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Plane Partitions and a Problem of Josephus

Mircea Merca [1,2]

[1] Department of Mathematical Methods and Models, Fundamental Sciences Applied in Engineering Research Center, National University of Science and Technology Politehnica Bucharest, RO-060042 Bucharest, Romania; mircea.merca@upb.ro or mircea.merca@profinfo.edu.ro
[2] Academy of Romanian Scientists, RO-050044 Bucharest, Romania

Abstract: The Josephus Problem is a mathematical counting-out problem with a grim description: given a group of n persons arranged in a circle under the edict that every kth person will be executed going around the circle until only one remains, find the position $L(n,k)$ in which you should stand in order to be the last survivor. Let J_n be the order in which the first person is executed on counting when $k=2$. In this paper, we consider the sequence $(J_n)_{n\geqslant 1}$ in order to introduce new expressions for the generating functions of the number of strict plane partitions and the number of symmetric plane partitions. This approach allows us to express the number of strict plane partitions of n and the number of symmetric plane partitions of n as sums over partitions of n in terms of binomial coefficients involving J_n. Also, we introduce interpretations for the strict plane partitions and the symmetric plane partitions in terms of colored partitions. Connections between the sum of the divisors' functions and J_n are provided in this context.

Keywords: Josephus problem; partitions; plane partitions; divisors; binomial coefficients

MSC: 11P81; 11P82; 05A19; 05A20

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1. Introduction

According to ([1], pp. 341–366, 387–391), in the Romano–Jewish conflict of 67 A. D., the Romans took the town Jotapata which Flavius Josephus was commanding. Josephus and 40 of his comrades escaped and were trapped in a cave. Fearing capture, they decided to kill themselves. Josephus and a friend did not agree with this proposal but did not dare to speak out openly against it. They agreed that they should arrange them in a circle and that, always counting in the same sense around the circle, every third man should be killed until there was only one survivor who would kill himself. By choosing positions 31 and 16 in the circle, Josephus and his friend saved their lives and joined the Romans, i.e.,

$$3,6,9,12,15,18,21,24,27,30,33,36,39,1,5,10,14,19,23,28,32,$$
$$37,41,7,13,20,26,34,40,8,17,29,38,11,25,2,22,4,35,16,31.$$

In mathematics and computer science, the Josephus problem is usually stated as follows: given a group of n men arranged in a circle under the edict that every kth man will be executed going around the circle until only one remains, find the position $L(n,k)$ in which you should stand in order to be the last survivor.

We are interested in the case $k=2$ of the Josephus problem. Thus, for $k=2$, we denote by J_n the order in which the first person is executed on counting. For example, if there are $n=7$ persons to begin with, they are executed in the following order:

$$2,\ 4,\ 6,\ 1,\ 5,\ 3,\ 7.$$

So, the first person is eliminated as number 4. Therefore, $J_7=4$.

We remark that the sequence

$$(J_n)_{n\geqslant 1} = (1, 2, 2, 4, 3, 5, 4, 8, 5, 8, 6, 11, 7, 11, 8, 16, 9, 14, 10, \ldots)$$

is known and can be seen in the On-Line Encyclopedia of Integer Sequence ([2], A225381). The sequence $(J_n)_{n\geqslant 1}$ can be defined as follows:

$$J_n = \lceil n/2 \rceil + \frac{1+(-1)^n}{2} \cdot J_{n/2},$$

with the initial condition $J_1 = 1$. When n is odd, it is clear that $J_n = (n+1)/2$. We introduced the sequence J_n to provide new formulae for two types of plane partitions.

Recall that a plane partition π of the positive integer n is a 2-dimensional array $\pi = (\pi_{i,j})_{i,j\geqslant 1}$ of non-negative integers $\pi_{i,j}$ such that

$$n = \sum_{i,j\geqslant 1} \pi_{i,j},$$

which is weakly decreasing in rows and columns:

$$\pi_{i,j} \geqslant \pi_{i+1,j}, \qquad \pi_{i,j} \geqslant \pi_{i,j+1}, \qquad \text{for all } i,j \geqslant 1.$$

It can be seen as the filling of a Young diagram with weakly decreasing rows and columns, where the sum of all these numbers equals n. This is a natural generalization of the concept of classical partitions [3]. Different configurations are counted as different plane partitions. The plane partitions of 4 are presented in Figure 1.

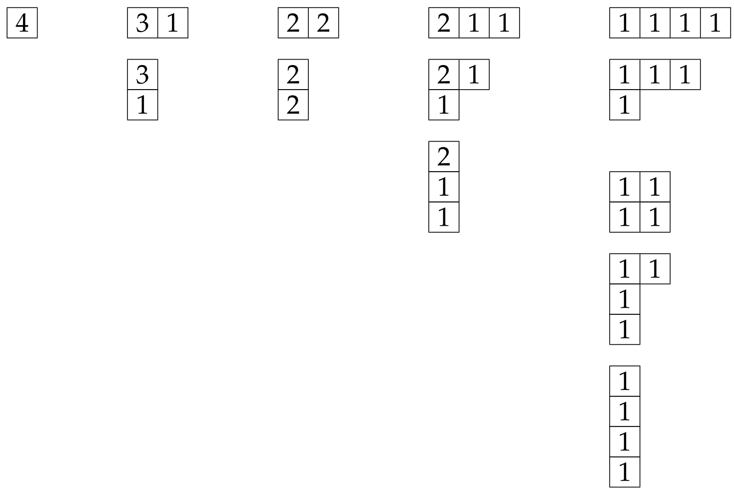

Figure 1. The plane partitions of 4.

