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Extropy and Some of Its More Recent Related Measures for Concomitants of *K*-Record Values in an Extended FGM Family

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Abstract: This study uses an effective, recently extended Farlie–Gumbel–Morgenstern (EFGM) family to derive the distribution of concomitants of *K*-record upper values (CKRV). For this CKRV, the negative cumulative residual extropy (NCREX), weighted NCREX (WNCREX), negative cumulative extropy (NCEX), and weighted NCEX (WNCEX) are theoretically and numerically examined. This study presents several beautiful symmetrical and asymmetric relationships that these inaccuracy measurements satisfy. Additionally, empirical estimations are provided for these measures, and their visualizations enable users to verify their accuracy.

Keywords: FGM family; concomitants; *K*-record values; weighted negative cumulative residual extropy; weighted negative cumulative extropy; non-parametric estimation

MSC: 60B12; 62G30

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1. Introduction

One of the most fundamental techniques in stochastic multivariate analysis is the concept of a family of multivariate distributions. Families of bivariate distributions with known marginals have drawn attention for many years. The Farlie–Gumbel–Morgenstern (FGM) family is regarded as one of the world's first bivariate distribution families. When the FGM family turns into a copula (i.e., when the marginals are uniform), the correlation coefficient between the FGM family's marginals reaches its minimum value of -0.33 and highest value of 0.33. The FGM distribution is, therefore, best suited for data with low correlation coefficients. Despite this constraining restriction, the FGM family has increasingly replaced conventional multivariate normal models in various applications and is now extensively used in several different fields. In a study by Ghosh et al. [1], the FGM family was applied to model the interdependence between environmental and biological variables. In another study by Shrahili and Alotaibi [2], variants of this copula were employed to simulate real-world datasets with symmetric characteristics.

Numerous modifications to the FGM copula that have been discussed in the literature aim to improve the correlation between the inner marginals. Ebaid et al. [3] presented the symmetric generalization of the FGM copula, and Barakat et al. [4] have since considered it. They discovered that the admissible range and correlation claims made by Ebaid et al. [3] were false. Barakat et al. [4] revised the copula's allowable range. The symbol for this extension is EFGM, and it has a more straightforward function than many known generalizations of the FGM family, such as Bairamov–Kotz–Becki–FGM, Huang–Kotz FGM (see [5,6]), and iterated FGM (see Barakat and Husseiny, [7]). Recently, Abd Elgawad et al. [8] revealed and discussed some distributional traits of concomitants of order statistics (OSs) arising from the EFGM family. The work by Abd Elgawad et al. [8] is expanded upon in this research to include $K-$record values in the perspective of some recent information measures. The cumulative distribution function (CDF) and probability density function (PDF) of the EFGM family, denoted by EFGM(c,d), are given, respectively, by (cf. [4])

$$H_{T,W}(t,w) = H_T(t)H_W(w)\left[1 + c\overline{H}_T(t)\overline{H}_W(w)(1 + dH_T(t))(1 + dH_W(w))\right] \tag{1}$$

and

$$h_{T,W}(t,w) = h_T(t)h_W(w)\left\{1 + c\left[1 + 2(d-1)H_T(t) - 3dH_T^2(t)\right]\left[1 + 2(d-1)H_W(w) - 3dH_W^2(w)\right]\right\}, \tag{2}$$

where the marginals $H_T(t)$ and $H_W(w)$ are continuous, and $\overline{H}_X(.) = 1 - H_X(.)$. Barakat et al. [4] clarified that the natural parameter space Λ (the admissible set of parameters c and d that ensure $H_{T,W}(t,w)$ is a bonafide CDF) is convex. The set Λ is given by $\Lambda = \Lambda^+ \cup \Lambda^-$, where

$$\begin{aligned} \Lambda^+ &= \left\{(c,d) : 0 \le d \le 1, -\frac{1}{(1+d)^2} \le c \le \frac{1}{(1+d)}; \text{ or } d > 1, -\frac{1}{(1+d)^2} \le c \le \frac{1}{(1+d)^2}\right\}, \\ \Lambda^- &= \left\{(c,d) : -2 \le d \le 0, \ -1 \le c \le 0; \text{ or } d < -2, -\frac{1}{(1+d)^2} \le c \le \frac{1}{(1+d)^2}\right\}. \end{aligned}$$

Let $\{T_i, i \ge 1\}$ be a set of independent random variables (RVs) with the same continuous CDF $H_T(.)$ and PDF $h_T(.)$. The observation T_j is called an upper record value when $T_j > T_i$ for every $i < j$. A similar definition can be given for lower record values. Due to the rarity of upper record values, which restricts their use in various applications, we can switch to a more flexible model, which is $K-$record upper values (KRVs), where we can always expect the occurrence of KRVs more frequently than upper record values. Considering the KRV model, refer to Dziubdziela and Kopociński [9]. For a fixed $K \ge 1$, the PDF of the Nth KRV is given by

$$g_{T_{N,k}}(t) = \frac{K^N}{\Gamma(N)}\left[-\ln \overline{H}_T(t)\right]^{N-1}\overline{H}_T^{K-1}(t)h_T(t), N \ge 1, \tag{3}$$

where $\Gamma(.)$ is the gamma function. For more details about this model and its applications, refer to [10–13].

Let a random bivariate sample $(T_i, W_i), i = 1, 2, \ldots$, have a common continuous CDF $H_{T,W}(t,w) = P(T \le t, W \le w)$. When the investigator is just interested in studying the sequence of $K-$records, $T_{N,K}$, of the first component T, the second component associated with the KRV of the first one is termed as the concomitant of that KRV, denoted by $W_{[N,K]}$. Several papers, including [14–17], discussed the PDF of CKRV $W_{[N,K]}$. This concomitant's PDF is provided by

$$h_{[N,K]}(w) = \int_{-\infty}^{\infty} h_{W|T}(w|t)g_{T_{N,K}}(t)dt, \tag{4}$$

where $h_{W|T}(w|t)$ is the conditional PDF of W, given T.

The extropy (EX) was proposed by Lad et al. [18]. Earlier in academic literature, EX was used to contrast with entropy. The EX refers to an organism's intelligence, functional order, vitality, energy, life, experience, and capacity for growth and improvement. The EX of an RV T with PDF $h_T(t)$ is defined as (see [18,19])

$$EX(T) = -\frac{1}{2}\int_0^{\infty} h_T^2(t)dt = -\frac{1}{2}\int_0^1 h_T(H_T^{-1}(u))du \le 0. \tag{5}$$

Qiu [20] reviewed several characterizations, as well as the EX-lower bounds for OSs and record values. Qiu and Jia [21] examined residual EX using OSs. Irshad et al. [22] refined the concept of past EX for concomitants of OSs from the FGM family. In addition, they studied the cumulative past EX and dynamic cumulative past EX for the concomitant of rth OS. There have been many studies of EX measures in conjunction with generalized OSs, such as Almaspoor et al. [23], Husseiny and Syam [24], and Husseiny et al. [25]. Additionally, Jahanshahi et al. [26] proposed a measure of uncertainty for RVs, known as cumulative residual extropy, abbreviated by CREX, which is given by

$$CREX(T) = -\frac{1}{2}\int_0^\infty \overline{H}_T^2(t)dt = -\frac{1}{2}\int_0^1 \frac{(1-u)^2}{h_T(H_T^{-1}(u))}du, \tag{6}$$

which is always negative. Consequently, the negative CREX (NCREX) shall be

$$\zeta R(T) = \frac{1}{2}\int_0^\infty \overline{H}_T^2(t)dt = \frac{1}{2}\int_0^1 \frac{(1-u)^2}{h_T(H_T^{-1}(u))}du. \tag{7}$$

Recently, Hashempour et al. [27] proposed a new information measure called weighted CREX (WCREX), which assigns more importance to large values of the considered RV, as well as EX and CREX; this measure is permanently negative and is defined by

$$CREX^w(T) = -\frac{1}{2}\int_0^\infty t\overline{H}_T^2(t)dt = -\frac{1}{2}\int_0^1 \frac{H_T^{-1}(u)(1-u)^2}{h_T(H_T^{-1}(u))}du. \tag{8}$$

Thus, the positive one would be called weighted negative cumulative residual extropy (WNCREX) and is expressed as

$$\zeta R^w(T) = \frac{1}{2}\int_0^\infty t\overline{H}_T^2(t)dt = \frac{1}{2}\int_0^1 \frac{H_T^{-1}(u)(1-u)^2}{h_T(H_T^{-1}(u))}du. \tag{9}$$

Also, a negative cumulative extropy (NCEX) has been introduced, similar to (7), by Tahmasebi and Toomaj [28]; that is,

$$\zeta(T) = \frac{1}{2}\int_0^\infty (1-H_T^2(t))dt = \frac{1}{2}\int_0^1 \frac{1-u^2}{h_T(H_T^{-1}(u))}du. \tag{10}$$

Furthermore, Chaudhary et al. [29] investigated another new information measure called weighted negative cumulative extropy (WNCEX), which is defined by

$$\zeta^w(T) = \frac{1}{2}\int_0^\infty t(1-H_T^2(t))dt = \frac{1}{2}\int_0^1 \frac{H_T^{-1}(u)(1-u^2)}{h_T(H_T^{-1}(u))}du. \tag{11}$$

Motivations of the Work

This study builds upon the work of Abd Elgawad et al. [8] regarding the OSs model, developing a significant parallel model about record values. Numerous real-world experiments lead to the concomitants of record values. These concomitants offer a practical and effective method for organizing and analyzing bivariate record data. One of the main motivations for this work is the practicality and realism of the KRV model, especially considering the rarity of record values. Another driving factor is the application of recent uncertainty measures to our model, which have broad implications across various scientific fields.

This paper is organized as follows: In Section 2, we derive the marginal distribution of CKRV based on the EFGM family and obtain the EX, NCREX, WNCREX, NCEX, and WNCEX for CKRV. In addition, in Section 3, numerical studies based on some well-known distributions are carried out. Moreover, Section 4 introduces the issue of non-parametric estimation of the mentioned measures through simulation studies. Finally, Section 5 presents the study's conclusion.

2. CKRV Based on EFGM(c,d) and EX, with Some of Its Associated Measures

In this section, we derive the marginal distribution of CKRV based on the EFGM family. Moreover, the EX, NCREX, WNCREX, NCEX, and WNCEX for CKRV are obtained.

2.1. The Marginal Distribution of CKRV Based on EFGM(c,d)

In the next theorem, we obtain a useful representation for the PDF of $W_{[N,K]}$. We use the notation $T \sim H_T$ to signify that T is distributed as H_T.

