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Closed-Loop Continuous-Time Subspace Identification with Prior Information

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Abstract: This paper presents a closed-loop continuous-time subspace identification method using prior information. Based on a rational inner function, a generalized orthonormal basis can be constructed, and the transformed noises have ergodicity features. The continuous-time stochastic system is converted into a discrete-time stochastic system by using generalized orthogonal basis functions. As is known to all, incorporating prior information into identification strategies can increase the precision of the identified model. To enhance the precision of the identification method, prior information is integrated through the use of constrained least squares, and principal component analysis is adopted to achieve the reliable estimate of the system. Moreover, the identification of open-loop models is the primary intent of the continuous-time system identification approaches. For closed-loop systems, the open-loop subspace identification methods may produce biased results. Principal component analysis, which reliably estimates closed-loop systems, provides a solution to this problem. The restricted least-squares method with an equality constraint is used to incorporate prior information into the impulse response following the principal component analysis. The input–output algebraic equation yielded an optimal multi-step-ahead predictor, and the equality constraints describe the prior information. The effectiveness of the proposed method is provided by numerical simulations.

Keywords: subspace identification; closed-loop identification; generalized orthonormal basis functions; principal component analysis; prior information; constrained least squares

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1. Introduction

In recent decades, subspace identification methods have garnered significant attention in the identification process [1–3]. The majority of identification approaches work with discrete-time, open-loop systems. Actually, the majority of systems are continuous in nature and run in closed-loop using a feedback controller. Moreover, many open-loop identification approaches could produce biased results for closed-loop system identification. Hence, investigating a novel identification approach for closed-loop continuous-time systems is significant.

To acquire the continuous-time systems, estimating discrete-time models and converting them into continuous-time systems are conventional methods. There are certain drawbacks regarding the system conversion. These drawbacks specifically include the complicated computation of the matrix logarithm and the difficult selection of the sampling time. Numerous identification approaches have been presented to address the aforementioned problems in order to acquire reliable models. Linear filtering, which includes Laguerre filters, generalized PMFs (GPMFs), and Poisson Moment Functionals (PMFs), is the first category [4–6]. In [7], a generalized singular-value decomposition (SVD) was used to compensate for the noise coloration for estimating the continuous-time system. Using the GPMF and nuclear norm minimization, Ref. [8] evaluated continuous-time models. In order to address two types of identification problems, Ref. [9] presented continuous-time system identification methods that made use of Laguerre filters and Laguerre projection. Based on fractional Laguerre generating functions, Ref. [10] provided the subspace identification approach for fractional-commensurate-order systems. The second category consists of modulating function methods, which were used in [11] to address the continuous-time state-space model-identification problem using generalized orthogonal basis functions. A transformation theory using generalized orthonormal basis functions for stochastic systems was established by [12]. The integral approaches fall into the third category. Ref. [13] used random distribution theory to illustrate time derivatives and to deduce the input–output algebra relationship in the time-domain. The subspace identification approaches were implemented in the aforementioned references with the open-loop assumption. For the future inputs that are correlated with the noise, these approaches have the drawback of giving biased estimates when the systems to be represented run in closed-loop. Ref. [14] presented a closed-loop subspace identification method based on subsequent SVD and orthogonal projection. Ref. [15] concentrated on the parity space rather than the observable subspace to produce a consistent estimation of the closed-loop systems. Several closed-loop subspace identification approaches during the past ten years were compiled by [16]. The closed-loop continuous-time subspace identification approaches are still thought of as parameterized black-box models, while the majority of the aforementioned closed-loop identification approaches concentrate on discrete-time systems [17–19]. Therefore, it is still difficult to incorporate prior information into closed-loop continuous-time identification algorithms.