Recently, Merca and Radu [4] considered the specialization of complete homogeneous symmetric functions and provided a new formula for $PL(n)$, which is the number of plane partitions of n:

$$PL(n) = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{1+t_2}{t_2}\binom{2+t_3}{t_3}\cdots\binom{n-1+t_n}{t_n}.$$

As can be seen, this formula expresses the number of plane partitions of n in terms of binomial coefficients as a sum over all the partitions of n, taking into account the multiplicity of the parts. They obtained similar results for the number of strict plane partitions of n and the number of symmetric plane partitions of n. In this paper, we consider the sequence $(J_n)_{\geqslant 1}$ and obtain new formulas for the number of strict plane partitions of n and the number of symmetric plane partitions of n as sums over partitions of n.

2. Strict Plane Partitions

Recall that a strict plane partition π of the positive integer n is a plane partition $\pi = (\pi_{i,j})_{i,j \geqslant 1}$ of n which is decreasing in rows, i.e.,

$$\pi_{i,j} > \pi_{i+1,j}, \qquad \text{for all } i,j \geqslant 1.$$

In [4], we denoted by $SPL(n)$ the number of strict plane partitions of n and, for convenience, we defined $SPL(0) = 1$. The strict plane partitions of 4 are presented in Figure 2. We see that $SPL(4) = 7$.

$$\boxed{4} \qquad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline\end{array}$$

$$\begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c}\hline 2 & \multicolumn{1}{|c|}{1} \\ \hline 1 & \\ \cline{1-1}\end{array}$$

Figure 2. The strict plane partitions of 4.

According to Gordon and Houten [5], the generating function for the number of strict plane partitions of n is given by

$$\sum_{n=0}^{\infty} SPL(n)\, q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{\lceil n/2 \rceil}}$$

and the expansion starts as

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{\lceil n/2 \rceil}} = 1 + q + 2\,q^2 + 4\,q^3 + 7\,q^4 + 12\,q^5 + 21\,q^6 + 34\,q^7 + 56\,q^8 + \cdots. \tag{1}$$

For any positive integer m, we denote by $SPL^{(m)}(n)$ the number of m-tuples of strict plane partitions of non-negative integers $n_1, n_2, \ldots, n_m$ where $n_1 + n_2 + \cdots + n_m = n$. It is clear that $SPL(n) = SPL^{(1)}(n)$ and

$$SPL^{(m)}(n) = \sum_{n_1+n_2+\cdots+n_m=n} SPL(n_1)\, SPL(n_2) \,\cdots\, SPL(n_m).$$

For $r \in \{-1, 0, 1\}$, we define the $SPL^{(m,r)}(n)$ as follows:

$$SPL^{(m,r)}(n) = \begin{cases} SPL^{(m)}(n), & \text{for } r = 0, \\ SPL^{(m)}(n) - PL^{(m)}(n-1), & \text{for } r = -1, \\ \sum\limits_{k=0}^{n} SPL^{(m)}(k), & \text{for } r = 1. \end{cases} \tag{2}$$

In [4], Merca and Radu considered specializations of complete homogeneous symmetric functions and provide the following formula for $SPL^{(m,r)}(n)$.

Theorem 1. *For $m \geqslant 1$, $r \in \{-1, 0, 1\}$ and $n \geqslant 0$,*

$$SPL^{(m,r)}(n) = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{m-1+r+t_1}{t_1} \prod_{j=2}^{n} \binom{\lceil j/2 \rceil m - 1 + t_j}{t_j}.$$

In this section, we shall provide another decomposition of $SPL^{(m,r)}(n)$ as a sum over partitions of n in terms of binomial coefficients. This time, in addition to the multiplicity of a part of size k, we also need to consider J_k.

Theorem 2. *For $m \geqslant 1$, $r \in \{-1,0,1\}$ and $n \geqslant 0$,*

$$SPL^{(m,r)}(n) = \sum_{t_1+2t_2+\cdots+nt_n=n} \prod_{k=1}^{n} \binom{J_k^{(m,r)}}{t_k},$$

where

$$J_n^{(m,r)} = \begin{cases} m \cdot J_n + r, & \text{for } n = 2^k,\, k = 0,1,2,\ldots, \\ m \cdot J_n, & \text{otherwise.} \end{cases}$$

Proof. Applying elementary techniques in the theory of partitions [3], we obtain the following generating function:

$$\sum_{n=0}^{\infty} SPL^{(m,r)}(n)\, q^n = \frac{1}{(1-q)^r} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{m\cdot\lceil n/2 \rceil}}, \qquad |q| < 1. \tag{3}$$

In order to prove our theorem, we consider the identity

$$\frac{1}{1-q} = \prod_{k=0}^{\infty} (1+q^{2^k}), \qquad |q| < 1. \tag{4}$$

In addition, by (4), with q replaced by q^n, we obtain

$$\frac{1}{1-q^n} = \prod_{k=0}^{\infty} (1+q^{2^k \cdot n}), \qquad |q| < 1. \tag{5}$$

For $|q| < 1$, considering (5), the generating function of $SPL^{(m,r)}(n)$ can be rewritten as follows:

$$\begin{aligned} &\sum_{n=0}^{\infty} SPL^{(m,r)}(n)\, q^n \\ &\quad = \prod_{k=0}^{\infty} (1+q^{2^k})^r \cdot \prod_{n=1}^{\infty} \prod_{k=0}^{\infty} (1+q^{2^k \cdot n})^{m\cdot\lceil n/2 \rceil} \\ &\quad = \prod_{n=1}^{\infty} (1+q^n)^{J_n^{(m,r)}} \\ &\quad = \prod_{n=1}^{\infty} \left(\sum_{k=0}^{J_n^{(m,r)}} \binom{J_n^{(m,r)}}{k} q^{k\cdot n} \right) \\ &\quad = \sum_{n=0}^{\infty} q^n \sum_{t_1+2t_2+\cdots+nt_n=n} \prod_{k=1}^{n} \binom{J_k^{(m,r)}}{t_k}, \end{aligned} \tag{6}$$

where we have invoked Cauchy multiplication of the power series. □

The cases $m = 1$ and $r = 0$ of Theorem 2 reads as follows.