Theorem 1. *Let $V_1 \sim H_W^2$ and $V_2 \sim H_W^3$. Then*

$$\begin{aligned} h_{[N,K]}(w) &= h_W(w)\Big\{1+\eta[1+2(d-1)H_W(w)-3dH_W^2(w)]\Big\} \\ &= (1+\eta)h_W(w)+(d-1)\eta h_{V_1}(w)-d\eta h_{V_2}(w), \end{aligned} \tag{12}$$

where

$$\eta = -c(d+1)+2c(2d+1)\left(\frac{K}{K+1}\right)^N - 3cd\left(\frac{K}{K+2}\right)^N.$$

Proof. The PDF of CKRV is derived, starting with (4) as follows:

$$\begin{aligned} h_{[N,K]}(w) &= \int_{-\infty}^{\infty} h_{W|T}(w|t)g_{T_{N,K}}(t)dt \\ &= \int_{-\infty}^{\infty} h_W(w)\Big\{1+c[1+2(d-1)H_T(t)-3dH_T^2(t)] \\ &\times \quad [1+2(d-1)H_W(w)-3dH_W^2(w)]\Big\}g_{T_{N,K}}(t)dt \\ &= h_W(w)\Big\{1+\eta[1+2(d-1)H_W(w)-3dH_W^2(w)]\Big\}, \end{aligned}$$

where

$$\eta = c\int_{-\infty}^{\infty}\Big[1+2(d-1)H_T(t)-3dH_T^2(t)\Big]g_{T_{N,K}}(t)dt.$$

Using $H_T(t) = 1-\overline{H}_T(t)$ and simple algebra, we have

$$\eta = -c(d+1)+2c(2d+1)I_1-3cdI_2, \tag{13}$$

such that, for $p = 1,2$

$$I_p = \int_{-\infty}^{\infty} \overline{H}_T^p(t)g_{T_{N,K}}(t)dt.$$

Taking the transformation $\overline{H}_T(t) = e^{-z}$, we have

$$I_p = \frac{K^N}{\Gamma(N)}\int_0^{\infty} z^{N-1}e^{-z(K+p)}dz = \left(\frac{K}{K+p}\right)^N. \tag{14}$$

Finally, by using (13) with (14). The proof is completed. □

Remark 1. *If $K = 1$ in Theorem 1, we obtain the case of upper record values.*

Remark 2. *When the value of K is large, we can use the approximation $\eta \approx c$. Moreover, when the value of N is large, we can use the approximation $\eta \approx -c(d+1)$. Finally, when both K and N are large, such that $K \sim N$, we have $\eta \approx -c(d+1)+2c(2d+1)e^{-1}-3cde^{-2}$.*

Corollary 1. *By using Theorem 1, the marginal CDF of CKRV and its survival function satisfy the following two elegant symmetry relationships:*

$$H_{[N,K]}(w) = (1+\eta)H_W(w) + (d-1)\eta H_{V_1}(w) - d\eta H_{V_2}(w) \tag{15}$$

and

$$\overline{H}_{[N,K]}(w) = (1+\eta)\overline{H}_W(w) + (d-1)\eta\overline{H}_{V_1}(w) - d\eta\overline{H}_{V_2}(w). \tag{16}$$

2.2. EX and Some of Its More Recent Related Measures

In this section, the measures EX, NCREX, WNCREX, NCEX, and WNCEX for CKRV $W_{[N,K]}$ based on EFGM(c,d) are derived.

2.2.1. EX of CKRV for EFGM(c,d)

Using (5) and (12), the EX of $W_{[N,K]}$ is given by

$$\begin{aligned} EX_{[N,K]}(W) &= -\frac{1}{2}\int_0^\infty h^2_{[N,K]}(w)dw \\ &= -\frac{1}{2}\int_0^\infty h^2_W(w)\Big[A_1^2 + A_2^2H_W^2(w) + A_3^2H_W^4(w) + 2A_1A_2H_W(w) \\ &+ 2A_1A_3H_W^2(w) + 2A_2A_3H_W^3(w)\Big]dw \\ &= A_1^2EX(W) - A_1A_2E[h_W(w)H_W(w)] - \left(\frac{1}{2}A_2^2 + A_1A_3\right)E[h_W(w)H_W^2(w)] \\ &- A_2A_3E[h_W(w)H_W^3(w)] - \frac{1}{2}A_3^2E[h_W(w)H_W^4(w)], \end{aligned} \tag{17}$$

where $A_1 = 1+\eta$, $A_2 = 2(d-1)\eta$, and $A_3 = -3d\eta$.

We can write $EX_{[N,K]}(W)$ in terms of the quantile function (QF). Let the QF be $Q(u) = H_W^{-1}(u)$, then the quantile density function is given by $q(u) = 1/h_W(Q(u))$, where the derivative of $Q(u)$ is respect to u and is denoted by $q(u)$ (i.e., $Q'(u) = q(u)$). Thus, $EX_{[N,K]}(W)$ is given by

$$\begin{aligned} EX_{[N,K]}(W) &= A_1^2EX(W) - A_1A_2E\left[\frac{U}{q(u)}\right] - \left(\frac{1}{2}A_2^2 + A_1A_3\right)E\left[\frac{U^2}{q(u)}\right] \\ &- A_2A_3E\left[\frac{U^3}{q(u)}\right] - \frac{1}{2}A_3^2E\left[\frac{U^4}{q(u)}\right], \end{aligned}$$

where $EX(W)$ is the EX of W, and U is a uniformly distributed RV on $(0,1)$.

Example 1. *Assume that the random vector (T,W) follows the extended Weibull family (denoted by EWF). As mentioned in [30], the EWF has its CDF and PDF described as follows*

$$\begin{aligned} H_T(t) &= 1 - e^{-\varrho G(t;\tau)}, \\ h_T(t) &= \varrho g(t;\tau)e^{-\varrho G(t;\tau)}, \end{aligned} \tag{18}$$

respectively, where $\varrho > 0$, τ is a vector of parameters, and $G(t,\tau)$ is a non-negative, continuous, monotonically increasing, differentiable function of t, dependent on the parameter vector τ, such that $G(t,\tau) \to 0^+$ as $t \to 0^+$ and $G(t,\tau) \to +\infty$ as $t \to +\infty$. $g(t;\tau)$ is the derivative of $G(t;\tau)$. Using (18) in (1), the CDF of EFGM with EWF (denoted by EFGM-EWF) is given by

$$\begin{aligned} H_{T,W}(t,w) &= \left(1 - e^{-\varrho_1 G(t;\tau_1)}\right)\left(1 - e^{-\varrho_2 G(w;\tau_2)}\right)\Big[1 + ce^{(-\varrho_1 G(t;\tau_1) - \varrho_2 G(w;\tau_2))} \\ &\times \left(1 + d\left(1 - e^{-\varrho_1 G(t;\tau_1)}\right)\right)\left(1 + d\left(1 - e^{-\varrho_2 G(w;\tau_2)}\right)\right)\Big]. \end{aligned} \tag{19}$$

According to (17), the EX of $W_{[N,K]}$ is

$$\begin{aligned} EX_{[N,K]}(W) &= A_1^2 EX(W) - A_1 A_2 E\left[\varrho_2 g(w;\tau_2) e^{-\varrho_2 G(w;\tau_2)}\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)\right] \\ &- \left(\frac{1}{2}A_2^2 + A_1 A_3\right) E\left[\varrho_2 g(w;\tau_2) e^{-\varrho_2 G(w;\tau_2)}\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^2\right] \\ &- A_2 A_3 E\left[\varrho_2 g(w;\tau_2) e^{-\varrho_2 G(w;\tau_2)}\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^3\right] \\ &- \frac{1}{2}A_3^2 E\left[\varrho_2 g(w;\tau_2) e^{-\varrho_2 G(w;\tau_2)}\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^4\right]. \end{aligned} \tag{20}$$

Example 2. *Based on Example 1, by using $\varrho_i G(x,\tau_i) = -log(1-x)$ (i.e., $\varrho_i = 1$) and $\varrho_i g(x,\tau_i) = \frac{1}{1-x}$ for $x = t, w$ and $i = 1,2$, respectively, with (19), we obtain the joint uniform distribution with parameters 0 and 1 as EFGM (denoted by EFGM-UD), which is given by*

$$H_{T,W}(t,w) = tw[1 + c(1-t)(1-w)(1+dt)(1+dw)]. \tag{21}$$

According to (20), the EX of $W_{[N,K]}$ is

$$\begin{aligned} EX_{[N,K]}(W) &= -\frac{1}{2}A_1^2 - \frac{1}{6}A_2^2 - \frac{1}{10}A_3^2 - \frac{1}{2}A_1A_2 - \frac{1}{3}A_1A_3 - \frac{1}{4}A_2A_3 \\ &= -\frac{1}{2} - \left(\frac{1}{6} + \frac{1}{6}d + \frac{1}{15}d^2\right)\eta^2. \end{aligned}$$

Example 3. *Based on Example (1), by putting $\varrho_i G(x,\tau_i) = \frac{x}{\lambda_i}$ (i.e., $\varrho_i = \frac{1}{\lambda_i}$) for $x = t, w$ and $i = 1,2$, respectively, and $\varrho_i g(x,\tau) = \frac{1}{\lambda_i}$, in (19), we obtain the joint exponential distribution with parameters $\lambda_i > 0$ as the EFGM family (denoted by EFGM-ED), which is given as*

$$H_{T,W}(t,w) = \left(1-e^{-\frac{t}{\lambda_1}}\right)\left(1-e^{-\frac{w}{\lambda_2}}\right)\left[1 + ce^{-\left(\frac{t}{\lambda_1}+\frac{w}{\lambda_2}\right)}\left(1+d\left(1-e^{-\frac{t}{\lambda_1}}\right)\right)\left(1+d\left(1-e^{-\frac{w}{\lambda_2}}\right)\right)\right]. \tag{22}$$

Moreover, in view of (20), we have

$$\begin{aligned} EX_{[N,K]}(W) &= -\lambda_2\left[\frac{1}{4}A_1^2 + \frac{1}{24}A_2^2 + \frac{1}{60}A_3^2 - \frac{1}{6}A_1A_2 + \frac{1}{12}A_1A_3 + \frac{1}{20}A_2A_3\right] \\ &= -\lambda_2\left[\frac{1}{4} + \left(\frac{1}{6} + \frac{1}{12}d\right)\eta + \left(\frac{1}{12} + \frac{1}{20}d + \frac{1}{60}d^2\right)\eta^2\right]. \end{aligned}$$

2.2.2. NCREX of CKRV for EFGM(c,d)

We can obtain $\zeta R_{[N,K]}(W)$ by using (7) and (16) as

$$\begin{aligned} \zeta R_{[N,K]}(W) &= \frac{1}{2}\int_0^\infty \overline{H}^2_{[N,K]}(w)dw \\ &= \frac{1}{2}\int_0^\infty \Big[(1+\eta)^2\overline{H}^2_W(w) + (d-1)^2\eta^2\overline{H}^2_{V_1}(w) + d^2\eta^2\overline{H}^2_{V_2}(w) \\ &+ 2(d-1)(1+\eta)\eta\overline{H}_W(w)\overline{H}_{V_1}(w) - 2d(1+\eta)\eta\overline{H}_W(w)\overline{H}_{V_2}(w) \\ &- 2d(d-1)\eta^2\overline{H}_{V_1}(w)\overline{H}_{V_2}(w)\Big]dw. \end{aligned}$$

Then,

$$
\begin{aligned}
\zeta R_{[N,K]}(W) &= (1+\eta)^2\zeta R(W) + (d-1)^2\eta^2\zeta R(V_1) + d^2\eta^2\zeta R(V_2) \\
&+ (d-1)(1+\eta)\eta E\left[\frac{\overline{H}_W(w)\overline{H}_{V_1}(w)}{h_W(w)}\right] - d(1+\eta)\eta E\left[\frac{\overline{H}_W(w)\overline{H}_{V_2}(w)}{h_W(w)}\right] \\
&- d(d-1)\eta^2 E\left[\frac{\overline{H}_{V_1}(w)\overline{H}_{V_2}(w)}{h_W(w)}\right].
\end{aligned} \tag{23}
$$

Also, it can be expressed in another form using QF as

$$
\begin{aligned}
\zeta R_{[N,K]}(W) &= \frac{1}{2}(1+\eta)^2 E\Big[(1-U)^2 q(u)\Big] + \frac{1}{2}(d-1)^2\eta^2 E\Big[\Big(1-U^2\Big)^2 q(u)\Big] \\
&+ \frac{1}{2}d^2\eta^2 E\Big[\Big(1-U^3\Big)^2 q(u)\Big] + (d-1)(1+\eta)\eta E\Big[(1-U)\Big(1-U^2\Big)q(u)\Big] \\
&- d(1+\eta)\eta E\Big[(1-U)\Big(1-U^3\Big)q(u)\Big] - d(d-1)\eta^2 E\Big[\Big(1-U^2\Big)\Big(1-U^3\Big)q(u)\Big].
\end{aligned}
$$

Example 4. *Let (T,W) follow EFGM.*

- *According to EFGM-EWF, which is defined by (19), using (23), the NCREX of $W_{[N,K]}$ is given by*

$$
\begin{aligned}
\zeta R_{[N,K]}(W) &= \frac{1}{2}(1+\eta)^2 E\left[\frac{e^{-\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\right] \\
&+ \frac{1}{2}(d-1)^2\eta^2 E\left[\frac{e^{\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^2\right)^2\right] \\
&+ \frac{1}{2}d^2\eta^2 E\left[\frac{e^{\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^3\right)^2\right] \\
&+ (d-1)(1+\eta)\eta E\left[\frac{1}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^2\right)\right] \\
&- d(1+\eta)\eta E\left[\frac{1}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^3\right)\right] \\
&- d(d-1)\eta^2 E\left[\frac{e^{\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^2\right)\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^3\right)\right].
\end{aligned} \tag{24}
$$

