mathematics

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Primal Structure with Closure Operators and Their Applications

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Abstract: Acharjee et al. have created a new structure in mathematics called a primal. Therefore, the primary goal of this research was to introduce and explore more primal space features. Additionally, we studied some of the fundamental characteristics of two novel operators that we define using primal spaces. Using these new operators, we were able to create a weaker version of the original topology. Finally, we provide some examples to further illustrate our discussion of some of their characteristics.

Keywords: primal; grill; Kuratowski closure operator; suitable; primal topological space

MSC: 54A05; 54A10

Citation: Al-Omari, A.; Alqahtani, M.H. Primal Structure with Closure Operators and Their Applications. *Mathematics* **2023**, *11*, 4946. https://doi.org/10.3390/math11244946

Academic Editors: Lakshmi Kanta Dey and Pratulananda Das

Received: 10 November 2023
Revised: 8 December 2023
Accepted: 10 December 2023
Published: 13 December 2023

1. Introduction

Topology is important in many fields of mathematics and computer science. Many topological principles have been applied to solve numerous natural problems, attracting scholars from various disciplines of the natural and social sciences. In topology, several novel ideas have been introduced, resulting in various new areas of research. Filters are among the most significant structures in classical topology [1], alongside ideals [2] and grills [3]. Ideals were defined for the first time by Kuratowski [2]. On the other hand, the concept of a grill was introduced in [3]. It is worth noting that the concept of an ideal is the dual of a filter; however, ideal topological spaces have assisted scholars in introducing several new fields of topological space [4].

To the best of our knowledge, before [5], there was no literature on the dual construction of a grill.

Recently, Acharjee et al. [5] introduced a new structure called a 'primal'. This structure presents not only several primal-related fundamental features, but also some links between primal topological spaces and topological spaces. Primals [5] appear to be the dual of the concept of grills, while the dual of filters are ideals. Later, in [6], we used primals to establish several new operators in primal topological spaces. Proximity space is one of the common topics mathematics, computer science, and pattern recognition. Recently, Al-Omari et al. [7] introduced a new structure named a primal proximity space. In addition, two new operators were implemented via primal proximity spaces to define and investigate some of their fundamental properties.

As a logical extension of the primal crisp topologies defined in [5], Al-shami et al. [8] proposed the new structure of a primal soft topology, and Ameen et al. [9] introduced a novel fuzzy structure called a fuzzy primal. The rationale for the creation of a unique framework that enables the establishment of new soft ideas and attributes is to enhance research on soft settings. Next, we devise a novel method for creating a soft topology, drawing inspiration from certain soft operators. In conclusion, we validate the significance of soft environments in offering several types of analogs for each classical notion. That is, different forms of belonging connections between soft sets and ordinary points may be used to create different kinds of soft operators and then generate certain types of soft topologies.

In Section 3 of this document, we discuss an innovative category of operator called a primal local closure operator. We define the primal local closure operator and investigate some of its fundamental properties in Section 4. In addition, we describe some of its basic topological features that are appropriate for a primal and define one more operator via the local closure operator. Furthermore, a weaker topology compared to the previous one is obtained via these new operators.

The class of θ-open sets was established by Veličko [10] in 1968. A set U is a θ-open set if each point in U has an open neighborhood and U contains its closure. The union of all θ-open subsets of U in G is the θ-interior of a U, denoted as $Int_\theta(U)$. Naturally, a θ-open set's counterpart is referred to as a θ-closed set. A set U is θ-closed iff $Cl_\theta(U) = \{x \in G : \overline{V} \cap U \neq \emptyset$ for every $V \in \mathcal{T}(x)\}$ specifically if $U = Cl_\theta(U)$, which is the complement of a θ-open set, is said to be θ-closed. It should be emphasized that a space $\tau = \tau_\theta$ iff (G, τ) is regular. Moreover, all θ-open sets form a topology on G that is coarser than τ and is denoted by τ_θ. More fundamental properties of primal spaces and primal soft topological spaces were introduced in [7,8,11–14].

2. Preliminaries

Throughout this entire document, $(G, \mathcal{T})$ and (X, σ) (briefly, G and X) represent topological spaces unless otherwise stated. For any subset A of a space G, $cl(A) = \overline{A}$ and $int(A)$ denote the closure and interior of A, respectively. The powerset of a set G will be symbolized by 2^G. The group of all open neighborhoods of a point x of G is denoted by $\mathcal{T}(x)$. Also, the family of all closed subsets of a space G will be symbolized by $C(G)$. Now, we procure the following notions and results, which will be required in the next section:

Definition 1 ([3])**.** *A collection $\mathcal{G}$ of 2^G is called a grill on G if $\mathcal{G}$ fulfills the requirements listed below:*

1. $\emptyset \notin \mathcal{G}$;
2. *If $H \cup K \in \mathcal{G}$, then $H \in \mathcal{G}$ or $K \in \mathcal{G}$;*
3. *If $H \in \mathcal{G}$ and $H \subseteq K$, then $K \in \mathcal{G}$.*

Definition 2 ([5])**.** *A collection $\mathcal{P} \subseteq 2^G$ is called a primal on G, where G is a nonempty set, if the below conditions hold:*

1. $G \notin \mathcal{P}$;
2. *If $H \cap K \in \mathcal{P}$, then $K \in \mathcal{P}$ or $H \in \mathcal{P}$;*
3. *If $K \in \mathcal{P}$ and $H \subseteq K$, then $H \in \mathcal{P}$.*

Corollary 1 ([5])**.** *A collection $\mathcal{P} \subseteq 2^G$ is a primal on G iff the conditions below hold:*

1. $G \notin \mathcal{P}$;
2. *If $K \notin \mathcal{P}$ and $H \notin \mathcal{P}$, then $H \cap K \notin \mathcal{P}$;*
3. *If $K \notin \mathcal{P}$ and $K \subseteq H$, then $H \notin \mathcal{P}$.*

A primal $\mathcal{P}$ [5] on G with a topological space $(G, \mathcal{T})$ is called a primal topological space $(G, \mathcal{T}, \mathcal{P})$ and denoted by PTS.

Definition 3 ([5])**.** *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. We consider a map $(\cdot)^\diamond : 2^G \to 2^G$ as $A^\diamond(G, \mathcal{T}, \mathcal{P}) = \{x \in G : (\forall U \in \mathcal{T}(x))(A^c \cup U^c \in \mathcal{P})\}$ for any subset A of G and $\mathcal{T}(x)$ is the family of all open neighbourhoods of $x \in G$. We can also write $A^\diamond_{\mathcal{P}}$ as $A^\diamond(G, \mathcal{T}, \mathcal{P})$ to specify the primal as per our requirements.*

Definition 4 ([5])**.** *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. We consider a map $cl^\diamond : 2^G \to 2^G$ as $cl^\diamond(A) = A \cup A^\diamond$, where A is any subset of G.*

Definition 5 ([5])**.** *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. Then, the family $\mathcal{T}^\diamond = \{A \subseteq G | cl^\diamond(A^c) = A^c\}$ is a topology on G induced by topology $\mathcal{T}$ and primal $\mathcal{P}$. It is called a primal topology on G. More details on $\mathcal{T}^\diamond$ can be found in [5].*

3. Primal Local Closure Operators

This section is allocated to displaying a novel primal structure, namely a primal local closure operator. The basic characteristics of this structure are demonstrated.

Definition 6. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. For $K \subseteq G$, we define a map $\Pi : 2^G \to 2^G$ as $\Pi(K)(\mathcal{P}, \mathcal{T})$ = $\{g \in G : K^c \cup (\overline{V})^c \in \mathcal{P}$ for every $V \in \mathcal{T}(g)\}$, where $\mathcal{T}(g) = \{V \in \mathcal{T} : g \in V\}$. To be clear, $\Pi(K)(\mathcal{P}, \mathcal{T})$ is denoted as $\Pi(K)$ for brevity and is called the primal local closure operator of K with respect to $\mathcal{T}$ and $\mathcal{P}$.*

Lemma 1. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. Then, for any $K \subseteq G$ we have $K^{\diamond}_{\mathcal{P}} \subseteq \Pi(K)$.*

Proof. Let $g \in K^{\diamond}_{\mathcal{P}}$. Then, $K^c \cup V^c \in \mathcal{P}$ for all $V \in \mathcal{T}$ and $g \in V$. Since $K^c \cup (\overline{V})^c \subseteq K^c \cup V^c$, we obtain $K^c \cup (\overline{V})^c \in \mathcal{P}$ and hence $g \in \Pi(K)$. □

Example 1. *Let $\mathcal{T} = \{\emptyset, G, \{a\}, \{b\}, \{a, b\}\}$ with $G = \{a, b, c\}$ and $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Let $K = \{a, c\}$. We have $\Pi(K) = \{a, b, c\}$ and $K^{\diamond} = \{c\}$.*

Example 2. *Let $\mathbb{R}$ be the real numbers with topology $\mathcal{T} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. Let $\mathcal{P}_f$ be the primal of all finite subsets of the real line whose complement is not finite. Let $H = \mathbb{R} - \{0, 1\}$. Then, $\Pi(H) = \{a \in \mathbb{R} : H^c \cup (\overline{V})^c = H^c \in \mathcal{P}_f$ for all $V \in \mathcal{T}(a)\} = \mathbb{R}$ and $-1 \notin H^{\diamond}$, which shows that $H^{\diamond} \subset \Pi(H)$.*

Lemma 2. *Let $(G, \mathcal{T})$ be a topological space. If the subset $H \subseteq G$ is:*

1. *Open, then $\overline{H} = Cl_\theta(H)$.*
2. *Closed, then $Int(H) = Int_\theta(H)$.*

Theorem 1. *Let $(G, \mathcal{T}, \mathcal{P})$ and $(G, \mathcal{T}, \mathcal{J})$ be two PTSs and let $K, H \subseteq G$. Thus, the properties below hold:*

(1) If $H \subseteq K$, then $\Pi(H) \subseteq \Pi(K)$.

