mathematics

MDPI

Review

A Survey on the Theory of n-Hypergroups

Bijan Davvaz [1,*], Violeta Leoreanu-Fotea [2] and Thomas Vougiouklis [3]

[1] Department of Mathematical Sciences, Yazd University, Yazd 8975818411 , Iran
[2] Faculty of Mathematics, "Al.I. Cuza" University, Bd Carol I, no.11, 6600 Iaşi, Romania
[3] School of Education, Democritus University of Thrace, 68100 Alexandroupolis, Greece
* Correspondence: davvaz@yazd.ac.ir

Abstract: This paper presents a series of important results from the theory of n-hypergroups. Connections with binary relations and with lattices are presented. Special attention is paid to the fundamental relation and to the commutative fundamental relation. In particular, join n-spaces are analyzed.

Keywords: n-hypergroup; fundamental relation; join n-space

MSC: 20N20

check for updates

Citation: Davvaz, B.; Leoreanu-Fotea, V.; Vougiouklis, T. A Survey on the Theory of n-Hypergroups. *Mathematics* **2023**, *11*, 551. https://doi.org/10.3390/math11030551

Academic Editor: Adolfo Ballester-Bolinches

Received: 19 December 2022
Revised: 10 January 2023
Accepted: 16 January 2023
Published: 19 January 2023

1. Introduction

The theory of n-ary hypergroups, also called n-hypergroups, contains two generalizations of the notion of a group: n-groups and hypergroups, which are briefly presented in the next paragraph. The two concepts were introduced around the same time.

n-groups, also called polyadic groups, were introduced in 1928 by W. Dörnte [1], and they are a generalization of classical groups. An important role in n-group theory is the paper written by E.L. Post of 143 pages [2]. Such operations are used then in the study of (m, n)-rings. Among those who made recently important contributions in the theory of n-groups, we mention W. Dudek and his collaborators; see for instance [3–5]. Let $n > 2$, and denote the chain $x_i, \ldots, x_j$ by x_i^j (for $j < i$, the above sequence is the empty symbol). For a nonempty set G with one n-operation, $f : G^n \to G$ is a *n-groupoid*, which is a *n-quasigroup*, if, for all $a_1^n, b \in G$, there is exactly one $x_i \in G$ such that $f(a_1^{i-1}, x_i, a_{i+1}^n) = b$. An n-quasigroup with an associative operation is called an n-ary group.

Hypergroup theory is a field of algebra that appeared in 1934 and was introduced by the French mathematician Marty [6]. The theory has known various periods of flourishing: the 1940s, then 1970s, and especially after the 1990s, the theory has been studied on all continents, both theoretically and for a multitude of applications in various fields of knowledge: various chapters of mathematics, computer science, biology, physics, chemistry, and sociology. Several books have been written in this field, which highlight both the theoretical aspects and the applications; for instance, see [7]. Figure 1 suggestively shows the connections between the previously mentioned domains.

This survey is structured as follows: First, basic notions in the field of algebraic hyperstructures are recalled, followed by results, in particular characterizations in the field of n-hypergroups. Special attention is given to the connections with binary relations and fundamental relations. Finally, join n-spaces with connections to lattice theory are presented.

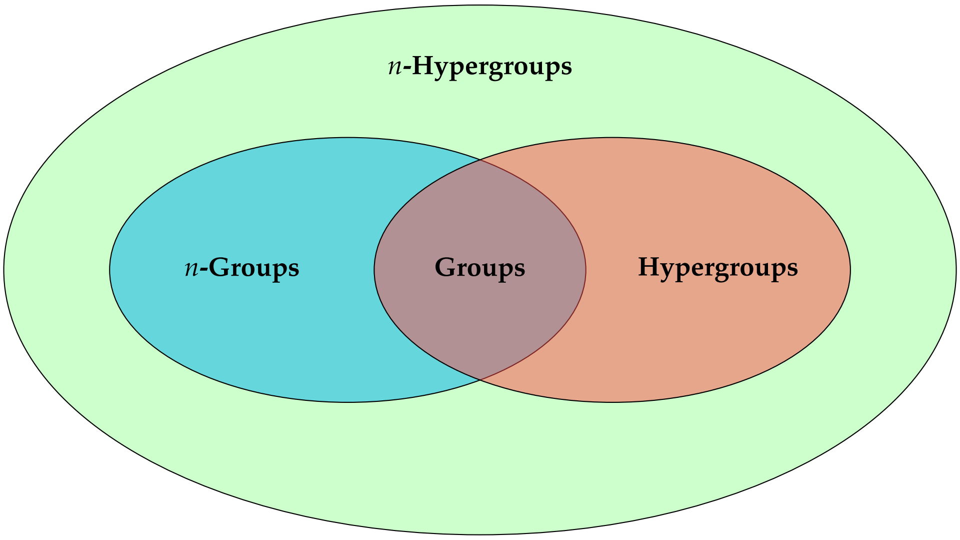

Figure 1. The connections between groups, n-groups, hypergroups, and n-hypergroups.

2. Hypergroups

An algebraic hyperstructure is a nonempty set H together with one or some functions from $H \times H$ to the set $\mathcal{P}^*(H)$ of nonempty subsets of H. For all $(x, y) \in H^2$, one denotes by $x \circ y$ the image $f(x, y)$, where f is the function $f : H \times H \to \mathcal{P}^*(H)$. Then, $(H, \circ)$ is called a hypergroupoid.

If $S, T \in \mathcal{P}^*(H)$, $S \circ T$ denotes the set $\bigcup_{s \in S, t \in T} s \circ t$.

Definition 1. The pair $(H, \circ)$ is called a *semihypergroup* if

$$\forall (r, s, t) \in H^3, \ \ (r \circ s) \circ t = r \circ (s \circ t),$$

where $(r \circ s) \circ t$ denotes the union

$$\bigcup_{a \in r \circ s} a \circ t.$$

Analogously,

$$r \circ (s \circ t) = \bigcup_{b \in s \circ t} r \circ b.$$

Definition 2. A *hypergroup* $(H, \circ)$ is a semihypergroup such that

$$\forall (a, b) \in H^2, \ \exists (x, y) \in H^2 \text{ such that}$$

$$a \in b \circ x \text{ and } a \in y \circ b$$

Several types of hypergroup homomorphisms are analyzed. We refer to [8]. Furthermore, several classes of subhypergroups are introduced and studied, such as canonical hypergroups, join spaces, and complete hypergroups. Join Spaces were introduced by Prenowitz.

Definition 3. Let $(H, \circ)$ be a commutative hypergroup. Then, $(H, \circ)$ is a join space if the following implication is satisfied:

$$\forall (r, s, t, w) \in H^4,$$

$$r/s \cap t/w \neq \emptyset \Rightarrow r \circ w \cap s \circ t \neq \emptyset,$$

where r/s denotes the set

$$\{a \in H \mid r \in a \circ s\}.$$

Example 1. *Suppose that* $(L, \vee, \wedge)$ *is a lattice. Then, L is a distributive lattice if and only if* $(L, \star)$ *is a join space, where* $a \star b = \{x \in L | a \wedge b \leq x \leq a \vee b\}$.

Example 2. *Suppose that* $(L, \vee, \wedge)$ *is a lattice. Then, L is a modular lattice if and only if* $(L, \circ)$ *is a join space, where* $a \circ b = \{x \in L | a \vee b = b \vee x = a \vee x\}$. *Clearly,* $a \vee b \in a \circ b$.

Canonical hypergroups have a structure close to that of a commutative group: they are commutative, have a scalar identity e (that is, $\forall x \in H, x \circ e = e \circ x = x$), every element has a unique inverse, and they are reversible (that is, if $x \in y \circ z$, then $z \in y^{-1} \circ x,\ y \in x \circ z^{-1}$).

An important result is the next one:

Theorem 1. *Let* $(H, \circ)$ *be a commutative hypergroup. Then, it is a canonical hypergroup iff it is a join space with a scalar identity.*

One of the most-investigated hypergroups associated with binary relations is that introduced by Rosenberg [9] in 1998. It represents a theme of research of numerous papers. Rosenberg associated a partial hypergroupoid $H_\rho = (H, \circ)$ with a binary relation ρ defined on a set H, where, for any $x, y \in H$, we have $x \circ x = \{z \in H | (x, z) \in \rho\}$ and $x \circ y = x \circ x \cup y \circ y$.

Definition 4. *An element b in H is an outer element of* ρ *if there exists* $a \in H$ *such that* $(a, b) \notin \rho^2$.

Theorem 2. $(H, \circ)$ *is a hypergroup iff:*

(1) *ρ has full domain;*
(2) *ρ has full range;*
(3) $\rho \subseteq \rho^2$;
(4) *If* $(a, b) \in \rho^2$, *then* $(a, b) \in \rho$, *where b is an outer element of* ρ.

Special attention is paid to the fundamental β relation, which leads to a group quotient structure.

