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Numerical Linear Algebra for the Two-Dimensional Bertozzi–Esedoglu–Gillette–Cahn–Hilliard Equation in Image Inpainting

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Abstract: In this paper, we present a numerical linear algebra analytical study of some schemes for the Bertozzi–Esedoglu–Gillette–Cahn–Hilliard equation. Both 1D and 2D finite difference discretizations in space are proposed with semi-implicit and implicit discretizations on time. We prove that our proposed numerical solutions converge to continuous solutions.

Keywords: Cahn–Hilliard equation; image inpainting; finite difference method; numerical linear algebra; stability; steady state

MSC: 65M06; 94A08; 65N06; 65N12

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1. Introduction

Image inpainting involves the completion of missing portions within an image or video by utilizing information from the surrounding areas. It essentially functions as a form of interpolation and finds applications in diverse domains, such as the restoration of aging paintings by museum artists [1], elimination of scratches from vintage photographs [2], manipulation of scenes in photographs [3], and the restoration of motion pictures [4].

The seminal work by Bertalmio et al. in [5] holds significant importance, presenting a novel approach to image inpainting through the incorporation of Partial Differential Equation (PDE) models. In this context, the authors introduced boundary conditions for PDE image inpainting models. These conditions involve the constant grayscale image intensity and the direction of isophote vectors at the boundary of the inpainting region. The isophote vector $\nabla^{\perp} u$ represents the orthogonal gradient vector.

In 1958, John Cahn and John Hilliard proposed the chemical energy for the Cahn–Hilliard equation to characterize phase separation phenomena, specifically phase coarsening in binary alloys. This phenomenon becomes apparent when a binary alloy is sufficiently cooled, leading to the emergence of nucleides in the material (partial or total nucleation). This process, known as spinodal decomposition, results in rapid material inhomogeneity, forming a fine-grained structure with alternating appearances of the two components. Subsequently, during a slower time scale phase called coarsening, these microstructures enlarge. The Cahn–Hilliard equation finds applications in various fields, including image processing, biology, and population dynamics. In 2007, Bertozzi et al. proposed the modified Cahn–Hilliard equation for binary image inpainting in [6], where the fidelity term is added to the Cahn–Hilliard equation. The authors in [7] demonstrated the existence of a unique local solution over time, while those in [8,9] established both the global solution in time and the existence of a finite-dimensional global attractor. The model has also been studied with singular (logarithmic) nonlinear terms, resulting in faster and more efficient image inpainting. The Cahn–Hilliard equation is not limited to binary image inpainting; various adaptations are employed for color image inpainting (see [8]) and grayscale image inpainting (see [9]). The modified Cahn–Hilliard for binary image inpainting is given by

$$u_t = \Delta\left(-\varepsilon\Delta u + \frac{1}{\varepsilon}f(u)\right) + \lambda_0\chi_{\Omega\backslash D}(x)(h-u),\ with\ \varepsilon > 0,\ \ \lambda_0 > 0, \tag{1}$$

which can be written in the following system:

$$\left\{\begin{array}{c} u_t = \Delta\mu + \lambda_0\chi_{\Omega\backslash D}(x)(h-u),\ in\ \Omega\times[0,T], \\ \mu = -\varepsilon\Delta u + \frac{1}{\varepsilon}f(u),\ in\ \Omega\times[0,T], \\ u = \Delta u = 0\ on\ \partial\Omega\times[0,T], \\ u(x,0) = u_0(x)\ in\ \Omega, \end{array}\right. \tag{2}$$

where Ω represents a bounded domain in R^n for $n \leq 3$, where $h = h(x)$ stands for a given (damaged) image, and $D \subset\subset \Omega$ denotes the inpainting region (total region). The term $\lambda_0\chi_{\Omega\backslash D}(x)(h-u)$ constitutes the fidelity term, representing the indicator function. The choice of this term, as opposed to a condition such as $u = h$ outside the inpainting domain, is motivated by several factors, especially in light of the model's analysis. Notably, there is no need for regularity assumptions on D, and h outside D does not need to be perfectly known (it could be, for example, noisy).

Additionally, nonlinear term f is defined as $f(s) = 4s^3 - 6s^2 + 2s$, for all $s \in \mathbb{R}$. It is essential to note that this nonlinear term f corresponds to the derivative of potential $F(s) = s^2(s-2)^2$. The underlying idea in this model is to solve (1) until a steady state is reached, thereby obtaining inpainted version $u(x)$ of the original damaged image $h(x)$. The equation has been examined with Neumann boundary conditions, as discussed in [8,9].

Our objective in this paper is to propose and analytically study various numerical linear algebra schemes for solving Problem (1) with Dirichlet boundary conditions. In the first part, we introduce a 1D finite difference discretization in space along with a semi-implicit discretization in time. We demonstrate the convergence of our proposed numerical solution to the continuous solution. Additionally, we present a 1D finite difference discretization in space with an implicit discretization in time and provide proof of convergence to the continuous solution once more. In the second part, we introduce 2D schemes based on finite difference discretization in space, utilizing both semi-implicit and implicit discretizations in time. Finally, we conduct a thorough analysis of the convergence of these schemes to the continuous problem in both cases.

2. 1D Discretization

2.1. 1D—Semi-Implicit Fully Discretized Scheme

Consider one-dimensional system

$$\left\{\begin{array}{c} u_t = \Delta\mu + \lambda(x)(h-u) \\ \mu = -\varepsilon\Delta u + \frac{1}{\varepsilon}f(u) \end{array}\right.,\ in\ \Omega\times[0,T], \tag{3}$$

where $u \equiv u(x,t)$, $h \equiv h(x)$, $\lambda(x) = \lambda_0\chi_{\Omega\backslash D}(x)$ for $a \leq x \leq b$, $D = [c,d]$, where $c > a$ and $d < b$, and χ denotes the indicator function such that

$$\chi_{\Omega\backslash D}(x) = \left\{\begin{array}{c} 0\ if\ x \in D, \\ 1\ if\ x \in \Omega\backslash D. \end{array}\right.$$

Using the centered difference and Euler's backward difference methods, let $\zeta = \frac{b-a}{M+1}$ where $M \in \mathbb{N}\cup\{0\}$ be the uniform step on the x-axis such that $a = x_0 < x_1 < \cdots <$

$x_M < x_{M+1} = b$, and let τ be the time step such that $t_{n+1} = t_n + \tau$ for $t \in \mathbb{N} \cup \{0\}$. Then, the semi-implicit 1D system scheme is as follows:

$$\begin{cases} \dfrac{u_i^{n+1} - u_i^n}{\tau} = \dfrac{\mu_{i+1}^{n+1} - 2\mu_i^{n+1} + \mu_{i-1}^{n+1}}{\zeta^2} + \lambda(x_i)\big(h(x_i) - u_i^n\big), \\ \mu_i^{n+1} = -\varepsilon \dfrac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\zeta^2} + \frac{1}{\varepsilon} f\big(u_i^n\big), \end{cases}$$

which is similar to

$$\begin{cases} u_i^{n+1} = u_i^n + \dfrac{\tau}{\zeta^2}\Big(\mu_{i+1}^{n+1} - 2\mu_i^{n+1} + \mu_{i-1}^{n+1}\Big) + \tau\lambda(x_i)\big(h(x_i) - u_i^n\big), \\ \mu_i^{n+1} = -\dfrac{\varepsilon}{\zeta^2}\Big(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\Big) + \frac{1}{\varepsilon} f\big(u_i^n\big). \end{cases}$$

The matrix form of the scheme is as follows:

$$\begin{cases} U^{n+1} = U^n + A\mu^{n+1} + L(U^n), \\ \mu^{n+1} = BU^{n+1} + G(U^n), \end{cases} \tag{4}$$

where A and B are symmetric matrices such that $A = \frac{\tau}{\zeta^2} diag(1, -2, 1)$, $B = -\dfrac{\varepsilon}{\tau} A$, $L(U^n) = \tau\lambda(x_i)\big(h(x_i) - u_i^n\big)$, $G(U^n) = \frac{1}{\varepsilon} f\big(u_i^n\big)$, and with boundary conditions $u_0^{n+1} = u_{M+1}^{n+1} = 0$, $\mu_0^{n+1} = \mu_{M+1}^{n+1} = 0$.

Definition 1. *Consider vector $v = (v_1,\ v_2, \ldots,\ v_M)$ in $\mathbb{R}^M$. Say that v is positive and write $v \geqslant 0$ if and only if $v_i \geqslant 0$, for all $i = 1, \ldots, m$.*

Theorem 1. *B is positive definite.*

Proof. Let $u = (u_1, u_2, \ldots, u_M)^T \in \mathbb{R}^M$, then

$$\begin{aligned} \frac{\zeta^2}{\varepsilon}\langle Bu, u\rangle = u^T Bu &= 2\sum_{i=1}^{M} u_i^2 - \sum_{i=2}^{M} u_{i-1}u_i - \sum_{i=1}^{M-1} u_i u_{i+1} \\ &= u_M^2 + \sum_{i=1}^{M-1} u_i^2 + \sum_{i=1}^{M-1} u_{i+1}^2 - 2\sum_{i=1}^{M-1} u_i u_{i+1} \\ &= u_M^2 + \sum_{i=1}^{M-1}\Big(u_i^2 - 2u_i u_{i+1} + u_{i+1}^2\Big) \\ &= u_M^2 + \sum_{i=1}^{M-1}(u_i - u_{i+1})^2 \geq 0. \end{aligned}$$

Moreover, if $\langle Bu, u\rangle = 0$, then $\sum\limits_{i=1}^{M-1}(u_i - u_{i+1})^2 + u_M^2 = 0$, which implies that $u_M = 0$ and $u_i = u_{i+1}$ for all $i = 1, 2, \ldots, M-1$, and hence $u_1 = u_2 = \cdots = u_{M-1} = u_M$. Since $u_M = 0$, infer that $u_1 = u_2 = \cdots = u_{M-1} = u_M = 0$ so that $u = 0_{\mathbb{R}^M}$. Therefore, B is a symmetric positive definite matrix. □

Corollary 1. *B and B^{-1} are symmetric positive definite matrices.*

Corollary 2. *A and A^{-1} are symmetric negative definite matrices.*

2.1.1. Existence of the Steady State

Consider steady state system

$$\begin{cases} U^* = U^* + A\mu^* + L(U^*) \\ \mu^* = BU^* + G(U^*) \end{cases}, \text{ where } B = -\frac{\varepsilon}{\tau}A. \tag{5}$$

So,

$$\begin{cases} \frac{\tau}{\varepsilon}B\mu^* = L(U^*), \\ \mu^* = BU^* + G(U^*). \end{cases}$$

But

$$B\mu^* = B(BU^* + G(U^*)),$$

which implies that

$$L(U^*) = \frac{\tau}{\varepsilon}B^2U^* + \frac{\tau}{\varepsilon}BG(U^*), \tag{6}$$

and hence

$$\frac{\tau}{\varepsilon}B^2U^* = -\frac{\tau}{\varepsilon}BG(U^*) + L(U^*),$$

which is similar to

$$BU^* = -G(U^*) + \frac{\varepsilon}{\tau}B^{-1}L(U^*).$$

Consequently, since B is invertible and positive definite, and $B = -\Delta_\zeta$, the discrete form, then the minimum eigenvalue of $-\Delta_\zeta$ is equal to $\frac{2\pi^2}{(b-2)^2}$, and $\left\|B^{-1}\right\|$ equals to the inverse of the least eigenvalue of operator $-\Delta_\zeta$, such that

$$\left\|B^{-1}\right\| = \frac{(b-a)^2}{2\pi^2}.$$

Solving (6) for U^*, obtain the following two equivalent forms:

$$U^* = \left(\lambda + \frac{1}{\lambda}B^2\right)^{-1}\left(\lambda h + \frac{1}{\varepsilon}Bf(U^*)\right),$$

or

$$U^* = -B^{-1}G(U^*) + \frac{\varepsilon}{\tau}B^{-2}L(U^*).$$

Now, define operator H as

$$H(V) = \left(\lambda + \frac{1}{\lambda}B^2\right)^{-1}\left(\lambda h + \frac{1}{\varepsilon}Bf(V)\right),$$

so that

$$U^* = H(U^*).$$

In the following, consider sequence

$$V^{k+1} = H\left(V^k\right), \text{ with } V^0 = 1.$$

Then,

$$
\begin{aligned}
V^1 &= H\left(V^0\right) \\
&= \left(\lambda + \frac{1}{\lambda}B^2\right)^{-1}\left(\lambda h + \frac{1}{\varepsilon}Bf\left(V^0\right)\right) \\
&= \left(\lambda + \frac{1}{\lambda}B^2\right)^{-1}\lambda\, h \\
&\geq 0.
\end{aligned}
$$

Suppose that $V^k \geq 0$, then

$$V^{k+1} = H\left(V^k\right) = \left(\lambda + \frac{1}{\lambda}B^2\right)^{-1}\left(\lambda h + \frac{1}{\varepsilon}Bf\left(V^k\right)\right) \geq 0, \qquad \text{if } \lambda \text{ is large enough.}$$

Hence, $\left\{V^K\right\}$ is a positive sequence for a large enough λ.

Lemma 1. *Suppose that λ is large enough; then, $\left\{V^K\right\}$ is a decreasing sequence such that $0 \leq \left\|V^K\right\|_\infty \leq 1$.*

Proof. Suppose that $V^k \leq V^{k+1}$. Then,

$$
\begin{aligned}
H\left(V^{k+1}\right) - H\left(V^k\right) &= \left(\lambda + \frac{1}{\varepsilon}B^2\right)^{-1}\left(\lambda h - \frac{1}{\varepsilon}Bf\left(V^{k+1}\right)\right) - \left(\lambda + \frac{1}{\varepsilon}B^2\right)^{-1}\left(\lambda h - \frac{1}{\varepsilon}Bf\left(V^k\right)\right) \\
&= -\frac{1}{\varepsilon}B\left(\lambda + \frac{1}{\varepsilon}B^2\right)^{-1}\left[f\left(V^{k+1}\right) - f\left(V^k\right)\right] \\
&= -\frac{1}{\varepsilon}B\left(\lambda + \frac{1}{\varepsilon}B^2\right)^{-1}\left(V^{k+1} - V^k\right)\left[4\left(V^{k+1} - \frac{3}{4}\right)^2 + 4\left(V^k - \frac{3}{4}\right)^2\right. \\
&\qquad \left. + 4V^{k+1}V^k + \frac{14}{16}\right] \\
&\leq 0.
\end{aligned}
$$

Hence, $\left\{V^K\right\}$ is a decreasing sequence for all $k \in N$.
However, $V^0 = 1$, then $V^k \leqslant 1$. It then follows that

$$0 \leq \left\|V^K\right\|_\infty \leq 1.$$

□

Corollary 3. *Under assumption (H) that $\lambda_0 \geq \frac{4\pi^4\delta}{\varepsilon(1-\delta)(b-a)^4}$ for some $\delta \in (0,1)$, system*

$$BU^* = -G(U^*) + \frac{\varepsilon}{\tau}B^{-1}L(U^*)$$

admits solution U^* such that $0 \leq U^* < 1$.