- *For EFGM-UD, which is given by (21), using (24), NCREX of $W_{[N,K]}$ is given by*

$$
\zeta R_{[N,K]}(W) = \frac{1}{6} - \left(\frac{1}{12} + \frac{1}{30}d\right)\eta + \left(\frac{1}{60} + \frac{1}{60}d + \frac{1}{210}d^2\right)\eta^2.
$$

- *By choosing $\varrho_i G(x,\tau_i) = -log(1-w^{\beta_i})$ (i.e., $\varrho_i = 1$) in EFGM-EWF, we obtain the EFGM with power function distribution marginals (denoted by EFGM-PFD), which is given by*

$$
H_{T,W}(t,w) = t^{\beta_1}w^{\beta_2}[1 + c\left(1-t^{\beta_1}\right)\left(1-w^{\beta_2}\right)(1+dt^{\beta_1})(1+dw^{\beta_2})],\ \beta_i > 0,\ i = 1,2. \tag{25}
$$

Also, by using $\varrho_i G(x,\tau_i)$ and $\varrho_i g(x,\tau_i)$ into (24), we have

$$\begin{aligned}
\zeta R_{[N,K]}(W) &= \beta_2^2\Bigg\{\frac{1}{1+3\beta_2+2\beta_2^2} \\
&+ \left[\frac{-2}{1+6\beta_2+11\beta_2^2+6\beta_2^3}-\frac{2}{1+9\beta_2+26\beta_2^2+24\beta_2^3}d\right]\eta \\
&+ \left[\frac{1}{1+9\beta_2+26\beta_2^2+24\beta_2^3}+\frac{1}{1+12\beta_2+47\beta_2^2+60\beta_2^3}d\right. \\
&+ \left.\frac{1}{1+15\beta_2+74\beta_2^2+120\beta_2^3}d^2\right]\eta^2\Bigg\}.
\end{aligned}$$

- *For EFGM-ED with parameters $\lambda_i, i=1,2$, whose CDF is (22), the $\zeta R_{[N,K]}(W)$ is given by*

$$\zeta R_{[N,K]}(W)=\frac{1}{\lambda_2}\left[\frac{1}{4}-\left(\frac{1}{6}+\frac{1}{12}d\right)\eta+\left(\frac{1}{24}+\frac{1}{20}d+\frac{1}{60}d^2\right)\eta^2\right].$$

2.2.3. WNCREX of CKRV for EFGM(c,d)

Using (9) and (16), the WNCREX of $W_{[N,K]}$ can be simply obtained as follows:

$$\begin{aligned}
\zeta R^w_{[N,K]}(W) &= \frac{1}{2}\int_0^\infty w\overline{H}^2_{[N,K]}(w)dw \\
&= \frac{1}{2}\int_0^\infty w[(1+\eta)^2\overline{H}^2_W(w)+(d-1)^2\eta^2\overline{H}^2_{V_1}(w)+d^2\eta^2\overline{H}^2_{V_2}(w) \\
&+ 2(d-1)(1+\eta)\eta\overline{H}_W(w)\overline{H}_{V_1}(w)-2d(1+\eta)\eta\overline{H}_W(w)\overline{H}_{V_2}(w) \\
&- 2d(d-1)\eta^2\overline{H}_{V_1}(w)\overline{H}_{V_2}(w)]dw.
\end{aligned}$$

Then,

$$\begin{aligned}
\zeta R^w_{[N,K]}(W) &= (1+\eta)^2\zeta R^w(W)+(d-1)^2\eta^2\zeta R^w(V_1)+d^2\eta^2\zeta R^w(V_2) \\
&+ (d-1)(1+\eta)\eta E\left[\frac{W\overline{H}_W(w)\overline{H}_{V_1}(w)}{h_W(w)}\right]-d(1+\eta)\eta E\left[\frac{W\overline{H}_W(w)\overline{H}_{V_2}(w)}{h_W(w)}\right] \\
&- d(d-1)\eta^2E\left[\frac{W\overline{H}_{V_1}(w)\overline{H}_{V_2}(w)}{h_W(w)}\right].
\end{aligned} \tag{26}$$

In addition, it can be written in terms of QF. Thus, the corresponding $\zeta R^w_{[N,K]}(W)$ based on the QF is given by

$$\begin{aligned}
\zeta R^w_{[N,K]}(W) &= \frac{(1+\eta)^2}{2}E\left[(1-U)^2q(u)Q(u)\right]+\frac{(d-1)^2\eta^2}{2}E\left[\left(1-U^2\right)^2q(u)Q(u)\right] \\
&+ \frac{d^2\eta^2}{2}E\left[\left(1-U^3\right)^2q(u)Q(u)\right]+(d-1)(1+\eta)\eta E\left[(1-U)\left(1-U^2\right)q(u)Q(u)\right] \\
&- d(1+\eta)\eta E\left[(1-U)\left(1-U^3\right)q(u)Q(u)\right]-d(d-1)\eta^2E\left[\left(1-U^2\right)\left(1-U^3\right)q(u)Q(u)\right].
\end{aligned}$$

Example 5. *Let T and W follow EFGM.*

- *According to EFGM-EWF, which is defined by (19), and by using (26), the WNCREX of $W_{[N,K]}$ is*

$$
\begin{aligned}
\zeta R^{w}_{[N,K]}(W) &= \frac{1}{2}(1+\eta)^2 E\left[\frac{We^{-\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\right] \\
&+ \frac{1}{2}(d-1)^2\eta^2 E\left[\frac{We^{\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^2\right)^2\right] \\
&+ \frac{1}{2}d^2\eta^2 E\left[\frac{We^{\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^3\right)^2\right] \\
&+ (d-1)(1+\eta)\eta E\left[\frac{W}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^2\right)\right] \\
&- d(1+\eta)\eta E\left[\frac{W}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^3\right)\right] \\
&- d(d-1)\eta^2 E\left[\frac{We^{\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^2\right)\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^3\right)\right].
\end{aligned} \tag{27}
$$

- *For EFGM-UD, which is defined by (21), and by using (27), the WNCREX of $W_{[N,K]}$ is*

$$\zeta R^{w}_{[N,K]}(W) = \frac{1}{24} - \left(\frac{1}{30} + \frac{1}{60}d\right)\eta + \left(\frac{1}{120} + \frac{1}{105}d + \frac{1}{336}d^2\right)\eta^2.$$

- *For EFGM-PFD, which is defined in (25), and by using (27), the WNCREX of $W_{[N,K]}$ would be*

$$
\begin{aligned}
\zeta R^{w}_{[N,K]}(W) &= \beta_2^2\Bigg\{\frac{1}{8+12\beta_2+4\beta_2^2} \\
&+ \left[\frac{-1}{4+12\beta_2+11\beta_2^2+3\beta_2^3} - \frac{1}{4+18\beta_2+26\beta_2^2+12\beta_2^3}d\right]\eta \\
&+ \Bigg[\frac{1}{8+36\beta_2+52\beta_2^2+24\beta_2^3} + \frac{1}{4+24\beta_2+47\beta_2^2+30\beta_2^3}d \\
&+ \frac{1}{8+60\beta_2+148\beta_2^2+120\beta_2^3}d^2\Bigg]\eta^2\Bigg\}.
\end{aligned}
$$

- *According to EFGM-ED, which is described in (22), and by using (27), the WNCREX of $W_{[N,K]}$ is given by*

$$\zeta R^{w}_{[N,K]}(W) = \frac{1}{\lambda_2^2}\left[\frac{1}{8} - \left(\frac{5}{36} + \frac{13}{144}d\right)\eta + \left(\frac{13}{288} + \frac{77}{1200}d + \frac{29}{1200}d^2\right)\eta^2\right].$$

- *Putting $\varrho_i G(x,\tau_i) = \frac{x^2}{2\lambda_i}$ (i.e., $\varrho_i = \frac{1}{2\lambda_i}$) in EFGM-EWF, we obtain EFGM with Rayleigh distribution marginals (denoted by EFGM-RD), which is given by*

$$H_{T,W}(t,w) = \left(1-e^{-\frac{t^2}{2\lambda_1}}\right)\left(1-e^{-\frac{w^2}{2\lambda_2}}\right)\left[1+ce^{-\left(\frac{t^2}{2\lambda_1}+\frac{w^2}{2\lambda_2}\right)}\left(1+d\left(1-e^{-\frac{t^2}{2\lambda_1}}\right)\right)\left(1+d\left(1-e^{-\frac{w^2}{2\lambda_2}}\right)\right)\right].$$

Therefore, the WNCREX of $W_{[N,K]}$ would be

$$\zeta R^{w}_{[N,K]}(W) = \lambda_2^2\left[\frac{1}{4} + \left(\frac{-1}{6} - \frac{1}{12}d\right)\eta + \left(\frac{1}{24} + \frac{1}{20}d + \frac{1}{60}d^2\right)\eta^2\right].$$

- *Choosing $\varrho_i G(x,\tau_i) = -\alpha_i log\left(\frac{\sigma_i}{x}\right)$ (i.e., $\varrho_i = -\alpha_i$), in EFGM- EWF, we obtain EFGM with Pareto type-I distribution marginals (denoted by EFGM-PID), as follows*

$$H_{T,W}(t,w) = \left(1-\left(\frac{\sigma_1}{t}\right)^{\alpha_1}\right)\left(1-\left(\frac{\sigma_2}{w}\right)^{\alpha_2}\right)\left[1+c\frac{\sigma_1^{\alpha_1}\sigma_2^{\alpha_2}}{t^{\alpha_1}w^{\alpha_2}}\left(1+d\left(1-\left(\frac{\sigma_1}{t}\right)^{\alpha_1}\right)\right)\left(1+d\left(1-\left(\frac{\sigma_2}{w}\right)^{\alpha_2}\right)\right)\right], \quad (28)$$

Further, by using (27), we have

$$\begin{aligned}\zeta R^{w}_{[N,K]}(W) &= \sigma_2^2\Bigg\{\frac{1}{-4+4\alpha_2} + \left[\frac{-4\alpha_2}{4-10\alpha_2+6\alpha_2^2} - \frac{\alpha_2^2}{-4+18\alpha_2-26\alpha_2^2+12\alpha^3}d\right]\eta \\ &+ \left[\frac{\alpha_2^2}{-8+36\alpha_2-52\alpha_2^2+24\alpha^3} + \frac{3\alpha_2^3}{8-56\alpha_2+142\alpha_2^2-154\alpha^3+60\alpha_2^4}d \right.\\ &+ \left.\frac{3\alpha_2^4}{-8+80\alpha_2-310\alpha_2^2+580\alpha^3-522\alpha_2^4+180\alpha_2^5}d^2\right]\eta^2\Bigg\}.\end{aligned}$$

Figure 1a,b depicts the WNCREX of $W_{[N,K]}$ from EFGM-PFD for various values of N and K at $d = -3$. The following properties can be extracted from Figure 1.