Definition 5. *Suppose that* $(H, \circ)$ *is a semihypergroup and n is a natural number greater than* 1. *We can consider the relation* β_n *on H as follows:* $x \beta_n y$ *if there exist* $a_1, a_2, \ldots, a_n$ *in H, such that* $\{x, y\} \subseteq \prod_{i=1}^{n} a_i$, *and assume that* $\beta = \bigcup_{n \geq 1} \beta_n$, *where* $\beta_1 = \{(r, r) | r \in H\}$.

In [10], Freni showed that, in every hypergroup, the relation β is transitive, so the following result holds:

Theorem 3. *If* $(H, \circ)$ *is a hypergroup, then* $(H/\beta, \cdot)$ *is a group, where* $\bar{x} \cdot \bar{y} = \bar{z}$, *where z is an arbitrary element of* $x \circ y$. *Moreover, the canonical projection* $\varphi : H \to H/\beta$ *is a good homomorphism.*

3. *n*-Hypergroups

Davvaz and Vougiouklis [11] defined the notion of n-hypergroups for the first time. This concept is a generalization of n-groups, as well as hypergroups in the sense of Marty. Some properties of such hyperstructures were investigated in [12–18]. Moreover, some researchers have pointed out the relation between n-hypergroup and fuzzy sets.

Suppose that H is a nonempty set. A function $f : \underbrace{H \times \ldots \times H}_{n \text{ times}} \to \mathcal{P}^*(H)$ is called an *n-hyperoperation*. As usual, we may write $H^n = H \times \ldots \times H$, where H appears n times. An element of H^n is denoted by $(x_1, \ldots, x_n)$, where $x_i \in H$ for any i with $1 \leq i \leq n$. Let $P_1, \ldots, P_n$ be nonempty subsets of H. We define

$$f(P_1, \ldots, P_n) = \bigcup \{f(p_1, \ldots, p_n) \mid p_i \in P_i, i = 1, \ldots, n\}$$

The pair (H, f) is called an *n-hypergroupoid*. An n-hypergroupoid (H, f) is called an *n-semihypergroup* iff

$$f(h_1^{i-1}, f(h_i^{n+i-1}), h_{n+i}^{2n-1}) = f(h_1^{j-1}, f(h_j^{n+j-1}), h_{n+j}^{2n-1}),$$

for all $1 \leq i, j \leq n$ and $h_1, h_2, \ldots, h_{2n-1} \in H$. An n-semihypergroup (H, f) in which the equation:

$$b \in f(h_1^{i-1}, x_i, h_{i+1}^n) \tag{1}$$

has the solution $x_i \in H$ for every $h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_n, b \in H$, and $1 \leq i \leq n$ is called an *n-hypergroup*. If the value of $f(h_1, \ldots, h_n)$ is independent of the permutation of elements $h_1, \ldots, h_n$, then we have a *commutative n-hypergroup*.

Example 3. *If $(H, \star)$ is a hypergroup, then obtain an n-hypergroup by defining $f(h_1, \ldots, h_n) = h_1 \star \ldots \star h_n$, for all $h_1, \ldots, h_n \in H$.*

Example 4. *Let $\mathbb{Z}$ be the set of integer numbers. If we define*

$$f(h_1, \ldots, h_n) = \{m_1 h_1 + \ldots + m_n h_n \mid m_1, \ldots, m_n \in \mathbb{Z}\},$$

then $(\mathbb{Z}, f)$ is an n-hypergroup.

Example 5. *Assume $(L, \vee, \wedge)$ is a modular lattice. For every $h_1, \ldots, h_n \in L$ and $i \in \{1, \ldots, n\}$, we define*

$$A_n^{(i)} = h_1 \vee \ldots \vee h_{i-1} \vee h_{i+1} \vee \ldots \vee h_n,$$

$$A_n = h_1 \vee \ldots \vee h_n.$$

If we define:

$$f(h_1, \ldots, h_n) = \{x \in L \mid x \vee A_n^{(i)} = A_n, \text{ for all } 1 \leq i \leq n\},$$

then (L, f) is a commutative n-hypergroup.

Theorem 4. *Suppose that (H, f) is an n-semihypergroup. Then, (H, f) is an n-hypergroup iff Equation (1) is solvable at the first place and at the last place or at least one place $1 < i < n$.*

Proof. If Equation (1) is solvable at the place $i = 1$ and $i = n$, then, for every $h_1, \ldots, h_n, b \in H$, there are $x_0, z_0 \in H$ such that

$$b \in f(x_0, h_2^n) \text{ and } x_0 \in f(h_1^{n-1}, z_0).$$

If $j \in \{1, \ldots, n\}$ is arbitrary, then we have

$$b \in f(f(h_1^{n-1}, z_0), h_2^n) = f(h_1^{j-1}, f(h_j^{n-1}, z_0, h_2^j), h_{j+1}^n).$$

Hence, there is $x \in f(h_j^{n-1}, z_0, h_2^j)$ such that $b \in f(h_1^{j-1}, x, h_{j+1}^n)$.

Now, assume that Equation (1) is solvable at place $1 < i < n$. Assume that $j < i$, then for every $a_1, \ldots, a_n, b \in H$, there is $y_1 \in H$ such that

$$b \in f(h_1^{i-1}, y_1, f(\underbrace{h_1, \ldots, h_1}_{n-(i-j+1)}, h_{j+1}^{i+1}), h_{i+2}^n).$$

This implies that

$$b \in f(h_1^{j-1}, f(h_j^{i-1}, y_1, \underbrace{h_1, \ldots, h_1}_{n-(i-j+1)}), h_{j+1}^n).$$

Hence, there is $x \in f(h_j^{i-1}, y_1, h_1, \ldots h_1)$ such that $b \in f(h_1^{j-1}, x, h_{j+1}^n)$. If we consider $i < j$, then in a similar, way we can prove that Equation (1) is solvable. $\square$

An n-hyperoperation f is called *weakly* (i,j)*-associative* if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) \cap f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}) \neq \emptyset,$$

and (i,j)*-associative* if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}),$$

holds for fixed $1 \leq i < j \leq n$ and all $x_1, x_2, \ldots, x_{2n-1} \in H$.

We say that the element $a \in H$ is in the *center* of an n-hypergroupoid (G, f), if

$$f(a, x_2^n) = f(x_2, a, x_3^n) = f(x_2^3, a, x_4^n) = \ldots = f(x_2^n, a),$$

for all $x_2, \ldots, x_n \in H$. An $(i, i+k)$-associative n-hypergroupoid (G, f) containing cancelable elements in the center (cancelable elements belong to the center) is $(1, n)$-associative [12].

Theorem 5 ([12])**.** *An n-hypergroupoid containing cancellative elements in the center is an n-semihypergroup iff it is (i,j)-associative for some $1 \leq i < j \leq n$.*

An n-hypergroupoid (H, f) is called a *b-derived* from a binary hypergroupoid $(G, \star)$ [12], and denote this fact by $(H, f) = der_b(H, \star)$ if the hyperoperation f has the form

$$f(x_1^n) = (x_1 \star x_2 \star \ldots \star x_n) \star b.$$

Theorem 6 ([12])**.** *An n-semihypergroup has a neutral element iff it is derived from a binary semihypergroup with the identity.*

Theorem 7 ([12])**.** *An n-semihypergroup derived from a binary semihypergroup has a neutral polyad iff it has a neutral element.*

Consequently, if an n-semihypergroup without neutral elements is derived from a binary semihypergroup, then it does not possess any neutral polyad.

Theorem 8 ([12])**.** *If an n-semihypergroup (H, f) does not contain any neutral elements, then to (H, f), we can adjoint the neutral element if and only if (H, f) is derived from a binary semihypergroup.*

Theorem 9 ([12])**.** *To an n-semihypergroup (H, f) we can adjoint the neutral element iff (H, f) is derived from a binary semihypergroup.*

Theorem 10 ([12])**.** *For any n-semihypergroup (H, f) with a right neutral polyad, there is a semihypergroup $(H, \star)$ with a right identity and an endomorphism φ of $(H, \star)$ such that*

$$f(x_1^n) = x_1 \star \varphi(x_2) \star \varphi^2(x_3) \star \ldots \star \varphi^{n-1}(x_n) \star b,$$

for some $b \in H$.

Theorem 11 ([12])**.** *For any n-semihypergroup (H, f) with a left neutral polyad, there is a semihypergroup $(H, \star)$ with a left identity and an endomorphism ψ of $(H, \star)$ such that*

$$f(x_1^n) = b \star \psi^{n-1}(x_1) \star \psi^{n-2}(x_2) \star \ldots \star \phi^2(x_{n-2}) \star \phi(x_{n-1}) \star x_n$$

for some $b \in H$.