(2) If $\mathcal{J} \subseteq \mathcal{P}$, then $\Pi(H)(\mathcal{J}) \subseteq \Pi(H)(\mathcal{P})$.

(3) $\Pi(H) = \overline{\Pi(H)} \subseteq Cl_\theta(H)$, and $\Pi(H)$ is closed.

(4) If $K \subseteq \Pi(K)$ and $\Pi(K)$ is open, then $\Pi(K) = Cl_\theta(K)$.

(5) If $H^c \notin \mathcal{P}$, then $\Pi(H) = \emptyset$.

Proof.

(1) Let $g \notin \Pi(K)$. Then, there exists $V \in \mathcal{T}(g)$ such that $K^c \cup (\overline{V})^c \notin \mathcal{P}$. Since $K^c \cup (\overline{V})^c \subseteq H^c \cup (Cl(V))^c$, $H^c \cup (\overline{V})^c \notin \mathcal{P}$. Hence, $g \notin \Pi(H)$. Thus, $G \setminus \Pi(K) \subseteq G \setminus \Pi(H)$ or $\Pi(H) \subseteq \Pi(K)$.

(2) Let $g \notin \Pi(H)(\mathcal{P})$. Now, there exists $V \in \mathcal{T}(g)$ such that $H^c \cup (\overline{V})^c \notin \mathcal{P}$. Since $\mathcal{J} \subseteq \mathcal{P}$, $H^c \cup (\overline{V})^c \notin \mathcal{J}$ and $g \notin \Pi(H)(\mathcal{J})$. Therefore, $\Pi(H)(\mathcal{J}) \subseteq \Pi(H)(\mathcal{P})$.

(3) We have $\Pi(H) \subseteq \overline{\Pi(H)}$ in general. Let $g_1 \in \overline{\Pi(H)}$. Then, $\Pi(H) \cap V \neq \emptyset$ for every $V \in \mathcal{T}(g_1)$. Therefore, there exist some $g_2 \in \Pi(H) \cap V$ and $V \in \mathcal{T}(g_2)$. Since $g_2 \in \Pi(H)$, $H^c \cup (\overline{V})^c \in \mathcal{P}$, and hence $g_1 \in \Pi(H)$. Therefore, we have $\overline{\Pi(H)} \subseteq \Pi(H)$, and hence $\overline{\Pi(H)} = \Pi(H)$. Again, let $g_1 \in \overline{\Pi(H)} = \Pi(H)$; then, $H^c \cup (\overline{V})^c \in \mathcal{P}$ for all $V \in \mathcal{T}(g_1)$. This means that $H \cap \overline{V} \neq \emptyset$ for all $V \in \mathcal{T}(g_1)$. Therefore, $g_1 \in Cl_\theta(H)$. This shows that $\Pi(H) = \overline{\Pi(H)} \subseteq Cl_\theta(H)$.

(4) For any subset K of G, by (3) we have $\overline{\Pi(K)} = \Pi(K) \subseteq Cl_\theta(K)$. Since $\Pi(K)$ is open and $K \subseteq \Pi(K)$, by Lemma 2, $Cl_\theta(K) \subseteq Cl_\theta(\Pi(K)) = \overline{\Pi(K)} = \Pi(K) \subseteq Cl_\theta(K)$, and hence $\Pi(K) = Cl_\theta(K)$.

(5) Suppose that $g \in \Pi(H)$. Then, for all $V \in \mathcal{T}(g)$, $H^c \cup (\overline{V})^c \in \mathcal{P}$. However, $H^c \notin \mathcal{P}$ and $H^c \cup (\overline{V})^c \notin \mathcal{P}$ for all $V \in \mathcal{T}(g)$. This is a logical contradiction. Hence, $\Pi(H) = \emptyset$. □

Lemma 3. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. If $K \in \mathcal{T}_\theta$, then $K \cap \Pi(M) = K \cap \Pi(K \cap M) \subseteq \Pi(K \cap M)$ for any $M \subseteq G$.*

Proof. Let $g \in K \cap \Pi(M)$ and $K \in \mathcal{T}_\theta$. Then, $g \in K$ and $g \in \Pi(M)$. Since $K \in \mathcal{T}_\theta$, then there exists $H \in \mathcal{T}$ such that $g \in H \subseteq \overline{H} \subseteq K$. Let V be an open set, such that $g \in V$. Then, $V \cap H \in \mathcal{T}(g)$ and $[\overline{V \cap H}]^c \cup M^c \in \mathcal{P}$, and $(\overline{V})^c \cup (K \cap M)^c = K^c \cup (\overline{V})^c \cup M^c \subseteq (\overline{H})^c \cup (\overline{V})^c \cup M^c \subseteq M^c \cup (\overline{V} \cap \overline{H})^c \subseteq [\overline{V \cap H}]^c \cup M^c$; hence, $(K \cap M)^c \cup (\overline{V})^c \in \mathcal{P}$. We obtain $g \in \Pi(K \cap M)$, and as a result we have $K \cap \Pi(M) \subseteq \Pi(K \cap M)$. Also, $K \cap \Pi(M) \subseteq K \cap \Pi(K \cap M)$, and by Theorem 1 $\Pi(M \cap K) \subseteq \Pi(M)$ and $\Pi(K \cap M) \cap K \subseteq \Pi(M) \cap K$. Thus, $\Pi(M) \cap K = \Pi(K \cap M) \cap K$. □

Theorem 2. *Let $K, H \subseteq G$ and $(G, \mathcal{T}, \mathcal{P})$ be a PTS. The subsequent properties hold:*

1. $\Pi(\emptyset) = \emptyset$.
2. $\Pi(H) \cup \Pi(K) = \Pi(H \cup K)$.

Proof.
(1) The proof is obvious.
(2) According to Theorem 1, we have $\Pi(H \cup K) \supseteq \Pi(H) \cup \Pi(K)$. Let us demonstrate the reverse inclusion, if $g \notin \Pi(H) \cup \Pi(K)$. Then, g belongs to neither $\Pi(H)$ nor $\Pi(K)$. So, there exist $U_g, V_g \in \mathcal{T}(g)$ such that $[\overline{U_g}]^c \cup H^c \notin \mathcal{P}$ and $[\overline{V_g}]^c \cup K^c \notin \mathcal{P}$. Since $\mathcal{P}$ is additive, $([\overline{U_g}]^c \cup H^c) \cap ([\overline{V_g}]^c \cup K^c) \notin \mathcal{P}$. Moreover, since $\mathcal{P}$ is hereditary and

$$\begin{aligned}
([\overline{U_g}]^c \cup H^c) \cap ([\overline{V_g}]^c \cup K^c) &= [([\overline{U_g}]^c \cup H^c) \cap [\overline{V_g}]^c] \cup [([\overline{U_g}]^c \cup H^c) \cap K^c] \\
&= [[\overline{U_g}]^c \cap [\overline{V_g}]^c] \cup [H^c \cap [\overline{V_g}]^c] \cup [[\overline{U_g}]^c \cap K^c] \cup [H^c \cap K^c] \\
&\subseteq [[\overline{U_g}]^c \cap [\overline{V_g}]^c] \cup [\overline{V_g}]^c \cup [\overline{U_g}]^c \cup [H^c \cap K^c] \\
&\subseteq [\overline{U_g \cap V_g}]^c \cup (H \cup K)^c,
\end{aligned}$$

then $[\overline{U_g \cap V_g}]^c \cup (H \cup K)^c \notin \mathcal{P}$. Since $U_g \cap V_g \in \mathcal{T}(g)$, $g \notin \Pi(H \cup K)$. Hence, $(G \setminus \Pi(H)) \cap (G \setminus \Pi(K) \subseteq G \setminus \Pi(H \cup K)$ or $\Pi(H \cup K) \subseteq \Pi(H) \cup \Pi(K)$. Thus, we obtain $\Pi(H) \cup \Pi(K) = \Pi(H \cup K)$. □