In general, the model's quality is determined by the quality of the input–output data. To guarantee the precision and consistency of the estimated models, the subspace identification approaches also require the confirmation of a specific excitation. However, in many circumstances, the input–output data might not provide enough information for the inadequate excitation or noise component. Obviously, these factors may influence the precision of the estimated models. To improve the quality of the identified model, the prior information can be incorporated into the models throughout the identification process. The prior information was integrated into the state space model realization algorithm by [20], which estimated the model parameters using Kung's singular-value decomposition realization [21]. Ref. [22] investigated the subspace identification approach using constrained least squares while taking into account prior information. The subspace identification method was enhanced as a multi-step forward predictor by resolving an optimization problem with equality constraints. Ref. [23] developed a new recursive-subspace-identification method that takes into account prior information, based on constrained recursive least squares. The prior information was taken into consideration by [24] as a linear equality and inequality constraints on the impulse response, which was resolved using active-set-optimization approaches. When comparing the precision of the identified model, subspace identification with prior information proved to be more accurate than classical subspace algorithms. Ref. [25] introduced the closed-loop subspace identification method, which took advantage of prior knowledge by eliminating the correlation between the future input and past innovation. By utilizing prior information, Ref. [26] developed a new closed-loop subspace-identification method based on principal component analysis (PCA). For batch operations exposed to repetitive disturbances, Ref. [27] developed a prior-knowledge-based subspace identification approach that offered unbiased parameter estimation and enhanced robustness to white measurement noises. Very few studies have focused on closed-loop continuous-time subspace system identification, despite the fact that including prior information in the subspace identification algorithm can improve the quality of the identified model.

In this paper, we propose to use generalized orthonormal basis functions to achieve closed-loop continuous-time subspace identification with prior information. The main contributions can be summarized as follows:

(1) The continuous-time stochastic system is converted into a discrete-time stochastic system by using generalized orthogonal basis functions.

(2) For closed-loop systems, the open-loop subspace identification methods may produce biased results. The principal component analysis, which reliably estimates closed-loop systems, provides a solution to this problem.

(3) The proposed approach converts the prior information into an equality constraint, and a weighting mechanism is used to solve the constrained least-squares issues. With increased computational performance, the proposed method can provide unbiased process models.

The remainder of this paper is organized as follows: The preliminaries are presented in Section 2. In Section 3, the system transformation is provided. Section 4 presents the main closed-loop subspace identification algorithm. The numerical simulations used to assess the proposed approach are shown in Section 5. Section 6 offers conclusions.

2. Preliminaries

Let $L^2(-\infty,\infty)$ denote the space of square integrable functions over $(-\infty,\infty)$, and the inner product is obtained.

$$\langle u,v\rangle = \int_{-\infty}^{\infty} v(t)^* u(t)dt, \tag{1}$$

where superscript * denotes the conjugate transpose.

Let $L^2(j\mathbb{R})$ denote the space of square integrable functions of frequency $j\omega \in j\mathbb{R}$, and the inner product is given.

$$\langle f,g\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)^* f(\omega)d\omega. \tag{2}$$

A continuous-time multiplication operator Λ_h: $L^2(j\mathbb{R}) \to L^2(j\mathbb{R})$ is defined as

$$(\Lambda_h f)(j\omega) = h(j\omega)f(j\omega), \tag{3}$$

where $h \in L^2(j\mathbb{R})$. Inner function $\phi(s)$ denotes a continuous-time, single-input, single-output rational transfer function.

The Hardy space H^∞ is the space of matrix-valued bounded analytic functions in the right half plane. For a continuous-time inner function $\phi(s)$ in H^∞, the multiplication operator $\Lambda_\phi : L^2(j\mathbb{R}) \to L^2(j\mathbb{R})$ is unitary. Let $\phi(s) \in H^\infty$ be a non-constant inner function. Consider the orthogonal complement of the invariant subspace $\Lambda_\phi H^2$ in H^2, that is $S = H^2 \ominus \Lambda_\phi H^2$; the Hardy space H^2 is the space of analytical functions on the right half plane.

$$H^2 = \bigoplus_{m=0}^{\infty} \Lambda_\phi^m S,\ \ L^2(j\mathbb{R}) = \bigoplus_{m=-\infty}^{\infty} \Lambda_\phi^m S, \tag{4}$$

where $\bigoplus$ denotes the direct sum.

A function $f \in H^2$ can be represented as follows.

$$f = \sum_{m=0}^{\infty} \Lambda_\phi^m f_m,\ \ f = \sum_{m=-\infty}^{\infty} \Lambda_\phi^m f_m,\ \ f_m \in S. \tag{5}$$

where m-power denotes the product of m elements.

Let $\{v_1,\cdots,v_{n_\phi}\}$ be an orthonormal basis for S and

$$\left\{v_1,\cdots,v_{n_\phi},\Lambda_\phi v_1,\cdots,\Lambda_\phi v_{n_\phi},\cdots,\Lambda_\phi^m v_1,\cdots,\Lambda_\phi^m v_{n_\phi},\cdots\right\} \tag{6}$$

be an orthonormal basis for H^2. Based on the the inverse Fourier transform, the set is identified with an orthonormal basis for $L^2(0,\infty)$. This set is called a generalized orthonormal basis. A generalized orthonormal basis can be constructed by any finite-dimensional Blashke product.