Proof. It is clear that

$$
\begin{aligned}
\|H(\theta)\|_\infty &= \left\|\frac{\varepsilon}{\tau}B^{-2}L(\theta) - \frac{1}{\varepsilon}B^{-1}G(\theta)\right\|_\infty \\
&\geq \frac{\varepsilon}{\tau}\left\|B^{-2}L(\theta)\right\|_\infty - \frac{1}{\varepsilon}\left\|B^{-1}G(\theta)\right\|_\infty.
\end{aligned}
$$

But $0 \le \theta_i \le 1$, for all $1 \le i \le M$, which implies that $-6 \le 4\theta_i^3 - 6\theta_i^2 + 2\theta_i \le 6$. Therefore,

$$\|G(\theta)\|_\infty = \frac{1}{\varepsilon}\|f(\theta)\|_\infty = \frac{1}{\varepsilon}\left(\max_{1\le i\le M}\left|4\theta_i^3 - 6\theta_i^2 + 2\theta_i\right|\right) \le \frac{6}{\varepsilon}.$$

In addition, $0 \le \theta_i \le 1$ implies that $0 \le \|\theta - h\|_\infty \le 1-\delta$, where $\delta \in (0,1)$. Thus,

$$0 \le U^* < 1.$$

In addition, there is

$$\|L(\theta)\|_\infty = \left\|\tau\lambda_0\chi_{\Omega\setminus D}(x)(\theta - h)\right\|_\infty \le \tau\lambda_0\chi_{\Omega\setminus D}(1-\delta).$$

It then follows from assumption (H) that

$$\begin{aligned}\|H(\theta)\|_\infty &\ge \frac{\varepsilon}{\tau}\left\|B^{-1}\right\|_\infty^2\|L(\theta)\|_\infty - \frac{1}{\varepsilon}\left\|B^{-1}\right\|_\infty\|G(\theta)\|_\infty \\ &\ge \frac{\varepsilon}{\tau}\frac{(b-a)^4}{4\pi^4}\tau\lambda_0\chi_{\Omega\setminus D}(1-\delta) - \frac{6}{\varepsilon^2}\frac{(b-a)^2}{2\pi^2} \\ &\ge \varepsilon\lambda_0\chi_{\Omega\setminus D}\frac{(b-a)^4}{4\pi^4}(1-\delta) - \frac{3}{\varepsilon^2}\frac{(b-a)^2}{\pi^2} \ge \delta.\end{aligned}$$

□

2.1.2. Convergence of the Solution

Suppose that $\left\|B^{-1}\right\| = \frac{(b-a)^2}{2\pi^2}$, and let U^* and μ^* be the steady states of system

$$\begin{cases} U^{n+1} = U^n + A\mu^{n+1} + L(U^n) \\ \mu^{n+1} = BU^{n+1} + G(U^n) \end{cases}, \text{ where } A = -\frac{\tau}{\varepsilon}B. \tag{7}$$

Then,

$$\begin{cases} 0 = -\frac{\tau}{\varepsilon}B\mu^* + L(U^*), \\ \mu^* = BU^* + G(U^*). \end{cases} \tag{8}$$

Subtracting (7) from (8), obtain the following system:

$$\begin{cases} U^{n+1} = U^n - \frac{\tau}{\varepsilon}B(\mu^{n+1} - \mu^*) + (L(U^n) - L(U^*)), \\ \mu^{n+1} - \mu^* = B(U^{n+1} - U^*) + (G(U^n) - G(U^*)). \end{cases}$$

This implies that

$$\begin{aligned}U^{n+1} &= U^n - \frac{\tau}{\varepsilon}B\left[B\left(U^{n+1} - U^*\right) + (G(U^n) - G(U^*))\right] + (L(U^n) - L(U^*)) \\ &= U^n - \frac{\tau}{\varepsilon}B^2\left(U^{n+1} - U^*\right) - \frac{\tau}{\varepsilon}B(G(U^n) - G(U^*)) + (L(U^n) - L(U^*)).\end{aligned}$$

Hence,

$$\begin{aligned}\left(U^{n+1} - U^*\right) - (U^n - U^*) = &-\frac{\tau}{\varepsilon}B^2\left(U^{n+1} - U^*\right) - \frac{\tau}{\varepsilon}B(G(U^n) - G(U^*)) \\ &+ (L(U^n) - L(U^*)).\end{aligned}$$

Let $V^{n+1} = U^{n+1} - U^*$ and $V^n = U^n - U^*$. Then,

$$V^{n+1} - V^n = -\frac{\tau}{\varepsilon}B^2V^{n+1} - \frac{\tau}{\varepsilon}B(G(U^n) - G(U^*)) + (L(U^n) - L(U^*)),$$

which is equivalent to

$$\begin{aligned}\left(B^{-2}+\frac{\tau}{\varepsilon}I\right)V^{n+1} &= B^{-2}V^{n}-\frac{\tau}{\varepsilon}B^{-1}(G(U^{n})-G(U^{*}))+B^{-2}(L(U^{n})-L(U^{*}))\\ &= B^{-2}V^{n}+\tau\left[\frac{-1}{\varepsilon}B^{-1}(G(U^{n})-G(U^{*}))\right]+\frac{\varepsilon}{\tau}B^{-2}(L(U^{n})-L(U^{*}))\\ &= B^{-2}V^{n}+\tau[H(U^{n})-H(U^{*})].\end{aligned}$$

Now, since B^{-2} is symmetric and positive definite, then $B^{-2}+\frac{\tau}{\varepsilon}I$ is also symmetric positive definite and invertible such that $\left(B^{-2}+\frac{\tau}{\varepsilon}I\right)^{-1}\geq 0$.

Hence,

$$V^{n+1}=\left(B^{-2}+\frac{\tau}{\varepsilon}I\right)^{-1}\left[B^{-2}V^{n}+\tau(H(U^{n})-H(U^{*}))\right].$$

Moreover, since $H(U)$ is an increasing function and $0\leq U^{n}\leq U^{*}<1$, then

$$0\leq H(U^{n})\leq H(U^{*})<1.$$

Therefore,

$$0\leq V^{n+1}<1 \ \text{ and } \ 0\leq U^{n+1}<U^{*}.$$

Moreover, since $\left(B^{-2}+\frac{\tau}{\varepsilon}I\right)^{-1}$ is positive definite and $0\leq U^{n}<1$, then $0\leq U^{n+1}<1$. This implies that U^{n} is well defined and $0\leq U^{n}\leq U^{*}<1$ for all $n\in\mathbb{N}\cup\{0\}$.

Theorem 2. $\{V^{n+1}\}$ *converges to* 0 *if*

$$\lambda_0>\frac{4\pi^2}{\varepsilon^2(1-\delta)}\left(\frac{\varepsilon\pi^2-3(b-a)^2}{(b-a)^4}\right).$$

Proof. Consider sequence $\{V^{n+1}\}$, matrix $K=I+\dfrac{\tau}{\varepsilon}B^2$, Euclidean inner product $\langle.\rangle$, and associate norm $\|.\|$. Then,

$$\begin{aligned}\left\langle KV^{n+1},V^{n+1}\right\rangle &= \left\langle V^{n}-\frac{\tau}{\varepsilon}B(G(U^{n})-G(U^{*}))+(L(U^{n})-L(U^{*})),V^{n+1}\right\rangle\\ &\leq \left\|V^{n}-\frac{\tau}{\varepsilon}B(G(U^{n})-G(U^{*}))+(L(U^{n})-L(U^{*}))\right\|_{\infty}\left\|V^{n+1}\right\|_{\infty}.\end{aligned}$$

Hence,

$$\begin{aligned}\left(1+\frac{\tau}{\varepsilon}\left(\frac{2\pi^2}{(b-a)^2}\right)^2\right)\left\|V^{n+1}\right\|_{\infty} &\leq \|\ V^{n}-\tfrac{\tau}{\varepsilon}B(G(U^{n})-G(U^{*}))+(L(U^{n})-L(U^{*}))\ \|_{\infty}\\ &\leq \|V^{n}\|_{\infty}+\frac{\tau}{\varepsilon}\|B\|_{\infty}\|G(U^{n})-G(U^{*})\|_{\infty}+\|L(U^{n})-L(U^{*})\|_{\infty}\\ &\leq \|V^{n}\|_{\infty}+\frac{\tau}{\varepsilon}\frac{2\pi^2}{(b-a)^2}\|G(V^{n})\|_{\infty}+\|L(V^{n})\|_{\infty}\\ &\leq \|V^{n}\|_{\infty}+\frac{\tau}{\varepsilon}\frac{2\pi^2}{(b-a)^2}\|G\|_{\infty}\|V^{n}\|_{\infty}+\|L\|_{\infty}\|V^{n}\|_{\infty}\\ &\leq \|V^{n}\|_{\infty}\left(1+\frac{\tau}{\varepsilon^2}\frac{12\pi^2}{(b-a)^2}+\tau\lambda_0(1-\delta)\right).\end{aligned}$$

This implies that

$$\left\|V^{n+1}\right\|_{\infty}\leq\left(1+\frac{\tau}{\varepsilon}\frac{4\pi^4}{(b-a)^4}\right)^{-1}\left(1+\frac{\tau}{\varepsilon^2}\frac{12\pi^2}{(b-a)^2}+\tau\lambda_0(1-\delta)\right)\|V^{n}\|_{\infty}.$$

Therefore, $\{V^{n+1}\}$ converges to 0 if

$$\left(1+\frac{\tau}{\varepsilon}\frac{4\pi^4}{(b-a)^4}\right)^{-1}\left(1+\frac{\tau}{\varepsilon^2}\frac{12\pi^2}{(b-a)^2}+\tau\lambda_0\chi_{\Omega\setminus D}(1-\delta)\right)<1,$$

which is equivalent to

$$1+\frac{\tau}{\varepsilon^2}\frac{12\pi^2}{(b-a)^2}+\tau\lambda_0\chi_{\Omega\setminus D}(1-\delta)>1+\frac{\tau}{\varepsilon}\frac{4\pi^4}{(b-a)^4},\ for\ \ \chi_{\Omega\setminus D}<1.$$

Therefore,

$$\lambda_0>\frac{4\pi^2}{\varepsilon^2(1-\delta)}\left(\frac{\varepsilon\pi^2-3(b-a)^2}{(b-a)^4}\right).$$

□

2.2. 1D—Implicit Fully Discretized Scheme

The fully discretized implicit 1D system scheme, (3), can be written as follows:

$$\begin{cases} \dfrac{u_i^{n+1}-u_i^n}{\tau}=\dfrac{\mu_{i+1}^{n+1}-2\mu_i^{n+1}+\mu_{i-1}^{n+1}}{\zeta^2}+\lambda(x_i)\left(h(x_i)-u_i^{n+1}\right), \\ \mu_i^{n+1}=-\varepsilon\dfrac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{\zeta^2}+\frac{1}{\varepsilon}f\left(u_i^{n+1}\right), \end{cases}$$

which is similar to

$$\begin{cases} u_i^{n+1}=u_i^n+\dfrac{\tau}{\zeta^2}\left(\mu_{i+1}^{n+1}-2\mu_i^{n+1}+\mu_{i-1}^{n+1}\right)+\tau\lambda(x_i)\left(h(x_i)-u_i^{n+1}\right), \\ \mu_i^{n+1}=-\dfrac{\varepsilon}{\zeta^2}\left(u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}\right)+\frac{1}{\varepsilon}f\left(u_i^{n+1}\right). \end{cases}$$

The matrix form of the scheme is as follows:

$$\begin{cases} U^{n+1}=U^n+A\mu^{n+1}+L\left(U^{n+1}\right), \\ \mu^{n+1}=BU^{n+1}+G\left(U^{n+1}\right), \end{cases}$$

where B is an invertible and positive definite matrix, and $A=-\frac{\tau}{\varepsilon}B$ is a negative definite matrix. Moreover, $B^{-1}\geq 0$ and $A^{-1}\leq 0$. The system is equivalent to

$$\begin{cases} U^{n+1}-U^n+\frac{\tau}{\varepsilon}B\mu^{n+1}-L\left(U^{n+1}\right)=0, \\ \mu^{n+1}-BU^{n+1}-G\left(U^{n+1}\right)=0, \end{cases}$$

and hence

$$\begin{cases} Q\left(U^{n+1},\mu^{n+1}\right)=0, \\ H\left(U^{n+1},\mu^{n+1}\right)=0, \end{cases}$$

where

$$\begin{cases} Q\left(U^{n+1},\mu^{n+1}\right)=U^{n+1}-U^n+\frac{\tau}{\varepsilon}B\mu^{n+1}-L\left(U^{n+1}\right), \\ H\left(U^{n+1},\mu^{n+1}\right)=\mu^{n+1}-BU^{n+1}-G\left(U^{n+1}\right), \end{cases}$$

with $u_0^{n+1}=u_1^{n+1}=0$, $\mu_0^{n+1}=\mu_1^{n+1}=0$, $L\left(U^{n+1}\right)=\tau\lambda(x_i)\left(h(x_i)-u_i^{n+1}\right)$, and $G\left(U^{n+1}\right)=\frac{1}{\varepsilon}f\left(u_i^{n+1}\right)$.

Existence of Roots of Q and H Such That $Q(U^*,\mu^*)=0$ and $H(U^*,\mu^*)=0$

Let U^* and μ^* be the roots of Q and H such that

$$\begin{cases} Q(U^*,\mu^*)=0, \\ H(U^*,\mu^*)=0, \end{cases}$$

which is equivalent to

$$\begin{cases} U^*=U^*-\frac{\tau}{\varepsilon}B\mu^*+L(U^*), \\ \mu^*=BU^*+G(U^*), \end{cases}$$

and implies that

$$\frac{\tau}{\varepsilon}\Big(B^2U^*+BG(U^*)\Big)-L(U^*)=0,$$

where B and B^2 are positive definite matrices. Hence,

$$U^*=-B^{-1}G(U^*)+\frac{\varepsilon}{\tau}B^{-2}L(U^*)\,. \tag{9}$$

Theorem 3. *Assume that $\lambda_0>\frac{4\pi^2\delta}{\varepsilon(1-\delta)(b-a)^4}$ for some $\delta\in(0,1)$, then Equation (9) admits solution U^* such that $0\le U^*<1$.*

Proof. Conducted in Corollary 3. □

Theorem 4. *Assume that*

$$\lambda_0\ge 2\sqrt{\frac{2}{3}}\left(\frac{\pi^3\delta}{(1-\delta)(b-a)^3}\right)$$

for some $\delta\in(0,1)$, then $\mu^*=BU^*+G(U^*)$ admits solution μ^*.