Lemma 4. *Let $K, H \subseteq G$ and $(G, \mathcal{T}, \mathcal{P})$ be a PTS. Then, $\Pi(H) - \Pi(K) = \Pi(H - K) - \Pi(K)$.*

Proof. We have by Theorem 2 $\Pi(H) = \Pi[(H - K) \cup (H \cap K)] = \Pi(H - K) \cup \Pi(H \cap K) \subseteq \Pi(H - K) \cup \Pi(K)$. Thus, $\Pi(H) - \Pi(K) \subseteq \Pi(H - K) - \Pi(K)$. By Theorem 1, $\Pi(H - K) \subseteq \Pi(H)$, and hence $\Pi(H - K) - \Pi(K) \subseteq \Pi(H) - \Pi(K)$. Therefore, $\Pi(H) - \Pi(K) = \Pi(H - K) - \Pi(K)$. □

Corollary 2. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS and $H, K \subseteq G$ with $K^c \notin \mathcal{P}$. Then, $\Pi(H \cup K) = \Pi(H) = \Pi(H - K)$.*

Proof. Since $K^c \notin \mathcal{P}$, by Theorem 1 $\Pi(K) = \emptyset$. By Lemma 4, $\Pi(H) = \Pi(H - K)$, and by Theorem 2 $\Pi(H \cup K) = \Pi(H) \cup \Pi(K) = \Pi(H)$. □

4. Topology Suitable for a Primal Space

This section serves to introduce the topology suitable for a primal in a PTS and investigate some of its properties.

Definition 7 ([6])**.** *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. Then, $\mathcal{T}$ is said to be suitable for the primal $\mathcal{P}$ if $H^c \cup H^\diamond \notin \mathcal{P}$ for all $H \subseteq G$.*

Definition 8. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. We say that $\mathcal{T}$ is Π-suitable for the primal $\mathcal{P}$ if, for every $H \subseteq G$ and $g \in H$, there exists $V \in \mathcal{T}(g)$, such that $\overline{V}^c \cup H^c \notin \mathcal{P}$; then, $H^c \notin \mathcal{P}$.*

If $\mathcal{T}$ is suitable for $\mathcal{P}$, then $\mathcal{T}$ is Π-suitable for $\mathcal{P}$.

Example 3. *Let $G = \{a, b, c\}$ with topology $\mathcal{T} = \{\emptyset, G, \{a\}, \{b\}, \{a, b\}\}$ and the primal $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. It is clear that $\mathcal{T}$ is Π-suitable for the primal $\mathcal{P}$, as shown by the following table. If $A \subseteq G$:*

A	$\Pi(A)$	$A - \Pi(A)$	$[A - \Pi(A)]^c$	$\in \mathcal{P}$ or $\notin \mathcal{P}$
$\emptyset$	$\emptyset$	$\emptyset$	G	$\notin \mathcal{P}$
G	G	$\emptyset$	G	$\notin \mathcal{P}$
$\{a\}$	$\emptyset$	$\{a\}$	$\{b, c\}$	$\notin \mathcal{P}$
$\{b\}$	$\emptyset$	$\{b\}$	$\{a, c\}$	$\notin \mathcal{P}$
$\{c\}$	G	$\emptyset$	$\{G\}$	$\notin \mathcal{P}$
$\{a, b\}$	$\emptyset$	$\{a, b\}$	$\{c\}$	$\notin \mathcal{P}$
$\{a, c\}$	G	$\emptyset$	$\{G\}$	$\notin \mathcal{P}$
$\{b, c\}$	G	$\emptyset$	$\{G\}$	$\notin \mathcal{P}$

We now give some equivalent descriptions of this definition.

Theorem 3. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. The following properties are equivalent for Π-suitable:*

(1) $\mathcal{T}$ is Π-suitable for the primal $\mathcal{P}$;

(2) If a subset $Q \subseteq G$ includes a cover of open sets, and whose complements of its own closure union with Q^c are in $\mathcal{P}$, then $Q^c \notin \mathcal{P}$;

(3) For every $Q \subseteq G$, $Q \cap \Pi(Q) = \emptyset$ implies that $Q^c \notin \mathcal{P}$;

(4) For every $Q \subseteq G$, $(Q - \Pi(Q))^c \notin \mathcal{P}$;

(5) For every $Q \subseteq G$, if there is no nonempty subset K in Q with $K \subseteq \Pi(K)$, then $Q^c \notin \mathcal{P}$.

Proof.

(1) $\Rightarrow$ (2): The proof is obvious.

(2) $\Rightarrow$ (3): Let $x \in Q \subseteq G$. Since $Q \cap \Pi(Q) = \emptyset$, then $x \notin \Pi(Q)$ and there exists $V_x \in \mathcal{T}(x)$ such that $(\overline{V_x})^c \cup Q^c \notin \mathcal{P}$. Thus, we have $Q \subseteq \cup\{V_x : x \in Q\}$ and $V_x \in \mathcal{T}(x)$, and by (2) $Q^c \notin \mathcal{P}$.

(3) $\Rightarrow$ (4): For any $Q \subseteq G$, $Q - \Pi(Q) \subseteq Q$ and $(Q - \Pi(Q)) \cap \Pi(Q - \Pi(Q)) \subseteq (Q - \Pi(Q)) \cap \Pi(Q) = \emptyset$. By (3), $(Q - \Pi(Q))^c \notin \mathcal{P}$.

(4) $\Rightarrow$ (5): By (4), for every $Q \subseteq G$, $(Q - \Pi(Q))^c \notin \mathcal{P}$. Let $Q - \Pi(Q) = J \notin \mathcal{P}$; then, $Q = J \cup (Q \cap \Pi(Q))$, and by Theorem 2 (2) and Theorem 1 (5), $\Pi(Q) = \Pi(J) \cup \Pi(Q \cap \Pi(Q)) = \Pi(Q \cap \Pi(Q))$. Therefore, we have $K = Q \cap \Pi(Q) = Q \cap \Pi(Q \cap \Pi(A)) \subseteq \Pi(Q \cap \Pi(Q)) = \Pi(K)$ and $K = Q \cap \Pi(Q) \subseteq Q$. By the assumption $Q \cap \Pi(Q) = \emptyset$, $(Q - \Pi(Q))^c = Q^c \notin \mathcal{P}$.

(5) $\Rightarrow$ (1): Let $Q \subseteq G$ and assume for $x \in Q$ that there exists $U \in \mathcal{T}(x)$ such that $\overline{U}^c \cup Q^c \notin \mathcal{P}$. Then, $Q \cap \Pi(Q) = \emptyset$ (if $x \in Q \cap \Pi(Q)$, then for every $U \in \mathcal{T}(x)$ we have $\overline{U}^c \cup Q^c \in \mathcal{P}$, which is a contradiction). Suppose that Q contains K such that $K \subseteq \Pi(K)$. Then, $K = K \cap \Pi(K) \subseteq Q \cap \Pi(Q) = \emptyset$. Thus, Q contains no nonempty subset K with $K \subseteq \Pi(K)$. Hence, $Q^c \notin \mathcal{P}$. Thus, $\mathcal{T}$ is Π-suitable for the primal $\mathcal{P}$. $\square$

Theorem 4. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS if $\mathcal{T}$ is Π-suitable for the primal $\mathcal{P}$. The following are equivalent:*

(1) For every $M \subseteq G$, $M \cap \Pi(M) = \emptyset$ implies that $\Pi(M) = \emptyset$;

(2) For every $M \subseteq G$, $\Pi(M - \Pi(M)) = \emptyset$;

(3) For every $M \subseteq G$, $\Pi(M \cap \Pi(M)) = \Pi(M)$.

Proof. First, we demonstrate that (1) holds if $\mathcal{T}$ is Π-suitable for the primal $\mathcal{P}$. Let $M \subseteq G$ and $M \cap \Pi(M) = \emptyset$. Then, by Theorem 3, $M^c \notin \mathcal{P}$, and by Theorem 1 (5) $\Pi(M) = \emptyset$.

(1) $\Rightarrow$ (2): Assume that for every $M \subseteq G$, $M \cap \Pi(M) = \emptyset$ implies $\Pi(M) = \emptyset$. Let $K = M - \Pi(M)$; then,

$$\begin{aligned} K \cap \Pi(K) &= (M - \Pi(M)) \cap \Pi(M - \Pi(M)) \\ &= (M \cap (G - \Pi(M))) \cap \Pi(M \cap (G - \Pi(M))) \\ &\subseteq [M \cap (G - \Pi(M))] \cap [\Pi(M) \cap (\Pi(G - \Pi(M)))] = \emptyset. \end{aligned}$$

By (1), we have $\Pi(K) = \emptyset$. Hence, $\Pi(M - \Pi(M)) = \emptyset$.