Corollary 1. *For $n \geqslant 0$,*

$$SPL(n) = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{J_1}{t_1} \binom{J_2}{t_2} \cdots \binom{J_n}{t_n}.$$

The sum in the right-hand side of this equation runs over all the partitions of n, but not all terms are nonzero. Since for $t_k > J_k$ we have $\binom{J_k}{t_k} = 0$, in this sum, we can consider only the partitions of n into, at most, the J_k copy of parts of size k, for each $k \in \{1,2,\ldots,n\}$. For example, the partitions of 4 that satisfy this condition can be rewritten as:

$$\begin{aligned}&1\cdot 0+2\cdot 0+3\cdot 0+4\cdot 1,\\ &1\cdot 1+2\cdot 0+3\cdot 1+4\cdot 0,\\ &1\cdot 0+2\cdot 2+3\cdot 0+4\cdot 0.\end{aligned} \tag{7}$$

So, the case $n=4$ of Corollary 1 reads as follows:

$$\begin{aligned}SPL(4) &= \binom{1}{0}\binom{2}{0}\binom{2}{0}\binom{4}{1}+\binom{1}{1}\binom{2}{0}\binom{2}{1}\binom{4}{0}+\binom{1}{0}\binom{2}{2}\binom{2}{0}\binom{4}{0}\\ &= 4+2+1=7.\end{aligned}$$

The cases $m=2$ and $r=0$ of Theorem 2 gives the following identity.

Corollary 2. *For $n \geqslant 0$,*

$$\sum_{k=0}^{n} SPL(k)\,SPL(n-k) = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{2J_1}{t_1}\binom{2J_2}{t_2}\cdots\binom{2J_n}{t_n}.$$

Considering the partitions of 4 with the property $t_k \leqslant 2J_k$, the case $n=4$ of Corollary 2 reads as follows:

$$\begin{aligned}\sum_{k=0}^{4} SPL(k)\,SPL(4-k) &= \binom{2}{0}\binom{4}{0}\binom{4}{0}\binom{8}{1}+\binom{2}{1}\binom{4}{0}\binom{4}{1}\binom{8}{0}\\ &+\binom{2}{0}\binom{4}{2}\binom{4}{0}\binom{8}{0}+\binom{2}{2}\binom{4}{1}\binom{4}{0}\binom{8}{0}\\ &= 8+8+6+4=26.\end{aligned}$$

On the other hand, according to the expansion (1), we can write:

$$\begin{aligned}\sum_{k=0}^{4} SPL(k)\,SPL(4-k) &= 1\cdot 7+1\cdot 4+2\cdot 2+4\cdot 1+7\cdot 1\\ &= 7+4+4+4+7=26.\end{aligned}$$

By Corollary 2, we easily deduce the following congruence identity.

Corollary 3. *For $n \geqslant 0$,*

$$\sum_{t_1+2t_2+\cdots+nt_n=n} \binom{2J_1}{t_1}\binom{2J_2}{t_2}\cdots\binom{2J_n}{t_n} \equiv SPL\left(\frac{n}{2}\right) \pmod 2,$$

where $SPL(x)=0$ if x is not a positive integer.

The following identity can be easily derived as an immediate consequence of Theorems 1 and 2.

Corollary 4. *For $m \geqslant 1$, $r \in \{-1,0,1\}$ and $n \geqslant 0$,*

$$\sum_{t_1+2t_2+\cdots+nt_n=n} \binom{m-1+r+t_1}{t_1} \prod_{k=2}^{n} \binom{\lceil k/2\rceil m-1+t_k}{t_k} = \sum_{t_1+2t_2+\cdots+nt_n=n} \prod_{k=1}^{n} \binom{J_k^{(m,r)}}{t_k}.$$

3. Symmetric Plane Partitions

A symmetric plane partition π of the positive integer n is a plane partition $\pi = (\pi_{i,j})_{i,j\geqslant 1}$ of n such that

$$\pi_{i,j} = \pi_{j,i}, \qquad \text{for all } i, j \geqslant 1.$$

We denote by $sPL(n)$ the number of symmetric plane partitions of n. The symmetric plane partitions of 6 are presented in Figure 3. We see that $sPL(6) = 6$. For convenience, we define $sPL(0) = 1$.

$$\begin{array}{|c|}\hline 6\\\hline\end{array}\qquad \begin{array}{|c|c|}\hline 4&1\\\hline 1\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|}\hline 3&1\\\hline 1&1\\\hline\end{array}\qquad \begin{array}{|c|c|}\hline 2&2\\\hline 2\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|}\hline 2&1&1\\\hline 1\\\cline{1-1} 1\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 1&1\\\cline{1-2} 1\\\cline{1-1}\end{array}$$

Figure 3. The symmetric plane partitions of 6.