Proof. It is clear that

$$\begin{aligned}
\|\mu^*\|_\infty &= \|BU^*-G(U^*)\|_\infty \\
&\ge \|BU^*\|_\infty-\|G(\theta_i)\|_\infty,\ for\ \ 0\le\theta_i\le 1 \\
&\ge \|B\|_\infty\|U^*\|_\infty-\frac{1}{\varepsilon}\|f(\theta_i)\|_\infty,\ with\ \ 0\le\|U^*\|_\infty\le 1 \\
&\ge \|B\|_\infty-\frac{1}{\varepsilon}\|f(\theta_i)\|_\infty \\
&\ge \frac{2\pi^2}{(b-a)^2}-\frac{1}{\varepsilon}\max_{1\le i\le N}\Big|4\theta_i^3-6\theta_i^2+2\theta_i\Big| \\
&\ge \frac{2\pi^2}{(b-a)^2}-\frac{6}{\varepsilon^2}.
\end{aligned}$$

But

$$\begin{aligned}
\|\mu^*\|_\infty &= \left\|-\frac{\varepsilon}{\zeta^2}(u^*_{i+1}-2u^*_i+u^*_{i-1})+\frac{1}{\varepsilon}f(u^*_i)\right\|_\infty \\
&\le \frac{\varepsilon}{\zeta^2}\|u^*_{i+1}-2u^*_i+u^*_{i-1}\|_\infty+\frac{1}{\varepsilon}\|f(u^*_i)\|_\infty \\
&\le \frac{6}{\varepsilon^2}.
\end{aligned}$$

This implies that

$$\frac{6}{\varepsilon^2}\ge\frac{2\pi^2}{(b-a)^2}-\frac{6}{\varepsilon^2},$$

and hence

$$\frac{12}{\varepsilon^2} \geq \frac{2\pi^2}{(b-a)^2}.$$

But

$$\frac{1}{\varepsilon} \leq \frac{\lambda_0(1-\delta)(b-a)^4}{4\pi^2\delta},$$

which implies that

$$\frac{12\lambda_0^2(1-\delta)^2(b-a)^4}{16\pi^4\delta^2} \geq \frac{2\pi^2}{(b-a)^2}.$$

Therefore,

$$\lambda_0^2 \geq \frac{8\pi^6\delta^2}{3(1-\delta)^2(b-a)^6}\ .$$

Now, in order to find U^{n+1} and μ^{n+1} such that

$$\begin{cases} Q\big(U^{n+1},\mu^{n+1}\big) = U^{n+1} - U^n - \dfrac{\tau}{\varepsilon}B\mu^{n+1} + \tau\lambda(x_i)\Big(h(x_i) - u_i^{n+1}\Big), \\ H\big(U^{n+1},\mu^{n+1}\big) = \mu^{n+1} - BU^{n+1} - \frac{1}{\varepsilon}f\Big(u_i^{n+1}\Big), \end{cases}$$

which is equivalent to

$$\begin{cases} Q\Big(u_i^{n+1},\mu_i^{n+1}\Big) = u_i^{n+1} - u_i^n - \dfrac{\tau}{\zeta^2}\Big(\mu_{i+1}^{n+1} - 2\mu_i^{n+1} + \mu_{i-1}^{n+1}\Big) + \tau\lambda(x_i)\Big(h(x_i) - u_i^{n+1}\Big) = 0, \\ H\Big(u_i^{n+1},\mu_i^{n+1}\Big) = \mu_i^{n+1} + \dfrac{\varepsilon}{\zeta^2}\Big(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\Big) - \frac{1}{\varepsilon}f\Big(u_i^{n+1}\Big) = 0, \end{cases}$$

let

$$\begin{aligned} F\Big(u_i^{n+1},\mu_i^{n+1}\Big) &= Q\Big(u_i^{n+1},\mu_i^{n+1}\Big) + H\Big(u_i^{n+1},\mu_i^{n+1}\Big) \\ &= u_i^{n+1} - u_i^n - \frac{\tau}{\zeta^2}\Big(\mu_{i+1}^{n+1} - 2\mu_i^{n+1} + \mu_{i-1}^{n+1}\Big) + \tau\lambda(x_i)\Big(h(x_i) - u_i^{n+1}\Big) \\ &+ \mu_i^{n+1} + \frac{\varepsilon}{\zeta^2}\Big(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\Big) - \frac{1}{\varepsilon}f\Big(u_i^{n+1}\Big). \end{aligned}$$

Let $U^{n+1} - U^n = \psi$, and $\mu^{n+1} - \mu^n = \varphi$ for all $n \in \mathbb{N} \cup \{0\}$. Then, using Newton's method, it is obtained that the sequence (U^{n+1},μ^{n+1}) for all $n \in \mathbb{N}$ converges to $(U^\star,\mu^\star)$such that $(U^\star,\mu^\star)$ is the root of F, so that $F(U^\star,\mu^\star) = 0$. $\square$

Moreover, by Newton's Method, there is

$$\Big(U^{n+1},\mu^{n+1}\Big) = -DF^{-1}(U^n,\mu^n)F(U^n,\mu^n) + (U^n,\mu^n).$$

Hence, in order to find U^{n+1} and μ^{n+1}, the Jacobian block matrix of $F(U^n,\mu^n)$ has to be computed at each iteration of $n \in \mathbb{N}$. This implies that

$$\begin{aligned} F\Big(u_i^{n+1},\mu_i^{n+1}\Big) &= Q\Big(U^{n+1},\mu^{n+1}\Big) + H\Big(U^{n+1},\mu^{n+1}\Big) \\ &= u_i^{n+1} - u_i^n - \frac{\tau}{\zeta^2}\Big(\mu_{i+1}^{n+1} - 2\mu_i^{n+1} + \mu_{i-1}^{n+1}\Big) + \tau\lambda(x_i)\Big(h(x_i) - u_i^{n+1}\Big) \\ &+ \mu_i^{n+1} + \frac{\varepsilon}{\zeta^2}\Big(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\Big) - \frac{1}{\varepsilon}\bigg(4\Big(u_i^{n+1}\Big)^3 - 6\Big(u_i^{n+1}\Big)^2 + 2\Big(u_i^{n+1}\Big)\bigg). \end{aligned}$$

The Jacobian of F is

$$\frac{\partial F_i}{\partial u_j} = \begin{cases} 1 - \tau\lambda(x_i) - 2\frac{\varepsilon}{\zeta^2} - \frac{1}{\varepsilon}\left(12\left(u_i^{n+1}\right)^2 - 12u_i^{n+1} + 2\right), & \text{if } j = i, \\ \frac{\varepsilon}{\zeta^2}, & \text{if } j = i \pm 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\frac{\partial F_i}{\partial \mu_j} = \begin{cases} 1 + 2\frac{\tau}{\zeta^2}, & \text{if } j = i, \\ -\frac{\tau}{\zeta^2}, & \text{if } j = i \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

Now, let $0 \leq U^*, \ \mu^* \leq 1$ be the roots of F such that $F(U^*, \mu^*) = 0$. This implies that

$$\begin{aligned} F_i(U^*, \mu^*) = \quad & -\frac{\tau}{\zeta^2}\left(\mu_{i+1}^* - 2\mu_i^* + \mu_{i-1}^*\right) \quad + \tau\lambda(x_i)\left(h(x_i) - u_i^*\right) + \mu_i^* \\ & + \frac{\varepsilon}{\zeta^2}\left(u_{i+1}^* - 2u_i^* + u_{i-1}^*\right) - \frac{1}{\varepsilon}f\left(u_i^*\right) = 0, \end{aligned}$$

and

$$DF_i(U^*, \mu^*) = \frac{\partial F_i}{\partial u_j}\partial u + \frac{\partial F_i}{\partial \mu_j}\partial \mu = \left(1 - \tau\lambda(x_i) - 2\frac{\varepsilon}{\zeta^2} + 2\frac{\tau}{\zeta^2}\right)I - J,$$

where I is the identity matrix with $\|I\|_\infty = 1$, and

$$J = diag\left(\frac{\tau}{\zeta^2} - \frac{\varepsilon}{\zeta^2}, \frac{1}{\varepsilon}\left(12(u^*)^2 - 12U^* + 2\right), \frac{\tau}{\zeta^2} - \frac{\varepsilon}{\zeta^2}\right),$$

with

$$\begin{aligned} \|J\|_\infty &= \max\left\{2\frac{\tau}{\zeta^2} - 2\frac{\varepsilon}{\zeta^2}, \frac{1}{\varepsilon}\left(12(u^*)^2 - 12u^* + 2\right)\right\} \\ &= 2\frac{\tau}{\zeta^2} - 2\frac{\varepsilon}{\zeta^2} + \frac{1}{\varepsilon}, \end{aligned}$$

since $\max\limits_{0 \leq U^* \leq 1}\left\{12(u^*)^2 - 12u^* + 2\right\} = 1.$

Theorem 5. *$DF(U^*, \mu^*)$ is invertible if and only if $\lambda_0 > \dfrac{1 - 2\varepsilon}{\varepsilon\tau}$.*

Proof.

$$\begin{aligned} DF(U^*, \mu^*) \text{ is invertible} \quad iff \quad & \left\|2 - \tau\lambda_0\chi_{\Omega\setminus D} - 2\frac{\varepsilon}{\zeta^2} + 2\frac{\tau}{\zeta^2}\right\|_\infty < \|J\|_\infty \\ iff \quad & 2 - \tau\lambda_0\chi_{\Omega\setminus D} - 2\frac{\varepsilon}{\zeta^2} + 2\frac{\tau}{\zeta^2} < 2\tfrac{\tau}{\zeta^2} - 2\tfrac{\varepsilon}{\zeta^2} + \frac{1}{\varepsilon} \\ iff \quad & \tau\lambda_0\chi_{\Omega\setminus D} > \frac{1}{\varepsilon} - 2, \ \text{where } \chi_{\Omega\setminus D} \leq 1 \\ iff \quad & \lambda_0 > \frac{1 - 2\varepsilon}{\varepsilon\tau}. \end{aligned}$$

Moreover,

$$\left\|DF(U^*,\mu^*)^{-1}\right\|_\infty = \frac{1}{1-\|J\|_\infty^{-1}\left\|2-\tau\lambda_0\chi_{\Omega\setminus D}-2\frac{\varepsilon}{\zeta^2}+2\frac{\tau}{\zeta^2}\right\|_\infty}$$
$$= \frac{1}{1-\left(\frac{2(\tau-\varepsilon)}{\zeta^2}+\frac{1}{\varepsilon}\right)^{-1}\left\|2-\tau\lambda_0\chi_{\Omega\setminus D}-2\frac{\varepsilon}{\zeta^2}+2\frac{\tau}{\zeta^2}\right\|_\infty}.$$

□

Corollary 4. *If $F \in C^2(a,b)$ for all $0 \le U^* \le r_1$, then $DF(U,\mu)$ is invertible and $\left\|DF(U,\mu)^{-1}\right\|_\infty \le \frac{k}{k-1}\left\|DF(U^*,\mu^*)^{-1}\right\| = r_1$, where $k > 1$.*

Proof.

$$\begin{aligned} DF(U,\mu) &= DF(U,\mu) - DF(U^*,\mu^*) + DF(U^*,\mu^*) \\ &= DF(U^*,\mu^*)\left[I + DF(U^*,\mu^*)^{-1}(DF(U,\mu) - DF(U^*,\mu^*))\right]. \end{aligned}$$

Since $F \in C^2(a,b)$, then $DF \in C^1(a,b)$; in particular, DF is continuous at (U^*,μ^*), and by the definition of continuity of DF at (U^*,μ^*), there exist $v > 0$, and $r_1 > 0$ such that $v = \frac{1}{k\left\|DF(U^*,\mu^*)^{-1}\right\|_\infty}$; and for any $0 \le U^* \le r_1$, there is

$$\|DF(U,\mu) - DF(U^*,\mu^*)\|_\infty < v,$$

and

$$\begin{aligned} &\left\|DF(U^*,\mu^*)^{-1}(DF(U,\mu) - DF(U^*,\mu^*))\right\|_\infty \\ &\le \left\|DF(U^*,\mu^*)^{-1}\right\|_\infty \|(DF(U,\mu) - DF(U^*,\mu^*))\|_\infty \\ &\le \frac{1}{k} < 1, \end{aligned}$$

hence, $I + DF(U^*,\mu^*)^{-1}(DF(U,\mu) - DF(U^*,\mu^*))$ is invertible (Von-Neumann Lemma). Then, for every $U^* \le U \le r_1$, there is

$$\begin{aligned} \left\|DF(U,\mu)^{-1}\right\|_\infty &= \left\|DF(U^*,\mu^*)\left[I + DF(U^*,\mu^*)^{-1}(DF(U,\mu) - DF(U^*,\mu^*))\right]^{-1}\right\|_\infty \\ &= \left\|I + DF(U^*,\mu^*)^{-1}(DF(U,\mu) - DF(U^*,\mu^*))\right\|_\infty^{-1} \|F(U^*,\mu^*)\|_\infty^{-1} \\ &= \frac{1}{\left\|I + DF(U^*,\mu^*)^{-1}(DF(U,\mu) - DF(U^*,\mu^*))\right\|_\infty} \|F(U^*,\mu^*)\|_\infty^{-1} \\ &\le \frac{1}{1 - \|DF(U^*,\mu^*)\|_\infty \|DF(U,\mu) - DF(U^*,\mu^*)\|_\infty} \|F(U^*,\mu^*)\|_\infty^{-1} \\ &\le \frac{1}{1-\frac{1}{K}} \|F(U^*,\mu^*)\|_\infty^{-1} \\ &\le \frac{k}{k-1}\left\|DF(U^*,\mu^*)^{-1}\right\|_\infty. \end{aligned}$$

□

Lemma 2. *If $F \in C^2(a,b)$ and $U^* \leq U^n \leq r_1$, then*

$$\left\| DF(U^n,\mu^*)\left(U^{n+1}-U^*\right)\right\|_\infty \leq r_2|U^n-U^*|,$$

where $r_2 = \frac{1}{2}\sup\limits_{U^*\leq\theta\leq r_1}\left\|D^2F(\theta,\mu^*)\right\|_\infty$.