(2) $\Rightarrow$ (3): Assume for every $M \subseteq G$, $\Pi(M - \Pi(M)) = \emptyset$.

$$\begin{aligned} M &= (M - \Pi(M)) \cup (M \cap \Pi(M)) \\ \Pi(M) &= \Pi[(M - \Pi(M)) \cup (M \cap \Pi(M))] \\ &= \Pi(M - \Pi(M)) \cup \Pi(M \cap \Pi(M)) \\ &= \Pi(M \cap \Pi(M)). \end{aligned}$$

(3) $\Rightarrow$ (1): Assume for every $M \subseteq G$, $\Pi(M) \cap M = \emptyset$ and $\Pi(\Pi(M) \cap M) = \Pi(M)$. This implies that $\emptyset = \Pi(\emptyset) = \Pi(M)$. □

Theorem 5. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS, so the following properties are equivalent:*

(a) $\mathcal{T} - \{G\} \subseteq \mathcal{P}$;

(b) *If $I^c \notin \mathcal{P}$, then $Int_\theta(I) = \emptyset$;*

(c) *For every clopen H, $H \subseteq \Pi(H)$;*

(d) $G = \Pi(G)$.

Proof.

(a) $\Rightarrow$ (b): Let $I^c \notin \mathcal{P}$ and $\mathcal{T} - \{G\} \subseteq \mathcal{P}$. Assume that $x \in Int_\theta(I)$. Then, there exists $U \in \mathcal{T}$ such that $x \in U \subseteq \overline{U} \subseteq I$ and $I^c \subseteq (\overline{U})^c$. Since $I^c \notin \mathcal{P}$, $(\overline{U})^c \notin \mathcal{P}$. This is contrary to the statement that $\mathcal{T} - \{G\} \subseteq \mathcal{P}$. Therefore, $Int_\theta(I) = \emptyset$.

(b) $\Rightarrow$ (c): Let $x \in H$. Suppose $x \notin \Pi(H)$; then, there exists $U_x \in \mathcal{T}(x)$ such that $H^c \cup (\overline{U_x})^c \notin \mathcal{P}$, and hence $(H \cap \overline{U_x})^c \notin \mathcal{P}$. Since H is clopen, by (b) and Lemma 2, $x \in H \cap U_x = Int(U_x \cap H) \subseteq Int(\overline{U_x} \cap H) = Int_\theta(\overline{U_x} \cap H) = \emptyset$. This is a logical contradiction. Hence, $x \in \Pi(H)$ and $H \subseteq \Pi(H)$.

(c) $\Rightarrow$ (d): Since G is clopen, we have $G = \Pi(G)$.

(d) $\Rightarrow$ (a): $G = \Pi(G) = \{a \in G : (\overline{U})^c \cup G^c = (\overline{U})^c \in \mathcal{P}$ for each $a \in U \in \mathcal{T}\}$. Hence, $\mathcal{T} - \{G\} \subseteq \mathcal{P}$. □

Theorem 6. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. If $\mathcal{T} - \{G\} \subseteq \mathcal{P}$, then $U \subseteq \Pi(U)$ for all $U \in \mathcal{T}_\theta$.*

Proof. In the case $U = \emptyset$, we obviously have $\Pi(U) = \emptyset = U$. Now note that if $\mathcal{T} - \{G\} \subseteq \mathcal{P}$, then $\Pi(G) = G$. In fact, since $x \notin \Pi(G)$, then there exists $V \in \mathcal{T}(x)$ such that $(\overline{V})^c \cup G^c \notin \mathcal{P}$. Hence, $(\overline{V})^c \notin \mathcal{P}$ is a contradiction. Now, by using Lemma 3, we have for any $U \in \mathcal{T}_\theta$, $U = \Pi(G) \cap U \subseteq \Pi(G \cap U) = \Pi(U)$. Thus, $U \subseteq \Pi(U)$. □

Theorem 7. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. $\mathcal{T}$ is Π-suitable for the primal $\mathcal{P}$. Then, for every $H \in \mathcal{T}_\theta$ and any subset Q of G, $\overline{\Pi(H \cap Q)} = \Pi(H \cap Q) \subseteq \Pi(H \cap \Pi(Q)) \subseteq Cl_\theta(H \cap \Pi(Q))$.*

Proof. By Theorem 1 and (3) of Theorem 4, we determine that $\Pi(Q \cap H) = \Pi((Q \cap H) \cap \Pi(Q \cap H)) \subseteq \Pi(H \cap \Pi(Q))$. Moreover, by Theorem 1, $\overline{(\Pi(H \cap Q)} = \Pi(H \cap Q) \subseteq \Pi(H \cap \Pi(Q)) \subseteq Cl_\theta(H \cap \Pi(Q))$. □

5. New Primal Space Operator

In this section, the new operator in primal space is presented, denoted as $\overrightarrow{\Pi}$. The basic characteristics of this structure are demonstrated.

Definition 9. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. An operator $\overrightarrow{\Pi} : 2^G \to 2^G$ is defined as $\overrightarrow{\Pi}(A) = \{x \in G : \exists U \in \mathcal{T}(x)$ and $(\overline{U} - A)^c \notin \mathcal{P}\}$ for every $A \subseteq G$.*

Example 4. *Let $G = \{a, b, c\}$ with topology $\mathcal{T} = \{\emptyset, G, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and the primal $\mathcal{P} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. It is clear that:*

1. *If $A = \{a, b\}$, then $\overrightarrow{\Pi}(A) = G$.*
2. *If $A = \{c\}$, then $\overrightarrow{\Pi}(A) = \{b, c\}$.*

The following theorem includes a number of fundamental truths about the behavior of the operator $\overrightarrow{\Pi}$.

Theorem 8. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. Then, the below characteristics hold:*

(1) *If $Q \subseteq G$, then $\overrightarrow{\Pi}(Q) = [\Pi(Q^c)]^c$.*

(2) *If $Q \subseteq G$, then $\overrightarrow{\Pi}(Q)$ is open.*

(3) *If $Q \subseteq K$, then $\overrightarrow{\Pi}(Q) \subseteq \overrightarrow{\Pi}(K)$.*

(4) *If $Q, K \subseteq G$, then $\overrightarrow{\Pi}(Q \cap K) = \overrightarrow{\Pi}(Q) \cap \overrightarrow{\Pi}(K)$.*

(5) *If $Q \subseteq G$, then $\overrightarrow{\Pi}(Q) = \overrightarrow{\Pi}(\overrightarrow{\Pi}(Q))$ iff $\Pi(Q^c) = \Pi(\Pi(Q^c))$.*

(6) *If $Q^c \notin \mathcal{P}$, then $\overrightarrow{\Pi}(Q) = G - \Pi(G)$.*

(7) *If $Q \subseteq G$ and $I^c \notin \mathcal{P}$, then $\overrightarrow{\Pi}(Q - I) = \overrightarrow{\Pi}(Q)$.*

(8) *If $Q \subseteq G$ and $I^c \notin \mathcal{P}$, then $\overrightarrow{\Pi}(Q \cup I) = \overrightarrow{\Pi}(Q)$.*

(9) *If $[(Q - K) \cup (K - Q)]^c \notin \mathcal{P}$, then $\overrightarrow{\Pi}(Q) = \overrightarrow{\Pi}(K)$.*

Proof.

(1) Let $x \in \overrightarrow{\Pi}(Q)$. Then, there exists $U \in \mathcal{T}(x)$ such that $(\overline{U})^c \cup Q = (\overline{U} \cap (Q^c))^c = (\overline{U} - Q)^c \notin \mathcal{P}$. Thus, $x \notin \Pi(Q^c)$ and $x \in [\Pi(Q^c)]^c$. Conversely, let $x \in [\Pi(Q^c)]^c$; then, $x \notin \Pi(Q^c)$, and there exists $U \in \mathcal{T}(x)$ such that $(\overline{U})^c \cup (Q^c)^c = (\overline{U} - Q)^c \notin \mathcal{P}$. Hence, $x \in \overrightarrow{\Pi}(Q)$ and $\overrightarrow{\Pi}(Q) = [\Pi(Q^c)]^c$.

(2) This derives from Theorem 1 (3).

(3) This derives from Theorem 1 (1).

