha} - rT + \frac{1}{2}U_T + k\lambda_T^* - n\mu_1 - \alpha\left(U_T + n\sigma_1^2\right)}{\sqrt{U_T + n\sigma_1^2}}. \end{aligned}$$

For (ii), when $T \in ((m-1+\rho)h, mh)$, the time-zero price of the point-to-point EIA (19) becomes

$$\begin{aligned} P_{PP}(S_0, T; \xi) = &\int_{[0,T]^N} \widetilde{V}(S_0, \lambda_T^*, U_T, \xi_0)\phi(J_1, J_2, \ldots, J_N)dJ_1 dJ_2 \ldots dJ_N \times {}_T p_x \\ &+ \sum_{t=1}^{T} \int_{[0,T]^N} \widetilde{V}(S_0, \lambda_t^*, U_t, \xi_0)\phi(J_1, J_2, \ldots, J_N)dJ_1 dJ_2 \ldots dJ_N \times {}_{t-1}p_x q_{x+t-1}, \end{aligned} \tag{29}$$

where

$$\widetilde{V}(S_0, \lambda_T^*, U_T; \xi_0) = e^{-rT} \sum_{n=1}^{\infty} \frac{e^{-\lambda_T^*}(\lambda_T^*)^n}{n!} \widetilde{\Psi}(0, T), \tag{30}$$

$$\widetilde{\Psi}(0,T) = e^{\gamma T}\Phi(d_1) + e^{\beta T}\Phi(d_2) + e^{\alpha\left(rT - \frac{1}{2}U_T - k\lambda_T^* - (r-\mu)h + n\mu_1\right) + \frac{\alpha^2}{2}\left(U_T - n\sigma_1^2\right)}[\Phi(d_3) - \Phi(d_4)],$$

$$d_1 = \frac{\frac{\gamma T}{\alpha} - \left(rT - \frac{1}{2}U_T - k\lambda_T^*\right) + (r-\mu)h - n\mu_1}{\sqrt{U_T + n\sigma_1^2}},$$
$$d_2 = \frac{\frac{-\beta T}{\alpha} + rT - \frac{1}{2}U_T - k\lambda_T' - (r-\mu)h + n\mu_1}{\sqrt{U_T + n\sigma_1^2}},$$
$$d_3 = \frac{\frac{\beta T}{\alpha} - rT + \frac{1}{2}U_T + k\lambda_T' + (r-\mu)h - n\mu_1 - \alpha\left(U_T + n\sigma_1^2\right)}{\sqrt{U_T + n\sigma_1^2}},$$
$$d_4 = \frac{\frac{\gamma T}{\alpha} - rT + \frac{1}{2}U_T + k\lambda_T' + (r-\mu)h - n\mu_1 - \alpha\left(U_T + n\sigma_1^2\right)}{\sqrt{U_T + n\sigma_1^2}}.$$

4.2. Pricing the Annual Reset EIA

With this policy, its payoffs are modified or reset each year. The return for one unit of such EIA in year t is as follows

$$C_{ar}(t) = \prod_{i=1}^{t} \max\{\min(e^{\alpha \tilde{Y}_i}, e^{\beta t}), e^{\gamma t}\}, \tag{31}$$

where $\tilde{Y}_i = Y(i) - Y(i-1)$. Just like the point-to-point EIA, the variable $Y(i)$ in the expression above has several forms, including the forms defined in (20) and (23). The valuations of these EIAs are similar to the pricing used in the point-to-point EIA. Meanwhile the return of the annual ratchet EIA is same for Theorem 3, except it replaces $C_{pp}(t)$ with $C_{ar}(t)$. According to the independence between mortality risk and financial risk, the time-zero prices of the annual ratchet EIA with maturity T years under the pricing measure $\mathbf{Q}$ are as follows
when $T \in ((m-1)h, (m+\rho)h)$,

$$\begin{aligned} P(S_0, T; \xi_0) &= \mathbf{E_Q}\left[e^{-rT} C_{ar}(T) I(\kappa(x) > T)\right] + \sum_{t=1}^{T} \mathbf{E_Q}\left[e^{-rt} C_{ar}(t) I(t-1 < \kappa(x) \le t)\right] \\ &= \mathbf{E_Q}\left[e^{-rT} C_{ar}(T)\right]{}_T p_x + \sum_{t=1}^{T} \mathbf{E_Q}\left[e^{-rt} C_{ar}(t)\right]{}_{t-1} p_x \times q_{x+t-1}, \end{aligned} \tag{32}$$

when $T \in ((m-1+\rho)h, mh)$,

$$P(S_0, T; \xi_0) = \mathbf{E_Q}\left[e^{-rT} C_{ar}(T)\right]{}_T p_x + \sum_{t=1}^{T} \mathbf{E_Q}\left[e^{-rt} C_{ar}(t)\right]{}_{t-1} p_x \times q_{x+t-1}. \tag{33}$$

5. Discussion and Recommendation

This paper extends the results of [5]. Although the dividend process is drawn by an exponential Lévy process, we generalize the diffusion process to a jump diffusion process with regime switching. Regime-switching models have the advantage of flexibility in describing the effects of structural changes in economic conditions and have been applied to a variety of practical problems in finance and insurance. We also use the Esscher transform to obtain a concrete pricing measure. Different from the assumption in [7], where dividends are declared and paid at the same time, we obtain the pricing formulas for the point-to-point EIA and the annual reset EIA in the case where dividends are declared earlier than the time of payment. It is clear that our results are more consistent with the reality of market transactions.

6. Conclusions

In this paper, we investigate the valuation of point-to-point EIA and the annual reset EIA when the dividend process of their reference stock is driven by the jump diffusion model with regime switching. Instead of the approach of modeling the risky assets directly, this paper starts by modeling the dividend process. Our results fully take into account the impact that dividends have on the pricing of EIAs. The stock usually pays dividends at discrete times after the payment dates are announced. Therefore, it is more practically useful to consider the pricing of EIAs when dividends are declared earlier than they are paid.

Although the pricing formulas are well derived mathematically under the proposed model, there are several limitations in this article. Therefore, we present the following research directions to overcome these limitations. First, since EIA itself is a long-term derivative instrument, it is unreasonable to assume that interest rates will remain constant for such a long time. Thus, it is necessary to consider the pricing of EIAs under stochastic dividends and interest rates. Second, the number of dividends per distribution of the risky asset is not constant. In addition to the jump diffusion process, we can also try to use the stable process to describe the dividend process. The large and small jump parts of the stable process can portray the variation of dividends more carefully. Third, the occurrence of dividend jumps in the price process of risky assets makes pricing various path-dependent derivatives an interesting problem. For example, pricing the barrier option under stochastic dividends is a challenging problem. Because the number of dividends affects the timing of the exercise, it is tricky to carve out the joint distribution of the dividend process and the stopping time. The above issues will be considered in our future research.

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