Proof. Let

$$\phi(t) = F(U^n + t(U^*-U^n),\mu^*) - F(U^n,\mu^*) - t\,DF(U^n,\mu^*)(U^*-U^n),$$

so that

$$\phi'(t) = DF(U^n + t(U^*-U^n),\mu^*)(U^*-U^n) - DF(U^n,\mu^*)(U^*-U^n).$$

By the fundamental theorem of calculus, there is $\phi(1)-\phi(0) = \int_0^1 \phi'(t)dt$. Hence,

$$\begin{aligned}
&F(U^*,\mu^*) - F(U^n,\mu^*) - DF(U^n,\mu^*)(U^*-U^n)\\
&= \int_0^1 [DF(U^n + t(U^*-U^n),\mu^*)(U^*-U^n) - DF(U^n,\mu^*)(U^*-U^n)]\,dt.
\end{aligned}$$

But $F(U^*,\mu^*) = 0$, so if norms of the previous equality are taken, there appears

$$\begin{aligned}
&\|-F(U^n,\mu^*) - DF(U^n,\mu^*)(U^*-U^n)\|_\infty\\
&\leq \int_0^1 \|DF(U^n + t(U^*-U^n),\mu^*)(U^*-U^n) - DF(U^n,\mu^*)(U^*-U^n)\|_\infty\,dt\\
&\leq \int_0^1 \|DF(U^n + t(U^*-U^n),\mu^*) - DF(U^n,\mu^*)\|_\infty\|(U^*-U^n)\|_\infty\,dt,
\end{aligned}$$

since $DF \in C^1(a,b)$; then, by the mean value theorem, there is

$$\|DF(U^n + t(U^*-U^n),\mu^*) - DF(U^n,\mu^*)\|_\infty \leq \|t\,(U^*-U^n)\|_\infty . \sup_{U^*\leq\theta\leq r_1}\left\|D^2F(\theta,\mu^*)\right\|_\infty.$$

Hence,

$$\begin{aligned}
\|-F(U^n,\mu^*) - DF(U^n,\mu^*)(U^*-U^n)\|_\infty &\leq \|(U^*-U^n)\|_\infty^2 \sup_{U^*\leq\theta\leq r_1}\left\|D^2F(\theta,\mu^*)\right\|_\infty \int_0^1 t\,dt\\
&\leq \frac{1}{2}\|(U^*-U^n)\|_\infty^2 \sup_{U^*\leq\theta\leq r_1}\left\|D^2F(\theta,\mu^*)\right\|_\infty.
\end{aligned}$$

If $r_2 = \frac{1}{2}\sup\limits_{U^*\leq\theta\leq r_1}\left\|D^2F(\theta,\mu^*)\right\|_\infty$ is taken, then

$$\|-F(U^n,\mu^*) - DF(U^n,\mu^*)(U^*-U^n)\|_\infty \leq r_2\|(U^*-U^n)\|_\infty^2.$$

By Newton's Raphson method, there is

$$\left(U^{n+1},\mu^*\right) = -DF(U^n,\mu^*)^{-1}F(U^n,\mu^*) + (U^n,\mu^*),$$

which implies that

$$U^{n+1}-U^n = -DF(U^n,\mu^*)^{-1}F(U^n,\mu^*),$$

and

$$DF(U^n,\mu^*)\left(U^{n+1}-U^n\right) = -F(U^n,\mu^*).$$

Hence,

$$\left\|DF(U^n,\mu^*)\left(U^{n+1}-U^n\right)-DF(U^n,\mu^*)(U^*-U^n)\right\|_\infty \le r_2\|U^*-U^n\|_\infty^2,$$

which is equivalent to

$$\left\|DF(U^n,\mu^*)\left(U^{n+1}-U^*\right)\right\|_\infty \le r_2\|U^*-U^n\|_\infty^2.$$

□

Theorem 6. *Let $U^* \le U^0 \le r_3 \le r_1$. Then, $DF\left(U^{n+1},\mu^*\right)$ is invertible and $\left\{\left(U^{n+1},\mu^*\right)\right\}$ converges to (U^*,μ^*).*

Proof. If $U^* \le U^0 \le r_3 \le r_1$, then $DF\left(U^0,\mu^*\right)$ is invertible and $\left\|DF\left(U^0,\mu^*\right)^{-1}\right\|_\infty \le r_1$. Thus, $\left(U^1,\mu^*\right)$ is well defined. By Newton's method, obtain

$$\left(U^1,\mu^*\right)=\left(U^0,\mu^*\right)-DF\left(U^0,\mu^*\right)^{-1}F\left(U^0,\mu^*\right),$$

which implies that

$$\left(U^1-U^0,\mu^*\right)DF\left(U^0,\mu^*\right)=-F\left(U^0,\mu^*\right).$$

Now, suppose that the previous equality holds for every $n\in\mathbb{N}$. So, $DF(U^n,\mu^*)$ is invertible and $\left\|DF(U^n,\mu^*)^{-1}\right\|_\infty \le r_1$, where $U^*\le U^n \le r_3$. This implies that $\left(U^{n+1},\mu^*\right)$ is well defined and

$$DF(U^n,\mu^*)\left(U^{n+1}-U^n\right)=-F(U^n,\mu^*).$$

But

$$\begin{aligned}
\left\|U^{n+1}-U^*\right\|_\infty &= \left\|DF(U^n,\mu^*)^{-1}DF(U^n,\mu^*)\left(U^{n+1}-U^*\right)\right\|_\infty\\
&\le \left\|DF(U^n,\mu^*)^{-1}\right\|_\infty\left\|DF(U^n,\mu^*)\left(U^{n+1}-U^*\right)\right\|_\infty\\
&\le r_1r_2\|U^*-U^n\|_\infty^2\\
&\le r_1r_2r_3^2,\ since (U^n,\mu^*)\in B((U^*,\mu^*),r_3)\\
&\le (r_1r_2r_3)r_3\\
&\le r_3.
\end{aligned}$$

Therefore, $U^*\le U^n\le r_3\le r_1$, and $DF\left(U^{n+1},\mu^*\right)$ is invertible such that

$$\left\|DF\left(U^{n+1},\mu^*\right)^{-1}\right\|_\infty \le r_1.$$

Moreover, $U^*\le U^n \le r_3\le r_1$ and $DF(U^n,\mu^*)$ is invertible for every $n\in\mathbb{N}$. So, sequence $\{(U^n,\mu^*)\}_{n\in\mathbb{N}}$ is well defined and

$$\left\|U^{n+1}-U^*\right\|_\infty \le r_1r_2\|U^n-U^*\|_\infty^2 \le \beta\|U^n-U^*\|_\infty^2, \text{ where } \beta=r_1r_2.$$

This implies that

$$\beta\left\|U^{n+1}-U^*\right\|_\infty \le \frac{1}{\beta}(\beta\|U^n-U^*\|_\infty)^2\le\ldots\le\frac{1}{\beta}\left(\beta\left\|U^0-U^*\right\|_\infty\right)^{2^n}.$$

But $U^*\le U^0\le r_3$, which implies that $\beta\left\|U^0-U^*\right\|_\infty \le \beta r_3<1$. Therefore, $\beta\left\|U^{n+1}-U^*\right\|_\infty \to 0$, and hence $\left\{\left(U^{n+1},\mu^*\right)\right\}$ converges to (U^*,μ^*). □

3. Two-Dimensional Discretization

3.1. 2D—Semi-Implicit Fully Discretized Scheme

Consider two-dimensional system

$$\begin{cases} u_t = \Delta\mu + \lambda(x,y)(h-u), \, in \, \Omega \times [0,T], \\ \mu = -\varepsilon\Delta u + \frac{1}{\varepsilon} f(u), \, in \, \Omega \times [0,T], \\ u = \Delta u = 0 \, on \, \partial\Omega \times [0,T], \\ u(0,x,y) = u_0(x,y) \, in \, \Omega, \end{cases} \tag{10}$$

where $u \equiv u(x,y,t)$, $h \equiv h(x,y)$, $\lambda(x,y) = \lambda_0 \chi_{\Omega \setminus D}(x,y)$ for $t \geq 0$, $a \leq x, y \leq b$, and $\Omega = \{(x,y) | a \leq x, y \leq b\}$. Let $M \in \mathbb{N} \cup \{0\}$ and let $\zeta = \gamma = \frac{b-a}{M+1}$ be the uniform step on the x and y axes, respectively, such that $x_{i+1} = x_i + \zeta \; \forall i = 0,1,\ldots M$, $y_{j+1} = y_j + \gamma$ $\forall j = 0,1,\ldots M$, $a = x_0 < x_1 < \cdots < x_M < x_{M+1} = b$, and $a = y_0 < y_1 < \cdots < y_M < y_{M+1} = b$. In addition, let τ be the time step such that $t_{n+1} = t_n + \tau$ for $t \in \mathbb{N} \cup \{0\}$. Then, the fully discretized semi-implicit 2D system scheme, (10), can be written in the following form:

$$\begin{cases} \dfrac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\tau} = \dfrac{\mu_{i+1,j}^{n+1} - 2\mu_{i,j}^{n+1} + \mu_{i-1,j}^{n+1}}{\zeta^2} + \dfrac{\mu_{i,j+1}^{n+1} - 2\mu_{i,j}^{n+1} + \mu_{i,j-1}^{n+1}}{\gamma^2} + \lambda_{i,j}\left(h_{i,j} - u_{i,j}^{n}\right), \\ \mu_{i,j}^{n+1} = -\varepsilon\left(\dfrac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{\zeta^2}\right) - \varepsilon\left(\dfrac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{\gamma^2}\right) + \frac{1}{\varepsilon} f\left(u_{i,j}^{n}\right), \end{cases} \tag{11}$$

with boundary conditions

$$\begin{cases} u_{0,j}^{n+1} = u_{M+1,j}^{n+1} = u_{i,0}^{n+1} = u_{i,M+1}^{n+1} = 0, \\ \mu_{0,j}^{n+1} = \mu_{M+1,j}^{n+1} = \mu_{i,0}^{n+1} = \mu_{i,M+1}^{n+1} = 0, \end{cases} \quad \forall\, i,j = 0,1,\ldots M,$$

where $\mu_{i,j}^{n} \approx \mu(t_n, x_i, y_j)$, $u_{i,j}^{n} \approx u(t_n, x_i, y_j)$, $\lambda_{i,j} \approx \lambda(x_i, y_j)$, and $h_{i,j} \approx h(x_i, y_j)$. Scheme (11) can also be written in the following form:

$$\begin{cases} u_{i,j}^{n+1} = u_{i,j}^{n} + \dfrac{\tau}{\zeta^2}\left(\mu_{i+1,j}^{n+1} - 2\mu_{i,j}^{n+1} + \mu_{i-1,j}^{n+1}\right) + \dfrac{\tau}{\gamma^2}\left(\mu_{i,j+1}^{n+1} - 2\mu_{i,j}^{n+1} + \mu_{i,j-1}^{n+1}\right) + \tau\lambda_{i,j}\left(h_{i,j} - u_{i,j}^{n}\right), \\ \mu_{i,j}^{n+1} = -\dfrac{\varepsilon}{\zeta^2}\left(u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}\right) - \dfrac{\varepsilon}{\gamma^2}\left(u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}\right) + \dfrac{1}{\varepsilon} f\left(u_{i,j}^{n}\right), \end{cases}$$

Since $\zeta = \gamma$, the system is equivalent to

$$\begin{cases} u_{i,j}^{n+1} - \dfrac{\tau}{\zeta^2}\left(\mu_{i+1,j}^{n+1} + \mu_{i-1,j}^{n+1} - 4\mu_{i,j}^{n+1} + \mu_{i,j+1}^{n+1} + \mu_{i,j-1}^{n+1}\right) = u_{i,j}^{n} + \tau\lambda_{i,j}\left(h_{i,j} - u_{i,j}^{n}\right), \\ \mu_{i,j}^{n+1} + \dfrac{\varepsilon}{\zeta^2}\left(u_{i+1,j}^{n+1} + u_{i-1,j}^{n+1} - 4u_{i,j}^{n+1} + u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}\right) = \dfrac{1}{\varepsilon} f\left(u_{i,j}^{n}\right). \end{cases} \tag{12}$$

The matrix form of Scheme (12) is as follows:

$$\begin{cases} U^{n+1} + A\mu^{n+1} = U^n + L(U^n), \\ \mu^{n+1} + EU^{n+1} = G(U^n), \end{cases}$$

with

$$U^{n+1} = \left(u_{1,1}^{n+1}, u_{2,1}^{n+1}, \ldots, u_{M,1}^{n+1}, \ldots, u_{1,M}^{n+1}, u_{2,M}^{n+1}, \ldots, u_{M,M}^{n+1}\right)^t,$$

$$\mu^{n+1} = \left(\mu_{1,1}^{n+1}, \mu_{2,1}^{n+1}, \ldots, \mu_{M,1}^{n+1}, \ldots, \mu_{1,M}^{n+1}, \mu_{2,M}^{n+1}, \ldots, \mu_{M,M}^{n+1}\right)^t,$$

where A and E are $M^2 \times M^2$ symmetric tridiagonal block matrices such that

$$A = \begin{bmatrix} B & C & 0 & & . & . & . & 0 \\ C & B & C & 0 & . & & & . \\ 0 & C & B & C & 0 & & & . \\ . & 0 & . & . & . & . & & . \\ . & & . & . & . & . & . & . \\ . & & & 0 & C & B & C & 0 \\ . & & & & 0 & C & B & C \\ 0 & . & . & . & . & 0 & C & B \end{bmatrix}_{M^2 \times M^2},$$

where $\left(B_{ij}\right)_{M\times M} = \frac{\tau}{\zeta^2}\begin{cases} -1 \ if \ \ i<j \\ 4 \ if \ i=j \\ -1 \ if \ \ i>j \end{cases}$, and $C = -\frac{\tau}{\zeta^2} I_{M\times M}$.

$$E = \begin{bmatrix} D & K & 0 & & . & . & . & 0 \\ K & D & K & 0 & . & & & . \\ 0 & K & D & K & 0 & & & . \\ . & 0 & . & . & . & . & & . \\ . & & . & . & . & . & . & . \\ . & & & 0 & K & D & K & 0 \\ . & & & & 0 & K & D & K \\ 0 & . & . & . & . & 0 & K & D \end{bmatrix}_{M^2 \times M^2},$$

where $\left(D_{ij}\right)_{M\times M} = \frac{\varepsilon}{\zeta^2}\begin{cases} 1 \ \ if \ \ i<j \\ -4 \ if \ \ i=j \\ 1 \ \ if \ \ i>j \end{cases}$, and $K = -\frac{\varepsilon}{\zeta^2} I_{M\times M}$.