(4) It derives from (3) that $\overrightarrow{\Pi}(Q \cap K) \subseteq \overrightarrow{\Pi}(Q)$ and $\overrightarrow{\Pi}(Q \cap K) \subseteq \overrightarrow{\Pi}(K)$. Hence, $\overrightarrow{\Pi}(Q \cap K) \subseteq \overrightarrow{\Pi}(Q) \cap \overrightarrow{\Pi}(K)$. Now, let $x \in \overrightarrow{\Pi}(Q) \cap \overrightarrow{\Pi}(K)$. Then, there exists $U, V \in \mathcal{T}(x)$ such that $(\overline{U} - Q)^c \notin \mathcal{P}$ and $(\overline{V} - K)^c \notin \mathcal{P}$. Let $M = U \cap V \in \mathcal{T}(x)$, and we obtain $(\overline{M} - Q)^c \notin \mathcal{P}$ and $(\overline{M} - K)^c \notin \mathcal{P}$ by heredity. Thus, $[\overline{M} - (Q \cap K)]^c = (\overline{M} - Q)^c \cap (\overline{M} - K)^c \notin \mathcal{P}$ by Corollary 1, and hence $x \in \overrightarrow{\Pi}(Q \cap K)$. We have shown that $\overrightarrow{\Pi}(Q) \cap \overrightarrow{\Pi}(K) \subseteq \overrightarrow{\Pi}(Q \cap K)$, and the proof is completed.

(5) This follows from the fact that:

(a) $\overrightarrow{\Pi}(Q) = [\Pi(Q^c)]^c$.

(b) $\overrightarrow{\Pi}(\overrightarrow{\Pi}(Q)) = G - \Pi[G - (G - \Pi(Q^c))] = [\Pi(\Pi(Q^c))]^c$.

(6) By Corollary 2, we determine that $\Pi(Q^c) = \Pi(G)$ if $Q^c \notin \mathcal{P}$. Then, $\overrightarrow{\Pi}(Q) = [\Pi(Q^c)]^c = [\Pi(G)]^c$.

(7) This follows from Corollary 2 and $\overrightarrow{\Pi}(Q - I) = G - \Pi[G - (Q - I)] = G - \Pi[(G - Q) \cup I] = G - \Pi(G - Q) = \overrightarrow{\Pi}(Q)$.

(8) This follows from Corollary 2 and $\overrightarrow{\Pi}(Q \cup I) = G - \Pi[G - (Q \cup I)] = G - \Pi[(G - Q) - I] = G - \Pi(G - Q) = \overrightarrow{\Pi}(Q)$.

(9) Assume that $[(Q - K) \cup (K - Q)]^c \notin \mathcal{P}$. Let $Q - K = I$ and $K - Q = J$. Observe that $I^c, J^c \notin \mathcal{P}$ by heredity. Also, we note that $K = (Q - I) \cup J$. Thus, $\overrightarrow{\Pi}(Q) = \overrightarrow{\Pi}(Q - I) = \overrightarrow{\Pi}[(Q - I) \cup J] = \overrightarrow{\Pi}(K)$ by (7) and (8). □

Corollary 3. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. Then, $A \subseteq \overrightarrow{\Pi}(A)$ for every $A \in \mathcal{T}_\theta$.*

Proof. We know that $\overrightarrow{\Pi}(A) = G - \Pi(G - A)$. Now, $\Pi(G - A) \subseteq Cl_\theta(G - A) = G - A$, since $G - A$ is θ-closed. Therefore, $A = G - (G - A) \subseteq G - \Pi(G - A) = \overrightarrow{\Pi}(A)$. □

Theorem 9. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS and $H \subseteq G$. Then, the below properties hold:*

1. $\overrightarrow{\Pi}(H) = \bigcup\{N \in \mathcal{T} : (\overline{N} - H)^c \notin \mathcal{P}\}$;
2. $\overrightarrow{\Pi}(H) \supseteq \bigcup\{N \in \mathcal{T} : (\overline{N} - H)^c \cup (H - \overline{N})^c \notin \mathcal{P}\}$.

Proof.

(1) This comes logically from the definition of the $\overrightarrow{\Pi}$-operator.

(2) Since $\mathcal{P}$ is hereditary, it is clear that $\bigcup\{N \in \mathcal{T} : (\overline{N} - H)^c \cup (H - \overline{N})^c \notin \mathcal{P}\} \subseteq \bigcup\{N \in \mathcal{T} : (\overline{N} - H)^c \notin \mathcal{P}\} = \overrightarrow{\Pi}(H)$ for every $H \subseteq G$. □

We will conclude this part with some technical results relating to the idempotency of the primal local closure operator and the $\overrightarrow{\Pi}$-operator.

Lemma 5. *For $H \subseteq G$ and a PTS $(G, \mathcal{T}, \mathcal{P})$, we have $\overrightarrow{\Pi}(H^c) \subseteq \overrightarrow{\Pi}[\overrightarrow{\Pi}(H^c)]$ iff $\Pi(\Pi(H)) \subseteq \Pi(H)$.*

Proof. For $H \subseteq G$, we have

$$\begin{aligned} \Pi(\Pi(H)) \subseteq \Pi(H) \ \text{ iff } & [\Pi(H)]^c \subseteq [\Pi(\Pi(H))]^c \\ \text{iff } & [\Pi((H^c)^c)]^c \subseteq [\Pi([\Pi((H^c)^c]^c)^c]^c \\ \text{iff } & \overrightarrow{\Pi}(H^c) \subseteq [\Pi(\overrightarrow{\Pi}(H^c))^c]^c \\ \text{iff } & \overrightarrow{\Pi}(H^c) \subseteq \overrightarrow{\Pi}[\overrightarrow{\Pi}(H^c)]. \end{aligned}$$

□

Corollary 4. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. The following criteria are equivalent:*

1. *For all $H \subseteq G$, we have $\Pi(\Pi(H)) \subseteq \Pi(H)$;*
2. *For all $H \subseteq G$, we have $\overrightarrow{\Pi}(H) \subseteq \overrightarrow{\Pi}(\overrightarrow{\Pi}(H))$.*

6. New Topology via Primal Spaces

Now, we introduce a new topology induced by the primal local closure operator.

Theorem 10. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. If $\beta = \{H \subseteq G : H \subseteq \overrightarrow{\Pi}(H)\}$, then β is a topology on G.*

Proof. Let $\beta = \{H \subseteq G : H \subseteq \overrightarrow{\Pi}(H)\}$. Since $G \notin \mathcal{P}$, by Theorem 1 (5) $\Pi(\emptyset) = \emptyset$ and $\overrightarrow{\Pi}(G) = G - \Pi(G - G) = G - \Pi(\emptyset) = G$. Moreover, $\overrightarrow{\Pi}(\emptyset) = G - \Pi(G - \emptyset) = G - G = \emptyset$. Therefore, we determine that $\emptyset \subseteq \overrightarrow{\Pi}(\emptyset)$ and $G \subseteq \overrightarrow{\Pi}(G) = G$, and thus $\emptyset$ and $G \in \beta$.

Now, if $H, K \in \beta$, then $H \cap K \subseteq \overrightarrow{\Pi}(H) \cap \overrightarrow{\Pi}(K) = \overrightarrow{\Pi}(K \cap H)$. This implies that $H \cap K \in \beta$.

If $\{H_\alpha : \alpha \in \Delta\} \subseteq \sigma$, then $H_\alpha \subseteq \overrightarrow{\Pi}(H_\alpha) \subseteq \overrightarrow{\Pi}(\bigcup_{\alpha \in \Delta} H_\alpha)$ for every $\alpha \in \Delta$, and hence $\bigcup H_\alpha \subseteq \overrightarrow{\Pi}(\bigcup_{\alpha \in \Delta} H_\alpha)$. This shows that β is a topology on G. □

Lemma 6. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. A set F is closed in (G, β) if and only if $\Pi(F) \subseteq F$.*

Proof. F is closed in (G, β) iff F^c is open in (G, β); $F^c \subseteq \overrightarrow{\Pi}(F^c)$; $F^c \subseteq [\Pi[(F^c)^c]]^c$; $F^c \subseteq [\Pi(F)]^c$; and $\Pi(F) \subseteq F$. □

The following example shows that the topology β exists.