According to Gordon [6], the generating function for the number of symmetric plane partitions of n is given by

$$\sum_{n=0}^{\infty} sPL(n)\, q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{a_n}}, \tag{8}$$

where

$$a_n = \begin{cases} 1, & n \text{ odd}, \\ \lfloor n/4 \rfloor, & n \text{ even}. \end{cases}$$

The expansion starts as

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{a_n}} = 1 + q + q^2 + 2\,q^3 + 3\,q^4 + 4\,q^5 + 6\,q^6 + 8\,q^7 + 12\,q^8 + \cdots . \tag{9}$$

We recall that the number of symmetric plane partitions of n is equal to the number of strict plane partitions of n into odd parts [6]. The strict plane partitions of 6 into odd parts are presented in Figure 4.

$$\begin{array}{|c|c|}\hline 5&1\\\hline\end{array}\qquad \begin{array}{|c|}\hline 5\\\hline 1\\\hline\end{array}\qquad \begin{array}{|c|c|}\hline 3&3\\\hline\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 3&1&1&1\\\hline\end{array}\qquad \begin{array}{|c|c|c|}\hline 3&1&1\\\hline 1\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&1\\\hline\end{array}$$

Figure 4. The strict plane partitions of 6 into odd parts.

For any positive integer m, we denote by $sPL^{(m)}(n)$ the number of m-tuples of symmetric plane partitions of non-negative integers $n_1, n_2, \ldots, n_m$ where $n_1 + n_2 + \cdots + n_m = n$. It is clear that $sPL(n) = sPL^{(1)}(n)$ and

$$sPL^{(m)}(n) = \sum_{n_1+n_2+\cdots+n_m=n} sPL(n_1)\, sPL(n_2) \,\cdots\, sPL(n_m).$$

For $r \in \{-1, 0, 1\}$, we define the $sPL^{(m,r)}(n)$ as follows:

$$sPL^{(m,r)}(n) = \begin{cases} sPL^{(m)}(n), & \text{for } r = 0, \\ sPL^{(m)}(n) - PL^{(m)}(n-1), & \text{for } r = -1, \\ \sum\limits_{k=0}^{n} sPL^{(m)}(k), & \text{for } r = 1. \end{cases} \tag{10}$$

In [4], Merca and Radu consider specializations of complete homogeneous symmetric functions and provide the following formula for $sPL^{(m,r)}(n)$.

Theorem 3. *For $m \geqslant 1$, $r \in \{-1,0,1\}$ and $n \geqslant 0$,*

$$sPL^{(m,r)}(n) = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{m-1+r+t_1}{t_1} \prod_{j=2}^{n} \binom{a_j \cdot m - 1 + t_j}{t_j}.$$

In this section, we shall provide another decomposition of $sPL^{(m,r)}(n)$ as a sum over partitions of n in terms of binomial coefficients. This time, in addition to the multiplicity of a part of size k, we also need the sequence $(\mathcal{J}_n)_{n\geqslant 1}$ defined as follows:

$$\mathcal{J}_n = \begin{cases} 1, & \text{for } n \text{ odd}, \\ J_{n/2}, & \text{for } n \text{ even}, \end{cases}$$

where $(J_n)_{n\geqslant 1}$ is the sequence introduced in the previous section in connection with the Josephus problem.

Theorem 4. *For $m \geqslant 1$, $r \in \{-1,0,1\}$ and $n \geqslant 0$,*

$$sPL^{(m,r)}(n) = \sum_{t_1+2t_2+\cdots+nt_n=n} \prod_{k=1}^{n} \binom{\mathcal{J}_k^{(m,r)}}{t_k},$$

where

$$\mathcal{J}_n^{(m,r)} = \begin{cases} m \cdot \mathcal{J}_n + r, & \text{for } n = 2^k, k = 0,1,2,\ldots, \\ m \cdot \mathcal{J}_n, & \text{otherwise}. \end{cases}$$

Proof. The proof of this theorem is quite similar to the proof of Theorem 2. Therefore, we omit some details:

$$\begin{aligned} &\sum_{n=0}^{\infty} sPL^{(m,r)}(n)\, q^n \\ &\quad = \frac{1}{(1-q)^r} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{m\cdot a_n}} \\ &\quad = \prod_{k=0}^{\infty} (1+q^{2^k})^r \cdot \prod_{n=1}^{\infty} \prod_{k=0}^{\infty} (1+q^{2^k(2n-1)})^m (1+q^{2^k(2n)})^{m\cdot \lfloor n/2 \rfloor} \\ &\quad = \prod_{k=0}^{\infty} (1+q^{2^k})^r \cdot \prod_{n=1}^{\infty} (1+q^n)^{m\cdot \mathcal{J}_n} \\ &\quad = \prod_{n=1}^{\infty} (1+q^n)^{\mathcal{J}_n^{(m,r)}} \qquad (11) \\ &\quad = \prod_{n=1}^{\infty} \left(\sum_{k=0}^{\mathcal{J}_n^{(m,r)}} \binom{\mathcal{J}_n^{(m,r)}}{k} q^{k\cdot n} \right) \\ &\quad = \sum_{n=0}^{\infty} q^n \sum_{t_1+2t_2+\cdots+nt_n=n} \prod_{k=1}^{n} \binom{\mathcal{J}_k^{(m,r)}}{t_k}, \end{aligned}$$

where we have invoked Cauchy multiplication of the power series. □

In analogy with Corollary 1, the cases $m = 1$ and $r = 0$ of Theorem 4 can be written as follows.