Proposition 1 ([28]). *Consider an inner function ϕ and the subspace $S = H^2 \ominus \Lambda_\phi H^2$. If $\phi(s) = \boldsymbol{D}_\phi + \boldsymbol{C}_\phi(s\boldsymbol{I} - \boldsymbol{A}_\phi)^{-1}\boldsymbol{B}_\phi$ is the balanced realization of an inner function for a continuous-time system such that $\boldsymbol{D}_\phi = \boldsymbol{I}$, $\boldsymbol{A}_\phi + \boldsymbol{A}_\phi^T + \boldsymbol{B}_\phi^T\boldsymbol{B}_\phi = \boldsymbol{0}$, then*

$$\begin{aligned} \boldsymbol{v}(s) &= [v_1(s), \cdots, v_{n_u n_\phi}(s)], \\ &= (\boldsymbol{I}_n \otimes \boldsymbol{C}_\phi)(s\boldsymbol{I} - (\boldsymbol{I}_n \otimes \boldsymbol{A}_\phi))^{-1} \end{aligned}$$

is an orthonormal basis for S^{n_u}, where $\otimes$ means the Kronecker product.

3. System Transformation

The generalized orthonormal basis can be constructed by the rational inner function. In addition, the continuous-time stochastic system is converted into a discrete-time stochastic system by using generalized orthogonal basis functions.

Consider a continuous-time system with process and observation noises:

$$\begin{aligned} d\boldsymbol{x} &= \boldsymbol{A}\boldsymbol{x}dt + \boldsymbol{B}_1 d\boldsymbol{w} + \boldsymbol{B}_2\boldsymbol{u}dt, \;\; \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ d\boldsymbol{\eta} &= \boldsymbol{C}\boldsymbol{x}dt + \boldsymbol{D}_1 d\boldsymbol{w} + \boldsymbol{D}_2\boldsymbol{u}dt, \end{aligned} \tag{7}$$

where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_1 \in \mathbb{R}^{n\times n_w}$, $\boldsymbol{B}_2 \in \mathbb{R}^{n\times n_u}$, $\boldsymbol{C} \in \mathbb{R}^{n_y\times n}$, $\boldsymbol{D}_1 \in \mathbb{R}^{n_y\times n_w}$, $\boldsymbol{D}_2 \in \mathbb{R}^{n_y\times n_u}$, $\boldsymbol{w}$ is a Wiener process, and $\boldsymbol{u}$ is a deterministic signal.

Define

$$\begin{aligned} \tilde{\boldsymbol{u}}_k &= \int_0^\infty \Lambda_\phi^k \boldsymbol{v}(t)^T \boldsymbol{u}(t)dt, \\ \tilde{\boldsymbol{w}}_k &= \int_0^\infty \Lambda_\phi^k \boldsymbol{v}(t)^T d\boldsymbol{w}(t), \\ \tilde{\boldsymbol{y}}_k &= \int_0^\infty \Lambda_\phi^k \boldsymbol{v}(t)^T d\boldsymbol{\eta}(\boldsymbol{t}). \end{aligned} \tag{8}$$

Notice that $\tilde{\boldsymbol{w}}_k$ and $\tilde{\boldsymbol{y}}_k$ are stochastic processes.

Theorem 1. *Consider a stochastic system (7), and define the deterministic sequence $\tilde{\boldsymbol{u}}_k$ and the stochastic processes $\tilde{\boldsymbol{w}}_k$, $\tilde{\boldsymbol{y}}_k$ by Equation (8). Then, they satisfy the following discrete-time stochastic system:*

$$\begin{aligned} \boldsymbol{\xi}(k+1) &= \tilde{\boldsymbol{A}}\boldsymbol{\xi}(k) + \tilde{\boldsymbol{B}}_1\tilde{\boldsymbol{w}}_k + \tilde{\boldsymbol{B}}_2\tilde{\boldsymbol{u}}_k, \;\; \boldsymbol{\xi}(0) = \boldsymbol{x}_0, \\ \boldsymbol{y}_k &= \tilde{\boldsymbol{C}}\boldsymbol{\xi}(k) + \tilde{\boldsymbol{D}}_1\tilde{\boldsymbol{w}}_k + \tilde{\boldsymbol{D}}_2\tilde{\boldsymbol{u}}_k, \end{aligned} \tag{9}$$