Example 5. *Let $G = \{a, b, c\}$ with topology $\mathcal{T} = \{\emptyset, G, \{a\}, \{b\}, \{a, b\}\}$ and the primal $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. It is clear that $\beta = \{\emptyset, G, \{c\}, \{b, c\}, \{a, c\}\}$, as shown by the following table. If $A \subseteq G$,*

A	$\Pi(G - A)$	$\overrightarrow{\Pi}(A)$
$\emptyset$	G	$\emptyset$
G	$\emptyset$	G
$\{a\}$	G	$\emptyset$
$\{b\}$	G	$\emptyset$
$\{c\}$	$\emptyset$	G
$\{a, b\}$	G	$\emptyset$
$\{a, c\}$	$\emptyset$	G
$\{b, c\}$	$\emptyset$	G

Corollary 5. *For a PTS $(G, \mathcal{T}, \mathcal{P})$, we have $\beta \subseteq \mathcal{T}^\diamond$.*

Proof. Consider the above lemma and the reality that for every H we have $H^\diamond \subseteq \Pi(H)$. □

Theorem 11. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. If for each $H \subseteq G$ we have $H^\diamond = \Pi(H)$, then $\beta = \mathcal{T}^\diamond$.*

Proof. F is closed in $\mathcal{T}^\diamond$ iff $F = F \cup F^\diamond$; $F^\diamond \subseteq F$; $\Pi(F) \subseteq F$; and F is closed in β by Lemma 6. □

Theorem 12. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. If there exists a set H such that $\Pi(\Pi(H)) \nsubseteq \Pi(H)$, then $\mathcal{T}^\diamond \nsubseteq \beta$, and therefore β and $\mathcal{T}^\diamond$ are not the same.*

Proof. Since $\Pi(H)$ is closed in $\mathcal{T}$ by Theorem 1, but for any subset H such that $\Pi(\Pi(H)) \nsubseteq \Pi(H)$, then by Lemma 6 $\Pi(H)$ is not closed in β, implying that $\mathcal{T}^\diamond \nsubseteq \beta$. □

If the primal closure operator is idempotent, then the closure operator in β can be defined similarly to the closure operator in $\mathcal{T}^\diamond$.

Theorem 13. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. If for each $H \subseteq G$ we have $\Pi(\Pi(H)) \subseteq \Pi(H)$, then $Cl_\beta(H) = H \cup \Pi(H)$.*

Proof. Since $\Pi(H \cup \Pi(H)) = \Pi(H) \cup \Pi(\Pi(H)) = \Pi(H) \subseteq H \cup \Pi(H)$, by Lemma 6 we know that $H \cup \Pi(H)$ is a closed set in topology β containing H. Let us prove that $H \cup \Pi(H)$ is a minimal closed set in topology β containing H. Let $x \in \Pi(H) \cup H$. If $x \in H$, then $x \in Cl_\beta(H)$. If $x \in \Pi(H)$, then for each open set $A \in \mathcal{T}(x)$, and $H^c \cup (\overline{A})^c \in \mathcal{P}$. From $[Cl_\beta(H)]^c \subseteq H^c$ and the property of a primal space we have $(\overline{A})^c \cup [Cl_\beta(H)]^c \in \mathcal{P}$. Therefore, $x \in \Pi[Cl_\beta(H)]$, and since $Cl_\beta(H)$ is closed in β, $\Pi[Cl_\beta(H)] \subseteq Cl_\beta(H)$, and we have $x \in Cl_\beta(H)$. Hence, $Cl_\beta(H) = H \cup \Pi(H)$ for each $H \subseteq G$. □

Theorem 14. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. Then, $\mathcal{T}$ is Π-suitable for the primal $\mathcal{P}$ if and only if $[\overrightarrow{\Pi}(H) - H]^c \notin \mathcal{P}$ for every $H \subseteq G$.*

Proof. *Necessity*. Assume that $\mathcal{T}$ is Π-suitable for the primal $\mathcal{P}$ and let $H \subseteq G$. Notice that $x \in \overrightarrow{\Pi}(H) - H$ iff $x \notin H$, $x \notin \Pi(G - H)$ iff $x \notin H$, and there exists $U_x \in \mathcal{T}(x)$ such that $(\overline{U_x} - H)^c = (\overline{U_x})^c \cup H \notin \mathcal{P}$ (since $\mathcal{T}$ is Π-suitable for the primal $\mathcal{P}$, then $H \notin \mathcal{P}$) iff there exists $U_x \in \mathcal{T}(x)$ such that $x \in [\overline{U_x} - H]^c \notin \mathcal{P}$. Now, for all $x \in \overrightarrow{\Pi}(H) - H$ and $U_x \in \mathcal{T}(x)$, $[\overline{U_x} \cap (\overrightarrow{\Pi}(H) - H)]^c = [\overline{U_x}]^c \cup [(\overrightarrow{\Pi}(H) - H)]^c \notin \mathcal{P}$ by heredity, and hence $[\overrightarrow{\Pi}(H) - H]^c \notin \mathcal{P}$ by the assumption that $\mathcal{T}$ is Π-suitable with the primal $\mathcal{P}$.

Sufficiency. Let $H \subseteq G$ and assume that for all $x \in H$, there exists $U_x \in \mathcal{T}(x)$ such that $(\overline{U_x})^c \cup H^c \notin \mathcal{P}$. Notice that $\overrightarrow{\Pi}(H^c) - (H^c) = H - \Pi(H) = \{x :$ there exists $U_x \in \mathcal{T}(x)$

such that $x \in (\overline{U_x})^c \cup H^c \notin \mathcal{P}\}$. Thus, we obtain $[H - \Pi(H)]^c = [\overrightarrow{\Pi}(H^c) - (H^c)]^c \notin \mathcal{P}$, and hence $\mathcal{T}$ is Π-suitable for the primal $\mathcal{P}$. □

Theorem 15. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS such that $\mathcal{T}$ is Π-suitable for the primal $\mathcal{P}$ and the primal closure operator is idempotent; then, $\beta = \{\overrightarrow{\Pi}(H) - I : H \subseteq G,\ I^c \notin \mathcal{P}\}$.*

Proof. By Theorem 8, we know that $\overrightarrow{\Pi}[\overrightarrow{\Pi}(H) - I] = \overrightarrow{\Pi}[\overrightarrow{\Pi}(H)] \supseteq \overrightarrow{\Pi}(H) \supseteq \overrightarrow{\Pi}(H) - I$ according to Corollary 4. Thus, all sets of the form $\overrightarrow{\Pi}(H) - I$ are in β according to Theorem 10.

Let $H \in \beta$. Therefore, $H \subseteq \overrightarrow{\Pi}(H)$. However, form $\mathcal{T}$ is Π-suitable for the primal $\mathcal{P}$. By Theorem 14, we have $[\overrightarrow{\Pi}(H) - H]^c \notin \mathcal{P}$, that is, there exists I such that $I = \overrightarrow{\Pi}(H) - H$. Hence, $H = \overrightarrow{\Pi}(H) - I$ and $I^c \notin \mathcal{P}$. Thus, $H \in \{\overrightarrow{\Pi}(H) - I : H \subseteq G,\ I^c \notin \mathcal{P}\} = \beta$. □

Lemma 7. *Let $(G, \mathcal{T})$ be a TS. If either $H \in \mathcal{T}$ or $K \in \mathcal{T}$, $Int((\overline{H \cap K})) = Int(\overline{H}) \cap Int(\overline{K})$.*

Proof. This is the direct result of Lemma 3.5 of [15]. □

Theorem 16. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. Let $\varphi = \{H \subseteq G : H \subseteq Int\left(\overline{\overrightarrow{\Pi}(H)}\right)\}$; then, φ is a form of topology on G.*

Proof. For Theorem 8, $H \subseteq G$, $\overrightarrow{\Pi}(H)$ is an open set, and $\beta \subset \varphi$. Thus, $\emptyset, G \in \varphi$. Let $H, K \in \varphi$. Then, using Theorem 8 and Lemma 7, we obtain $H \cap K \subset Int\left(\overline{\overrightarrow{\Pi}(H)}\right) \cap Int\left(\overline{\overrightarrow{\Pi}(K)}\right) = Int\left(\overline{\overrightarrow{\Pi}(H) \cap \overrightarrow{\Pi}(K)}\right) = Int\left(\overline{\overrightarrow{\Pi}(H \cap K)}\right)$. Therefore, $H \cap K \in \varphi$. Let $H_\alpha \in \varphi$ for each $\alpha \in \Delta$. By Theorem 8, for each $\alpha \in \Delta$, $H_\alpha \subseteq Int\left[\overline{\overrightarrow{\Pi}(H_\alpha)}\right] \subseteq Int\left[\overline{\overrightarrow{\Pi}(\cup H_\alpha)}\right]$, and hence $\cup H_\alpha \subset Int\left[\overline{\overrightarrow{\Pi}(\cup H_\alpha)}\right]$. Hence $\cup H_\alpha \in \varphi$. Therefore, φ is a topology on X. □

The strict inequality between these two topologies has a required condition, which is provided by the lemma below.