$$G(U^n) = \begin{bmatrix} u^n_{1,1} + \tau\lambda_{1,1}\left(u^n_{1,1} - h_{1,1}\right) \\ \vdots \\ u^n_{M,1} + \tau\lambda_{M,1}\left(u^n_{M,1} - h_{M,1}\right) \\ \vdots \\ \vdots \\ u^n_{1,M} + \tau\lambda_{1,M}\left(u^n_{1,M} - h_{1,M}\right) \\ \vdots \\ u^n_{M,M} + \tau\lambda_{M,M}\left(u^n_{M,M} - h_{M,M}\right) \end{bmatrix},$$

and

$$L(U^n) = \frac{1}{\varepsilon}\begin{bmatrix} f\left(u^n_{1,1}\right) \\ \vdots \\ f\left(u^n_{M,1}\right) \\ \vdots \\ \vdots \\ f\left(u^n_{1,M}\right) \\ \vdots \\ f\left(u^n_{M,M}\right) \end{bmatrix}.$$

Lemma 3. *Block matrix A is positive definite.*

Proof. Let $u = \left(u_1^1, u_2^1, \ldots, u_M^1 \; ; \; u_1^2, u_2^2, \ldots, u_M^2 \; ; \; u_1^M, u_2^M, \ldots, u_M^M\right)^T \in \mathbb{R}^{M^2}$; then, making some calculations, obtain

$$\begin{aligned}
\langle Au, u\rangle &= \frac{4\tau}{\zeta^2}\sum_{i=1}^{M}\sum_{j=1}^{M}\left(u_j^i\right)^2 - \frac{2\tau}{\zeta^2}\sum_{i=1}^{M}\sum_{j=1}^{M-1}u_j^i u_{j+1}^i - \frac{2\tau}{\zeta^2}\sum_{i=1}^{M-1}\sum_{j=1}^{M}u_j^i u_j^{i+1} \\
&= \frac{\tau}{\zeta^2}\left(4\sum_{i=1}^{M}\sum_{j=1}^{M}\left(u_j^i\right)^2 - 2\sum_{i=1}^{M}\sum_{j=1}^{M-1}u_j^i u_{j+1}^i - 2\sum_{i=1}^{M-1}\sum_{j=1}^{M}u_j^i u_j^{i+1}\right) \\
&= \frac{\tau}{\zeta^2}\left(\sum_{i=1}^{M}\sum_{j=1}^{M}\left(u_j^i\right)^2 - 2\sum_{i=1}^{M}\sum_{j=1}^{M-1}u_j^i u_{j+1}^i + \sum_{i=1}^{M}\sum_{j=1}^{M}\left(u_j^{i+1}\right)^2\right) \\
&+ \frac{\tau}{\zeta^2}\left(4\sum_{i=1}^{M}\sum_{j=1}^{M}\left(u_j^i\right)^2 - 2\sum_{i=1}^{M-1}\sum_{j=1}^{M}u_j^i u_j^{i+1} - 2\sum_{i=1}^{M}\sum_{j=1}^{M}\left(u_j^i\right)^2\right) \\
&= \frac{\tau}{\zeta^2}\left[\left(\sum_{i=1}^{M}\sum_{j=1}^{M-1}\left(u_j^i\right)^2 - 2\sum_{i=1}^{M}\sum_{j=1}^{M-1}u_j^i u_{j+1}^i + \sum_{i=1}^{M}\sum_{j=1}^{M-1}\left(u_{j+1}^i\right)^2\right) + \sum_{i=1}^{M}\left(u_1^i\right)^2 + \sum_{i=1}^{M}\left(u_M^i\right)^2\right] \\
&+ \frac{\tau}{\zeta^2}\left[\left(\sum_{i=1}^{M-1}\sum_{j=1}^{M}\left(u_j^i\right)^2 - 2\sum_{i=1}^{M-1}\sum_{j=1}^{M}u_j^i u_j^{i+1} + \sum_{i=1}^{M-1}\sum_{j=1}^{M}\left(u_j^{i+1}\right)^2\right) + \sum_{j=1}^{M}\left(u_j^M\right)^2 + \sum_{j=1}^{M}\left(u_j^1\right)^2\right] \\
&= \frac{\tau}{\zeta^2}\left[\sum_{i=1}^{M}\sum_{j=1}^{M-1}\left(u_j^i - u_{j+1}^i\right)^2 + \sum_{i=1}^{M-1}\sum_{j=1}^{M}\left(u_j^i - u_j^{i+1}\right)^2\right. \\
&\qquad\qquad \left. + \sum_{i=1}^{M}\left(u_1^i\right)^2 + \sum_{i=1}^{M}\left(u_M^i\right)^2 + \sum_{j=1}^{M}\left(u_j^M\right)^2 + \sum_{j=1}^{M}\left(u_j^1\right)^2 + \sum_{i=1}^{M}\right] \\
&\geq 0.
\end{aligned}$$

Now, if $\langle Au, u\rangle = 0$, then

$$\sum_{i=1}^{M}\left(u_1^i\right)^2 + \sum_{i=1}^{M}\left(u_M^i\right)^2 + \sum_{j=1}^{M}\left(u_j^M\right)^2 + \sum_{j=1}^{M}\left(u_j^1\right)^2 + \sum_{i=1}^{M}\sum_{j=1}^{M-1}\left(u_j^i - u_{j+1}^i\right)^2 + \sum_{i=1}^{M-1}\sum_{j=1}^{M}\left(u_j^i - u_j^{i+1}\right)^2 = 0,$$

which implies that $u_1^i = u_M^i = 0 \; \forall i = 0, 1, \ldots M$, $u_j^M = u_j^1 = 0 \; \forall j = 0, 1, \ldots M$, $u_j^i = u_{j+1}^i$ and $u_j^i = u_j^{i+1} \; \forall i, j = 0, 1, \ldots M$. So, $u_1^i = u_2^i = \cdots = u_{M-1}^i = u_M^i = 0 \; \forall i = 0, 1, \ldots M$, and hence $u = (0, 0, \cdots 0)^t_{1\times M^2}$.

Therefore, A is a symmetric positive definite block matrix. □

Corollary 5. *Using the fact that A is a symmetric positive definite block matrix, A is invertible and $A^{-1} \geq 0$.*

Corollary 6. *Since A is a symmetric positive definite block matrix, and $E = -\frac{\varepsilon}{\tau}A$, where $\frac{\varepsilon}{\tau} > 0$, E is invertible such that $E \leq 0$ and $E^{-1} \leq 0$.*

3.1.1. Existence of the Steady State

Consider the system which is the centered divided difference scheme of the steady state,

$$\begin{cases} U^* + A\mu^* = U^* + L(U^*), \\ \mu^* + EU^* = G(U^*), \end{cases}$$

which is equivalent to

$$A\mu^* = \tau\lambda_0\chi_{\Omega\setminus D}(h - U^*) \text{ with } \mu^* = \frac{\varepsilon}{\tau}AU^* + \frac{1}{\varepsilon}f(u^*).$$

But

$$A\mu^* = A\left(\frac{\varepsilon}{\tau}AU^* + \frac{1}{\varepsilon}f(u^*)\right) = \frac{\varepsilon}{\tau}A^2U^* + \frac{1}{\varepsilon}Af(u^*).$$

Hence,

$$A^2U^* = -\frac{\tau}{\varepsilon^2}Af(u^*) + \frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\setminus D}(h - U^*). \tag{13}$$

Since A is invertible and positive definite, A^{-1} is also positive definite and $A = -\Delta_{\zeta,\gamma}$ is the discrete form, where the minimum eigenvalue of $-\Delta_{\zeta,\gamma}$ is equal to $\frac{2\pi^2}{(b-2)^2}$. Hence,

$$\|A\|_\infty = \frac{2\pi^2}{(b-a)^2}.$$

Solving (13) for U^*, obtain two equivalent forms as follows:

$$U^* = -\frac{\tau}{\varepsilon^2}A^{-1}f(u^*) + \frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\setminus D}A^{-2}(h - U^*)$$

or

$$U^* = \left(I + \frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\setminus D}A^{-2}\right)^{-1}\left[\frac{\tau}{\varepsilon}A^{-1}\left(\tau\lambda_0\chi_{\Omega\setminus D}A^{-1}h - \frac{1}{\varepsilon}f(u^*)\right)\right].$$

Now, define operator H as

$$H(V) = \left(I + \frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\setminus D}A^{-2}\right)^{-1}\left[\frac{\tau}{\varepsilon}A^{-1}\left(\tau\lambda_0\chi_{\Omega\setminus D}A^{-1}h - \frac{1}{\varepsilon}f(V)\right)\right]$$

so that

$$U^* = H(U^*)$$

and consider sequence

$$V^{k+1} = H\left(V^k\right), \text{ with } V^0 = 1.$$

Thus,

$$\begin{aligned} V^1 = H\left(V^0\right) &= \left(I + \frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\setminus D}A^{-2}\right)^{-1}\left[\frac{\tau}{\varepsilon}A^{-1}\left(\tau\lambda_0\chi_{\Omega\setminus D}A^{-1}h - \frac{1}{\varepsilon}f\left(V^0\right)\right)\right] \\ &= \left(I + \frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\setminus D}A^{-2}\right)^{-1}\left[\frac{\tau^2}{\varepsilon}A^{-2}\lambda_0\chi_{\Omega\setminus D}h\right] \geq 0. \end{aligned}$$

Moreover, if $V^k \geq 0$, then

$$\begin{aligned} V^{k+1} &= H\left(V^k\right) \\ &= \left(I + \frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\setminus D}A^{-2}\right)^{-1}\left[\frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\setminus D}A^{-2}h + \frac{1}{\varepsilon}E^{-1}f\left(V^k\right)\right] \\ &\geq 0, \qquad \text{if } \lambda \text{ is large enough.} \end{aligned}$$

Hence, $\left\{V^k\right\}$ is a positive sequence for all $k \in N$ if λ is large enough.

Lemma 4. $\left\{V^k\right\}$ *is a decreasing sequence such that* $0 \leq \left\|V^k\right\| \leq 1$.

Proof. Suppose that $V^k \leq V^{k+1}$. Then,

$$
\begin{aligned}
H\left(V^{k+1}\right) - H\left(V^k\right) &= \left(I + \frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\backslash D}A^{-2}\right)^{-1}\left[\frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\backslash D}A^{-2}h + \frac{1}{\varepsilon}E^{-1}f\left(V^k\right)\right] - \\
&\quad - \left(I + \frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\backslash D}A^{-2}\right)^{-1}\left[\frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\backslash D}A^{-2}h + \frac{1}{\varepsilon}E^{-1}f\left(V^k\right)\right] \\
&= \frac{1}{\varepsilon}E^{-1}\left(I + \frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\backslash D}A^{-2}\right)^{-1}\left(f\left(V^{k+1}\right) - f\left(V^k\right)\right) \\
&= -\frac{\tau}{\varepsilon^2}A^{-1}\left(I + \frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\backslash D}A^{-2}\right)^{-1}\left(V^{k+1} - V^k\right)\left[4\left(V^{k+1} - \frac{3}{4}\right)^2\right. \\
&\quad \left. +4\left(V^k - \frac{3}{4}\right)^2 + 4V^{k+1}V^k + \frac{14}{16}\right] \\
&\leq 0.
\end{aligned}
$$

Hence, $\left\{V^k\right\}$ is a decreasing sequence for all $k \in N$. Note that $V^0 = 1$; infer that

$$0 \leq \left\|V^k\right\| \leq 1.$$

□

Now, let $0 \leq \theta \leq 1$, which implies that $0 \leq \|h(x) - \theta\|_\infty \leq 1 - \delta$, where $\delta \in (0,1)$, so that

$$0 \leq U^* < 1.$$

In addition, there is

$$-6 \leq 4\theta^3 - 6\theta^2 + 2\theta \leq 6,$$

which yields

$$\|G(\theta)\|_\infty = \frac{1}{\varepsilon}\|f(\theta)\|_\infty = \frac{1}{\varepsilon}\left(\max_{1\leq i\leq M}\left|4\theta^3 - 6\theta^2 + 2\theta\right|\right) \leq \frac{6}{\varepsilon},$$

and

$$
\begin{aligned}
\|H(\theta)\|_\infty &= \left\|-\frac{\tau}{\varepsilon^2}A^{-1}\,f(\theta) + \frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\backslash D}A^{-2}\,(h-\theta)\right\|_\infty \\
&\geq \frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\backslash D}\left\|A^{-2}\,(h-\theta)\right\|_\infty - \frac{\tau}{\varepsilon^2}\left\|A^{-1}\,f(\theta)\right\|_\infty \\
&\geq \frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\backslash D}\left\|A^{-2}\right\|_\infty\|(h-\theta)\|_\infty - \frac{\tau}{\varepsilon^2}\left\|A^{-1}\right\|_\infty\|f(\theta)\|_\infty \\
&\geq \frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\backslash D}\frac{(b-a)^4}{4\pi^4}(1-\delta) - \frac{\tau}{\varepsilon^2}\frac{3(b-a)^2}{2\pi^2} \\
&\geq \frac{\tau^2\lambda_0(1-\delta)(b-a)^4}{4\varepsilon\pi^4} - \frac{3\tau(b-a)^2}{2\varepsilon^2\pi^2} \\
&\geq \frac{\tau^2\lambda_0\chi_{\Omega\backslash D}(1-\delta)(b-a)^4}{4\varepsilon\pi^4} - \frac{3\tau(b-a)^2}{2\varepsilon^2\pi^2},
\end{aligned}
$$

where the assumption (H′) (given in the next corollary) is used as well as the fact that $\|H(\theta)\|_\infty > \delta$ for some $\delta \in (0,1)$, and $\chi_{\Omega\backslash D} < 1$.

Corollary 7. *Under assumption (H') that is* $\lambda_0 \geq \frac{4\pi^4\varepsilon\delta+12\pi^2\tau^2(b-a)^2}{\varepsilon\tau^2(1-\delta)(b-a)^4}$ *for some* $\delta \in (0,1)$, *find that*

$$A^2U^* = -\frac{\tau}{\varepsilon^2}A\, f(u^*) + \frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\setminus D}(U^* - h)$$

admits solution U^* *such that* $0 \leq U^* < 1$.