4. Binary Relations and Fundamental Relations

Suppose that R is a binary relation on a nonempty set H. We define a partial n-hypergroupoid (H, f_R), as follows:

$$f_R(\underbrace{w,\ldots,w}_{n}) = \{y \mid (w,y) \in R\},$$

for all w in H and

$$f_R(w_1, w_2, \ldots, w_n) = f_R(\underbrace{w_1,\ldots,w_1}_{n}) \cup f_R(\underbrace{w_2,\ldots,w_2}_{n}) \cup \ldots \cup f_R(\underbrace{w_n,\ldots,w_n}_{n}),$$

for every $w_1, w_2, \ldots, w_n \in H$. It is clear that (H, f_R) is commutative. The partial n-hypergroupoid (H, f_R) is a generalization of the Rosenberg partial hypergroupoid. We denote $f_R(w_1, w_2, \ldots, w_n)$ by $f_R(w_1^n)$. The relation R is transitive iff, for any w in H, we have

$$f_R(f_R(\underbrace{w,\ldots,w}_{n}), \underbrace{w,\ldots,w}_{n-1}) = f_R(\underbrace{w,\ldots,w}_{n}).$$

Moreover, (H, f_R) is an n-hypergroupoid if the domain of R is H.

Theorem 12 ([17])**.** *Suppose that R is a binary relation on H, with full domain. Then, (H, f_R) is an n-semihypergroup iff $R \subset R^2$ and for each outer element y of R, if $(x,y) \in R^2$ implies $(x,y) \in R$.*

It follows that:

Corollary 1. *Suppose that R is a binary relation with full domain. Then, (H, f_R) is an n-hypergroup iff the following hold:*

(1) *R has a full range;*
(2) *$R \subset R^2$;*
(3) *$(x,y) \in R^2$ implies $(x,y) \in R$ for every outer element $y \in R$.*

Note that if R is a subset of R^2, then x is an outer element of R iff $x \notin f_R(f_R(\underbrace{w,\ldots,w}_{n}), \underbrace{w,\ldots,w}_{n-1})$ for some $w \in H$.

If R is a subset of R^2, then there are no outer elements of R iff, for each $w \in H$, we have

$$f_R(f_R(\underbrace{w,\ldots,w}_{n}), \underbrace{w,\ldots,w}_{n-1}) = H.$$

Theorem 13 ([17])**.** *Suppose that the relation R is reflexive and symmetric. Then, (H, f_R) is an n-hypergroup iff, for every $u, w \in H$, we have*

$$f_R(f_R(\underbrace{u,\ldots,u}_{n}), \underbrace{u,\ldots,u}_{n-1}) - f_R(\underbrace{u,\ldots,u}_{n}) \subset f_R(f_R(\underbrace{w,\ldots,w}_{n}), \underbrace{w,\ldots,w}_{n-1}).$$

Corollary 2. *Suppose that the relation R is reflexive and symmetric, but not transitive. Then, (H, f_R) is an n-hypergroup iff $R^2 = H^2$.*

The concept of mutually associative hypergroupoids was introduced by Corsini [19]. We generalize this concept to n-hypergroupoids. Two partial n-hypergroupoids (H, f_1) and (H, f_2) are *mutually associative* if, for every $w_1, \ldots, w_{2n-1} \in H$, we have:

(i_1) $f_2(f_1(w_1^n), w_{n+1}^{2n-1}) = f_1(w_1^{n-1}, f_2(w_n^{2n-1}))$;

(i$_2$) $f_2(w_1, f_1(w_2^{n+1}), w_{n+2}^{2n-1}) = f_1(w_1^{n-2}, f_2(w_{n-1}^{2n-2}), w_{2n-1})$;
(i$_3$) $f_2(w_1, w_2, f_1(w_3^{n+2}), w_{n+3}^{2n-1}) = f_1(w_1^{n-3}, f_2(w_{n-2}^{2n-3}), w_{2n-2}, w_{2n-1})$;
...
(i$_{n-1}$)$f_2(w_1^{n-2}, f_1(w_{n-1}^{2n-2}), w_{2n-1}) = f_1(w_1, f_2(w_2^{n+1}), w_{n+2}^{2n-1})$;
(i$_n$) $f_2(w_1^{n-1}, f_1(w_n^{2n-1})) = f_1(f_2(w_1^n), w_{n+1}^{2n-1})$.

Let f_1 and f_2 be two ordinary hyperoperations. Then, we obtain two mutually associative partial hypergroupoids. If R is a binary relation on H and $A \subset H$, we denote

$$R(A) = \{b \mid (a,b) \in R, \text{ for some } a \in A\}.$$

If $A = \{w_1, w_2, \ldots, w_k\}$, we write $R(w_1^k)$ for $R(A)$. If R and S are binary relations on H, then we denote by SR the relation $\{(a,c) \in H^2 \mid (a,b) \in R \text{ and } (b,c) \in S, \text{ for some } b \in H\}$.

Theorem 14 ([17]). *Let R and S be two relations on H with full domains. Then, (H, f_R) and (H, f_S) are mutually associative iff, for every $w_1, w_2, \ldots, w_{2n-1} \in H$, we have*

$$SR(w_1^n) \cup S(w_{n+1}^{2n-1}) = RS(w_n^{2n-1}) \cup R(w_1^{n-1}).$$

Theorem 15 ([17]). *If (H, f_R) and (H, f_S) are mutually associative n-hypergroups, then $(H, f_{R\cup S})$ is also an n-hypergroup.*

Theorem 16. *Let R and S be relations on H, such that $R \subset SR$. If (H, f_R) is an n-hypergroup, (H, f_R) and (H, f_S) are mutually associative and one of the following two conditions holds:*
(1) $RS \cap \{(w,w) \mid w \in H\} = \emptyset$;
(2) *The domain (RS) of RS is different from H.*
Then, (H, f_{SR}) is an n-hypergroup, as well.

Now, suppose that (H, f) is an n-semihypergroup. We denote

$$\begin{aligned} f_{(1)} &= \{f(w_1^n) \mid w_i \in H,\ 1 \le i \le n\}\}, \\ f_{(2)} &= \{f(f(u_1^n), w_2^n) \mid u_i \in H,\ w_j \in H,\ \ 1 \le i \le n, \\ &\qquad \forall\, 2 \le j \le n\}, \\ f_{(3)} &= \{f(f(f(v_1^n), u_2^n), w_2^n) \mid v_i \in H,\ u_j \in H,\ w_j \in H, \\ &\qquad \forall 1 \le i \le n,\ \forall 2 \le j \le n\}, \end{aligned}$$

and so on. Denote $\mathcal{U} = \bigcup\limits_{k \in \mathbb{N}^*} f_{(k)}$. We define $\beta = \bigcup\limits_{k \ge 1} \beta_k$, where, for all x, y of H,

$$a\beta_k y \ \Leftrightarrow \exists u \in f_{(k)}, \text{ such that } \{x,y\} \subseteq u.$$

Denote $\bigcup\limits_{\substack{a \in u \\ u \in \mathcal{U}}} u$ by $\mathcal{C}_1(a)$, which means

$$\mathcal{C}_1(w) = \{a \mid \text{ there exists } u \in \mathcal{U} \text{ such that } w \in u,\ a \in u\}.$$

For every $n \in \mathbb{N}^*$, denote

$$\mathcal{C}_{n+1}(w) = \{a \mid \text{ there exists } u \in \mathcal{U} \text{ such that } \mathcal{C}_n(w) \cap u \ne \emptyset,\ a \in u\}.$$

A subsets B is a *complete part* of (H, f) if, for every $u \in \mathcal{U}$,

$$B \cap u = \emptyset \Longrightarrow u \subset B.$$

Suppose that $\mathcal{C}(w)$ is the complete closure of w. We have $\mathcal{C}(w) = \bigcup_{i\in\mathbb{N}^*} \mathcal{C}_i(w)$, for all $w \in H$.

Theorem 17 ([17])**.** *Suppose that (H, f) is an n-semihypergroup. The relation β is transitive iff $\mathcal{C}(w) = \mathcal{C}_1(w)$, for all $w \in H$.*

Theorem 18 ([17])**.** *If (H, f) is an n-hypergroup, then β is transitive.*

Suppose that (H_1, f) and (H_2, g) are n-hypergroups. We define $(f, g) : (H_1 \times H_2)^n \longrightarrow \mathcal{P}^*(H_1 \times H_2)$ by $(f, g)((u_1, v_1), \ldots, (u_n, v_n)) = \{(u, v) \mid u \in f(u_1, \ldots, u_n), v \in g(v_1, \ldots, v_n)\}$. Clearly, $(H_1 \times H_2, (f, g))$ is an n-hypergroup, and it is the *direct hyperproduct* of H_1 and H_2.