Lemma 8. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. If $\beta \subsetneqq \varphi$, then there exist a set H and a point $x \in H$ such that*

1. *$[\overline{K} - H]^c \in \mathcal{P}$ for each $K \in \mathcal{T}(x)$;*
2. *There exist $F \in \mathcal{T}(x)$ and an open set $M \subseteq F$ such that: $[\overline{M} - H]^c \notin \mathcal{P}$.*

Proof. If $\beta \subsetneqq \varphi$, then there exists $H \in \varphi - \beta$. Since $H \notin \beta$, there exists $x \in H$ such that

$$\begin{aligned}
x \notin \overrightarrow{\Pi}(H) &\iff x \notin G - \Pi[G - H] \\
&\iff x \in \Pi[G - H] \\
&\iff \forall\ K \in \mathcal{T}(x),\ \ \overline{K}^c \cup H \in \mathcal{P} \\
&\iff \forall\ K \in \mathcal{T}(x),\ \ [\overline{K} \cap H^c]^c \in \mathcal{P} \\
&\iff \forall\ K \in \mathcal{T}(x),\ \ [\overline{K} - H]^c \in \mathcal{P}.
\end{aligned}$$

Since $H \in \varphi$, for each $y \in H$, we have

$$
\begin{aligned}
& y \in Int\left(\overline{\overrightarrow{\Pi}(H)}\right) \\
\Longleftrightarrow\ & \exists D \in \mathcal{T}(y),\ \ D \subseteq \overline{\overrightarrow{\Pi}(H)} \\
\Longleftrightarrow\ & \exists D \in \mathcal{T}(y),\ \ \forall\ z \in D, \forall\ F \in \mathcal{T}(z),\ \ F \cap \overrightarrow{\Pi}(H) \neq \emptyset \\
\Longleftrightarrow\ & \exists D \in \mathcal{T}(y),\ \ \forall\ F \subseteq D,\ \ [F \in \mathcal{T} \Rightarrow\ \ F \cap \overrightarrow{\Pi}(H) \neq \emptyset] \\
\Longleftrightarrow\ & \exists D \in \mathcal{T}(y),\ \ \forall\ F \subseteq D,\ \ [F \in \mathcal{T} \Rightarrow\ \ F \cap [G - \Pi(G - H)] \neq \emptyset] \\
\Longleftrightarrow\ & \exists D \in \mathcal{T}(y),\ \ \forall\ F \subseteq D,\ \ [F \in \mathcal{T} \Rightarrow\ \ F - \Pi(G - H) \neq \emptyset] \\
\Longleftrightarrow\ & \exists D \in \mathcal{T}(y), \forall F \subseteq D, [F \in \mathcal{T} \Rightarrow [\exists M \subseteq F\ (M \in \mathcal{T} \Rightarrow [\overline{M} - H]^c \notin \mathcal{P})].
\end{aligned}
$$

□

7. Π-Suitability via Primal Spaces

Now, we consider certain characteristics of a suitable structure via primal spaces and explore its major properties.

Proposition 1. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS, where $\mathcal{T}$ is Π-suitable for the primal $\mathcal{P}$ and $H \subseteq G$. If $N \subseteq \Pi(H) \cap \overrightarrow{\Pi}(H)$ and $N \neq \emptyset$ is open, then $[N - H]^c \notin \mathcal{P}$ and $(\overline{N})^c \cup H^c \in \mathcal{P}$.*

Proof. If $N \subseteq \Pi(H) \cap \overrightarrow{\Pi}(H)$, then $[\overrightarrow{\Pi}(H) - H]^c \subseteq [N - H]^c$ by Theorem 14, and hence $[N - H]^c \notin \mathcal{P}$ by heredity. Since $N \in \mathcal{T} - \{\emptyset\}$ and $N \subseteq \Pi(H)$, we have $(\overline{N})^c \cup H^c \in \mathcal{P}$ by the definition of $\Pi(H)$. □

We note that $H = K$ [mod $\mathcal{P}$] if $[(H - K) \cup (K - H)]^c \notin \mathcal{P}$, where $=$ [mod $\mathcal{P}$] is an equivalence relation. By (9) of Theorem 8, we determine that if $H = K$ [mod $\mathcal{P}$], then $\overrightarrow{\Pi}(H) = \overrightarrow{\Pi}(K)$.

Lemma 9. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS such that $\mathcal{T}$ is Π-suitable for the primal $\mathcal{P}$. If $N, M \in \mathcal{T}_\theta$, and $\overrightarrow{\Pi}(N) = \overrightarrow{\Pi}(M)$, then $N = M$ [mod $\mathcal{P}$].*

Proof. Since $N \in \mathcal{T}_\theta$, by Corollary 3 we have $N \subseteq \overrightarrow{\Pi}(N)$, and hence $N - M \subseteq \overrightarrow{\Pi}(N) - M = \overrightarrow{\Pi}(M) - M$ and $[\overrightarrow{\Pi}(M) - M]^c \notin \mathcal{P}$ by Theorem 14. Therefore, $[N - M]^c \notin \mathcal{P}$. Similarly, $[M - N]^c \notin \mathcal{P}$. Now, $(N - M)^c \cap (M - N)^c = [(N - M) \cup (M - N)]^c \notin \mathcal{P}$ by additivity. Hence, $N = M$ [mod $\mathcal{P}$]. □

Definition 10. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS. A subset A of G is a Baire set with respect to $\mathcal{T}$ and $\mathcal{P}$, symbolized by $A \in \mathcal{B}_\theta$, if there exists a θ-open set U such that $A = U$ [mod $\mathcal{P}$].*

Example 6. *Let $G = \{a, b, c\}$ with topology $\mathcal{T} = \{\emptyset, G, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and the primal $\mathcal{P} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then, $\mathcal{T}_\theta = \{\emptyset, G\}$. It is clear that for any $A \notin \mathcal{P}$, A is a Baire set with respect to $\mathcal{T}$ and $\mathcal{P}$. That is:*

1. *If $A = \{a, b\}$, the only θ-open set U is G and $[(G - A) \cup (A - G)]^c = A \notin \mathcal{P}$; hence, $A = U$ [mod $\mathcal{P}$]. Thus, A is a Baire set with respect to $\mathcal{T}$ and $\mathcal{P}$.*
2. *If $A = \{b, c\}$, the only θ-open set U is G and $[(G - A) \cup (A - G)]^c = A \in \mathcal{P}$; hence, $A \neq U$ [mod $\mathcal{P}$]. Thus, A is not a Baire set with respect to $\mathcal{T}$ and $\mathcal{P}$.*

Theorem 17. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS such that $\mathcal{T}$ is Π-suitable for the primal $\mathcal{P}$. If $H, K \in \mathcal{B}_\theta$, and $\overrightarrow{\Pi}(H) = \overrightarrow{\Pi}(K)$, then $H = K$ [mod $\mathcal{P}$].*

Proof. Let $N, M \in \mathcal{T}_\theta$ such that $H = N$ [mod $\mathcal{P}$] and $K = M$ [mod $\mathcal{P}$]. Now, $\overrightarrow{\Pi}(H) = \overrightarrow{\Pi}(N)$ and $\overrightarrow{\Pi}(K) = \overrightarrow{\Pi}(M)$ by Theorem 8 (9). $\overrightarrow{\Pi}(H) = \overrightarrow{\Pi}(K)$ implies that $\overrightarrow{\Pi}(N) = \overrightarrow{\Pi}(M)$, and hence $N = M$ [mod $\mathcal{P}$] by Lemma 9. Thus, $H = K$ [mod $\mathcal{P}$] by transitivity. □

Proposition 2. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS:*

1. *If $B \in \mathcal{B}_\theta$ and $B^c \in \mathcal{P}$, then there exists $H \in \mathcal{T}_\theta - \{\emptyset\}$ such that $B = H$ [mod $\mathcal{P}$].*
2. *Let $\mathcal{T} - \{G\} \subseteq \mathcal{P}$; then, $B \in \mathcal{B}_\theta$ and $B^c \in \mathcal{P}$ iff there exists $H \in \mathcal{T}_\theta - \{\emptyset\}$ such that $B = H$ [mod $\mathcal{P}$].*

Proof.

(1) Assume that $B \in \mathcal{B}_\theta$ and $B^c \in \mathcal{P}$. Hence, there exists $H \in \mathcal{T}_\theta$ such that $B = H$ [mod $\mathcal{P}$]. If $H = \emptyset$, then we obtain $B = \emptyset$ [mod $\mathcal{P}$] and $[(B - \emptyset) \cup (\emptyset - B)]^c \notin \mathcal{P}$. This means that $B^c \notin \mathcal{P}$, which contradicts itself.