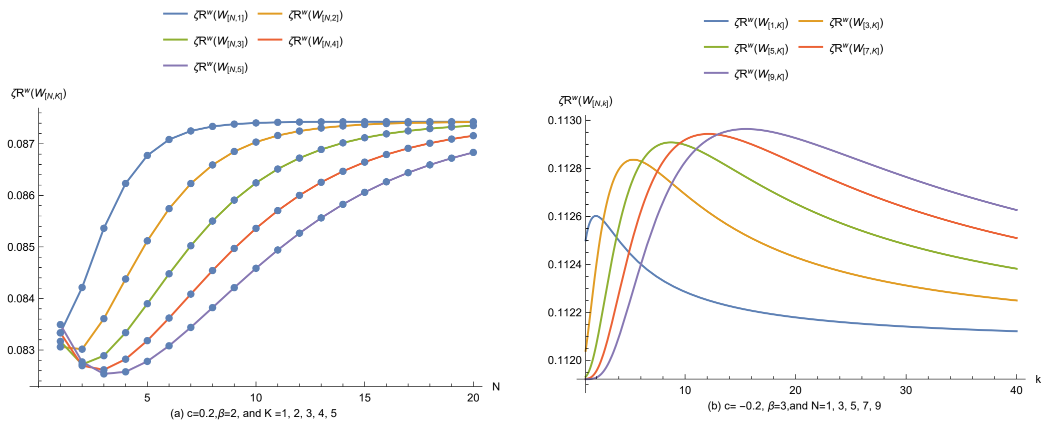

Figure 1. WNCREX of $W_{[N,K]}$ from EFGM-PFD.

1. With fixed N, c, and β, the value of $\zeta R^{w}_{[N,K]}(W)$ increases as K decreases (see Figure 1a) and stability occurs for large N.
2. For the fixed large K, the value of $\zeta R^{w}_{[N,K]}(W)$ increases with the increasing N; see Figure 1b.

Figure 2a,b depicts the WNCREX of $W_{[N,K]}$ from EFGM-RD for various values of N and K at $d = 2$. The following properties can be extracted from Figure 2.

1. Stability occurs for large N and K, see Figure 2a,b.
2. With fixed c and σ, the values of $\zeta R^{w}_{[N,K]}(W)$ are very near to each other as N and K rise.

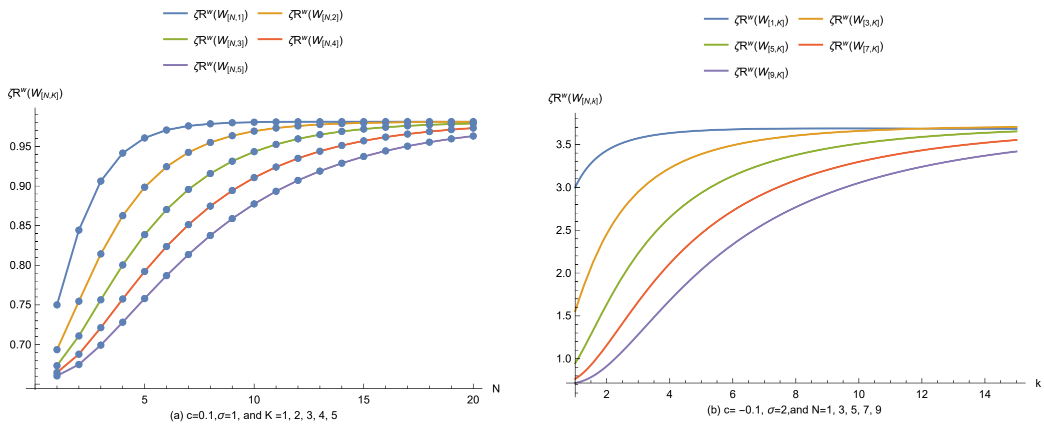

Figure 2. WNCREX of $W_{[N,K]}$ from EFGM-RD.

2.2.4. NCEX of CKRV for EFGM(c,d)

We can calculate the NCEX of CKRV $W_{[N,K]}$ as follows:

$$\begin{aligned}\zeta_{[N,K]}(W) &= \frac{1}{2}\int_0^\infty \left[1 - H^2_{[N,K]}(w)\right]dt \\ &= \int_0^\infty \overline{H}_{[N,K]}(w)dw - \zeta R_{[N,K]}(W) \\ &= \int_0^\infty [(1+\eta)\overline{H}_W(w) + (d-1)\eta\overline{H}_{V_1}(w) - d\eta\overline{H}_{V_2}(w)]dw - \zeta R_{[N,K]}(W).\end{aligned}$$

Using (23) and simple algebra, we have

$$\begin{aligned}\zeta_{[N,K]}(W) &= (1+\eta)E\left[\frac{\overline{H}_W(w)}{h_W(w)}\right] + (d-1)\eta E\left[\frac{\overline{H}_{V_1}(w)}{h_W(w)}\right] - d\eta E\left[\frac{\overline{H}_{V_2}(w)}{h_W(w)}\right] \\ &- (1+\eta)^2\zeta R(W) - (d-1)^2\eta^2\zeta R(V_1) - d^2\eta^2\zeta R(V_2) \\ &- (d-1)(1+\eta)\eta E\left[\frac{\overline{H}_W(w)\overline{H}_{V_1}(w)}{h_W(w)}\right] + d(1+\eta)\eta E\left[\frac{\overline{H}_W(w)\overline{H}_{V_2}(w)}{h_W(w)}\right] \\ &+ d(d-1)\eta^2 E\left[\frac{\overline{H}_{V_1}(w)\overline{H}_{V_2}(w)}{h_W(w)}\right].\end{aligned}$$

According to QF, $\zeta_{[N,K]}(W)$ is given by

$$\begin{aligned}\zeta_{[N,K]}(W) &= (1+\eta)E[(1-U)q(u)] + (d-1)\eta E\left[\left(1-U^2\right)q(u)\right] - d\eta E\left[\left(1-U^3\right)q(u)\right] \\ &- \frac{1}{2}(1+\eta)^2 E\left[(1-U)^2 q(u)\right] - \frac{1}{2}(d-1)^2\eta^2 E\left[\left(1-U^2\right)^2 q(u)\right] \\ &- \frac{1}{2}d^2\eta^2 E\left[\left(1-U^3\right)^2 q(u)\right] - (d-1)(1+\eta)\eta E\left[(1-U)\left(1-U^2\right)q(u)\right] \\ &+ d(1+\eta)\eta E\left[(1-U)\left(1-U^3\right)q(u)\right] + d(d-1)\eta^2 E\left[\left(1-U^2\right)\left(1-U^3\right)q(u)\right]. \qquad (29)\end{aligned}$$

Example 6. *Let T and W follow the EFGM family*

- *For EFGM-EWF, which is defined by (19), and by using (29), the NCEX of $W_{[N,K]}$ is*

$$
\begin{aligned}
\zeta_{[N,K]}(W) &= (1+\eta)E\left[\frac{1}{\varrho_2 g(w;\tau_2)}\right] \\
&+ (d-1)\eta E\left[\frac{e^{\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^2\right)\right] \\
&- d\eta E\left[\frac{e^{\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^3\right)\right] - \frac{1}{2}(1+\eta)^2 E\left[\frac{e^{-\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\right] \\
&- \frac{1}{2}(d-1)^2\eta^2 E\left[\frac{e^{\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^2\right)^2\right] \\
&- \frac{1}{2}d^2\eta^2 E\left[\frac{e^{\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^3\right)^2\right] \\
&- (d-1)(1+\eta)\eta E\left[\frac{1}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^2\right)\right] \\
&+ d(1+\eta)\eta E\left[\frac{1}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^3\right)\right] \\
&+ d(d-1)\eta^2 E\left[\frac{e^{\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^2\right)\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^3\right)\right].
\end{aligned} \tag{30}
$$

- *For EFGM-UD, as clarified in (21), the following equation is derived using (30):*

$$\zeta_{[N,K]}(W) = \frac{1}{3} - \left(\frac{1}{12} + \frac{1}{20}d\right)\eta - \left(\frac{1}{60} + \frac{1}{60}d + \frac{1}{210}d^2\right)\eta^2.$$

- *For EFGM-ED, with parameters as defined in (22), the equation using (30) is given by*

$$\zeta_{[N,K]}(W) = \frac{1}{\lambda_2}\left[\frac{3}{4} - \left(\frac{1}{3} + \frac{1}{4}d\right)\eta - \left(\frac{1}{24} + \frac{1}{20}d + \frac{1}{60}d^2\right)\eta^2\right].$$

- *For EFGM-PID, with parameters as mentioned in (28), by using (30), then*

$$
\begin{aligned}
\zeta_{[N,K]}(W) &= \sigma_2\Bigg\{\frac{-1+3\alpha_2}{2-6\alpha_2+4\alpha_2^2}\left[\frac{2\alpha_2^2}{-1+6\alpha_2-11\alpha_2^2+6\alpha_2^3} + \frac{6\alpha_2^3}{1-10\alpha_2+35\alpha_2^2-50\alpha^3+24\alpha_2^4}d\right]\eta \\
&- \Bigg[\frac{\alpha_2^2}{-1+9\alpha_2-26\alpha_2^2+24\alpha^3} + \frac{6\alpha_2^3}{1-14\alpha_2+71\alpha_2^2-154\alpha^3+120\alpha_2^4}d \\
&+ \frac{3\alpha_2^4}{-1+20\alpha_2-155\alpha_2^2+580\alpha^3-1044\alpha_2^4+720\alpha_2^5}d^2\Bigg]\eta^2\Bigg\}.
\end{aligned}
$$

2.2.5. WNCEX of CKRV for EFGM(c,d)

Similar to ζ, we can obtain ζ^w of $W_{[N,K]}$ as

$$
\begin{aligned}
\zeta^{w}_{[N,K]}(W) &= (1+\eta)E[(1-U)q(u)Q(u)] + (d-1)\eta E\Big[\Big(1-U^2\Big)q(u)Q(u)\Big] \\
&- d\eta E\Big[\Big(1-U^3\Big)q(u)Q(u)\Big] - \frac{1}{2}(1+\eta)^2 E\Big[(1-U)^2 q(u)Q(u)\Big] \\
&- \frac{1}{2}(d-1)^2\eta^2 E\Big[\Big(1-U^2\Big)^2 q(u)Q(u)\Big] - \frac{1}{2}d^2\eta^2 E\Big[\Big(1-U^3\Big)^2 q(u)Q(u)\Big] \\
&- (d-1)(1+\eta)\eta E\Big[(1-U)\Big(1-U^2\Big)q(u)Q(u)\Big] \\
&+ d(1+\eta)\eta E\Big[(1-U)\Big(1-U^3\Big)q(u)Q(u)\Big] \\
&+ d(d-1)\eta^2 E\Big[\Big(1-U^2\Big)\Big(1-U^3\Big)q(u)Q(u)\Big].
\end{aligned} \tag{31}
$$

Example 7. *Suppose that T and W follow the EFGM family*

- *For EFGM-EWF, which is defined by (19), by using (31), the WNCEX of* $W_{[N,K]}$ *is*

$$
\begin{aligned}
\zeta^{w}_{[N,K]}(W) &= (1+\eta)E\left[\frac{W}{\varrho_2 g(w;\tau_2)}\right] \\
&+ (d-1)\eta E\left[\frac{We^{\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^2\right)\right] \\
&- d\eta E\left[\frac{We^{\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^3\right)\right] - \frac{1}{2}(1+\eta)^2 E\left[\frac{We^{-\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\right] \\
&- \frac{1}{2}(d-1)^2\eta^2 E\left[\frac{We^{\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^2\right)^2\right] \\
&- \frac{1}{2}d^2\eta^2 E\left[\frac{We^{\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^3\right)^2\right] \\
&- (d-1)(1+\eta)\eta E\left[\frac{W}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^2\right)\right] \\
&+ d(1+\eta)\eta E\left[\frac{W}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^3\right)\right] \\
&+ d(d-1)\eta^2 E\left[\frac{We^{\varrho_2 G(w;\tau_2)}}{\varrho_2 g(w;\tau_2)}\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^2\right)\left(1-\left(1-e^{-\varrho_2 G(w;\tau_2)}\right)^3\right)\right].
\end{aligned} \tag{32}
$$

- *For EFGM-UD, we have*

$$\zeta^{w}_{[N,K]}(W) = \frac{1}{8} - \left(\frac{1}{20}+\frac{1}{30}d\right)\eta - \left(\frac{1}{120}+\frac{1}{105}d+\frac{1}{336}d^2\right)\eta^2.$$

- *For EFGM-ED, we have*

$$\zeta^{w}_{[N,K]}(W) = \frac{1}{\lambda_2^2}\left[\frac{7}{8} - \left(\frac{11}{18}+\frac{25}{48}d\right)\eta - \left(\frac{13}{288}+\frac{77}{1200}d+\frac{29}{1200}d^2\right)\eta^2\right].$$

- *For EFGM-RD, we have*

$$\zeta R^{w}_{[N,K]}(W) = \lambda_2^2\left[\frac{3}{4} - \left(\frac{1}{3}+\frac{1}{4}d\right)\eta - \left(\frac{1}{24}+\frac{1}{20}d+\frac{1}{60}d^2\right)\eta^2\right].$$

3. Numerical Study for the EX, NCREX, WNCREX, NCEX, and WNCEX

Tables 1–6 display the EX, NCREX, and WNCREX of $W_{[N,K]}$ from EFGM. The following properties can be extracted:

Table 1 displays the EX of $W_{[N,K]}$ from the EFGM copula.