Theorem 19 ([11])**.** *Let (H_1, f) and (H_2, g) be two n-hypergroups, and let β_1^*, β_2^*, and β^* be fundamental equivalence relations on H_1, H_2, and $H_1 \times H_2$, respectively. Then,*

$$(H_1 \times H_2)/\beta^* \;\cong\; H_1/\beta_1^* \times H_2/\beta_2^*.$$

Let (H, f) be an n-semihypergroup and ρ be an equivalence relation on H; we define

$$X\overline{\overline{\rho}}Y \Longleftrightarrow x\rho y \text{ for all } x \in X, y \in Y.$$

The relation ρ is a *strongly regular relation* if $x_i\rho y_i$ for all $1 \le i \le n$, then,

$$f(x_1, \ldots, x_n)\overline{\overline{\rho}}f(y_1, \ldots, y_n).$$

If ρ is a strongly regular relation on an n-semihypergroup (H, f), then the quotient $(H/\rho, f/\rho)$ is an n-semigroup such that

$$f/\rho(\rho(x_1), \ldots, \rho(x_n)) = \rho(z) \;\text{ for all } z \in f(x_1, \ldots, x_n)$$

where $x_1, \ldots, x_n \in H$.

Similar to the relation defined by Freni [20,21] on semihypergroups, Davvaz et al. [13] introduced the following relation on an n-semihypergroup so that the quotient is a commutative n-semigroup. Let (H, f) be an n-semihypergroup. Then, $\widehat{\gamma}$ denotes the transitive closure of the relation $\gamma = \bigcup_{k\ge 1} \gamma_k$, where $\gamma_1 = \{(w, w) \mid w \in H\}$, and for every integer $k > 1$, we define

$$x\gamma_k y \iff x \in u_{(k)} \text{ and } y \in u^{\sigma}_{(k)}.$$

When $m = k(n-1) + 1$, there are $a_1^m \in H^m$ and $\sigma \in \mathbb{S}_m$ such that $u_{(k)} = f_{(k)}(a_1^m)$ and $u^{\sigma}_{(k)} = f_{(k)}(a^{\sigma(m)}_{\sigma(1)})$. $x\gamma_1 y$ (i.e., $x = y$), then we write $x \in u_{(0)}$ and $y \in u^{\sigma}_{(0)} = u_{(0)}$. We define γ^* as the smallest equivalence relation such that the quotient $(H/\gamma^*, f/\gamma^*)$ is a commutative n-semigroup.

Theorem 20 ([13])**.** *The fundamental relation γ^* is the transitive closure of the relation γ.*

Proof. The n-operation $f/\widehat{\gamma}$ in $H/\widehat{\gamma}$ is defined in the usual manner:

$$f/\widehat{\gamma}(\widehat{\gamma}(x_1), \ldots, \widehat{\gamma}(x_n)) = \{\widehat{\gamma}(y) \mid y \in f(\widehat{\gamma}(x_1), \ldots, \widehat{\gamma}(x_n))\}$$

for all $x_1, \ldots, x_n \in H$. Let $a_1 \in \widehat{\gamma}(x_1), \ldots, a_n \in \widehat{\gamma}(x_n)$. Then, we have:

$a_1\widehat{\gamma}x_1$ iff there exist $x_{11},\ldots x_{1m_1+1}$ with $x_{11}=a_1$, $x_{1m_1+1}=x_1$ such that

$$x_{1i_1}\in u_{(k_1)}\ \ (1\le i_1\le m_1-1),$$
$$x_{1i_1+1}\in u^{\sigma_1}_{(k_1)}\ \ (2\le i_1\le m_1).$$

...

$a_n\widehat{\gamma}x_n$ iff there exist $x_{n1},\ldots x_{nm_n+1}$ with $x_{n1}=a_n$, $x_{nm_n+1}=x_n$ such that

$$x_{ni_n}\in u_{(k_n)}\ \ (1\le i_n\le m_n-1),$$
$$x_{ni_n+1}\in u^{\sigma_n}_{(k_n)}\ \ (2\le i_n\le m_n).$$

Therefore, we obtain

$$\begin{array}{ll}
f(x_{1i_1},x_{21},\ldots,x_{n1})\subseteq u_{(k_1)} & 1\le i_1\le m_1-1,\\
f(x_{1i_1+1},x_{21},\ldots,x_{n1})\subseteq u^{\sigma_1}_{(k_1)} & 2\le i_1\le m_1,\\
f(x_{1m_1+1},x_{2i_2},\ldots,x_{n1})\subseteq u_{(k_2)} & 1\le i_2\le m_2-1,\\
f(x_{1m_1+1},,x_{2i_2+1},\ldots,x_{n1})\subseteq u^{\sigma_2}_{(k_2)} & 2\le i_2\le m_2,\\
\ldots & \ldots\\
f(x_{1m_1+1},x_{2m_2+1},\ldots,x_{ni_n})\subseteq u_{(k_n)} & 1\le i_n\le m_n-1,\\
f(x_{1m_1+1},x_{2m_2+1},\ldots,x_{ni_n+1})\subseteq u^{\sigma_n}_{(k_n)} & 2\le i_n\le m_n.
\end{array}$$

This yields that $f/\widehat{\gamma}(\widehat{\gamma}(x_1),\ldots,\widehat{\gamma}(x_n))$ is singleton. Therefore, we can write

$$f/\widehat{\gamma}(\widehat{\gamma}(x_1),\ldots,\widehat{\gamma}(x_n))=\widehat{\gamma}(z)\ \text{ for all } z\in f(\widehat{\gamma}(x_1),\ldots,\widehat{\gamma}(x_n)).$$

Moreover, since f is associative, we obtain that $f/\widehat{\gamma}$ is associative, and consequently, $H/\widehat{\gamma}$ is an n-semigroup.

$(H/\widehat{\gamma},f/\widehat{\gamma})$ is commutative because, if $\sigma\in\mathbb{S}_n$ and $a\in f(x_1^n)$ and $b\in f(x^{\sigma(n)}_{\sigma(1)})$, then $a\gamma b$, and so, $\widehat{\gamma}(a)=\widehat{\gamma}(b)$. Therefore, $f/\widehat{\gamma}(\widehat{\gamma}(x_1),\ldots,\widehat{\gamma}(x_n))=f/\widehat{\gamma}(\widehat{\gamma}(x_{\sigma(1)}),\ldots,\widehat{\gamma}(x_{\sigma(n)}))$; thus $(H/\widehat{\gamma},f/\widehat{\gamma})$ is commutative.

Now, assume that θ is an equivalence relation on H such that H/θ is a commutative n-semigroup. Then, for all $w_1,\ldots,w_n\in H$,

$$f/\theta(\theta(w_1),\ldots,\theta(w_n))=\theta(z)\ \text{ for all } z\in f(\theta(w_1),\ldots,\theta(w_n)).$$

However, for any $\sigma\in\mathbb{S}_n$ and $w_1,\ldots,w_n\in H$ and $X_i\subseteq\theta(w_i)$ $(i=1,\ldots,n)$, we have

$$f/\theta(\theta(w_1),\ldots,\theta(w_n))=\theta(f(w_{\sigma(1)},\ldots,w_{\sigma(n)}))=\theta(f(X_{\sigma(1)},\ldots,X_{\sigma(n)})).$$

Therefore,

$$\theta(w)=\theta(u^{\sigma}_{(k)})\ \text{ for all } k\ge 0 \text{ and for all } w\in u_{(k)}.$$

This gives that, for all $y\in H$,

$$w\in\gamma(y)\ \text{ implies }\ w\in\theta(y).$$

Since θ is transitively closed, it follows that

$$w\in\widehat{\gamma}(y)\ \text{ implies }\ w\in\theta(y).$$

Consequently, we obtain $\widehat{\gamma}=\gamma^*$. □

Relation γ is a strongly regular relation.

Now, we present some necessary and sufficient conditions such that the relation γ is transitive. These conditions are analogous to those determined in [20] for the transitivity of relation γ in hypergroups. Let M be a nonempty subset of n-semihypergroup (H,f). We say that M is a γ-*part* if, for any $k \in \mathbb{N}$, $i = 1, 2, \ldots, m = k(n-1)+1$, $\forall (w_1, w_2, \ldots, w_m) \in H^m$, $\forall \sigma \in \mathbb{S}_m$, we have

$$f_{(k)}(w_1^m) \cap M \neq \emptyset \Longrightarrow f_{(k)}(w_{\sigma(1)}^{\sigma(m)}) \subseteq M.$$

Theorem 21. *Suppose that M is a nonempty subset of an n-semihypergroup H. Then, the following statements are equivalent:*

(1) *M is a γ-part of H;*
(2) *$x \in M, x \,\gamma\, y$ implies that $y \in M$;*
(3) *$x \in M, x \,\gamma^*\, y$ implies that $y \in M$.*

Proof. $(1 \Rightarrow 2)$: If $(x, y) \in H^2$ is a pair such that $x \in M$ and $x \,\gamma\, y$, then $\exists k \in \mathbb{N}$ for $i = 1, \ldots, m = k(n-1)+1$, $\exists \sigma \in \mathbb{S}_m$ and $\exists (z_1, \ldots, z_m) \in H^m$, such that $x \in f_{(k)}(z_1^m) \bigcap M$ and $y \in f_{(k)}(z_{\sigma(1)}^{\sigma(m)})$. Since M is a γ-part of H, we have $f_{(k)}(z_{\sigma(1)}^{\sigma(m)}) \subseteq M$ and $y \in M$.