3.1.2. Convergence of the Solution

Suppose that $\|A^{-1}\|_\infty = \frac{(b-a)^2}{2\pi^2}$, and let $U^*$and μ^* be steady states of the system

$$\begin{cases} U^{n+1} + A\mu^{n+1} = U^n + \tau\lambda_0\chi_{\Omega\setminus D}(h - U^n), \\ \mu^{n+1} + EU^{n+1} = \dfrac{1}{\varepsilon}\, f(u^n). \end{cases} \tag{14}$$

Then,

$$\begin{cases} A\mu^* = \tau\lambda_0\chi_{\Omega\setminus D}(h - U^*), \\ \mu^* + EU^* = \dfrac{1}{\varepsilon} f(u^*). \end{cases} \tag{15}$$

Subtracting (15) from (14), obtain the following system:

$$\begin{cases} U^{n+1} + A\left(\mu^{n+1} - \mu^*\right) = U^n + \tau\lambda_0\chi_{\Omega\setminus D}((h - U^n) - (h - U^*)), \\ \mu^{n+1} - \mu^* + E\left(U^{n+1} - U^*\right) = \dfrac{1}{\varepsilon}(f(u^n) - f(u^*)), \end{cases}$$

which is equivalent to the following difference equation:

$$U^{n+1} + A\left[-E\left(U^{n+1} - U^*\right) + \frac{1}{\varepsilon}(f(u^n) - f(u^*))\right] = U^n + \tau\lambda_0\chi_{\Omega\setminus D}((h - U^n) - (h - U^*)).$$

Hence,

$$U^{n+1} - U^n = A\left[E\left(U^{n+1} - U^*\right) - \frac{1}{\varepsilon}(f(u^n) - f(u^*))\right] + \tau\lambda_0\chi_{\Omega\setminus D}((h - U^n) - (h - U^*)),$$

which yields

$$\begin{aligned} \left(U^{n+1} - U^*\right) - (U^n - U^*) = -\frac{\varepsilon}{\tau}A^2\left(U^{n+1} - U^*\right) - \frac{1}{\varepsilon}A(f(u^n) - f(u^*)) \\ + \tau\lambda_0\chi_{\Omega\setminus D}((h - U^n) - (h - U^*)). \end{aligned}$$

Let $V^{n+1} = U^{n+1} - U^*$ and $V^n = U^n - U^*$, then

$$V^{n+1} - V^n = -\frac{\varepsilon}{\tau}A^2V^{n+1} - \frac{1}{\varepsilon}A(f(u^n) - f(u^*)) + \tau\lambda_0\chi_{\Omega\setminus D}((h - U^n) - (h - U^*)),$$

which is equivalent to

$$
\begin{aligned}
\Big(\frac{\varepsilon}{\tau}I + A^{-2}\Big)V^{n+1} &= A^{-2}V^n - \frac{1}{\varepsilon}A^{-1}(f(u^n) - f(u^*)) \\
&+ \tau\lambda_0\chi_{\Omega\backslash D}A^{-2}((h - U^n) - (h - U^*)) \\
&= A^{-2}V^n + \frac{\varepsilon}{\tau}\left[\begin{array}{c} -\dfrac{\tau}{\varepsilon^2}A^{-1}(f(u^n) - f(u^*)) \\ +\lambda_0\chi_{\Omega\backslash D}\dfrac{\tau^2}{\varepsilon}A^{-2}((h - U^n) - (h - U^*)) \end{array}\right] \\
&= A^{-2}V^n + \frac{\varepsilon}{\tau}\left[\begin{array}{c} \Big(-\dfrac{\tau}{\varepsilon^2}A^{-1}f(u^n) + \lambda_0\chi_{\Omega\backslash D}\dfrac{\tau^2}{\varepsilon}A^{-2}(h - U^n)\Big) \\ -\Big(-\dfrac{\tau}{\varepsilon^2}A^{-1}\overline{f}(u^*) + \lambda_0\chi_{\Omega\backslash D}\dfrac{\tau^2}{\varepsilon}A^{-2}(h - U^*)\Big) \end{array}\right] \\
&= A^{-2}V^n + \frac{\varepsilon}{\tau}[H(U^n) - H(U^*)].
\end{aligned}
$$

Now, since A^{-2} is a symmetric and positive definite block matrix, then $A^{-2} + \frac{\varepsilon}{\tau}I$ is also symmetric positive definite, and it is invertible such that $\left(A^{-2} + \frac{\varepsilon}{\tau}I\right)^{-1} \geq 0$. Therefore,

$$
V^{n+1} = \Big(A^{-2} + \frac{\varepsilon}{\tau}I\Big)^{-1}\Big(A^{-2}V^n + \frac{\varepsilon}{\tau}[H(U^n) - H(U^*)]\Big).
$$

Moreover, since $H(U^n)$ is a decreasing function, and $0 \leq U^* \leq U^n < 1$, then

$$
0 \leq H(U^n) \leq H(U^*) < 1.
$$

This implies that

$$
0 \leq U^{n+1} < 1 \text{ and } 0 \leq V^{n+1} < 1.
$$

Therefore, U^n is well defined and $0 \leq U^* \leq U^n < 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Theorem 7. $\{V^{n+1}\}$ *converges to* 0 *if*

$$
\lambda_0 > \frac{4\pi^4\varepsilon^2 + 12\pi^2\tau^2(b-a)^2}{\varepsilon\tau^2(1-\delta)(b-a)^4}.
$$

Proof. Consider sequence $\{V^{n+1}\}$, matrix $K = I + \dfrac{\varepsilon}{\tau}A^2$, Euclidean inner product $\langle . \rangle$, and associate norm $\|.\|_\infty$. Then, for $L(U^n) = \tau\lambda_0\chi_{\Omega\backslash D}(h - u^n)$, and $G(U^n) = \frac{1}{\varepsilon}f(u^n)$

$$
\begin{aligned}
\Big\langle KV^{n+1}, V^{n+1}\Big\rangle &= \Big\langle V^n - \frac{\tau}{\varepsilon}A(G(U^n) - G(U^*)) + \frac{\tau}{\varepsilon}(L(U^n) - L(U^*)), V^{n+1}\Big\rangle \\
&\leq \Big\|V^n - \frac{\tau}{\varepsilon}A(G(U^n) - G(U^*)) + \frac{\tau}{\varepsilon}(L(U^n) - L(U^*))\Big\|_\infty \Big\|V^{n+1}\Big\|_\infty.
\end{aligned}
$$

Hence,

$$\left(1+\frac{\varepsilon}{\tau}\left(\frac{2\pi^2}{(b-a)^2}\right)^2\right)\left\|V^{n+1}\right\|_\infty \leq \left\|V^n-\frac{\tau}{\varepsilon}A(G(U^n)-G(U^*))+\frac{\tau}{\varepsilon}(L(U^n)-L(U^*))\right\|_\infty$$
$$\leq \|V^n\|_\infty+\frac{\tau}{\varepsilon}\|A\|_\infty\|G(U^n)-G(U^*)\|_\infty+\frac{\tau}{\varepsilon}\|L(U^n)-L(U^*)\|_\infty$$
$$\leq \|V^n\|_\infty+\frac{\tau}{\varepsilon}\frac{2\pi^2}{(b-a)^2}\|G(V^n)\|_\infty+\frac{\tau}{\varepsilon}\|L(V^n)\|_\infty$$
$$\leq \|V^n\|_\infty+\frac{\tau}{\varepsilon}\frac{2\pi^2}{(b-a)^2}\|G\|_\infty\|V^n\|_\infty+\frac{\tau}{\varepsilon}\|L\|_\infty\|V^n\|_\infty$$
$$\leq \|V^n\|_\infty\left(1+\frac{\tau}{\varepsilon^2}\frac{12\pi^2}{(b-a)^2}+\frac{\tau^2}{\varepsilon}\lambda_0(1-\delta)\right).$$

This implies that

$$\left\|V^{n+1}\right\|_\infty \leq \left(1+\frac{\varepsilon}{\tau}\frac{4\pi^4}{(b-a)^4}\right)^{-1}\left(1+\frac{\tau}{\varepsilon^2}\frac{12\pi^2}{(b-a)^2}+\frac{\tau^2}{\varepsilon}\lambda_0(1-\delta)\right)\|V^n\|_\infty.$$

Therefore, $\{V^{n+1}\}$ converges to 0 if

$$\left(1+\frac{\varepsilon}{\tau}\frac{4\pi^4}{(b-a)^4}\right)^{-1}\left(1+\frac{\tau}{\varepsilon^2}\frac{12\pi^2}{(b-a)^2}+\frac{\tau^2}{\varepsilon}\lambda_0(1-\delta)\right)<1,$$

which is equivalent to

$$1+\frac{\tau}{\varepsilon^2}\frac{12\pi^2}{(b-a)^2}+\frac{\tau^2}{\varepsilon}\lambda_0(1-\delta)>1+\frac{\varepsilon}{\tau}\frac{4\pi^4}{(b-a)^4},$$

and hence

$$\lambda_0>\frac{4\pi^4\varepsilon^2+12\pi^2\tau^2(b-a)^2}{\varepsilon\tau^2(1-\delta)(b-a)^4}.$$

□

3.2. 2D Implicit Fully Discretized Scheme

The semi-implicit 2D time discretization of System (1) is given as follows:

$$\begin{cases} \dfrac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}-\left(\dfrac{\mu_{i+1,j}^{n+1}-2\mu_{i,j}^{n+1}+\mu_{i-1,j}^{n+1}}{\zeta^2}\right)-\left(\dfrac{\mu_{i,j+1}^{n+1}-2\mu_{i,j}^{n+1}+\mu_{i,j-1}^{n+1}}{\gamma^2}\right) \\ \qquad -\lambda(x_i,y_i)\left(h_{i,j}-u_{i,j}^{n+1}\right)=0, \\ \mu_{i,j}^{n+1}+\varepsilon\left(\dfrac{u_{i+1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{\zeta^2}\right)+\varepsilon\left(\dfrac{u_{i,j+1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{\gamma^2}\right)-\dfrac{1}{\varepsilon}f\left(u_{i,j}^{n+1}\right)=0, \end{cases}$$

with $i,j=1,2,\cdots,M$, and boundary conditions

$$\begin{cases} u_{0,j}^{n+1}=u_{M+1,j}^{n+1}=u_{i,0}^{n+1}=u_{i,M+1}^{n+1}=0 \\ \mu_{0,j}^{n+1}=\mu_{M+1,j}^{n+1}=\mu_{i,0}^{n+1}=\mu_{i,M+1}^{n+1}=0 \end{cases}, \forall i,j=0,1,\dots M,$$

where $\mu_{i,j}^n\approx\mu(t_n,x_i,y_j)$, $u_{i,j}^n\approx u(t_n,x_i,y_j)$. Now, if $\zeta=\gamma$, then the scheme can be written in the following form:

$$\begin{cases} u_{i,j}^{n+1} - u_{i,j}^{n} - \dfrac{\tau}{\zeta^2}\left(\mu_{i+1,j}^{n+1} + \mu_{i-1,j}^{n+1} - 4\mu_{i,j}^{n+1} + \mu_{i,j+1}^{n+1} + \mu_{i,j-1}^{n+1}\right) - \tau\lambda_0\chi_{\Omega\backslash D}\left(h_{i,j} - u_{i,j}^{n+1}\right) = 0, \\ \mu_{i,j}^{n+1} + \dfrac{\varepsilon}{\zeta^2}\left(u_{i+1,j}^{n+1} + u_{i-1,j}^{n+1} - 4u_{i,j}^{n+1} + u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}\right) - \dfrac{1}{\varepsilon}f\left(u_{i,j}^{n}\right) = 0. \end{cases}$$

Since $\zeta = \gamma$, the system is equivalent to

$$\begin{cases} u_{i,j}^{n+1} + \dfrac{\tau}{\zeta^2}\left(-\mu_{i+1,j}^{n+1} - \mu_{i-1,j}^{n+1} + 4\mu_{i,j}^{n+1} - \mu_{i,j+1}^{n+1} - \mu_{i,j-1}^{n+1}\right) = u_{i,j}^{n} + \tau\lambda_0\chi_{\Omega\backslash D}\left(h_{i,j} - u_{i,j}^{n}\right), \\ \mu_{i,j}^{n+1} + \dfrac{\varepsilon}{\zeta^2}\left(u_{i+1,j}^{n+1} + u_{i-1,j}^{n+1} - 4u_{i,j}^{n+1}u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}\right) = \dfrac{1}{\varepsilon}f\left(u_{i,j}^{n}\right). \end{cases}$$

Now, if $L\left(u_{i,j}^{n+1}\right) = \tau\lambda_0\chi_{\Omega\backslash D}\left(h_{i,j} - u_{i,j}^{n+1}\right)$ and $G\left(u_{i,j}^{n+1}\right) = \dfrac{1}{\varepsilon}f\left(u_{i,j}^{n+1}\right)$, then the matrix form of the scheme is as follows:

$$\begin{cases} U^{n+1} - U^{n} + A\mu^{n+1} - L\left(U^{n+1}\right) = 0, \\ \mu^{n+1} + EU^{n+1} - G\left(U^{n+1}\right) = 0, \end{cases}$$

where A is symmetric positive definite block invertible matrix such that $A^{-1} \geq 0$, and $E = -\dfrac{\varepsilon}{\tau}A$ is also symmetric and negative definite invertible block matrix such that $E^{-1} \leq 0$. Define $Q_{i,j}\left(u_{i,j}^{n+1}, \mu_{i,j}^{n+1}\right) : [0,1]^{M^2} \longrightarrow R^{M^2}$ and $H_{i,j}\left(u_{i,j}^{n+1}, \mu_{i,j}^{n+1}\right) : [0,1]^{M^2} \longrightarrow R^{M^2}$ such that

$$\begin{cases} Q_{i,j}\left(u_{i,j}^{n+1}, \mu_{i,j}^{n+1}\right) = u_{i,j}^{n+1} - u_{i,j}^{n} + A\mu_{i,j}^{n+1} - L\left(u_{i,j}^{n+1}\right) = 0, \\ H_{i,j}\left(u_{i,j}^{n+1}, \mu_{i,j}^{n+1}\right) = \mu_{i,j}^{n+1} + Eu_{i,j}^{n+1} - G\left(u_{i,j}^{n+1}\right) = 0, \end{cases}$$

which have the following matrix form:

$$\begin{cases} Q_{i,j}\left(U^{n+1}, \mu^{n+1}\right) = 0, \\ H_{i,j}\left(U^{n+1}, \mu^{n+1}\right) = 0, \end{cases}$$

where

$$U^{n+1} = \left(u_{1,1}^{n+1}, u_{2,1}^{n+1}, \ldots, u_{M,1}^{n+1}, \ldots, u_{1,M}^{n+1}, u_{2,M}^{n+1}, \ldots, u_{M,M}^{n+1}\right)^t,$$
$$\mu^{n+1} = \left(\mu_{1,1}^{n+1}, \mu_{2,1}^{n+1}, \ldots, \mu_{M,1}^{n+1}, \ldots, \mu_{1,M}^{n+1}, \mu_{2,M}^{n+1}, \ldots, \mu_{M,M}^{n+1}\right)^t,$$

with

$$\begin{cases} Q(U,\mu) = \left(Q_{1,1}(U,\mu), Q_{2,1}(U,\mu), \ldots, Q_{1,M}(U,\mu); \ldots;\ Q_{1,M}(U,\mu), Q_{2,M}(U,\mu), \ldots, Q_{M,M}(U,\mu)\right)^t, \\ H(U,\mu) = \left(H_{1,1}(U,\mu), H_{2,1}(U,\mu), \ldots, H_{1,M}(U,\mu); \ldots;\ H_{1,M}(U,\mu), H_{2,M}(U,\mu), \ldots, H_{M,M}(U,\mu)\right)^t. \end{cases}$$