- The value of $EX(W_{[N,K]}; c) = EX(W_{[N,K]}; -c)$, at $d = -3, 0.9, 2$.
- For $N > 1$, large K ($K \geq 40$), and $d = -3, -1, 0.9$, the value of $EX(W_{[N,K]}; |c|)$ increases as the value of N increases.
- For $N > 1$, large K ($K \geq 40$), and $d = 2$, the value of $EX(W_{[N,K]}; |c|)$ decreases as the value of N increases.

Table 2 displays the EX of $W_{[N,K]}$ based on EFGM-UD.

- For $N > 1$ and large K ($K \geq 40$), the value of $EX(W_{[N,K]}; c)$ increases as the value of N increases along with the values of parameters (c, d) as $(-0.2, -3)$, $(0.2, 0.9)$, and $(-0.1, 2)$.
- For $N > 1$ and large K ($K \geq 40$), the value of $EX(W_{[N,K]}; c)$ decreases as the value of N increases along with the values of parameters (c, d) as $(0.2, -3)$, $(-0.75, -1)$, $(-0.25, -1)$, $(0.9, -0.2)$, and $(0.1, 2)$.

NCREX in $W_{[N,K]}$ based on EFGM copula, NCREX in $W_{[N,K]}$ based on EFGM-ED, WNCREX in $W_{[N,K]}$ based on EFGM copula, and WNCREX in $W_{[N,K]}$ based on EFGM-ED all satisfy the same asymmetry properties extracted for EX in $W_{[N,K]}$ from EFGM-ED (Table 2).

Moreover, in the earlier version of this paper, we presented an extra four tables for NCEX in $W_{[N,K]}$ based on EFGM copula, NCEX in $W_{[N,K]}$ based on EFGM-ED, WNCEX in $W_{[N,K]}$ based on EFGM copula, and WNCEX in $W_{[N,K]}$ based on EFGM-ED. Responding to the reviewers' comments about the excessive number of tables, we removed these extra tables, knowing that the same asymmetry properties, as extracted for EX in $W_{[N,K]}$ from EFGM-ED, also hold for these removed tables.

Table 1. EX in $W_{[N,K]}$ from the EFGM copula.

		$d=-3$		$d=-1$		$d=0.9$		$d=2$	
K	N	$c=-0.2$	$c=0.2$	$c=-0.75$	$c=-0.25$	$c=-0.2$	$c=0.2$	$c=-0.1$	$c=0.1$
2	1	−0.500296	−0.500296	−0.501042	−0.500116	−0.503464	−0.503464	−0.503407	−0.503407
2	3	−0.500280	−0.500280	−0.501775	−0.500197	−0.504957	−0.504957	−0.504749	−0.504749
2	5	−0.509920	−0.509920	−0.501079	−0.500120	−0.523053	−0.523053	−0.526828	−0.526828
2	7	−0.523524	−0.523524	−0.500329	−0.500037	−0.537641	−0.537641	−0.546456	−0.546456
6	1	−0.500340	−0.500340	−0.510762	−0.501196	−0.511352	−0.511352	−0.508801	−0.508801
6	3	−0.502672	−0.502672	−0.500001	−0.500000	−0.503523	−0.503523	−0.504500	−0.504500
6	5	−0.502570	−0.502570	−0.501708	−0.500190	−0.500037	−0.500037	−0.500315	−0.500315
6	7	−0.500417	−0.500417	−0.502927	−0.500325	−0.501888	−0.501888	−0.501237	−0.501237
40	1	−0.507091	−0.507091	−0.530776	−0.503420	−0.514590	−0.514590	−0.508321	−0.508321
40	3	−0.502546	−0.502546	−0.520221	−0.502247	−0.513887	−0.513887	−0.509327	−0.509327
40	5	−0.500485	−0.500485	−0.512740	−0.501416	−0.512936	−0.512936	−0.509918	−0.509918
40	7	−0.500003	−0.500003	−0.507577	−0.500842	−0.511804	−0.511804	−0.510114	−0.510114
100	1	−0.509078	−0.509078	−0.534630	−0.503848	−0.514752	−0.514752	−0.507955	−0.507955
100	3	−0.506407	−0.506407	−0.529423	−0.503269	−0.514562	−0.514562	−0.508484	−0.508484
100	5	−0.504327	−0.504327	−0.524863	−0.502763	−0.514319	−0.514319	−0.508947	−0.508947
100	7	−0.502751	−0.502751	−0.520883	−0.502320	−0.514028	−0.514028	−0.509341	−0.509341

Table 2. EX in $W_{[N,K]}$ from EFGM-ED at $\lambda_2 = 0.5$.

		$d=-3$		$d=-1$		$d=0.9$		$d=2$	
K	N	$c=-0.2$	$c=0.2$	$c=-0.75$	$c=-0.25$	$c=-0.2$	$c=0.2$	$c=-0.1$	$c=0.1$
2	1	−0.123657	−0.126435	−0.120182	−0.123307	−0.113982	−0.137343	−0.114444	−0.136667
2	3	−0.126394	−0.123693	−0.132466	−0.127341	−0.139923	−0.111974	−0.138892	−0.112657
2	5	−0.134587	−0.118514	−0.130705	−0.126812	−0.159545	−0.099277	−0.160551	−0.098197
2	7	−0.141051	−0.116300	−0.128049	−0.125989	−0.170707	−0.093696	−0.173601	−0.091548
6	1	−0.126541	−0.123565	−0.112295	−0.119868	−0.106026	−0.148318	−0.108578	−0.144292
6	3	−0.121247	−0.129588	−0.124808	−0.124936	−0.113894	−0.137455	−0.112965	−0.138502
6	5	−0.121311	−0.129492	−0.132310	−0.127294	−0.123794	−0.126220	−0.121671	−0.128432
6	7	−0.123417	−0.126714	−0.134828	−0.128032	−0.133986	−0.116737	−0.131897	−0.118506
40	1	−0.132902	−0.119314	−0.108231	−0.116846	−0.103818	−0.151764	−0.108993	−0.143720
40	3	−0.129469	−0.121327	−0.109635	−0.118193	−0.104269	−0.151045	−0.108138	−0.144903
40	5	−0.126852	−0.123299	−0.111563	−0.119459	−0.104902	−0.150048	−0.107661	−0.145573
40	7	−0.124863	−0.125138	−0.113794	−0.120633	−0.105696	−0.148821	−0.107506	−0.145792
100	1	−0.134106	−0.118731	−0.107956	−0.116433	−0.103716	−0.151929	−0.109320	−0.143274
100	3	−0.132459	−0.119543	−0.108353	−0.116999	−0.103836	−0.151736	−0.108851	−0.143916
100	5	−0.130984	−0.120368	−0.108878	−0.117554	−0.103990	−0.151489	−0.108454	−0.144463
100	7	−0.129662	−0.121198	−0.109511	−0.118097	−0.104177	−0.151191	−0.108126	−0.144920

Table 3. NCREX in $W_{[N,K]}$ based on EFGM copula.

		$d=-3$		$d=-1$		$d=0.9$		$d=2$	
K	N	$c=-0.2$	$c=0.2$	$c=-0.75$	$c=-0.25$	$c=-0.2$	$c=0.2$	$c=-0.1$	$c=0.1$
2	1	0.167233	0.166122	0.172991	0.168758	0.177954	0.156043	0.176974	0.156974
2	3	0.166137	0.167217	0.158634	0.163961	0.154035	0.180249	0.155289	0.178900
2	5	0.163806	0.170236	0.160383	0.164555	0.140612	0.197140	0.141023	0.197142
2	7	0.162557	0.172457	0.163179	0.165499	0.134158	0.206390	0.133927	0.207775
6	1	0.166084	0.167274	0.187525	0.173449	0.187588	0.147921	0.183531	0.151388
6	3	0.168430	0.165094	0.166897	0.166744	0.178054	0.155955	0.178564	0.155580
6	5	0.168395	0.165122	0.158786	0.164013	0.167808	0.165532	0.169737	0.163653
6	7	0.167341	0.166022	0.156399	0.163198	0.158759	0.174937	0.160752	0.172804
40	1	0.164202	0.169638	0.202837	0.178235	0.190550	0.145580	0.183043	0.151789
40	3	0.165129	0.168386	0.195648	0.176006	0.189934	0.146061	0.184051	0.150962
40	5	0.165973	0.167395	0.189434	0.174054	0.189079	0.146734	0.184621	0.150499
40	7	0.166722	0.166612	0.184065	0.172346	0.188022	0.147574	0.184806	0.150349
100	1	0.163916	0.170066	0.205177	0.178954	0.190690	0.145471	0.182663	0.152104
100	3	0.164312	0.169479	0.201985	0.177973	0.190526	0.145599	0.183210	0.151651
100	5	0.164698	0.168944	0.198977	0.177042	0.190314	0.145764	0.183677	0.151268
100	7	0.165072	0.168458	0.196142	0.176160	0.190059	0.145963	0.184065	0.150951

Table 4. NCREX in $W_{[N,K]}$ based on EFGM-ED at $\lambda_2 = 0.5$.

		$d=-3$		$d=-1$		$d=0.9$		$d=2$	
K	N	$c=-0.2$	$c=0.2$	$c=-0.75$	$c=-0.25$	$c=-0.2$	$c=0.2$	$c=-0.1$	$c=0.1$
2	1	0.505648	0.494537	0.521094	0.506973	0.548594	0.455150	0.546296	0.457407
2	3	0.494686	0.505489	0.473245	0.490983	0.446783	0.558576	0.450112	0.555050
2	5	0.470954	0.535246	0.479067	0.492962	0.391923	0.632996	0.389872	0.639289
2	7	0.457850	0.556852	0.488380	0.496108	0.366321	0.674366	0.361141	0.689354
6	1	0.494154	0.506059	0.569655	0.522620	0.590719	0.421552	0.576212	0.433355
6	3	0.517518	0.484152	0.500769	0.500256	0.549026	0.454782	0.553521	0.451371
6	5	0.517167	0.484440	0.473750	0.491155	0.504873	0.495168	0.513692	0.486650
6	7	0.506724	0.493537	0.465809	0.488440	0.466523	0.535518	0.473890	0.527455
40	1	0.475038	0.529394	0.620934	0.538602	0.603777	0.411993	0.573977	0.435068
40	3	0.484511	0.517080	0.596844	0.531158	0.601059	0.413952	0.578600	0.431538
40	5	0.493045	0.507258	0.576044	0.524640	0.597284	0.416699	0.581216	0.429564

Table 4. *Cont.*

K	N	$d=-3$		$d=-1$		$d=0.9$		$d=2$	
		$c=-0.2$	$c=0.2$	$c=-0.75$	$c=-0.25$	$c=-0.2$	$c=0.2$	$c=-0.1$	$c=0.1$
40	7	0.500552	0.499450	0.558084	0.518940	0.592630	0.420129	0.582068	0.428926
100	1	0.472086	0.533588	0.628780	0.541003	0.604398	0.411549	0.572233	0.436414
100	3	0.476169	0.527835	0.618079	0.537725	0.603670	0.412070	0.574742	0.434480
100	5	0.480122	0.522583	0.607998	0.534618	0.602737	0.412741	0.576881	0.432844
100	7	0.483931	0.517788	0.598501	0.531674	0.601610	0.413553	0.578665	0.431489

Table 5. WNCREX in $W_{[N,K]}$ based on EFGM copula.