$(2 \Rightarrow 3)$: Assume that $(x, y) \in H^2$ such that $x \in M$ and $x \,\gamma^*\, y$. Then, there exist $p \in \mathbb{N}$ and $(x = w_0, w_1, \ldots, w_{p-1}, w_p = y) \in H^{p+1}$ such that $x = w_0 \,\gamma\, w_1 \,\gamma \ldots \gamma\, w_{p-1} \,\gamma\, w_p = y$. Since $x \in M$, applying (2) p times, it follows that $y \in M$.

$(3 \Rightarrow 1)$: Suppose that $f_{(k)}(z_1^m) \bigcap M \neq \emptyset$, and $x \in f_{(k)}(z_1^m) \bigcap M$. For any $\sigma \in \mathbb{S}_m$ and $y \in f_{(k)}(z_{\sigma(1)}^{\sigma(m)})$, we have $x \,\gamma\, y$. This yields that $x \in M$ and $x \,\gamma^*\, y$. Finally, by (3), we obtain $y \in M$. This means that $f_{(k)}(z_{\sigma(1)}^{\sigma(m)}) \subseteq M$. $\square$

For every element x of an n-semihypergroup (H, f), set:

$$T_k(w) = \{(w_1, \ldots, w_m) \in H^m | m = k(n-1)+1, w \in f_{(k)}(x_1^m)\}$$

$$P_k(w) = \bigcup \{f_{(k)}(w_{\sigma(1)}^{\sigma(m)}) | \sigma \in \mathbb{S}_m, (w_1, \ldots, w_m) \in T_k(w), m = k(n-1)+1\}$$

$$P_\sigma(w) = \bigcup_{k \geq 1} P_k(w).$$

From the preceding notations and definitions, it follows that

Corollary 3 ([13])**.** *For every $x \in H$, $P_\sigma(x) = \{y \in H | x \,\gamma\, y\}$.*

Theorem 22 ([13])**.** *Let (H, f) be an n-semihypergroup. The following statements are equivalent:*

(1) *γ is transitive;*
(2) *For any $w \in H, \gamma^*(w) = P_\sigma(w)$;*
(3) *For any $w \in H$, $P_\sigma(w)$ is a γ-part of H.*

Let (H, f) be an n-hypergroup; we consider the canonical projection $\varphi : H \to H/\gamma^*$ with $\varphi(x) = \gamma^*(x)$.

Corollary 4 ([13])**.** *Let (H, f) be an n-hypergroup and $\delta \in H/\gamma^*$, then $\varphi^{-1}(\delta)$ is a γ-part of H.*

Corollary 5 ([13])**.** *If (H, f) is a commutative n-semihypergroup, then $\gamma = \beta$.*

Theorem 23 ([13])**.** *For every nonempty subset M of an n-hypergroup (H, f), we have:*

(1) *If H/γ^* has a neutral element ε and $D = \varphi^{-1}(\varepsilon)$, then for every $i = 1, \ldots, n$,*

$$f(D^{i-1}, M, D^{n-i}) \subseteq \varphi^{-1}(\varphi(M));$$

(2) *Moreover if H/γ^* is one-cancellative, then $f(D^{i-1}, M, D^n) = \varphi^{-1}(\varphi(M))$;*
(3) *If M is a γ-part of H, then $\varphi^{-1}(\varphi(M)) = M$.*

Proof. (1) For any $x \in f(D^{i-1}, M, D^{n-i})$, there exist $d_2, \ldots, d_n \in D$ and $b \in M$ such that $x \in f(d_2^{i-1}, b, d_{i+1}^n)$, so $\varphi(x) = f/\gamma^*(\varepsilon^{i-1}, \varphi(b), \varepsilon^{n-i}) = \varphi(b)$; therefore, $x \in \varphi^{-1}(\varphi(x)) \subseteq \varphi^{-1}(\varphi(M))$.

(2) For any $x \in \varphi^{-1}(\varphi(M))$, an element $b \in M$ exists such that $\varphi(x) = \varphi(b)$. Let $d \in D$. Then, there exists $a \in H$ such that $x \in f(a, b, d^{n-2})$. Therefore,

$$\varphi(b) = \varphi(x) = f/\gamma^*(\varphi(a), \varphi(d)^{i-2}, \varphi(b), \varphi(d)^{n-i}) = f/\gamma^*(\varphi(a), \varepsilon^{i-2}, \varphi(b), \varepsilon^{n-2}).$$

However, $f(\varepsilon^{i-1}, \varphi(b), \varepsilon^{n-i}) = \varphi(b)$, and since (H, f) is one-cancellative, thus $\varphi(a) = \varepsilon$ and $a \in \varphi^{-1}(\varepsilon) = D$. Therefore, $x \in f(a, b, d^{n-2}) = f(D^{i-1}, M, D^{n-i})$. This and (1) prove that $\varphi^{-1}(\varphi(M)) = f(D^{i-1}, M, D^{n-i})$.

(3) Clearly, we have $M \subseteq \varphi^{-1}(\varphi(M))$. Furthermore, if $x \in \varphi^{-1}(\varphi(M))$, then there exists $b \in M$ such that $\varphi(x) = \varphi(b)$. This yields that $x \in \gamma^*(x) = \gamma^*(b) \subseteq M$ and $\varphi^{-1}(\varphi(M)) \subseteq M$. $\square$

Theorem 24 ([13]). *If (H, f) is an n-hypergroup with neutral (identity) e, such that $H/\gamma*$ is j-cancellative, then we have:*

(1) *If $x \in P_\sigma(e)$ and $x\ \gamma\ y$, then $y \in P_\sigma(e)$;*
(2) *γ is transitive.*

Proof. (1) If $x \in P_\sigma(e)$ and $x\ \gamma\ y$, then $\exists (k, k') \in \mathbb{N} \times \mathbb{N}, m = k(n-1) + 1, m' = k'(n-1) + 1$, $\exists\ (x_1, \ldots, x_m) \in H^m$, $\exists\ (y_1, \ldots, y_{m'}) \in H^{m'}$, $\exists\ \sigma \in \mathbb{S}_m$ and $\exists\ \sigma' \in \mathbb{S}_{m'}$, such that $e \in f_{(k)}(x_1^m)$, $x \in f_{(k)}(x_{\sigma(1)}^{\sigma(m)})$, $x \in f_{(k')}(y_1^{m'})$, $y \in f_{(k')}(y_{\sigma'(1)}^{\sigma'(m')})$. Therefore, if x' is an element of H such that

$$\begin{aligned} e \in f(e^{n-2}, x, x') &\subseteq f(e^{n-2}, f(e^{n-2}, x, e), x') \\ &\subseteq f(e^{n-2}, f(e^{n-2}, f_{(k')}(y_1^{m'}), f_{(k)}(x_1^m)), x'). \end{aligned}$$

Moreover, we have

$$\begin{aligned} y \in f(e^{n-2}, y, e) &\subseteq f(e^{n-2}, y, f(e^{n-2}, x, x')) \\ &\subseteq f(e^{n-2}, f_{(k')}(y_{\sigma'(1)}^{\sigma'(m')}), f(e^{n-2}, f_{(k)}(x_{\sigma(1)}^{\sigma(m)}), x')). \end{aligned}$$

Thus, $y \in P_\sigma(e)$.

(2) By (1), we have $P_\sigma(e) = \gamma^*(e) = D$. Moreover, if $x\ \gamma^*\ y$, then $x \in \gamma^*(y)$, so $x \in \varphi^{-1}(\varphi(y)) = f(D^{i-1}, y, D^{n-i})$. Therefore, there exist $(a_2^n) \in D^n$ such that $x \in f(a_2^{i-1}, y, a_{i+1}^n)$. Thus, there exist $k_i \in \mathbb{N}$, and there are $(x_{i1}, \ldots, x_{im_i}) \in H^{m_i}$, where $m_i = k_i(n-1) + 1$, and $\sigma_i \in \mathbb{S}_{m_i}$ such that $e \in f_{(k_i)}(x_{i1}^{im_i}) = A_i$ and $a_i \in f_{(k)}(x_{i\sigma_i(1)}^{i\sigma_i(m_i)}) = A_{\sigma(i)}$, where $i = 2, \ldots, n$. If $j \in \{1, \ldots, n\}$, it follows that

$$x \in f(a_2^{j-1}, y, a_{j+1}^n) \subseteq f(A_{\sigma(2)}^{\sigma(j-1)}, y, A_{\sigma(j+1)}^{\sigma(n)}) \text{ and } y \in f(e^{j-1}, y, e^{j-n}) \subseteq f(A_2^{j-1}, y, A_{j+1}^n).$$

Whence $x\ \gamma\ y$ and $\gamma^* = \gamma$. $\square$

5. Join *n*-Spaces

Let $(L, \leq, \vee)$ be a join semi-lattice and a_1^n be elements of L. We denote

$$\begin{aligned} &A_n = a_1 \vee a_2 \vee \ldots \vee a_n, \quad A_n^{(1)} = a_2 \vee \ldots \vee a_n, \\ &A_n^{(n)} = a_1 \vee \ldots \vee a_{n-1}, \quad A_n^{(i)} = a_1 \vee \ldots \vee a_{i-1} \vee a_{i+1} \vee \ldots \vee a_n, \end{aligned}$$

for any $2 \leq i \leq n-1$. For any a_1^n of L, we define the following n-hyperoperation:

$$f(a_1^n) = \{x \mid x \vee A_n^{(i)} = A_n, \text{ for any } i \in \{1, 2, \ldots, n\}\}.$$

Notice that $A_n \in f(a_1^n)$. Notice also that the n-hyperoperation f is commutative.