Existence of Roots of Q and H Such That $Q(U^*, \mu^*) = 0$ and $H(U^*, \mu^*) = 0$

Let U^* and μ^* be the roots of Q and H such that

$$\begin{cases} Q_{i,j}(U^*, \mu^*) = 0, \\ H_{i,j}(U^*, \mu^*) = 0, \end{cases}$$

and

$$\begin{cases} u_{i,j}^{*} - u_{i,j}^{*} + A\mu_{i,j}^{*} - \tau\lambda_0\chi_{\Omega\backslash D}\left(h_{i,j} - u_{i,j}^{*}\right) = 0, \\ \mu^{*} = -Eu_{i,j}^{*} + \dfrac{1}{\varepsilon}f\left(u_{i,j}^{*}\right). \end{cases}$$

Hence,

$$A\left(-Eu_{i,j}^{*} + \frac{1}{\varepsilon}f\left(u_{i,j}^{*}\right)\right) - \tau\lambda_0\chi_{\Omega\backslash D}\left(h_{i,j} - u_{i,j}^{*}\right) = 0.$$

If $E = -\frac{\varepsilon}{\tau}A$ is substituted, the following difference equation can be obtained:

$$\frac{\varepsilon}{\tau}A^2 u^*_{i,j} + \frac{1}{\varepsilon}Af\left(u^*_{i,j}\right) - \tau\lambda_0\chi_{\Omega\setminus D}\left(h_{i,j} - u^*_{i,j}\right) = 0,$$

where A and A^2 are positive definite matrices. Hence,

$$u^*_{i,j} = -\frac{\tau}{\varepsilon}A^{-1}\, f\left(u^*_{i,j}\right) + \frac{\tau^2}{\varepsilon}\lambda_0\chi_{\Omega\setminus D}A^{-2}\left(u^*_{i,j} - h_{i,j}\right). \tag{16}$$

Theorem 8. *Assume that* $\lambda_0 > \frac{4\pi^4\varepsilon^2 + 12\pi^2\tau^2(b-a)^2}{\varepsilon\tau^2(1-\delta)(b-a)^4}$ *for some* $\delta \in (0,1)$*; then,* (16) *admits solution* (U^*, μ^*) *such that* $0 \le U^* < 1$.

Proof. Performed in Corollary 7. □

Theorem 9. *Assume that* $\lambda_0 \ge \frac{2\pi^4\left(\pi^2\varepsilon^2\delta + 3\tau^2(b-a)^2\right)}{3\tau^3(1-\delta)(b-a)^6}$ *for some* $\delta \in (0,1)$*; then,* $\mu^* = -Eu^*_{i,j} + \frac{1}{\varepsilon}f\left(u^*_{i,j}\right)$ *admits solution* (U^*, μ^*) *such that* $0 \le U^* < 1$.

Proof. Since $\mu^*_{i,j} = -Eu^*_{i,j} + \frac{1}{\varepsilon}f\left(u^*_{i,j}\right)$,

$$\begin{aligned}
\|\mu^*\|_\infty &= \left\|-Eu^*_{i,j} + \frac{1}{\varepsilon}f\left(u^*_{i,j}\right)\right\|_\infty \\
&\ge \left\|Eu^*_{i,j}\right\|_\infty - \frac{1}{\varepsilon}\|f(\theta_{i,j})\|_\infty, \text{ for } 0 \le \theta_{i,j} \le 1 \\
&\ge \|E\|_\infty\left\|u^*_{i,j}\right\|_\infty - \frac{1}{\varepsilon}\|f(\theta_i)\|_\infty, \text{ with } 0 \le \left\|u^*_{i,j}\right\|_\infty \le 1 \\
&\ge \frac{2\pi^2}{(b-a)^2} - \frac{1}{\varepsilon}\max_{1\le i\le N}\left|4\theta_i^3 - 4\theta_i^2 + 2\theta_i\right| \\
&\ge \frac{2\pi^2}{(b-a)^2} - \frac{6}{\varepsilon}.
\end{aligned}$$

But

$$\begin{aligned}
\|\mu^*\|_\infty &= \left\|-\frac{\varepsilon}{\zeta^2}\left(u^*_{i+1,j} + u^*_{i-1,j} - 4u^*_{i,j} + u^*_{i,j+1} + u^*_{i,j-1}\right) + \frac{1}{\varepsilon}f(u^*_i)\right\|_\infty \\
&\le \frac{\varepsilon}{\zeta^2}\left\|u^*_{i,j} + u^*_{i,j} - 4u^*_{i,j} + u^*_{i,j} + u^*_{i,j}\right\|_\infty + \frac{1}{\varepsilon}\|f(u^*_i)\|_\infty \\
&\le \frac{6}{\varepsilon}.
\end{aligned}$$

This implies that

$$\frac{6}{\varepsilon} \ge \frac{2\pi^2}{(b-a)^2} - \frac{6}{\varepsilon},$$

and hence

$$\frac{12}{\varepsilon} \ge \frac{2\pi^2}{(b-a)^2}.$$

But

$$\frac{1}{\varepsilon} \le \frac{\lambda_0(1-\delta)(b-a)^4\tau^2}{4\pi^4\varepsilon^2\delta + 12\pi^2\tau^2(b-a)^2},$$

which implies that

$$\frac{12\lambda_0(1-\delta)(b-a)^4\tau^2}{4\pi^4\varepsilon^2\delta + 12\pi^2\tau^2(b-a)^2} \ge \frac{2\pi^2}{(b-a)^2}.$$

Therefore,

$$\lambda_0 \geq \frac{2\pi^2\left(4\pi^4\varepsilon^2\delta + 12\pi^2\tau^2(b-a)^2\right)}{12(1-\delta)(b-a)^6\tau^2}$$
$$\geq \frac{2\pi^4\left(\pi^2\varepsilon^2\delta + 3\tau^2(b-a)^2\right)}{3(1-\delta)(b-a)^6\tau^2}.$$

□

Now, in order to find U^{n+1} and μ^{n+1} such that $Q_{i,j}(U^{n+1},\mu^{n+1}) = 0$ and $H_{i,j}(U^{n+1},\mu^{n+1}) = 0$, let

$$\begin{aligned} F_{i,j}\left(U^{n+1},\mu^{n+1}\right) &= Q_{i,j}\left(U^{n+1},\mu^{n+1}\right) + H_{i,j}\left(U^{n+1},\mu^{n+1}\right) \\ &= u_{i,j}^{n+1} - u_{i,j}^{n} + A\mu_{i,j}^{n+1} - \tau\lambda_0\chi_{\Omega\setminus D}\left(h_{i,j} - u_{i,j}^{n+1}\right) \\ &+ \mu_{i,j}^{n+1} + Eu_{i,j}^{n+1} - \frac{1}{\varepsilon}f\left(u_{i,j}^{n+1}\right) = 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} F_{i,j}\left(U^{n+1},\mu^{n+1}\right) &= u_{i,j}^{n+1} - u_{i,j}^{n} - \frac{\tau}{\zeta^2}\left(+\mu_{i+1,j}^{n+1} + \mu_{i-1,j}^{n+1} - 4\mu_{i,j}^{n+1} + \mu_{i,j+1}^{n+1} + \mu_{i,j-1}^{n+1}\right) \\ &- \tau\lambda_0\chi_{\Omega\setminus D}\left(h_{i,j} - u_{i,j}^{n}\right) + \mu_{i,j}^{n+1} + \frac{\varepsilon}{\zeta^2}\left(u_{i+1,j}^{n+1} + u_{i-1,j}^{n+1} - 4u_{i,j}^{n+1}u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}\right) \\ &- \frac{1}{\varepsilon}f\left(u_{i,j}^{n}\right) = 0. \end{aligned}$$

Let $u_{i,j}^{n+1} - u_{i,j}^{n} = \psi_{i,j}$, and $\mu_{i,j}^{n+1} - \mu_{i,j}^{n} = \varphi_{i,j}$ for all $n \in \mathbb{N} \cup \{0\}$. Then, using Taylor series expansion, obtain

$$\begin{aligned} F_{i,j}\left(u_{i,j}^{n+1},\mu_{i,j}^{n+1}\right) &= F_{i,j}\left(u_{i,j}^{n} + \psi_{i,j},\mu_{i,j}^{n} + \varphi_{i,j}\right) \\ &\approx F_{i,j}\left(u_{i,j}^{n},\mu_{i,j}^{n}\right) + \left(\psi_{i,j}DQ_{i,j}\left(u_{i,j}^{n},\mu_{i,j}^{n}\right) + \varphi_{i,j}DH_{i,j}\left(u_{i,j}^{n},\mu_{i,j}^{n}\right)\right) \\ &\approx F_{i,j}\left(u_{i,j}^{n},\mu_{i,j}^{n}\right) + \Phi_{i,j}DF\left(u_{i,j}^{n},\mu_{i,j}^{n}\right), \end{aligned}$$

where $\Phi_{i,j}$ is a function in $\psi_{i,j}$ and $\varphi_{i,j}$.

But $u_{i,j}^{n+1}$ and $\mu_{i,j}^{n+1}$ are roots of $F_{i,j}(U^{n+1},\mu^{n+1})$ so that $F_{i,j}\left(u_{i,j}^{n+1},\mu_{i,j}^{n+1}\right) = 0$. So, using Newton's Method, obtain

$$\left(u_{i,j}^{n+1},\mu_{i,j}^{n+1}\right) = -DF^{-1}\left(u_{i,j}^{n},\mu_{i,j}^{n}\right)F_{i,j}\left(u_{i,j}^{n},\mu_{i,j}^{n}\right) + \left(u_{i,j}^{n},\mu_{i,j}^{n}\right).$$

So, in order to find U^{n+1} and μ^{n+1}, compute the Jacobian block matrix of $F\left(u_{i,j}^{n},\mu_{i,j}^{n}\right)$ at each iteration of $n \in \mathbb{N}$. Since

$$\begin{aligned} F_{i,j}\left(u_{i,j}^{n+1},\mu_{i,j}^{n+1}\right) = \quad & u_{i,j}^{n+1} - u_{i,j}^{n} - \frac{\tau}{\zeta^2}\left(\mu_{i+1,j}^{n+1} + \mu_{i-1,j}^{n+1} - 4\mu_{i,j}^{n+1}\mu_{i,j+1}^{n+1} + \mu_{i,j-1}^{n+1}\right) + \\ & \tau\lambda_0\chi_{\Omega\setminus D}\left(h_{i,j} - u_{i,j}^{n}\right) + \mu_{i,j}^{n+1} + \frac{\varepsilon}{\zeta^2}\left(u_{i+1,j}^{n+1} + u_{i-1,j}^{n+1} - 4u_{i,j}^{n+1}u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}\right) \\ & -\frac{1}{\varepsilon}\left(4\left(u_{i,j}^{n+1}\right)^3 - 6\left(u_{i,j}^{n+1}\right)^2 + 2\left(u_{i,j}^{n+1}\right)\right), \end{aligned}$$

the Jacobian of F is

$$J_{i,j,r,s} = \begin{cases} \dfrac{\partial F_{i,j}}{\partial u_{r,s}} = \begin{cases} 1 - 4\dfrac{\varepsilon}{\zeta^2} - \tau\lambda_0\chi_{\Omega\setminus D} \\ -\dfrac{1}{\varepsilon}\left(12\left(u_{i,j}^{n+1}\right)^2 - 12u_{i,j}^{n+1} + 2\right), & if\ r = i\ and\ s = j, \\ \dfrac{\varepsilon}{\zeta^2}, & if\ r = i \pm 1, s = j, \\ & and\ if\ r = i, s = j \pm 1, \\ 0, & otherwise. \end{cases} \\ \dfrac{\partial F_{i,j}}{\partial \mu_{r,s}} = \begin{cases} 1 + 4\dfrac{\tau}{\zeta^2}, & if\ r = i\ and\ s = j, \\ -\dfrac{\tau}{\zeta^2}, & if\ r = i \pm 1, s = j, \\ & and\ if\ r = i, s = j \pm 1, \\ 0, & otherwise. \end{cases} \end{cases}$$

This implies that for all $i, j = 1, 2, \ldots, M$, there is

$$\begin{aligned} DF_{i,j}(U^*, \mu^*) &= \left(\frac{\partial F_{i,j}}{\partial u_r}\partial u_r + \frac{\partial F_{i,j}}{\partial u_s}\partial u_s\right) + \left(\frac{\partial F_{i,j}}{\partial \mu_r}\partial \mu_r + \frac{\partial F_{i,j}}{\partial \mu_s}\partial \mu_s\right) \\ &= \left(2 - \tau\lambda_0\chi_{\Omega\setminus D} - 4\frac{\varepsilon}{\zeta^2} + 4\frac{\tau}{\zeta^2}\right)I - K, \end{aligned}$$

where

$$K = \begin{bmatrix} L_{i,1} & \left(\frac{\tau}{\zeta^2} - \frac{\varepsilon}{\zeta^2}\right)I_{M\times M} & 0 & . & . & 0 \\ \left(\frac{\tau}{\zeta^2} - \frac{\varepsilon}{\zeta^2}\right)I_{M\times M} & L_{i,2} & . & . & & . \\ 0 & . & . & . & . & . \\ & & . & . & . & 0 \\ & & & . & L_{i,M-1} & \left(\frac{\tau}{\zeta^2} - \frac{\varepsilon}{\zeta^2}\right)I_{M\times M} \\ & & & 0 & \left(\frac{\tau}{\zeta^2} - \frac{\varepsilon}{\zeta^2}\right)I_{M\times M} & L_{i,M} \end{bmatrix}_{M^2\times M^2}$$

$$= diag\left(\left(\frac{\tau}{\zeta^2} - \frac{\varepsilon}{\zeta^2}\right)I_{M\times M}, L_{i,j}, \left(\frac{\tau}{\zeta^2} - \frac{\varepsilon}{\zeta^2}\right)I_{M\times M}\right),$$

such that $\forall i,\ j = 1, 2, \ldots, M$,

$$L_{i,j} = \begin{bmatrix} \frac{-\left(12(u^*)^2 - 12u^* + 2\right)}{\varepsilon} & \frac{\tau}{\zeta^2} - \frac{\varepsilon}{\zeta^2} & 0 & . & . & 0 \\ \frac{\tau}{\zeta^2} - \frac{\varepsilon}{\zeta^2} & . & . & . & & . \\ 0 & . & . & . & . & . \\ & & . & . & . & 0 \\ & & & . & . & \frac{\tau}{\zeta^2} - \frac{\varepsilon}{\zeta^2} \\ & & & & \frac{\tau}{\zeta^2} - \frac{\varepsilon}{\zeta^2} & \frac{-\left(12(u^*)^2 - 12u^* + 2\right)}{\varepsilon} \end{bmatrix}_{M^2\times M^2}$$

$$= diag\left(\frac{\tau}{\zeta^2} - \frac{\varepsilon}{\zeta^2}, -\frac{1}{\varepsilon}\left(12(u^*)^2 - 12U^* + 2\right), \frac{\tau}{\zeta^2} - \frac{\varepsilon}{\zeta^2}\right).$$