Corollary 5. *For $n \geqslant 0$,*

$$sPL(n) = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{\mathcal{J}_1}{t_1} \binom{\mathcal{J}_2}{t_2} \cdots \binom{\mathcal{J}_n}{t_n}.$$

The sum in the right-hand side of this equation runs over all the partitions of n, but not all terms are nonzero. Since for $t_k > \mathcal{J}_k$ we have $\binom{\mathcal{J}_k}{t_k} = 0$, in this sum, we can consider only the partitions of n into, at most, the $\mathcal{J}_k$ copy of parts of size k, for each $k \in \{1, 2, \ldots, n\}$. For example, the partitions of 6 that satisfy this condition can be rewritten as:

$$
\begin{aligned}
&1 \cdot 0 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 0 + 5 \cdot 0 + 6 \cdot 1, \\
&1 \cdot 1 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 0 + 5 \cdot 1 + 6 \cdot 0, \\
&1 \cdot 0 + 2 \cdot 1 + 3 \cdot 0 + 4 \cdot 1 + 5 \cdot 0 + 6 \cdot 0, \\
&1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 + 4 \cdot 0 + 5 \cdot 0 + 6 \cdot 0.
\end{aligned}
$$

So, the case $n = 6$ of Corollary 5 reads as follows:

$$
\begin{aligned}
sPL(6) = &\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{2}{0}\binom{1}{0}\binom{2}{1} + \binom{1}{1}\binom{1}{0}\binom{1}{0}\binom{2}{0}\binom{1}{1}\binom{2}{0} \\
&+ \binom{1}{0}\binom{1}{1}\binom{1}{0}\binom{2}{1}\binom{1}{0}\binom{2}{0} + \binom{1}{1}\binom{1}{1}\binom{1}{1}\binom{2}{0}\binom{1}{0}\binom{2}{0} \\
&= 2 + 1 + 2 + 1 = 6.
\end{aligned}
$$

In analogy with Corollary 2, the cases $m = 2$ and $r = 0$ of Theorem 4 gives the following identity.

Corollary 6. *For $n \geqslant 0$,*

$$
\sum_{k=0}^{n} sPL(k)\, sPL(n-k) = \sum_{t_1 + 2t_2 + \cdots + nt_n = n} \binom{2\mathcal{J}_1}{t_1}\binom{2\mathcal{J}_2}{t_2} \cdots \binom{2\mathcal{J}_n}{t_n}.
$$

For example, the partitions of 6 with the property $t_k \leqslant 2\mathcal{J}_k$ are:

$$
\begin{aligned}
&1 \cdot 0 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 0 + 5 \cdot 0 + 6 \cdot 1, \\
&1 \cdot 1 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 0 + 5 \cdot 1 + 6 \cdot 0, \\
&1 \cdot 0 + 2 \cdot 1 + 3 \cdot 0 + 4 \cdot 1 + 5 \cdot 0 + 6 \cdot 0, \\
&1 \cdot 2 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 1 + 5 \cdot 0 + 6 \cdot 0, \\
&1 \cdot 0 + 2 \cdot 0 + 3 \cdot 2 + 4 \cdot 0 + 5 \cdot 0 + 6 \cdot 0, \\
&1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 + 4 \cdot 0 + 5 \cdot 0 + 6 \cdot 0, \\
&1 \cdot 2 + 2 \cdot 2 + 3 \cdot 0 + 4 \cdot 0 + 5 \cdot 0 + 6 \cdot 0.
\end{aligned}
$$

Thus, the case $n = 6$ of Corollary 6 reads as follows:

$$
\begin{aligned}
\sum_{k=0}^{6} sPL(k)\, sPL(6-k) = &\binom{2}{0}\binom{2}{0}\binom{2}{0}\binom{4}{0}\binom{2}{0}\binom{4}{1} + \binom{2}{1}\binom{2}{0}\binom{2}{0}\binom{4}{0}\binom{2}{1}\binom{4}{0} \\
&+ \binom{2}{0}\binom{2}{1}\binom{2}{0}\binom{4}{1}\binom{2}{0}\binom{4}{0} + \binom{2}{2}\binom{2}{0}\binom{2}{0}\binom{4}{1}\binom{2}{0}\binom{4}{0} \\
&+ \binom{2}{0}\binom{2}{0}\binom{2}{2}\binom{4}{0}\binom{2}{0}\binom{4}{0} + \binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{4}{0}\binom{2}{0}\binom{4}{0} \\
&+ \binom{2}{2}\binom{2}{2}\binom{2}{0}\binom{4}{0}\binom{2}{0}\binom{4}{0} \\
&= 4 + 4 + 8 + 4 + 1 + 8 + 1 = 30.
\end{aligned}
$$

On the other hand, according to expansion (9), we can write:

$$\begin{aligned}\sum_{k=0}^{6} SPL(k)\,SPL(6-k) &= 1\cdot 6+1\cdot 4+1\cdot 3+2\cdot 2+3\cdot 1+4\cdot 1+6\cdot 1\\ &= 6+4+3+4+3+4+6=30.\end{aligned}$$

By Corollary 6, in analogy with Corollary 3, we easily deduce the following congruence identity.

Corollary 7. *For $n \geqslant 0$,*

$$\sum_{t_1+2t_2+\cdots+nt_n=n} \binom{2\mathcal{J}_1}{t_1}\binom{2\mathcal{J}_2}{t_2}\cdots\binom{2\mathcal{J}_n}{t_n} \equiv sPL\left(\frac{n}{2}\right) \pmod 2,$$

where $sPL(x)=0$ if x is not a positive integer.

As a consequence of Theorems 3 and 4, in analogy with Corollary 4, we remark the following identity.