If $(L, \leq, \vee)$ is a join semi-lattice, then the following statements hold:

(1) For any b, a_1^{n-1} of L, there is $x = b \vee A_{n-1}$ such that $b \in f(x, a_1^{n-1})$.
(2) If L has a 0, then 0 is a scalar identity for (L, f).
(3) If $n \geq 3$, then any $x \in L$ is an identity for (L, f).
(4) For any a, x, b_1^{n-1} of L, we have the equivalence:

$$a \in f(x, b_1^{n-1}) \text{ iff } x \in f(a, b_1^{n-1}).$$

(5) For any a, b_1^{n-1} of L, we have $a/b_1^{n-1} = f(a, b_1^{n-1})$.

Theorem 25 ([18])**.** *If (L, f) is an n-semihypergroup, then for any $a, c, b_1^{n-1}, d_1^{n-1}$ of L, we have*

$$f(a, b_1^{n-1}) \cap f(c, d_1^{n-1}) \neq \emptyset \Longrightarrow f(a, d_1^{n-1}) \cap f(c, b_1^{n-1}) \neq \emptyset.$$

Theorem 26 ([18])**.** *For any a_1^{2n-1} of L, if we denote*

$$S = \{y \mid A_{2n-1} = A_{2n-1}^{(i)} \vee y, \textit{ for any } i \in \{1, 2, \ldots, 2n-1\}\},$$

then $f(a_1^{n-1}, f(a_n^{2n-1})) \subset S$.

Theorem 27 ([18])**.** *If $(L, \vee, \wedge)$ is a modular lattice, then $S \subset f(a_1^{n-1}, f(a_n^{2n-1}))$.*

Proof. Let $y \in S$. Set $z \in (y \vee A_{n-1}) \wedge (a_n \vee \ldots \vee a_{2n-1})$. We check $z \in f(a_n^{2n-1})$ and $y \in f(a_1^{n-1}, z)$. Indeed, for any $i \in \{1, 2, \ldots, n-2\}$, we have

$$\begin{aligned}
&a_n \vee \ldots \vee a_{n+i-1} \vee z \vee a_{n+i+1} \vee \ldots \vee a_{2n-1} = \\
&= (a_n \vee \ldots \vee a_{n+i-1} \vee a_{n+i-1} \vee \ldots \vee a_{2n-1}) \vee [(y \vee A_{n-1}) \wedge (a_n \vee \ldots \vee a_{2n-1})] = \\
&= (A_{2n-1}^{(n+i)} \vee y) \wedge (a_n \vee \ldots \vee a_{2n-1}) = A_{2n-1} \wedge (a_n \vee \ldots \vee a_{2n-1}) = a_n \vee \ldots \vee a_{2n-1}.
\end{aligned}$$

Similarly, we have

$$z \vee a_{n+1} \vee \ldots \vee a_{2n-1} = a_n \vee \ldots \vee a_{2n-2} \vee z = a_n \vee \ldots \vee a_{2n-1}.$$

Hence, $z \in f(a_n^{2n-1})$. On the other hand,

$$\begin{aligned}
&A_{n-1} \vee z = A_{n-1} \vee [(y \vee A_{n-1}) \wedge (a_n \vee \ldots \vee a_{2n-1})] = \\
&= (y \vee A_{n-1}) \wedge A_{2n-1} = (y \vee A_{n-1}) \wedge (A_{2n-2} \vee y) = y \vee A_{n-1}
\end{aligned}$$

and for any $i \in \{1, 2, \ldots, n-1\}$, we have

$$\begin{aligned}
&A_n^{(i)} \vee y \vee z = (A_{n-1}^{(i)} \vee y) \vee [(y \vee A_{n-1}) \wedge (a_n \vee \ldots \vee a_{2n-1}) = \\
&= (y \vee A_{n-1}) \wedge (A_{n-1}^{(i)} \vee y \vee a_n \vee \ldots \vee a_{2n-1}) = \\
&= (y \vee A_{n-1}) \wedge (A_{2n-1}^{(i)} \vee y) = \\
&= (y \vee A_{n-1}) \wedge (A_{2n-2} \vee y) = y \vee A_{n-1}.
\end{aligned}$$

Therefore, $y \in f(a_1^{n-1}, z) \subset f(a_1^{n-1}, f(a_n^{2n-1}))$. □

Corollary 6 ([18])**.** *If $(L, \vee, \wedge)$ is a modular lattice, then (L, f) is an n-semihypergroup.*

Theorem 28 ([18])**.** *If $(L, \vee, \wedge)$ is a lattice and (L, f) is an n-semihypergroup, then the lattice $(L, \vee, \wedge)$ is modular.*

Proof. Assume that L is not modular. Hence, L contains a five-element sublattice, isomorphic to this one: $\{m, a, b, c, M\}$, where $m < b < a < M$, $m < c < M$, a, c, and b, c, respectively, are not comparable. We have $c \in f(a, \underbrace{b, \ldots, b}_{n-2}, M)$ and $M \in f(b, \underbrace{c, \ldots, c}_{n-1})$, since $a \vee c = b \vee c = M$. Hence,

$$c \in f(a, \underbrace{b, \ldots, b}_{n-2}, f(b, \underbrace{c, \ldots, c}_{n-1})) = f(f(a, \underbrace{b, \ldots, b}_{n-1}), \underbrace{c, \ldots, c}_{n-1}).$$

Therefore, there exists $x \in f(a, \underbrace{b, \ldots, b}_{n-1})$, such that $c \in f(x, \underbrace{c, \ldots, c}_{n-1})$. We have $a = a \vee b = b \vee x = a \vee x \vee b = a \vee x$ and $c \vee x = c$, whence $x \leq a$ and $x \leq c$, that is $x \leq a \wedge c = m$. Hence, $x < b$, which contradicts $a = b \vee x$. Therefore, $(L, \vee, \wedge)$ is modular. □

Corollary 7 ([18]). *A lattice $(L, \vee, \wedge)$ is modular iff (L, f) is an n-semihypergroup.*

Corollary 8 ([18]). *The lattice $(L, \vee, \wedge)$ is modular iff the n-hypergroupoid (L, f) is a join n-space.*

Now, we can consider the following dual-n-hyperoperation f° on a meet semilattice $(L, \leq, \wedge)$, defined by: for any a_1^n of L, we have:

$$f^\circ(a_1^n) = \{x \in L \mid x \wedge B_n^{(i)} = B_n, \text{ for any } i \in \{1, 2, \ldots, n\}\},$$

where $B_n = a_1 \wedge a_2 \wedge \ldots \wedge a_n$, $B_n^{(1)} = a_2 \wedge \ldots \wedge a_n$, $B_n^{(n)} = a_1 \wedge \ldots \wedge a_{n-1}$ and for any $i \in \{2, \ldots, n-1\}$, $B_n^{(i)} = a_1 \wedge \ldots \wedge a_{i-1} \wedge a_{i+1} \wedge \ldots \wedge a_n$. By duality, the following result holds:

Theorem 29 ([18]). *A lattice $(L, \vee, \wedge)$ is modular iff the n-hypergroupoid (L, f°) is a join n-space:*

- If L has the greatest element 1, then 1 is a scalar identity for (L, f°).
- If $n \geq 3$, then any $x \in L$ is an identity for (L, f°).

Theorem 30 ([18]). *Let $(L, \vee, \wedge)$ be a modular lattice:*

(1) *A subset I of L is an n-subhypergroup of (L, f) iff I is an ideal of L.*
(2) *A subset I of L is an n-subhypergroup of (L, f°) iff I is a filter of L.*

Proof. (1) Let (I, f) be an n-subhypergroupoid of (L, f). Then, for any $a_1, a_2 \in I$, we have

$$a_1 \vee a_2 \in f(a_1, \underbrace{a_2, \ldots, a_2}_{n-1}) \subset I.$$

If $a \in I$ and $x \leq a$, then $x \in f(\underbrace{a, \ldots, a}_{n}) \subset I$. "$\Longleftarrow$" Let a_1^n be elements of I. If $z \in f(a_1^n)$, then $A_n = z \vee A_n^{(i)}$, for any $i \in \{1, 2, \ldots, n\}$, whence $z \leq A_n$. Since $A_n \in I$, it follows that $z \in I$. On the other hand, for any a, a_1^{i-1}, a_{i+1}^n of I and $1 \leq i \leq n$, there is $x_i = a \vee A_n^{(i)}$ such that $a \in f(a_1^{i-1}, x_i, a_{i+1}^n)$. Hence, I is an n-subhypergroup of (L, f).