Moreover, for $i = 1, M$ and $j = 1, 2, \ldots, M$, $s = 1, M$ and $t = 2, 3, \ldots, M-1$, $d = 2, \ldots, M-1$ and $k = 1, 2, \ldots, M$, and $\max\limits_{0 \le U^* \le 1}\left\{12(u^*)^2 - 12u^* + 2\right\} = 1$, obtain

$$\begin{aligned} \|K\|_\infty &= \max\left\{ \begin{array}{c} -\frac{1}{\varepsilon}\left(12\left(u_{i,j}^*\right)^2 - 12u_{i,j}^* + 2\right) + 2\left(\frac{\tau}{\zeta^2} - \frac{\varepsilon}{\zeta^2}\right);\ -\frac{1}{\varepsilon}\left(12\left(u_{l,k}^*\right)^2 - 12u_{l,k}^* + 2\right) \\ +4\left(\frac{\tau}{\zeta^2} - \frac{\varepsilon}{\zeta^2}\right);\ -\frac{1}{\varepsilon}\left(12\left(u_{l,k}^*\right)^2 - 12u_{l,k}^* + 2\right) + 4\left(\frac{\tau}{\zeta^2} - \frac{\varepsilon}{\zeta^2}\right) \end{array} \right\} \\ &= 4\frac{\tau}{\zeta^2} - 4\frac{\varepsilon}{\zeta^2} + \frac{1}{\varepsilon}. \end{aligned}$$

Theorem 10. *$DF(U^*, \mu^*)$ is invertible if and only if $\lambda_0 > \dfrac{2\varepsilon - 1}{\varepsilon\tau}$.*

Proof.

$$
\begin{aligned}
DF(U^*, \mu^*) \text{ is invertible} \quad & iff \quad || \, 2 - \tau\lambda_0\chi_{\Omega\setminus D} - 4\frac{\varepsilon}{\zeta^2} + 4\frac{\tau}{\zeta^2} I \, ||_\infty < ||K||_\infty \\
& iff \quad 2 - \lambda_0\tau - 4\frac{\varepsilon}{\zeta^2} + 4\frac{\tau}{\zeta^2} < \|K\|_\infty \\
& iff \quad 2 - \lambda_0\tau - 4\frac{\varepsilon}{\zeta^2} + 4\frac{\tau}{\zeta^2} < \frac{1}{\varepsilon} - 4\frac{\varepsilon}{\zeta^2} + 4\frac{\tau}{\zeta^2} \\
& iff \quad \lambda_0\tau > 2 - \frac{1}{\varepsilon} \\
& iff \quad \lambda_0 > \frac{2\varepsilon - 1}{\varepsilon\tau}.
\end{aligned}
$$

□

Corollary 8. *If $F \in C^2(a, b)$ for all $B((U^*, \mu^*), r_1)$, then $DF(U, \mu)$ is invertible and*

$$
\left\| DF(U, \mu)^{-1} \right\|_\infty \leq \frac{k}{k-1} \left\| DF(U^*, \mu^*)^{-1} \right\| = r_1, \text{ where } k > 1.
$$

Proof.

$$
\begin{aligned}
DF(U, \mu) &= DF(U, \mu) - DF(U^*, \mu^*) + DF(U^*, \mu^*) \\
&= DF(U^*, \mu^*) \left[I + DF(U^*, \mu^*)^{-1} (DF(U, \mu) - DF(U^*, \mu^*)) \right].
\end{aligned}
$$

Since $F \in C^2(a, b)$, then $DF \in C^1(a, b)$; in particular, DF is continuous at (U^*, μ^*), and by the definition of continuity of DF at (U^*, μ^*), there exist $\epsilon > 0$, and $r_1 > 0$ such that $\epsilon = \frac{1}{k\left\| DF(U^*, \mu^*)^{-1} \right\|_\infty}$, and for any $U \in B((U^*, \mu^*), r_1)$, there is

$$
\| DF(U, \mu) - DF(U^*, \mu^*) \|_\infty < \epsilon,
$$

and

$$
\begin{aligned}
& \left\| DF(U^*, \mu^*)^{-1} (DF(U, \mu) - DF(U^*, \mu^*)) \right\|_\infty \\
& \leq \left\| DF(U^*, \mu^*)^{-1} \right\|_\infty \| (DF(U, \mu) - DF(U^*, \mu^*)) \|_\infty \\
& \leq \frac{1}{k} < 1,
\end{aligned}
$$

and hence, $I + DF(U^*, \mu^*)^{-1}(DF(U, \mu) - DF(U^*, \mu^*))$ is invertible (Von-Neumann Lemma). Then, for every $(U, \mu) \in B((U^*, \mu^*), r_1)$, there is

$$
\begin{aligned}
\left\|DF(U,\mu)^{-1}\right\|_\infty &= \left\|DF(U^*,\mu^*)\left[I + DF(U^*,\mu^*)^{-1}(DF(U,\mu) - DF(U^*,\mu^*))\right]^{-1}\right\|_\infty \\
&= \left\|I + DF(U^*,\mu^*)^{-1}(DF(U,\mu) - DF(U^*,\mu^*))\right\|_\infty^{-1} \|F(U^*,\mu^*)\|_\infty^{-1} \\
&= \frac{1}{\left\|I + DF(U^*,\mu^*)^{-1}(DF(U,\mu) - DF(U^*,\mu^*))\right\|_\infty} \|F(U^*,\mu^*)\|_\infty^{-1} \\
&\le \frac{1}{1 - \|DF(U^*,\mu^*)\|_\infty \|DF(U,\mu) - DF(U^*,\mu^*)\|_\infty} \|F(U^*,\mu^*)\|_\infty^{-1} \\
&\le \frac{1}{1-\frac{1}{K}} \|F(U^*,\mu^*)\|_\infty^{-1} \\
&\le \frac{k}{k-1} \left\|DF(U^*,\mu^*)^{-1}\right\|_\infty.
\end{aligned}
$$

□

Lemma 5. *If $F \in C^2(a,b)$ and $(U^n,\mu^*) \in B((U^*,\mu^*),r_1)$, then*

$$\left\|DF(U^n,\mu^*)\left(U^{n+1} - U^*\right)\right\|_\infty \le r_2|U^n - U^*|,$$

where

$$r_2 = \frac{1}{2} \sup_{\theta \in B(U^*,r_1)} \left\|D^2F(\theta,\mu^*)\right\|_\infty.$$

Proof. Let

$$\phi(t) = F(U^n + t(U^* - U^n),\mu^*) - F(U^n,\mu^*) - t\, DF(U^n,\mu^*)(U^* - U^n),$$

so that

$$\phi'(t) = DF(U^n + t(U^* - U^n),\mu^*)(U^* - U^n) - DF(U^n,\mu^*)(U^* - U^n).$$

By the fundamental theorem of calculus, there is $\phi(1) - \phi(0) = \int_0^1 \phi'(t)dt$. Hence,

$$
\begin{aligned}
&F(U^*,\mu^*) - F(U^n,\mu^*) - DF(U^n,\mu^*)(U^* - U^n) \\
&= \int_0^1 [DF(U^n + t(U^* - U^n),\mu^*)(U^* - U^n) - DF(U^n,\mu^*)(U^* - U^n)]\, dt.
\end{aligned}
$$

But $F(U^*,\mu^*) = 0$, so if norms of the previous equality are taken, the following can be obtained:

$$
\begin{aligned}
&\|-F(U^n,\mu^*) - DF(U^n,\mu^*)(U^* - U^n)\|_\infty \\
&\le \int_0^1 \|DF(U^n + t(U^* - U^n),\mu^*)(U^* - U^n) - DF(U^n,\mu^*)(U^* - U^n)\|_\infty\, dt \\
&\le \int_0^1 \|DF(U^n + t(U^* - U^n),\mu^*) - DF(U^n,\mu^*)\|_\infty \|(U^* - U^n)\|_\infty\, dt,
\end{aligned}
$$

since $DF \in C^1(a,b)$, by the mean value theorem, there is

$$\|DF(U^n + t(U^* - U^n),\mu^*) - DF(U^n,\mu^*)\|_\infty \le \|t\,(U^* - U^n)\|_\infty . \sup_{\theta \in B(U^*,r_1)} \left\|D^2F(\theta,\mu^*)\right\|_\infty.$$

Hence,

$$\begin{aligned}\|-F(U^n,\mu^*)-DF(U^n,\mu^*)(U^*-U^n)\|_\infty &\le \|(U^*-U^n)\|_\infty^2 \sup_{\theta\in B(U^*,r_1)}\left\|D^2F(\theta,\mu^*)\right\|_\infty \int_0^1 t dt\\ &\le \frac{1}{2}\|(U^*-U^n)\|_\infty^2 \sup_{\theta\in B(U^*,r_1)}\left\|D^2F(\theta,\mu^*)\right\|_\infty.\end{aligned}$$

If $r_2=\frac{1}{2}\sup_{\theta\in B(U^*,r_1)}\left\|D^2F(\theta,\mu^*)\right\|_\infty$ is taken, then

$$\|-F(U^n,\mu^*)-DF(U^n,\mu^*)(U^*-U^n)\|_\infty \le r_2\|(U^*-U^n)\|_\infty^2.$$

By Newton's Raphson method, there is

$$\left(U^{n+1},\mu^*\right)=-DF(U^n,\mu^*)^{-1}F(U^n,\mu^*)+(U^n,\mu^*),$$

which implies that

$$U^{n+1}-U^n=-DF(U^n,\mu^*)^{-1}F(U^n,\mu^*),$$

and

$$DF(U^n,\mu^*)\left(U^{n+1}-U^n\right)=-F(U^n,\mu^*).$$

Hence,

$$\left\|DF(U^n,\mu^*)\left(U^{n+1}-U^n\right)-DF(U^n,\mu^*)(U^*-U^n)\right\|_\infty \le r_2\|U^*-U^n\|_\infty^2,$$

which is equivalent to

$$\left\|DF(U^n,\mu^*)\left(U^{n+1}-U^*\right)\right\|_\infty \le r_2\|U^*-U^n\|_\infty^2.$$

□

Theorem 11. *Let $\left(U^0,\mu^*\right)\in B((U^*,\mu^*),r_3)\subset B((U^*,\mu^*),r_1)$; then, $DF\left(U^{n+1},\mu^*\right)$ is invertible and $\{\left(U^{n+1},\mu^*\right)\}$ converges to (U^*,μ^*).*

Proof. Suppose that $\left(U^0,\mu^*\right)\in B((U^*,\mu^*),r_3)\subset B((U^*,\mu^*),r_1)$; then, $DF\left(U^0,\mu^*\right)$ is invertible and $\left\|DF\left(U^0,\mu^*\right)^{-1}\right\|_\infty \le r_1$. Thus, $\left(U^1,\mu^*\right)$ is well defined and, by Newton's method, there is $\left(U^1,\mu^*\right)=\left(U^0,\mu^*\right)-DF\left(U^0,\mu^*\right)^{-1}F\left(U^0,\mu^*\right)$ which implies that

$$\left(U^1-U^0,\mu^*\right)DF\left(U^0,\mu^*\right)=-F\left(U^0,\mu^*\right).$$

Suppose that the previous equality holds for $n\in\mathbb{N}$, so that $DF(U^n,\mu^*)$ is invertible and $\left\|DF(U^n,\mu^*)^{-1}\right\|_\infty \le r_1$, where $(U^n,\mu^*)\in B((U^*,\mu^*),r_3)$. Then, $\left(U^{n+1},\mu^*\right)$ is well defined and hence

$$DF(U^n,\mu^*)\left(U^{n+1}-U^n\right)=-F(U^n,\mu^*).$$

But

$$\begin{aligned}\left\|U^{n+1}-U^*\right\|_\infty &= \left\|DF(U^n,\mu^*)^{-1}DF(U^n,\mu^*)\left(U^{n+1}-U^*\right)\right\|_\infty\\ &\le \left\|DF(U^n,\mu^*)^{-1}\right\|_\infty\left\|DF(U^n,\mu^*)\left(U^{n+1}-U^*\right)\right\|_\infty\\ &\le r_1r_2\|U^*-U^n\|_\infty^2,\end{aligned}$$

and $(U^n, \mu^*) \in B((U^*, \mu^*), r_3)$, which implies that

$$\begin{aligned}\left\| U^{n+1} - U^* \right\|_\infty &\leq r_1 r_2 r_3^2 \\ &\leq (r_1 r_2 r_3) r_3 \\ &\leq r_3.\end{aligned}$$

Therefore, $U^{n+1} \in B((U^*, \mu^*), r_3) \subset B(U^*, r_1)$, and $DF\left(U^{n+1}, \mu^*\right)$ is invertible such that

$$\left\| DF\left(U^{n+1}, \mu^*\right)^{-1} \right\|_\infty \leq r_1.$$

Moreover, $(U^n, \mu^*) \in B((U^*, \mu^*), r_3) \subset B(U^*, r_1)$, and $DF(U^n, \mu^*)$ is invertible for every $n \in \mathbb{N}$. So, sequence $\{(U^n, \mu^*)\}_{n \in \mathbb{N}}$ is well defined and

$$\begin{aligned}\left\| U^{n+1} - U^* \right\|_\infty &\leq r_1 r_2 \|U^n - U^*\|_\infty^2 \\ &\leq \beta \|U^n - U^*\|_\infty^2, \text{ where } \beta = r_1 r_2,\end{aligned}$$

which implies that

$$\beta \left\| U^{n+1} - U^* \right\|_\infty \leq \frac{1}{\beta} (\beta \|U^n - U^*\|_\infty)^2 \leq \cdots \leq \frac{1}{\beta} \left(\beta \left\| U^0 - U^* \right\|_\infty \right)^{2^n}.$$

But $\left(U^0, \mu^*\right) \in B((U^*, \mu^*), r_3)$, which implies that $\beta \left\| U^0 - U^* \right\|_\infty \leq \beta r_3 < 1$. Therefore, $\beta \left\| U^{n+1} - U^* \right\|_\infty \to 0$, and hence $\left\{ \left(U^{n+1}, \mu^*\right) \right\}$ converges to (U^*, μ^*). □

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