K	N	$d=-3$		$d=-1$		$d=0.9$		$d=2$	
		$c=-0.2$	$c=0.2$	$c=-0.75$	$c=-0.25$	$c=-0.2$	$c=0.2$	$c=-0.1$	$c=0.1$
2	1	0.042230	0.041118	0.043778	0.042364	0.046519	0.037175	0.046286	0.037397
2	3	0.041133	0.042214	0.038994	0.040765	0.036335	0.047515	0.036663	0.047157
2	5	0.038696	0.045125	0.039575	0.040963	0.030814	0.054922	0.030571	0.055512
2	7	0.037294	0.047194	0.040505	0.041278	0.028226	0.059030	0.027637	0.060458
6	1	0.041080	0.042270	0.048651	0.043931	0.050717	0.033800	0.049261	0.034975
6	3	0.043401	0.040064	0.041744	0.041692	0.046563	0.037138	0.047005	0.036790
6	5	0.043366	0.040093	0.039045	0.040783	0.042154	0.041183	0.043035	0.040331
6	7	0.042336	0.041018	0.038253	0.040511	0.038315	0.045215	0.039052	0.044408
40	1	0.039123	0.044559	0.053815	0.045533	0.052016	0.032838	0.049039	0.035148
40	3	0.040101	0.043358	0.051387	0.044787	0.051746	0.033035	0.049498	0.034792
40	5	0.040968	0.042389	0.049294	0.044133	0.051370	0.033312	0.049758	0.034592
40	7	0.041722	0.041612	0.047489	0.043562	0.050907	0.033657	0.049842	0.034528
100	1	0.038814	0.044965	0.054607	0.045774	0.052078	0.032793	0.048865	0.035283
100	3	0.039241	0.044407	0.053527	0.045445	0.052006	0.032846	0.049115	0.035088
100	5	0.039650	0.043896	0.052511	0.045133	0.051913	0.032913	0.049327	0.034923
100	7	0.040041	0.043427	0.051554	0.044838	0.051801	0.032995	0.049504	0.034787

Table 6. WNCREX in $W_{[N,K]}$ based on EFGM-ED at $\lambda_2=0.5$.

K	N	$d=-3$		$d=-1$		$d=0.9$		$d=2$	
		$c=-0.2$	$c=0.2$	$c=-0.75$	$c=-0.25$	$c=-0.2$	$c=0.2$	$c=-0.1$	$c=0.1$
2	1	0.517904	0.482719	0.524627	0.508138	0.589698	0.419457	0.589988	0.419617
2	3	0.483191	0.517399	0.468815	0.489483	0.404717	0.608386	0.406127	0.607259
2	5	0.408642	0.612232	0.475596	0.491791	0.310868	0.750063	0.298787	0.776836
2	7	0.367995	0.681503	0.486449	0.495460	0.269140	0.830349	0.250938	0.880013
6	1	0.481509	0.519207	0.581443	0.526410	0.669099	0.360905	0.649309	0.375500
6	3	0.555642	0.449981	0.500897	0.500299	0.590505	0.418808	0.604236	0.408448
6	5	0.554522	0.450887	0.469404	0.489684	0.508890	0.491209	0.526360	0.474529
6	7	0.521319	0.479559	0.460159	0.486519	0.439646	0.565345	0.450411	0.553077
40	1	0.421397	0.593523	0.641603	0.545092	0.693981	0.344582	0.644848	0.378608
40	3	0.451110	0.554247	0.613322	0.536389	0.688792	0.347913	0.654080	0.372211
40	5	0.478006	0.523014	0.588930	0.528770	0.681594	0.352597	0.659311	0.368646
40	7	0.501747	0.498259	0.567890	0.522111	0.672733	0.358465	0.661016	0.367494
100	1	0.412172	0.606930	0.650820	0.547901	0.695166	0.343826	0.641371	0.381053
100	3	0.424935	0.588546	0.638249	0.544067	0.693777	0.344712	0.646376	0.377540
100	5	0.437322	0.571783	0.626413	0.540434	0.691994	0.345853	0.650645	0.374575
100	7	0.449287	0.556502	0.615266	0.536991	0.689844	0.347235	0.654209	0.372122

4. Non-Parametric Estimation of NCREX, WNCREX, NCEX, and WNCEX

In this section, we study the non-parametric estimators of NCREX, WNCREX, NCEX, and WNCEX of the CKRV, $W_{[N,K]}$. Furthermore, the mean and variance of the empirical measures (EMs) of the CKRV, $W_{[N,K]}$, are deduced. Let $W_i,\ i = 1, 2, \dots, n$, be a random sample from an absolutely continuous CDF H_W and $W_{1:n} \le W_{2:n} \le \dots \le W_{n:n}$ display the OSs of $W_1, W_2, \dots, W_n$. Then the EM of $H_W(w)$ is given by

$$\widehat{H}_W(w) = \begin{cases} 0, & w < W_{1:n}, \\ \frac{i}{n}, & W_{i:n} \le w \le W_{i+1:n}, \\ 1, & w > W_{n:n}. \end{cases} \tag{33}$$

From (33) into (15), we have the EM of $H_{[N,K]}(w)$ as

$$\begin{aligned} \widehat{H}_{[N,K]}(w) &= (1+\eta)\widehat{H}_W(w) + (d-1)\eta\widehat{H}^2_W(w) - d\eta\widehat{H}^3_W(w) \\ &= (1+\eta)\left(\frac{i}{n}\right) + (d-1)\eta\left(\frac{i}{n}\right)^2 - d\eta\left(\frac{i}{n}\right)^3. \end{aligned} \tag{34}$$

4.1. EM of NCREX in CKRV Based on EFGM(c,d)

According to (33), the EM of $\zeta R_{[N,K]}(W)$ is given by

$$\begin{aligned} \widehat{\zeta R}_{[N,K]}(W) &= \frac{1}{2}\int_0^\infty \left[1 - \widehat{H}_{[N,K]}(w)\right]^2 dw \\ &= \frac{1}{2}\int_0^\infty \left[1 - \left((1+\eta)\left(\frac{i}{n}\right) + (d-1)\eta\left(\frac{i}{n}\right)^2 - d\eta\left(\frac{i}{n}\right)^3\right)\right]^2 dw \\ &= \frac{1}{2}\sum_{i=1}^{n-1}\int_{W_{i:n}}^{W_{i+1:n}} \left[1 - (1+\eta)\left(\frac{i}{n}\right) - (d-1)\eta\left(\frac{i}{n}\right)^2 + d\eta\left(\frac{i}{n}\right)^3\right]^2 dw \\ &= \frac{1}{2}\sum_{i=1}^{n-1} U_i\left[1 - (1+\eta)\left(\frac{i}{n}\right) - (d-1)\eta\left(\frac{i}{n}\right)^2 + d\eta\left(\frac{i}{n}\right)^3\right]^2, \end{aligned}$$

where $U_i = W_{i+1:n} - W_{i:n},\ i = 1, 2, \dots, n-1$, are sample spacings. Thus, the expectation and variance of the empirical NCREX are given by

$$E\left[\widehat{\zeta R}_{[N,K]}(W)\right] = \frac{1}{2}\sum_{i=1}^{n-1} E[U_i]\left[1 - (1+\eta)\left(\frac{i}{n}\right) - (d-1)\eta\left(\frac{i}{n}\right)^2 + d\eta\left(\frac{i}{n}\right)^3\right]^2 \tag{35}$$

and

$$Var\left[\widehat{\zeta R}_{[N,K]}(W)\right] = \frac{1}{4}\sum_{i=1}^{n-1} Var[U_i]\left[1 - (1+\eta)\left(\frac{i}{n}\right) - (d-1)\eta\left(\frac{i}{n}\right)^2 + d\eta\left(\frac{i}{n}\right)^3\right]^4. \tag{36}$$

Example 8. *Assume that $(T_i, W_i),\ i = 1, 2, \dots, n$, is a random sample from EFGM-UD with parameters 0 and 1 for each T_i and W_i. By using (35) and (36), we have*

$$E\left[\widehat{\zeta R}_{[N,K]}(W)\right] = \frac{1}{2(n+1)}\sum_{i=1}^{n-1}\left[1 - (1+\eta)\left(\frac{i}{n}\right) - (d-1)\eta\left(\frac{i}{n}\right)^2 + d\eta\left(\frac{i}{n}\right)^3\right]^2$$

and

$$Var\left[\widehat{\zeta R}_{[N,K]}(W)\right] = \frac{n}{4(n+1)^2(n+2)}\sum_{i=1}^{n-1}\left[1 - (1+\eta)\left(\frac{i}{n}\right) - (d-1)\eta\left(\frac{i}{n}\right)^2 + d\eta\left(\frac{i}{n}\right)^3\right]^4.$$

Example 9. *Let* $(T_i, W_i),\ i = 1, 2, \ldots, n$*, be a random sample from EFGM-ED with parameters* λ_1 *and* λ_2*, respectively. Then, we have*

$$E\left[\widehat{\zeta R}_{[N,K]}(W)\right] = \frac{1}{2\lambda_2} \sum_{i=1}^{n-1} \frac{1}{n-i} \left[1 - (1+\eta)\left(\frac{i}{n}\right) - (d-1)\eta\left(\frac{i}{n}\right)^2 + d\eta\left(\frac{i}{n}\right)^3\right]^2$$

and

$$Var\left[\widehat{\zeta R}_{[N,K]}(W)\right] = \frac{1}{4\lambda_2^2} \sum_{i=1}^{n-1} \frac{1}{(n-i)^2} \left[1 - (1+\eta)\left(\frac{i}{n}\right) - (d-1)\eta\left(\frac{i}{n}\right)^2 + d\eta\left(\frac{i}{n}\right)^3\right]^4.$$

Figure 3 shows the relation between NCREX and the empirical NCREX in $W_{[N,K]}$ from EFGM-UD, at $n = 100$. It can be extracted that, at any value of N, the values of NCREX are very close to the values of the empirical NCREX, as long as n is large.

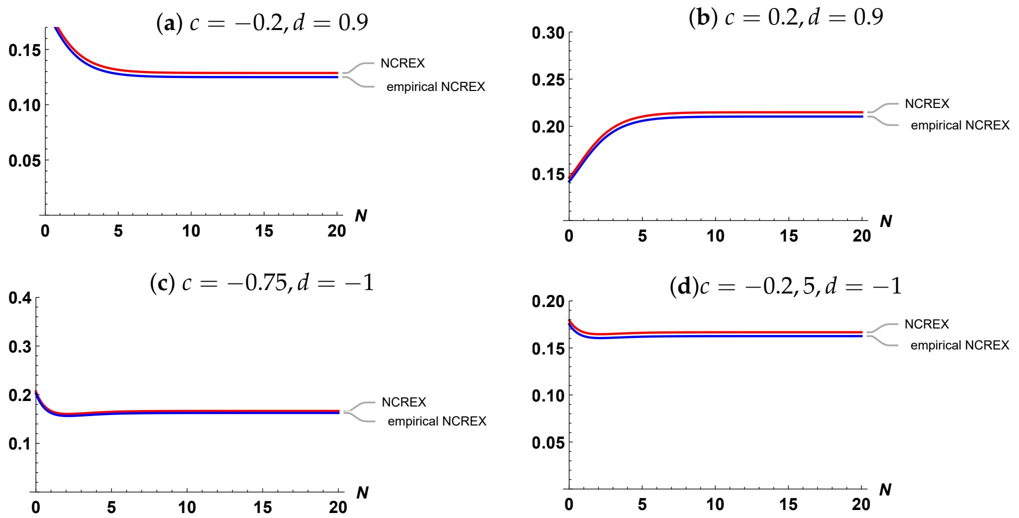

Figure 3. Representation of NCREX and empirical NCREX based on $W_{[N,K=1]}$ from EFGM-UD.