Corollary 8. *For $m \geqslant 1$, $r \in \{-1,0,1\}$ and $n \geqslant 0$,*

$$\sum_{t_1+2t_2+\cdots+nt_n=n} \binom{m-1+r+t_1}{t_1}\prod_{k=2}^{n}\binom{a_k\cdot m-1+t_k}{t_k} = \sum_{t_1+2t_2+\cdots+nt_n=n}\prod_{k=1}^{n}\binom{\mathcal{J}_k^{(m,r)}}{t_k}.$$

4. Connections between Divisors and J_n

This section is inspired by the following well-known connection between plane partitions and divisors

$$PL(n)=\frac{1}{n}\sum_{k=1}^{n}\sigma_2(k)\,PL(n-k),$$

where, for a real or complex number z, the sum of positive divisors' function $\sigma_z(n)$ is defined as the sum of the zth powers of the positive divisors of n, i.e.,

$$\sigma_z(n)=\sum_{d|n} d^z.$$

It is well known that the generating function of $\sigma_z(n)$ is given by the following Lambert series:

$$\sum_{n=1}^{\infty}\sigma_z(n)\,q^n=\sum_{n=1}^{\infty}\frac{n^z\,q^n}{1-q^n}, \qquad |q|<1.$$

Related to strict plane partitions, we remark the following analogous result.

Theorem 5. *Let n be a positive integer. Then*

$$SPL(n)=\frac{1}{n}\sum_{k=1}^{\infty}\overline{\sigma}(k)\,SPL(n-k),$$

where

$$\overline{\sigma}(n)=\sum_{d|n} d\,\lceil d/2\rceil.$$

Proof. The logarithmic differentiation of the generating function of $SPL(n)$ can be written as:

$$\frac{\partial}{\partial q}\ln\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{\lceil n/2\rceil}}=\sum_{n=1}^{\infty}\frac{\partial}{\partial q}\ln\frac{1}{(1-q^n)^{\lceil n/2\rceil}}=\sum_{n=1}^{\infty}\frac{n\lceil n/2\rceil\,q^{n-1}}{1-q^n}$$
$$=\sum_{n=1}^{\infty}\overline{\sigma}(n)\,q^{n-1}. \tag{12}$$

On the other hand, we have:

$$\begin{aligned}\frac{\partial}{\partial q}\ln\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{\lceil n/2\rceil}}&=\left(\frac{\partial}{\partial q}\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{\lceil n/2\rceil}}\right)\left(\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{\lceil n/2\rceil}}\right)^{-1}\\&=\left(\frac{\partial}{\partial q}\sum_{n=0}^{\infty}SPL(n)\,q^n\right)\left(\sum_{n=0}^{\infty}SPL(n)\,q^n\right)^{-1}\\&=\left(\sum_{n=0}^{\infty}n\,SPL(n)\,q^{n-1}\right)\left(\sum_{n=0}^{\infty}SPL(n)\,q^n\right)^{-1}.\end{aligned}$$

Thus, we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}n\,SPL(n)\,q^n&=\left(\sum_{n=1}^{\infty}\overline{\sigma}(n)\,q^n\right)\left(\sum_{n=0}^{\infty}SPL(n)\,q^n\right)\\&=\sum_{n=0}^{\infty}q^n\sum_{k=0}^{n}\overline{\sigma}(k)\,SPL(n-k),\end{aligned}$$

where we have invoked Cauchy multiplication of two power series. This concludes the proof. □

The sum of the divisors' function $\overline{\sigma}(n)$ can be expressed in terms of $\sigma_z(n)$ as we can see in the following result.

Theorem 6. *Let n be a positive integer. Then*

$$\overline{\sigma}(n)=\frac{\sigma_2(n)+\sigma_1(2n)}{2}-\sigma_1(n).$$

Proof. Considering the generation of $\sigma_z(n)$, we can write:

$$\begin{aligned}\sum_{n=1}^{\infty}\sigma_1(2n)\,q^{2n}&=\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{n\,q^n}{1-q^n}+\sum_{n=1}^{\infty}\frac{n\,(-q)^n}{1-(-q)^n}\right)\\&=\frac{1}{2}\sum_{\substack{n=1\\n\text{ odd}}}^{\infty}\left(\frac{n\,q^n}{1-q^n}+\frac{n\,(-q)^n}{1-(-q)^n}\right)+\frac{1}{2}\sum_{\substack{n=1\\n\text{ even}}}^{\infty}\left(\frac{n\,q^n}{1-q^n}+\frac{n\,(-q)^n}{1-(-q)^n}\right)\\&=\sum_{\substack{n=1\\n\text{ odd}}}^{\infty}\frac{n\,q^{2n}}{1-q^{2n}}+\sum_{\substack{n=1\\n\text{ even}}}^{\infty}\frac{n\,q^n}{1-q^n}\\&=\sum_{n=1}^{\infty}\frac{(2n-1)\,q^{2(2n-1)}}{1-q^{2(2n-1)}}+\sum_{n=1}^{\infty}\frac{2n\,q^{2n}}{1-q^{2n}}.\end{aligned}$$

By this relation, with q^2 replaced by q, we obtain:

$$\sum_{n=1}^{\infty}\frac{(2n-1)\,q^{2n-1}}{1-q^{2n-1}}=\sum_{n=1}^{\infty}\sigma_1(2n)\,q^n-2\sum_{n=1}^{\infty}\sigma_1(n)\,q^n. \tag{13}$$

The generating function of $\overline{\sigma}(n)$ can be written as follows:

$$\begin{aligned}\sum_{n=1}^{\infty} \overline{\sigma}(n)\, q^n &= \sum_{n=1}^{\infty} \frac{n \lceil n/2 \rceil\, q^n}{1-q^n} \\ &= \frac{1}{2} \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \frac{n^2\, q^n}{1-q^n} + \frac{1}{2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{n(n+1)\, q^n}{1-q^n} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{n^2\, q^n}{1-q^n} + \frac{1}{2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{n\, q^n}{1-q^n} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \sigma_2(n)\, q^n + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(2n-1)\, q^{2n-1}}{1-q^{2n-1}} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \sigma_2(n)\, q^n + \frac{1}{2} \sum_{n=1}^{\infty} \sigma_1(2n)\, q^n - \sum_{n=1}^{\infty} \sigma_1(n)\, q^n.\end{aligned}$$

This concludes the proof. □

In this context, we remark the following connection between $\overline{\sigma}(n)$ and J_n.