(2) Suppose that there exists $H \in \mathcal{T}_\theta - \{\emptyset\}$ such that $B = H$ [mod $\mathcal{P}$]. Thus, by Definition 10, $B \in \mathcal{B}_\theta$. Then, $H = (B - J) \cup I$, where $J = B - H$, $I = H - B$, and so $(H - B)^c$ and $(B - H)^c \notin \mathcal{P}$. If $B^c \notin \mathcal{P}$, then $(B - J)^c \notin \mathcal{P}$ and $H^c \notin \mathcal{P}$. Since $H \in \mathcal{T}_\theta - \{\emptyset\}$, $H \neq \emptyset$ and there exists $U \in \mathcal{T}$ such that $\emptyset \neq U \subseteq \overline{U} \subseteq H$. Since $H^c \notin \mathcal{P}$, then $\overline{U}^c \notin \mathcal{P}$ and $\overline{U}^c$ is open. This contradicts the statement that $\mathcal{T} - \{G\} \subseteq \mathcal{P}$. □

Proposition 3. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS with $\mathcal{T} - \{G\} \subseteq \mathcal{P}$. If $B \in \mathcal{B}_\theta$ and $B^c \in \mathcal{P}$, then $\overrightarrow{\Pi}(B) \cap Int_\theta(\Pi(B)) \neq \emptyset$.*

Proof. Assume that $B \in \mathcal{B}_\theta$ and $B^c \in \mathcal{P}$; then, by Proposition 2 (1), there exists $U \in \mathcal{T}_\theta - \{\emptyset\}$ such that $B = U$ [mod $\mathcal{P}$]. By Theorem 5 and Lemma 3, $U = U \cap G = U \cap \Pi(G) \subseteq \Pi(U \cap G) = \Pi(U)$. This means that $\emptyset \neq U \subseteq \Pi(U) = \Pi((B - J) \cup I) = \Pi(B)$, where $J^c = (B - U)^c, I^c = (U - B)^c \notin \mathcal{P}$ by Corollary 2. Since $U \in \mathcal{T}_\theta$, $U \subseteq Int_\theta(\Pi(B))$. Also, $\emptyset \neq U \subseteq \overrightarrow{\Pi}(U)$ by Corollary 3, and $B = U$ [mod $\mathcal{P}$]. This implies that $[(U - B) \cup (B - U)]^c \notin \mathcal{P}$, and hence $U \subseteq \overrightarrow{\Pi}(U) = \overrightarrow{\Pi}(B)$ by Theorem 8 (9). Consequently, we obtain $U \subseteq \overrightarrow{\Pi}(B) \cap Int_\theta(\Pi(B))$. □

Proposition 4. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS with $\mathcal{T} - \{G\} \subseteq \mathcal{P}$. If $\mathcal{T} = \mathcal{T}_\theta$, then the following statements are equivalent:*

(1) *There exist $B \in \mathcal{B}_\theta$ and $B^c \in \mathcal{P}$ such that $B \subseteq Q$;*

(2) $\overrightarrow{\Pi}(Q) \cap Int_\theta(\Pi(Q)) \neq \emptyset$;

(3) $\overrightarrow{\Pi}(Q) \cap \Pi(Q) \neq \emptyset$;

(4) $\overrightarrow{\Pi}(Q) \neq \emptyset$;

(5) $\overrightarrow{\Pi}(Q) \cap Q \neq \emptyset$;

(6) *There exists a nonempty open set M such that $[\overline{M} - Q]^c \notin \mathcal{P}$ and $[\overline{M} \cap Q]^c \in \mathcal{P}$.*

Proof.

(1) $\Rightarrow$ (2): Let $B \in \mathcal{B}_\theta$ and $B^c \in \mathcal{P}$ such that $B \subseteq Q$. Then, $Int_\theta(\Pi(B)) \subseteq Int_\theta(\Pi(Q))$ and $\overrightarrow{\Pi}(B) \subseteq \overrightarrow{\Pi}(Q)$, and hence $Int_\theta(\Pi(B)) \cap \overrightarrow{\Pi}(B) \subseteq Int_\theta(\Pi(Q)) \cap \overrightarrow{\Pi}(Q)$. By Proposition 3, $\overrightarrow{\Pi}(Q) \cap Int_\theta(\Pi(Q)) \neq \phi$.

(2) $\Rightarrow$ (3): The evidence is clear.

(3) $\Rightarrow$ (4): The evidence is clear.

(4) $\Rightarrow$ (5): If $\overrightarrow{\Pi}(Q) \neq \phi$, then there exists an open set $M \neq \emptyset$ such that $[\overline{M} - Q]^c \notin \mathcal{P}$. Since $\overline{M}^c \in \mathcal{P}$ and $\overline{M}^c = [(\overline{M} - Q) \cup (\overline{M} \cap Q)]^c = [\overline{M} - Q]^c \cap [\overline{M} \cap Q]^c$, we have $[\overline{M} \cap Q]^c \in \mathcal{P}$. By Theorem 8 and Corollary 3, $\emptyset \neq (\overline{M} \cap Q) \subseteq \overrightarrow{\Pi}(\overline{M}) \cap Q = \overrightarrow{\Pi}((\overline{M} - Q) \cup (\overline{M} \cap Q)) \cap Q = \overrightarrow{\Pi}(\overline{M} \cap Q) \cap Q \subseteq \overrightarrow{\Pi}(Q) \cap Q$. Hence, $\overrightarrow{\Pi}(Q) \cap Q \neq \emptyset$.

(5) $\Rightarrow$ (6): If $\overrightarrow{\Pi}(Q) \cap Q \neq \emptyset$, then $\overrightarrow{\Pi}(Q) \neq \emptyset$ and there exists an open set $M \neq \emptyset$ such that $[\overline{M} - Q]^c \notin \mathcal{P}$, $\overline{M}^c = [(\overline{M} - Q) \cup (\overline{M} \cap Q)]^c = [\overline{M} - Q]^c \cap [\overline{M} \cap Q]^c$ and $\overline{M}^c \in \mathcal{P}$. This means that $[\overline{M} \cap Q]^c \in \mathcal{P}$.

(6) $\Rightarrow$ (1): Let $B^c = [\overline{M} \cap Q]^c \in \mathcal{P}$, where M is a nonempty θ-open set and $(\overline{M} - Q)^c \notin \mathcal{P}$. Then, $B \in \mathcal{B}_\theta$ and $B^c \in \mathcal{P}$, since $[(B - \overline{M}) \cup (\overline{M} - B)]^c = [\overline{M} - Q]^c \notin \mathcal{P}$ and

$[(B - \overline{M}) \cup (\overline{M} - B)]^c \subseteq [(B - M) \cup (M - B)]^c$, that is, $B = M$ [mod $\mathcal{P}$]. Then, there exists $B \in \mathcal{B}_\theta$ and $B^c \in \mathcal{P}$ such that $B \subseteq Q$. $\square$

Theorem 18. *Let $(G, \mathcal{T}, \mathcal{P})$ be a PTS where $\mathcal{T} - \{G\} \subseteq \mathcal{P}$ and $\mathcal{T}$ is Π-suitable for the primal $\mathcal{P}$. Then, for any subset H, $\overrightarrow{\Pi}(H) \subseteq \Pi(H)$.*

Proof. Let $x \in \overrightarrow{\Pi}(H)$ and $x \notin \Pi(H)$. Then, there exists an open set $U_x \neq \emptyset$ such that $[\overline{U_x} \cap H]^c \notin \mathcal{P}$. Since $x \in \overrightarrow{\Pi}(H)$, by Theorem 9, $x \in \cup\{U \in \mathcal{T} : [\overline{U} - H]^c \notin \mathcal{P}\}$ and there exist $V \in \mathcal{T}(x)$ and $[\overline{V} - H]^c \notin \mathcal{P}$. Thus, $U_x \cap V \in \mathcal{T}(x)$, $[\overline{U_x \cap V} \cap H]^c \notin \mathcal{P}$ and $[\overline{U_x \cap V} - H]^c \notin \mathcal{P}$ by heredity. Therefore, by finite additivity we obtain $[\overline{U_x \cap V}]^c = [\overline{U_x \cap V} \cap H]^c \cap [\overline{U_x \cap V} - H]^c \notin \mathcal{P}$. Since $[\overline{U_x \cap V}]^c \in \mathcal{T}(x)$, this is in opposition to $\mathcal{T} - \{G\} \subseteq \mathcal{P}$. Thus, $x \in \Pi(H)$. This means that $\overrightarrow{\Pi}(H) \subseteq \Pi(H)$. $\square$

8. Conclusions and Future Work

The concept of a primal topology, as demonstrated by Acharjee et al. [5] and Al-Omari et al. [6,7], is a continuation of the classical (crisp) topology. This topological generalization is becoming increasingly interesting to research. Primal space areas formed the foundation of our study. Several basic operations on primal spaces were discussed. An outline of their features was provided, along with an outline of some new primal space operators. Additionally, we identified the fundamental qualities of appropriate spaces and the linkages among other factors. The results of this work are preliminary; other features of the primal resolvable space will be investigated in future studies. Through the integration of these two methodologies, our work opens up prospects for possible contributions to the resolvability of soft topologies [16] in classical and soft settings, as well as the resolvability of primal soft topologies and the primal hyper-connectedness and resolvability of structures with generalized rough approximation spaces.

Author Contributions: Writing—original draft, A.A.-O.; Writing—review & editing, A.A.-O. and M.H.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No data were used to support this study.

Conflicts of Interest: The authors declare no conflicts of interest.

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