(2) It follows by duality. □

Theorem 31 ([18]). *Let $(L, \vee, \wedge)$ be a lattice and $\varphi : L \to L$ a bijective map. The following conditions are equivalent:*

(1) *For any a_1^n of L, we have $\varphi(A_n) = \varphi(a_1) \wedge \ldots \wedge \varphi(a_n)$.*
(2) *φ is a morphism from (L, f) to (L, f°).*

Proof. (1$\Longrightarrow$2): For any a_1^n of L, we have $\varphi(f(a_1^n)) = \{\varphi(z) \mid z \in f(a_1^n)\} = \{\varphi(z) \mid A_n = z \vee A_n^{(i)}$, for any $i \in \{1, 2, \ldots, n\}\}$, whence $\varphi(a_1) \wedge \ldots \wedge \varphi(a_n) = \varphi(A_n) = \varphi(z \vee A_n^{(i)}) = \varphi(z) \wedge \varphi(a_1) \wedge \ldots \wedge \varphi(a_{i-1}) \wedge \varphi(a_{i+1}) \wedge \ldots \wedge \varphi(a_n)$, that is

$$\varphi(z) \in f^\circ(\varphi(a_1), \ldots, \varphi(a_n)).$$

Now, let $t \in f^\circ(\varphi(a_1), \ldots, \varphi(a_n))$. Since there is x such that $t = \varphi(x)$, it follows that

$$\varphi(x) \wedge [\varphi(a_1) \wedge \ldots \wedge \varphi(a_{i-1}) \wedge \varphi(a_{i+1}) \wedge \ldots \wedge \varphi(a_n)] = \varphi(a_1) \wedge \ldots \wedge \varphi(a_n),$$

for any $i \in \{1, 2, \ldots, n\}$, and according to (1), we obtain $\varphi(x \vee A_n^{(i)}) = \varphi(A_n)$, for any $i \in \{1, 2, \ldots, n\}$. Since φ is bijective, it follows that $x \vee A_n^{(i)} = A_n$, for any $i \in \{1, 2, \ldots, n\}$, that is $x \in f(a_1^n)$. Hence,

$$t = \varphi(x) \in \varphi(f(a_1^n)).$$

(2$\Longrightarrow$1): Let a_1^n be elements of L. If $z \in f(a_1^n)$, then

$$\varphi(z) \in f^\circ(\varphi(a_1), \ldots, \varphi(a_n))$$

that is

$$\varphi(z) \wedge \varphi(a_1) \wedge \ldots \wedge \varphi(a_{i-1}) \wedge \varphi(a_{i+1}) \wedge \ldots \wedge \varphi(a_n) = \varphi(a_1) \wedge \ldots \wedge \varphi(a_n),$$

for any $i \in \{1, 2, \ldots, n\}$. Hence,

$$\varphi(a_1) \wedge \ldots \wedge \varphi(a_n) \leq \varphi(z).$$

For $z = A_n \in f(a_1^n)$, we obtain $\varphi(a_1) \wedge \ldots \wedge \varphi(a_n) \leq \varphi(A_n)$. On the other hand, for any $i \in \{1, 2, \ldots, n\}$, $A_n \in f(a_i, \underbrace{A_n, \ldots, A_n}_{n-1})$, so

$$\varphi(A_n) \in \varphi(f(a_i, \underbrace{A_n, \ldots, A_n}_{n-1})) = f^\circ(\varphi(a_i), \underbrace{\varphi(A_n), \ldots, \varphi(A_n)}_{n-1})$$

whence $\varphi(A_n) = \varphi(a_i) \wedge \varphi(A_n)$, that is $\varphi(A_n) \leq \varphi(a_i)$. It follows that

$$\varphi(A_n) \leq \varphi(a_1) \wedge \ldots \wedge \varphi(a_n).$$

Therefore, the condition (1) holds. □

By duality, we obtain the following.

Theorem 32 ([18])**.** *Let $(L, \vee, \wedge)$ be a lattice and $\varphi : L \to L$ a bijective map. The following conditions are equivalent:*

(1) *For any a_1^n of L, we have*
$$\varphi(B_n) = \varphi(a_1) \vee \ldots \vee \varphi(a_n).$$

(2) *φ is a morphism from (L, f°) to (L, f).*

Let $(L, \vee, \wedge)$ be an arbitrary lattice. We define on L the following n-hyperoperation: for any a_1^n of L, we have

$$\begin{array}{rcl} g(a_1^n) & = & \{x \in L \mid B_n \leq x \leq A_n\}, \text{ where} \\ B_n & = & a_1 \wedge a_2 \wedge \ldots \wedge a_n \text{ and } A_n = a_1 \vee a_2 \vee \ldots \vee a_n. \end{array}$$

The n-hypergroupoid (L, g) has the following properties:

(1) g is commutative;

(2) For any $a \in L$, we have $g(\underbrace{a, \ldots, a}_{n}) = a$;
(3) for any a_1^n of L, we have $\{a_i^n\} \subset g(a_i^n)$;
(4) For any a_1^{n-1} of L, we have $b \in b/a_1^{n-1}$;
(5) For any $a \in L$, we have $a/\{\underbrace{a, \ldots, a}_{n-1}\} = L$;
(6) For any $a, b \in L$, we have $x \in a/\{\underbrace{b, \ldots, b}_{n-1}\} \cap b/\{\underbrace{a, \ldots, a}_{n-1}\}$ iff $a \wedge x = b \wedge x$ and $a \vee x = b \vee x$.

Theorem 33 ([18]). *If the lattice $(L, \vee, \wedge)$ is distributive, then for any a_1^{2n-1} of L, we have*

$$g(g(a_1^n), a_{n+1}^{2n-1}) = [B_{2n-1}, A_{2n-1}].$$

Proof. Indeed, for any a_1^{2n-1} of L, we have

$$g(g(a_1^n), a_{n+1}^{2n-1}) \subset [B_{2n-1}, A_{2n-1}].$$

Conversely, let $z \in [B_{2n-1}, A_{2n-1}]$. If $x = (z \wedge A_n) \vee B_n$, then $B_n \leq x \leq A_n$, that is $x \in g(a_1^n)$. On the other hand,

$$z \in g(x, a_{n+1}^{2n-1}).$$

Indeed, by distributivity, we have

$$\begin{aligned} &a_{n+1} \wedge \ldots \wedge a_{2n-1} \wedge x = a_{n+1} \wedge \ldots \wedge a_{2n-1} \wedge [(z \wedge A_n) \vee B_n] = \\ &= (z \wedge A_n \wedge a_{n+1} \wedge \ldots \wedge a_{2n-1}) \vee B_{2n-1} \leq z \end{aligned}$$

and

$$\begin{aligned} &a_{n+1} \vee \ldots \vee a_{2n-1} \vee x = a_{n+1} \vee \ldots \vee a_{2n-1} \vee (z \wedge A_n) \vee B_n = \\ &= (a_{n+1} \vee \ldots \vee a_{2n-1} \vee B_n \vee z) \wedge (a_{n+1} \vee \ldots \vee a_{2n-1} \vee B_n \vee A_n) = \\ &= A_{2n-1} \wedge (a_{n+1} \vee \ldots \vee a_{2n-1} \vee B_n \vee z) \geq z. \end{aligned}$$

Hence $z \in g(x, a_{n+1}^{2n-1})$, whence $z \in g(g(a_1^n), a_{n+1}^{2n-1})$. We obtain

$$g(g(a_1^n), a_{n+1}^{2n-1}) = [B_{2n-1}, A_{2n-1}].$$

□

Corollary 9 ([18]). *If $(L, \vee, \wedge)$ is a distributive lattice, then (L, g) is an n-hypergroup.*

Proof. Since the subset $[B_{2n-1}, A_{2n-1}]$ is invariant to any permutation $(a_{i_1}, \ldots, a_{i_{2n-1}})$ of $(a_1, \ldots, a_{2n-1})$, it follows that

$$[B_{2n-1}, A_{2n-1}] = g(g(a_{i_1}, \ldots, a_{i_n}), a_{i_{n+1}}, \ldots, a_{i_{2n-1}}).$$

Moreover, g is commutative, so it follows that g is associative. Therefore, we obtain that (L, g) is an n-hypergroup. □

Theorem 34 ([18]). *If $(L, \vee, \wedge)$ is a distributive lattice, then (L, g) is a join n-space.*