Theorem 7. *Let n be a positive integer. Then*

$$\overline{\sigma}(n) = \sum_{d|n} (-1)^{1+n/d}\, d\, J_d.$$

Proof. The cases $m = 1$ and $r = 0$ of (6) gives the following expression for the generating function of $SPL(n)$:

$$\sum_{n=1}^{\infty} SPL(n)\, q^n = \prod_{n=1}^{\infty} (1+q^n)^{J_n}.$$

The logarithmic differentiation of this generating function reads as follows:

$$\frac{\partial}{\partial q} \ln \prod_{n=1}^{\infty} (1+q^n)^{J_n} = \sum_{n=1}^{\infty} \frac{\partial}{\partial q} \ln(1+q^n)^{J_n} = \sum_{n=1}^{\infty} \frac{n\, J_n\, q^{n-1}}{1+q^n}.$$

According to (12), we obtain:

$$\begin{aligned}\sum_{n=1}^{\infty} \overline{\sigma}(n)\, q^n &= \sum_{n=1}^{\infty} \frac{n\, J_n\, q^n}{1-q^n} \\ &= \sum_{n=1}^{\infty} q^n \sum_{d|n} (-1)^{1+n/d} d\, J_d.\end{aligned}$$

This concludes the proof. □

For example, for $n = 6$, it has:

$$\overline{\sigma}(6) = 1 \cdot \lceil 1/2 \rceil + 2 \cdot \lceil 2/2 \rceil + 3 \cdot \lceil 3/2 \rceil + 6 \cdot \lceil 6/2 \rceil = 27.$$

The case $n = 6$ of Theorem 6 is given by:

$$\begin{aligned}\overline{\sigma}(6) &= \frac{(1^2+2^2+3^2+6^2) + (1+2+3+4+6+12)}{2} - (1+2+3+6) \\ &= \frac{50+28}{2} - 12 = 27.\end{aligned}$$

Taking into account Theorem 7, we can write:

$$\overline{\sigma}(6) = -1\cdot 1 + 2\cdot 2 - 3\cdot 2 + 6\cdot 5 = 27.$$

In analogy with Theorem 5, we have the following result.

Theorem 8. *Let n be a positive integer. Then*

$$sPL(n) = \frac{1}{n}\sum_{k=1}^{\infty} \overline{\sigma}^*(k)\, sPL(n-k),$$

where

$$\overline{\sigma}^*(n) = \sum_{\substack{d|n \\ d \text{ odd}}} d + \sum_{\substack{d|n \\ d \text{ even}}} d\,\lfloor d/4 \rfloor.$$

Proof. The proof is quite similar to the proof of Theorem 5 and invokes the logarithmic differentiation of (8). We omit the details. □

In analogy with Theorem 6, we have the following representation of $\overline{\sigma}^*(n)$ in terms of the sum of positive divisors' function $\sigma_z(n)$.

Theorem 9. *Let n be a positive integer. Then*

$$\overline{\sigma}^*(n) = \begin{cases} \sigma_1(n), & \text{for } n \text{ odd}, \\ \sigma_2(n/2), & \text{for } n \text{ even}. \end{cases}$$

Proof. The generating function of $\overline{\sigma}^*(n)$ can be written as:

$$\begin{aligned} \sum_{n=1}^{\infty} \overline{\sigma}^*(n)\, q^n &= \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{n\, q^n}{1-q^n} + \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \frac{n\,\lfloor n/4 \rfloor\, q^n}{1-q^n} \\ &= \sum_{n=1}^{\infty} \frac{(2n-1)\, q^{2n-1}}{1-q^{2n-1}} + 2\sum_{n=1}^{\infty} \frac{n\,\lfloor n/2 \rfloor\, q^{2n}}{1-q^{2n}}. \end{aligned} \tag{14}$$

On the other hand, we can write:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n\,\lfloor n/2 \rfloor\, q^n}{1-q^n} &= \frac{1}{2}\sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \frac{n^2\, q^n}{1-q^n} + \frac{1}{2}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{n(n-1)\, q^n}{1-q^n} \\ &= \frac{1}{2}\sum_{n=1}^{\infty} \frac{n^2\, q^n}{1-q^n} - \frac{1}{2}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{n\, q^n}{1-q^n} \\ &= \frac{1}{2}\sum_{n=1}^{\infty} \sigma_2(n)\, q^n - \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2n-1)\, q^{2n-1}}{1-q^{2n-1}}. \end{aligned} \tag{15}$$