Table 7 displays the values of $E\left[\widehat{\zeta R}_{[N,K]}(W)\right]$ and $Var\left[\widehat{\zeta R}_{[N,K]}(W)\right]$ for the EFGM-ED model at $K = 2$, $n = 10$, and $d = 0.5$. The following features can be extracted:

1. With fixed N and λ_2, $E\left[\widehat{\zeta R}_{[N,K]}(W)\right]$ and $Var\left[\widehat{\zeta R}_{[N,K]}(W)\right]$ increase as c increases.
2. With fixed N and c, $E\left[\widehat{\zeta R}_{[N,K]}(W)\right]$ and $Var\left[\widehat{\zeta R}_{[N,K]}(W)\right]$ increase as λ_2 increases.

Table 7. $E\left[\widehat{\zeta R}_{[N,K]}(W)\right]$ and $Var\left[\widehat{\zeta R}_{[N,K]}(W)\right]$ for EFGM-ED at $K = 2$, $n = 10$, and $d = 0.5$.

		$E\left[\widehat{\zeta R}_{[N,K]}(W)\right]$				$Var\left[\widehat{\zeta R}_{[N,K]}(W)\right]$			
N	λ_2	$c = -0.2$	$c = -0.1$	$c = 0.1$	$c = 0.2$	$c = -0.2$	$c = -0.1$	$c = 0.1$	$c = 0.2$
3	0.5	0.102492	0.107408	0.117768	0.123213	0.001551	0.001662	0.001911	0.002051
3	1	0.204985	0.214816	0.235537	0.246426	0.006204	0.006646	0.007643	0.008204
3	2	0.409970	0.429632	0.471073	0.492852	0.024818	0.026585	0.030574	0.032817
5	0.5	0.092915	0.102344	0.123384	0.134995	0.001350	0.001548	0.002056	0.002377
5	1	0.185829	0.204687	0.246767	0.269990	0.005400	0.006191	0.008222	0.009507
5	2	0.371659	0.409375	0.493535	0.539980	0.021598	0.024766	0.032889	0.038028
8	0.5	0.087208	0.099217	0.127056	0.142886	0.001238	0.001480	0.002154	0.002612
8	1	0.174415	0.198434	0.254113	0.285773	0.004954	0.005921	0.008616	0.010449
8	2	0.348831	0.396868	0.508226	0.571545	0.019815	0.023684	0.034463	0.041796
10	0.5	0.085830	0.098448	0.127985	0.144904	0.001212	0.001464	0.002179	0.002675
10	1	0.171660	0.196897	0.255971	0.289808	0.004850	0.005856	0.008717	0.010699
10	2	0.343320	0.393793	0.511941	0.579617	0.019399	0.023423	0.034869	0.042796

4.2. EM of WNCREX in CKRV Based on EFGM(c,d)

Using (9) and (34), the EM of $\zeta R^{w}_{[N,K]}(W)$ is given by

$$\begin{aligned}\widehat{\zeta R^{w}}_{[N,K]}(W) &= \frac{1}{2}\int_0^\infty w\Big[1-\widehat{H}_{[N,K]}(w)\Big]^2 dw \\ &= \frac{1}{2}\sum_{i=1}^{n-1} Z_i\left[1-(1+\eta)\left(\frac{i}{n}\right)-(d-1)\eta\left(\frac{i}{n}\right)^2+d\eta\left(\frac{i}{n}\right)^3\right]^2,\end{aligned}$$

where $Z_i = \frac{W^2_{i+1:n}-W^2_{i:n}}{2}$, $i = 1,2,\dots,n-1$.

Example 10. *Let (T_i, W_i), $i = 1,2,\dots,n$, be a random sample from the EFGM family. Furthermore, let W_i have a distribution with PDF $h_W(w) = 2w$, $0 < w < 1$. According to Chakraborty et al. [31], W_i^2 has a standard UD. Furthermore, the RVs $Z_i = \frac{W^2_{i+1:n}-W^2_{i:n}}{2}$, $i = 1,2,\dots,n-1$, follow beta distribution with a mean $\frac{1}{2(n+1)}$ and variance $\frac{n}{4(n+1)^2(n+2)}$. Thus,*

$$E\Big[\widehat{\zeta R^{w}}_{[N,K]}(W)\Big] = \frac{1}{4(n+1)}\sum_{i=1}^{n-1}\left[1-(1+\eta)\left(\frac{i}{n}\right)-(d-1)\eta\left(\frac{i}{n}\right)^2+d\eta\left(\frac{i}{n}\right)^3\right]^2$$

and

$$Var\Big[\widehat{\zeta R^{w}}_{[N,K]}(W)\Big] = \frac{n}{16(n+1)^2(n+2)}\sum_{i=1}^{n-1}\left[1-(1+\eta)\left(\frac{i}{n}\right)-(d-1)\eta\left(\frac{i}{n}\right)^2+d\eta\left(\frac{i}{n}\right)^3\right]^4.$$

Example 11. *Suppose (T_i, W_i), $i = 1,2,\dots,n$, is a random sample from the EFGM family. If W_i has RD with PDF $h_W(w) = 2\lambda w e^{-\lambda w^2}$; $w, \lambda > 0$. Then, the RVs $Z_i = \frac{W^2_{i+1:n}-W^2_{i:n}}{2}$, $i = 1,2,\dots,n-1$, follow the exponential distribution with a mean $\frac{1}{2\lambda(n-i)}$ and variance $\frac{1}{4\lambda^2(n-i)^2}$. Moreover, the mean and variance of $\widehat{\zeta R^{w}}_{[N,K]}(W)$ are, respectively, given by*

$$E\Big[\widehat{\zeta R^{w}}_{[N,K]}(W)\Big] = \frac{1}{4\lambda}\sum_{i=1}^{n-i}\frac{1}{n-i}\left[1-(1+\eta)\left(\frac{i}{n}\right)-(d-1)\eta\left(\frac{i}{n}\right)^2+d\eta\left(\frac{i}{n}\right)^3\right]^2$$

and

$$Var\Big[\widehat{\zeta R^{w}}_{[N,K]}(W)\Big] = \frac{1}{16\lambda^2}\sum_{i=1}^{n-1}\frac{1}{(n-i)^2}\left[1-(1+\eta)\left(\frac{i}{n}\right)-(d-1)\eta\left(\frac{i}{n}\right)^2+d\eta\left(\frac{i}{n}\right)^3\right]^4.$$

Table 8 presents $E\Big[\widehat{\zeta R^{w}}_{[N,K]}(W)\Big]$ and $Var\Big[\widehat{\zeta R^{w}}_{[N,K]}(W)\Big]$ for EFGM-ED at $K = 2$, $n = 10$, and $d = 0.5$. Table 8 shows the following features:

1. Generally, with fixed N and λ_2, $E\Big[\widehat{\zeta R^{w}}_{[N,K]}(W)\Big]$ and $Var\Big[\widehat{\zeta R^{w}}_{[N,K]}(W)\Big]$ increase with increasing c.
2. Generally, with fixed N and c, $E\Big[\widehat{\zeta R^{w}}_{[N,K]}(W)\Big]$ and $Var\Big[\widehat{\zeta R^{w}}_{[N,K]}(W)\Big]$ increase with increasing λ_2.

Table 8. $E\left[\widehat{\zeta R^w}_{[N,K]}(W)\right]$ and $Var\left[\widehat{\zeta R^w}_{[N,K]}(W)\right]$ for EFGM-ED at $K=2$, $n=10$, and $d=0.5$.

		$E\left[\widehat{\zeta R^w}_{[N,K]}(W)\right]$				$Var\left[\widehat{\zeta R^w}_{[N,K]}(W)\right]$			
N	λ_2	$c=-0.2$	$c=-0.1$	$c=0.1$	$c=0.2$	$c=-0.2$	$c=-0.1$	$c=0.1$	$c=0.2$
3	0.5	0.051246	0.053704	0.058884	0.061607	0.000388	0.000415	0.000478	0.000513
3	1	0.102492	0.107408	0.117768	0.123213	0.001551	0.001662	0.001911	0.002051
3	2	0.204985	0.214816	0.235537	0.246426	0.006204	0.006646	0.007643	0.008204
5	0.5	0.046457	0.051172	0.061692	0.067497	0.000337	0.000387	0.000514	0.000594
5	1	0.092915	0.102344	0.123384	0.134995	0.001350	0.001548	0.002056	0.002377
5	2	0.185829	0.204687	0.246767	0.269990	0.005400	0.006191	0.008222	0.009507
8	0.5	0.043604	0.049609	0.063528	0.071443	0.000310	0.000370	0.000538	0.000653
8	1	0.087208	0.099217	0.127056	0.142886	0.001238	0.001480	0.002154	0.002612
8	2	0.174415	0.198434	0.254113	0.285773	0.004954	0.005921	0.008616	0.010449
10	0.5	0.042915	0.049224	0.063993	0.072452	0.000303	0.000366	0.000545	0.000669
10	1	0.085830	0.098448	0.127985	0.144904	0.001212	0.001464	0.002179	0.002675
10	2	0.171660	0.196897	0.255971	0.289808	0.004850	0.005856	0.008717	0.010699

4.3. EM of NCEX in CKRV Based on EFGM(c,d)

From (34) and (10), we obtain the EM of $\zeta_{[N,K]}$ as

$$\begin{aligned}\widehat{\zeta}_{[N,K]}(W) &= \frac{1}{2}\int_0^\infty \left[1-\widehat{H}^2_{[N,K]}(w)\right]dw\\ &= \frac{1}{2}\sum_{i=1}^{n-1} U_i\left[1-\left((1+\eta)\left(\frac{i}{n}\right)+(d-1)\eta\left(\frac{i}{n}\right)^2-d\eta\left(\frac{i}{n}\right)^3\right)^2\right].\end{aligned}$$

Example 12. *Suppose that* (T_i, W_i), $i=1,2,\ldots,n$, *is a random sample from EFGM-UD with parameters 0 and 1. Thus,*

$$E\left[\widehat{\zeta}_{[N,K]}(W)\right]=\frac{1}{2(n+1)}\sum_{i=1}^{n-1}\left[1-\left((1+\eta)\left(\frac{i}{n}\right)+(d-1)\eta\left(\frac{i}{n}\right)^2-d\eta\left(\frac{i}{n}\right)^3\right)^2\right]$$

and

$$Var\left[\widehat{\zeta}_{[N,K]}(W)\right]=\frac{n}{4(n+1)^2(n+2)}\sum_{i=1}^{n-1}\left[1-\left((1+\eta)\left(\frac{i}{n}\right)+(d-1)\eta\left(\frac{i}{n}\right)^2-d\eta\left(\frac{i}{n}\right)^3\right)^2\right]^2.$$

Example 13. *Let* (T_i, W_i), $i=1,2,\ldots,n$, *be a random sample from EFGM-ED. Then, we have*

$$E\left[\widehat{\zeta}_{[N,K]}(W)\right]=\frac{1}{2\lambda_2}\sum_{i=1}^{n-1}\frac{1}{n-i}\left[1-\left((1+\eta)\left(\frac{i}{n}\right)+(d-1)\eta\left(\frac{i}{n}\right)^2-d\eta\left(\frac{i}{n}\right)^3\right)^2\right]$$

and

$$Var\left[\widehat{\zeta}_{[N,K]}(W)\right]=\frac{1}{4\lambda_2^2}\sum_{i=1}^{n-1}\frac{1}{(n-i)^2}\left[1-\left((1+\eta)\left(\frac{i}{n}\right)+(d-1)\eta\left(\frac{i}{n}\right)^2-d\eta\left(\frac{i}{n}\right)^3\right)^2\right]^2.$$

Figure 4 shows the relation between NCREX and the empirical NCEX in $W_{[N,K]}$ from EFGM-UD, at $n=100$. It can be concluded that NCREX and empirical NCREX have very similar values.