Proof. We still have to check the join n-space condition. Let $x \in a/b_1^{n-1} \cap c/d_1^{n-1}$, that is

$$\begin{aligned} &x \wedge b_1 \wedge \ldots \wedge b_{n-1} \leq a \leq x \vee b_1 \vee \ldots \vee b_{n-1} \quad \text{and} \\ &x \wedge d_1 \wedge \ldots \wedge d_{n-1} \leq c \leq x \vee d_1 \vee \ldots \vee d_{n-1}. \end{aligned}$$

We have to prove that there is $z \in g(a, d_1^{n-1}) \cap g(c, b_1^{n-1})$, that is

$$(a \wedge d_1 \wedge \ldots \wedge d_{n-1}) \vee (c \wedge b_1 \wedge \ldots \wedge b_{n-1}) \leq z \leq$$
$$\leq (a \vee d_1 \vee \ldots \vee d_{n-1}) \wedge (c \vee b_1 \vee \ldots \vee b_{n-1}).$$

We have $a \wedge d_1 \wedge \ldots \wedge d_{n-1} \leq (x \vee b_1 \vee \ldots \vee b_{n-1}) \wedge (d_1 \wedge \ldots \wedge d_{n-1}) = (x \wedge d_1 \wedge \ldots \wedge d_{n-1}) \vee [(b_1 \vee \ldots \vee b_{n-1}) \wedge d_1 \wedge \ldots \wedge d_{n-1}] \leq c \vee b_1 \vee \ldots \vee b_{n-1}$. Hence, $(a \wedge d_1 \wedge \ldots \wedge d_{n-1}) \vee (c \wedge b_1 \wedge \ldots \wedge b_{n-1}) \leq c \vee b_1 \vee \ldots \vee b_{n-1}$. Similarly, we have $(a \wedge d_1 \wedge \ldots \wedge d_{n-1}) \vee (c \wedge b_1 \wedge \ldots \wedge b_{n-1}) \leq a \vee d_1 \vee \ldots \vee d_{n-1}$. Therefore,

$$(a \wedge d_1 \wedge \ldots \wedge d_{n-1}) \vee (c \wedge b_1 \wedge \ldots \wedge b_{n-1}) \leq (a \vee d_1 \vee \ldots \vee d_{n-1}) \wedge (c \vee b_1 \vee \ldots \vee b_{n-1}),$$

that is
$$g(a, d_1^{n-1}) \cap g(c, b_1^{n-1}) \neq \emptyset.$$

□

Theorem 35 ([18]). *If $(L, \vee, \wedge)$ is a join n-space, then the lattice $(L, \vee, \wedge)$ is distributive.*

Proof. Suppose that L is not distributive. Then, L contains a five-element sublattice $\{m, a, b, c, M\}$, where $a \vee c = b \vee c = M$, $a \wedge c = b \wedge c = m$, and either $a > b$ or a, b, c are mutually non-comparable. We have $c \in a/\{\underbrace{b, \ldots, b}_{n-1}\} \cap b/\{\underbrace{a, \ldots, a}_{n-1}\}$, and since (L, g) is a join n-space, we obtain

$$g(\underbrace{a, \ldots\ a}_{n}) \cap g(\underbrace{b, \ldots, b}_{n}) \neq \emptyset,$$

that is $a = b$, which is a contradiction.

Therefore, $(L, \vee, \wedge)$ is distributive. □

Corollary 10 ([18]). *The n-hypergroupoid (L, g) is a join n-space iff the lattice $(L, \vee, \wedge)$ is distributive.*

Theorem 36 ([18]). *Let $(L, \vee, \wedge)$ be a distributive lattice. If I is an ideal and F is a filter of L, then (I, g) and (F, g) are n-subhypergroups of (L, g).*

Proof. Let I be an ideal of L. For any a_1^n of I, we have $g(a_1^n) = \{z \mid B_n \leq z \leq A_n\}$. Since $A_n = a_1 \vee \ldots \vee a_n \in I$ and $z \leq A_n$, it follows $z \in I$. Hence, $g(a_1^n) \subset I$. On the other hand, we have $a \in g(a, a_1^{n-1})$ for any a, a_1^{n-1} of I. Therefore, (I, g) is an n-subhypergroup of (L, g). Similarly, it follows that (F, g) is an n-subhypergroup of (L, g). □

The converse fails, as can be seen from the following example:

Example 6. *Let us consider the distributive lattice $(\mathcal{P}(M), \cup, \cap)$, where M is a set with at least three elements. Let $a, b \in M$, $a \neq b$ and $S = \{M - \{a\}, M - \{a, b\}\}$. Then, (S, g) is an n-subhypergroup of $(\mathcal{P}(M), g)$, but S is neither an ideal, nor a filter of $\mathcal{P}(M)$, since $\emptyset \notin S$ and $M \notin S$, respectively.*

Author Contributions: Conceptualization, B.D. and V.L.-F.; methodology, B.D. and V.L.-F.; formal analysis, B.D. and T.V.; investigation, B.D.; resources, B.D., V.L.-F. and T.V.; writing—original draft preparation, B.D. and V.L.-F.; writing—review and editing, B.D., V.L.-F. and T.V.; supervision, B.D. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Dörnte, W. Untersuchungen uber einen verallgemeinerten Gruppenbegriff. *Math. Z.* **1929**, *29*, 1–19. [CrossRef]
2. Post, E.L. Polyadic groups. *Trans. Amer. Math. Soc.* **1940**, *48*, 208–350. [CrossRef]
3. Dudek, W.A.; Michalski, J. On retracts of polyadic groups. *Demonstr. Math.* **1984**, *17*, 281–302. [CrossRef]
4. Dudek, W.A.; Glazek, K. Around the Hosszu-Gluskin theorem for n-ary groups. *Discret. Math.* **2008**, *308*, 4861–4876. [CrossRef]
5. Dudek, W.A. Varieties of polyadic groups. *Filomat* **1995**, *9*, 657–674.
6. Marty, F. Sur une generalisation de la notion de groupe. In Proceedings of the 8th Congres Des Mathematiciens Scandinaves, Stockholm, Sweden, 14–18 August 1934; pp. 45–49.
7. Davvaz, B.; Vougiouklis, T. *A Walk through Weak Hyperstructures: H_v-Structures*; World Scientific Publishing Co. Pte. Ltd.: Hackensack, NJ, USA, 2019.
8. Corsini, P. *Prolegomena of Hypergroup Theory*; Aviani Editore: Tricesimo, Italy, 1993.
9. Rosenberg, I.G. Hypergroups and join spaces determined by relations. *Ital. J. Pure Appl. Math.* **1998**, *4*, 93–101.
10. Freni, D. Une note sur le coeur d'un hypergroupe et sur la cloture β^* de β. *Mat. Pura Appl.* **1991**, *8*, 153–156.
11. Davvaz, B.; Vougiouklis, T. *N*-Ary hypergroups. *Iran. J. Sci. Technol. Trans. A* **2006**, *30*, 165–174.
12. Davvaz, B.; Dudek, W.A.; Vougiouklis, T. A generalization of n-ary algebraic systems. *Commun. Algebra* **2009**, *37*, 1248–1263. [CrossRef]
13. Davvaz, B.; Dudek, W.A.; Mirvakili, S. Neutral elements, fundamental relations and n-ary hypersemigroups. *Int. J. Algebra Comput.* **2009**, *19*, 567–583. [CrossRef]
14. Anvariyeh, S.M.; Momeni, S. n-ary hypergroups associated with n-ary relations. *Bull. Korean Math. Soc.* **2013**, *502*, 507–524. [CrossRef]
15. Davvaz, B.; Leoreanu-Fotea, V. Binary relations on ternary semihypergroups. *Commun. Algebra* **2010**, *38*, 3621–3636. [CrossRef]
16. Krehlik, S.; Novak, M.; Bolat, M. Properties of n-ary hypergroups relevant for modelling trajectories in HD maps. *An. Stiint. Univ. "Ovidius" Constanta Ser. Mat.* **2022**, *30*, 161–178.
17. Leoreanu-Fotea, V.; Davvaz, B. n-hypergroups and binary relations. *Eur. J. Comb.* **2008**, *29*, 1207–1218. [CrossRef]
18. Leoreanu-Fotea, V.; Davvaz, B. Join n-spaces and lattices. *J. Mult.-Valued Log. Soft Comput.* **2009**, *15*, 421–432.
19. Corsini, P. Mutually associative hypergroupoids. In *Algebraic Hyperstructures and Applications, Prague, 1996*; Democritus Univ. Thrace, Alexandroupolis: Komotini, Greece, 1997; pp. 25–33.
20. Freni, D. A new characterization of the derived hypergroup via strongly regular equivalences. *Commun. Algebra* **2002**, *30*, 3977–3989. [CrossRef]
21. Freni, D. Strongly transitive geometric spaces: Applications to hypergroups and semigroups theory. *Commun. Algebra* **2004**, *32*, 969–988. [CrossRef]