By (14) and (15), we obtain:

$$\begin{aligned} \sum_{n=1}^{\infty} \overline{\sigma}^*(n)\, q^n &= \sum_{n=1}^{\infty} \frac{(2n-1)\, q^{2n-1}}{1-q^{2n-1}} - \sum_{n=1}^{\infty} \frac{(2n-1)\, q^{2(2n-1)}}{1-q^{2(2n-1)}} + \sum_{n=1}^{\infty} \sigma_2(n)\, q^{2n} \\ &= \sum_{n=1}^{\infty} \frac{(2n-1)\, q^{2n-1}}{1-q^{2(2n-1)}} + \sum_{n=1}^{\infty} \sigma_2(n)\, q^{2n} \\ &= \sum_{n=1}^{\infty} \sigma_1(2n-1)\, q^{2n-1} + \sum_{n=1}^{\infty} \sigma_2(n)\, q^{2n}, \end{aligned}$$

where we have invoked the following bisection of the generating function of $\sigma_1(n)$:

$$\begin{aligned}\sum_{n=1}^{\infty} \sigma_1(2n-1)\, q^{2n-1} &= \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{n\, q^n}{1-q^n} - \sum_{n=1}^{\infty} \frac{n\,(-q)^n}{1-(-q)^n}\right) \\ &= \frac{1}{2}\left(\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{n\, q^n}{1-q^n} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{n\, q^n}{1+q^n}\right) \\ &= \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{n\, q^n}{1-q^{2n}}.\end{aligned}$$

This concludes the proof. □

In analogy with Theorem 7, we have the following connection between $\overline{\sigma}^*(n)$ and J_n.

Theorem 10. *Let n be a positive integer. Then*

$$\overline{\sigma}^*(n) = \sum_{\substack{d|n \\ d \text{ odd}}} (-1)^{1+n/d}\, d + \sum_{\substack{d|n \\ d \text{ even}}} (-1)^{1+n/d} d\, J_{d/2}.$$

Proof. The proof is quite similar to the proof of Theorem 7 and invokes the following expression for the generating function of $sPL(n)$:

$$\sum_{n=0}^{\infty} sPL(n)\, q^n = \prod_{n=1}^{\infty} (1+q^n)^{\mathcal{J}_n}, \qquad |q| < 1.$$

This follows from (11) with m replaced by 1 and r replaced by 0. We omit the details. □

For example, for $n = 6$, it has:

$$\overline{\sigma}^*(6) = 1 + 2 \cdot \lfloor 2/4 \rfloor + 3 + 6 \cdot \lfloor 6/4 \rfloor = 10.$$

The case $n = 6$ of Theorem 9 is given by:

$$\overline{\sigma}^*(6) = 1^2 + 3^2 = 10.$$

Taking into account Theorem 10, we can write:

$$\overline{\sigma}^*(6) = -1 - 3 + 2 \cdot J_1 + 6 \cdot J_3 = -1 - 3 + 2 + 12 = 10.$$

5. Concluding Remarks

An n-color partition of a positive integer m is a partition in which a part of size n can come in n different colors denoted by subscripts: $n_1, n_2, \ldots, n_n$. The parts satisfy the order:

$$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < \ldots$$

We remark that n-color partitions were introduced to mathematics in 1987 by A. K. Agarwal and G. E. Andrews [7]. For example, there are thirteen n-color partitions of 4:

$$\begin{gathered}(4_4),\ (4_3),\ (4_2),\ (4_1),\ (3_3, 1_1),\ (3_2, 1_1),\ (3_1, 1_1),\ (2_2, 2_2) \\ (2_2, 2_1),\ (2_1, 2_1),\ (2_2, 1_1, 1_1),\ (2_1, 1_1, 1_1),\ (1_1, 1_1, 1_1, 1_1).\end{gathered}$$

According to [7], the plane partitions and the n-color partitions have a common generating function. This is equivalent to the following result.

Theorem 11. *The number of plane partitions of m equals the number of n-color partitions of m.*

Similarly, we introduce the following definition.

Definition 1. *Let n be a positive integer.*

1. *A J_n-color partition of the first kind of a positive integer m is a partition in which a part of size n can come in J_n different colors denoted by subscripts: $n_1, n_2, \dots, n_{s_n}$. The parts satisfy the order:*

$$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 4_1 < 4_2 < 4_3 < 4_4 < \dots$$

2. *A J_n-color partition of the second kind of a positive integer m is a partition in which a part of size $2n$ can come in J_n different colors denoted by subscripts: $n_1, n_2, \dots, n_{s_n}$. The parts satisfy the order:*

$$1_1 < 2_1 < 3_1 < 4_1 < 4_2 < 5_1 < 6_1 < 6_2 < 6_3 < \dots$$

We denote by $QJ_1(m)$ the number of J_n-color partitions of the first kind of m into distinct parts. For convenience, we define $QJ_1(0) = 1$. For example, there are seven J_n-color partitions of the first kind into distinct parts of 4:

$$(4_4),\ (4_3),\ (4_2),\ (4_1),\ (3_2, 1_1),\ (3_1, 1_1),\ (2_2, 2_1).$$

We denote by $QJ_2(m)$ the number of J_n-color partitions of the second kind of m into distinct parts. For convenience, we define $QJ_2(0) = 1$. For example, there are six J_n-color partitions of the second kind into distinct parts of 6:

$$(6_2),\ (6_1),\ (5_1, 1_1),\ (4_2, 2_1),\ (4_1, 2_1),\ (3_1, 2_1, 1_1).$$

Applying elementary techniques in the theory of partitions [3], we obtain the following generating functions:

$$\sum_{n=0}^{\infty} QJ_1(n)\, q^n = \prod_{n=1}^{\infty} (1+q^n)^{J_n}, \qquad |q| < 1,$$

and

$$\sum_{n=0}^{\infty} QJ_2(n)\, q^n = \prod_{n=1}^{\infty} (1+q^{2n-1})(1+q^{2n})^{J_n}, \qquad |q| < 1.$$

In this way, we deduce the following results.

Theorem 12. *The number of strict plane partitions of m equals the number of J_n-color partitions of the first kind into distinct parts of m.*

Theorem 13. *The number of symmetric plane partitions of m equals the number of J_n-color partitions of the second kind into distinct parts of m.*

Combinatorial proofs of Theorems 12 and 13 would be very interesting.

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