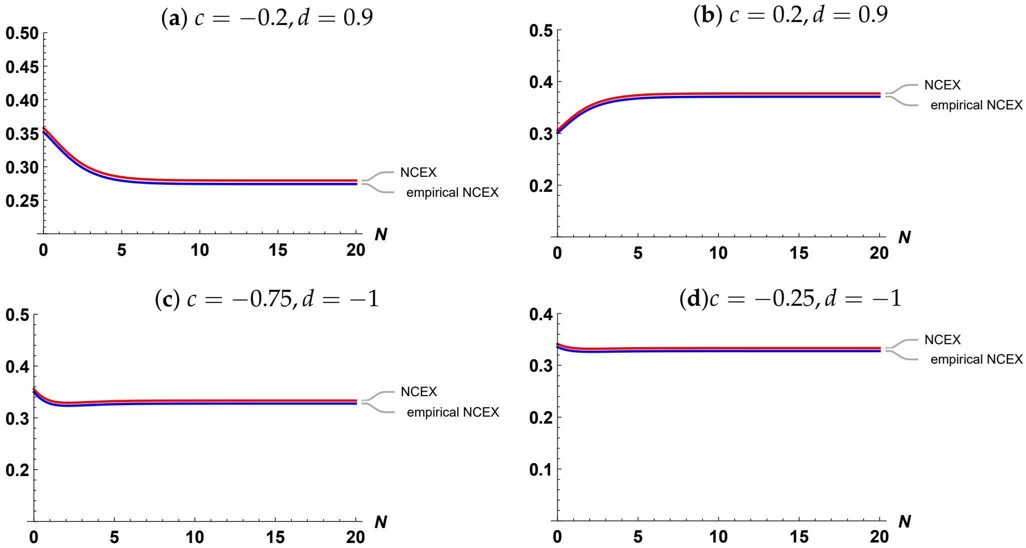

Figure 4. Representation of NCEX and empirical NCEX based on $W_{[N,K=1]}$ from EFGM-UD.

Table 9 shows $E\left[\widehat{\zeta}_{[N,K]}(W)\right]$ and $Var\left[\widehat{\zeta}_{[N,K]}(W)\right]$ for EFGM-ED at $K = 2$, $n = 10$, and $d = 0.5$. It is observed that:

1. At fixed N and λ_2, $E\left[\widehat{\zeta}_{[N,K]}(W)\right]$ and $Var\left[\widehat{\zeta}_{[N,K]}(W)\right]$ increase as the value of c increases.
2. At fixed N and c, $E\left[\widehat{\zeta}_{[N,K]}(W)\right]$ and $Var\left[\widehat{\zeta}_{[N,K]}(W)\right]$ increase as the value of λ_2 increases.

Table 9. $E\left[\widehat{\zeta}_{[N,K]}(W)\right]$ and $Var\left[\widehat{\zeta}_{[N,K]}(W)\right]$ for EFGM-ED at $K = 2$, $n = 10$, and $d = 0.5$.

		$E\left[\widehat{\zeta}_{[N,K]}(W)\right]$				$Var\left[\widehat{\zeta}_{[N,K]}(W)\right]$			
N	λ_2	$c = -0.2$	$c = -0.1$	$c = 0.1$	$c = 0.2$	$c = -0.2$	$c = -0.1$	$c = 0.1$	$c = 0.2$
3	0.5	0.317745	0.327711	0.347113	0.356549	0.011409	0.012208	0.013878	0.014747
3	1	0.635491	0.655422	0.694226	0.713098	0.045635	0.048830	0.055513	0.058989
3	2	1.270980	1.310840	1.388450	1.426200	0.182540	0.195321	0.222053	0.235954
5	0.5	0.296643	0.317435	0.356837	0.375448	0.009847	0.011385	0.014774	0.016601
5	1	0.593286	0.634870	0.713675	0.750895	0.039389	0.045538	0.059097	0.066403
5	2	1.186570	1.269740	1.427350	1.501790	0.157557	0.182153	0.236388	0.265614
8	0.5	0.282818	0.310796	0.362931	0.387088	0.008918	0.010875	0.015356	0.017820
8	1	0.565635	0.621591	0.725862	0.774177	0.035674	0.043501	0.061424	0.071280
8	2	1.131270	1.243180	1.451720	1.548350	0.142696	0.174003	0.245696	0.285121
10	0.5	0.279318	0.309126	0.364441	0.389948	0.008695	0.010750	0.015503	0.018129
10	1	0.558636	0.618252	0.728881	0.779895	0.034780	0.042999	0.062010	0.072515
10	2	1.117270	1.236500	1.457760	1.559790	0.139120	0.171997	0.248041	0.290061

4.4. EM of WNCEX in CKRV Based on EFGM(c,d)

Based on (11), the EM of $\zeta^{w}_{[N,K]}(W)$ is given by

$$\widehat{\zeta^{w}}_{[N,K]}(W) = \frac{1}{2}\int_0^\infty w\left[1 - \widehat{H}^2_{[N,K]}(w)\right]dw. \tag{37}$$

Using the CDF representation of CKRV that is established in (15) and substituting into (37), the empirical measure of $\zeta^{w}_{[N,K]}(W)$ can be calculated as

$$\begin{aligned}\widehat{\zeta^{w}}_{[N,K]}(W) &= \frac{1}{2}\int_0^\infty w\left[1-\left((1+\eta)\left(\frac{i}{n}\right)+(d-1)\eta\left(\frac{i}{n}\right)^2-d\eta\left(\frac{i}{n}\right)^3\right)^2\right]dw \\ &= \frac{1}{2}\sum_{i=1}^{n-1} Z_i\left[1-\left((1+\eta)\left(\frac{i}{n}\right)+(d-1)\eta\left(\frac{i}{n}\right)^2-d\eta\left(\frac{i}{n}\right)^3\right)^2\right].\end{aligned}$$

Example 14. *Assume (T_i, W_i), $i = 1, 2, \ldots, n$, is a random sample from the EFGM family and the RV W_i follows a distribution with the PDF $h_W(w) = 2w, \quad 0 < w < 1$. Therefore, we obtain*

$$E\left[\widehat{\zeta^{w}}_{[N,K]}(W)\right] = \frac{1}{4(n+1)}\sum_{i=1}^{n-1}\left[1-\left((1+\eta)\left(\frac{i}{n}\right)+(d-1)\eta\left(\frac{i}{n}\right)^2-d\eta\left(\frac{i}{n}\right)^3\right)^2\right]$$

and

$$Var\left[\widehat{\zeta^{w}}_{[N,K]}(W)\right] = \frac{n}{16(n+1)^2(n+2)}\sum_{i=1}^{n-1}\left[1-\left((1+\eta)\left(\frac{i}{n}\right)+(d-1)\eta\left(\frac{i}{n}\right)^2-d\eta\left(\frac{i}{n}\right)^3\right)^2\right]^2.$$

Example 15. *Suppose (T_i, W_i), $i = 1, 2, \ldots, n$, is a random sample from the EFGM family. If the RV W_i follows the Rayleigh distribution with the PDF $h_W(w) = 2\lambda w e^{-\lambda w^2}$; $w, \lambda > 0$, then, we have*

$$E\left[\widehat{\zeta^{w}}_{[N,K]}(W)\right] = \frac{1}{4\lambda}\sum_{i=1}^{n-i}\frac{1}{n-i}\left[1-\left((1+\eta)\left(\frac{i}{n}\right)+(d-1)\eta\left(\frac{i}{n}\right)^2-d\eta\left(\frac{i}{n}\right)^3\right)^2\right]$$

and

$$Var\left[\widehat{\zeta^{w}}_{[N,K]}(W)\right] = \frac{1}{16\lambda^2}\sum_{i=1}^{n-1}\frac{1}{(n-i)^2}\left[1-\left((1+\eta)\left(\frac{i}{n}\right)+(d-1)\eta\left(\frac{i}{n}\right)^2-d\eta\left(\frac{i}{n}\right)^3\right)^2\right]^2.$$

Table 10 clarifies a numerical application of Example 15 at $K = 2$, $n = 10$, and $d = 0.5$ and some distinct values of the parameters c, λ_2, and N. It is apparent that

1. For fixed N and λ_2, $E\left[\widehat{\zeta^{w}}_{[N,K]}(W)\right]$ and $Var\left[\widehat{\zeta^{w}}_{[N,K]}(W)\right]$ increase as c increases.
2. For fixed N and c, $E\left[\widehat{\zeta^{w}}_{[N,K]}(W)\right]$ and $Var\left[\widehat{\zeta^{w}}_{[N,K]}(W)\right]$ increase as λ_2 increases.

Table 10. $E\left[\widehat{\zeta^{w}}_{[N,K]}(W)\right]$ and $Var\left[\widehat{\zeta^{w}}_{[N,K]}(W)\right]$ for EFGM-ED at $K = 2$, $n = 10$, and $d = 0.5$.

		$E\left[\widehat{\zeta^{w}}_{[N,K]}(W)\right]$				$Var\left[\widehat{\zeta^{w}}_{[N,K]}(W)\right]$			
N	λ_2	$c = -0.2$	$c = -0.1$	$c = 0.1$	$c = 0.2$	$c = -0.2$	$c = -0.1$	$c = 0.1$	$c = 0.2$
3	0.5	0.158873	0.163855	0.173556	0.178275	0.002852	0.003052	0.003470	0.003687
3	1	0.317745	0.327711	0.347113	0.356549	0.011409	0.012208	0.013878	0.014747
3	2	0.635491	0.655422	0.694226	0.713098	0.045635	0.048830	0.055513	0.058989
5	0.5	0.148321	0.158718	0.178419	0.187724	0.002462	0.002846	0.003694	0.004150
5	1	0.296643	0.317435	0.356837	0.375448	0.009847	0.011385	0.014774	0.016601
5	2	0.593286	0.634870	0.713675	0.750895	0.039389	0.045538	0.059097	0.066403
8	0.5	0.141409	0.155398	0.181465	0.193544	0.002230	0.002719	0.003839	0.004455

Table 10. *Cont.*

		$E\left[\widehat{\zeta^w}_{[N,K]}(W)\right]$				$Var\left[\widehat{\zeta^w}_{[N,K]}(W)\right]$			
N	λ_2	$c=-0.2$	$c=-0.1$	$c=0.1$	$c=0.2$	$c=-0.2$	$c=-0.1$	$c=0.1$	$c=0.2$
8	1	0.282818	0.310796	0.362931	0.387088	0.008918	0.010875	0.015356	0.017820
8	2	0.565635	0.621591	0.725862	0.774177	0.035674	0.043501	0.061424	0.071280
10	0.5	0.139659	0.154563	0.182220	0.194974	0.002174	0.002687	0.003876	0.004532
10	1	0.279318	0.309126	0.364441	0.389948	0.008695	0.010750	0.015503	0.018129
10	2	0.558636	0.618252	0.728881	0.779895	0.034780	0.042999	0.062010	0.072515

5. Conclusions

Despite the fact that the EFGM family is as efficient as many other generalizations of the FGM family in terms of correlation level, its flexibility, and usability render it superior to many of these generalizations. Owing to this advantage, most PDFs in this paper are linear functions of other simpler distributions. This study has yielded useful representations of the PDF, CDF, and survival function of CKRV, along with some elegant symmetry relationships between them.

EX and its more recent related measures for CKRV were derived from the EFGM family, where a numerical study was carried out to reveal some features of these measures. Also, the QF based on these measures was derived. In addition, we derived non-parametric estimators of NCREX, WNCREX, NCEX, and WNCEX. An empirical analysis of the NCREX and NCEX has produced distinct results.

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Abbreviations

RVs	random variables
CDF	cumulative distribution function
PDF	probability density function
QF	quantile function
FGM	Farlie–Gumbel–Morgenstern
EFGM	extended Farlie–Gumbel–Morgenstern
OSs	order statistics
KRVs	*K*-record upper values
EX	extropy
CREX	cumulative residual extropy

CKRV	K-record upper values
NCREX	negative cumulative residual extropy
WNCREX	weighted negative cumulative residual extropy
NCEX	negative cumulative extropy
WNCEX	weighted negative cumulative extropy
EFGM-UD	EFGM family with uniform marginals
EFGM-ED	EFGM family with exponential marginals
EFGM-PFD	EFGM family with power function distribution marginals
EFGM-PID	EFGM family with Pareto type-I distribution marginals

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