where

$$\begin{aligned} &\tilde{\boldsymbol{A}} = \phi^\sim(\boldsymbol{A}), \tilde{\boldsymbol{B}}_1 = \boldsymbol{X}_1, \tilde{\boldsymbol{B}}_2 = \boldsymbol{X}_2, \tilde{\boldsymbol{C}} = \boldsymbol{Y}, \\ &\tilde{\boldsymbol{D}}_1 = \begin{bmatrix} h_{111}^\sim(\boldsymbol{A}_\phi^T) & \cdots & h_{11n_u}^\sim(\boldsymbol{A}_\phi^T) \\ \vdots & \vdots & \vdots \\ h_{1n_y1}^\sim(\boldsymbol{A}_\phi^T) & \cdots & h_{1n_yn_u}^\sim(\boldsymbol{A}_\phi^T) \end{bmatrix}, \\ &\tilde{\boldsymbol{D}}_2 = \begin{bmatrix} h_{211}^\sim(\boldsymbol{A}_\phi^T) & \cdots & h_{21n_u}^\sim(\boldsymbol{A}_\phi^T) \\ \vdots & \vdots & \vdots \\ h_{2n_y1}^\sim(\boldsymbol{A}_\phi^T) & \cdots & h_{2n_yn_u}^\sim(\boldsymbol{A}_\phi^T) \end{bmatrix}, \end{aligned} \tag{10}$$

$h_{1ij}(s)$ is the (i,j)-th element of the transfer function $h_1(s) = \boldsymbol{D}_1 + \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}_1$; $h_{2ij}(s)$ is the (i,j)-th element of the transfer function $h_2(s) = \boldsymbol{D}_2 + \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}_2$; $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, and $\boldsymbol{Y}$ are the unique solutions to the following Sylvester equations:

$$\begin{aligned} &\boldsymbol{A}\boldsymbol{X}_1 + \boldsymbol{X}_1(\boldsymbol{I}_{n_u} \otimes \boldsymbol{A}_\phi)^T + \boldsymbol{B}_1(\boldsymbol{I}_{n_u} \otimes \boldsymbol{B}_\phi)^T = 0, \\ &\boldsymbol{A}\boldsymbol{X}_2 + \boldsymbol{X}_2((\boldsymbol{I}_{n_u} \otimes \boldsymbol{A}_\phi)^T + \boldsymbol{B}_2(\boldsymbol{I}_{n_u} \otimes \boldsymbol{B}_\phi)^T = 0, \\ &(\boldsymbol{I}_{n_u} \otimes \boldsymbol{A}_\phi)^T\boldsymbol{Y} + \boldsymbol{Y}\boldsymbol{A} + (\boldsymbol{I}_{n_u} \otimes \boldsymbol{C}_\phi)^T\boldsymbol{C} = 0. \end{aligned} \tag{11}$$

Proof. Assume that $n_u = n_y = 1$. If $\boldsymbol{u} \in S$, from Proposition 1, $\boldsymbol{u} = \boldsymbol{C}_\phi(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{\eta}$ for some $\boldsymbol{\eta}$. In terms of $\boldsymbol{C}_\phi^T\boldsymbol{C}_\phi = (s\boldsymbol{I} - \boldsymbol{A}_\phi) - (s\boldsymbol{I} + \boldsymbol{A}_\phi^T)$, we have

$$\begin{aligned}
\Lambda_\phi^\sim \boldsymbol{u} &= [\boldsymbol{D}_\phi^T - \boldsymbol{B}_\phi^T(s\boldsymbol{I} + \boldsymbol{A}_\phi^T)^{-1}\boldsymbol{C}_\phi^T] \cdot \boldsymbol{C}_\phi(s\boldsymbol{I} - \boldsymbol{A}_\phi)^{-1}\boldsymbol{\eta} \\
&= \boldsymbol{D}_\phi^T\boldsymbol{C}_\phi(s\boldsymbol{I} - \boldsymbol{A}_\phi)^{-1}\boldsymbol{\eta} - \boldsymbol{B}_\phi^T(s\boldsymbol{I} + \boldsymbol{A}_\phi^T)^{-1}\boldsymbol{\eta} + \boldsymbol{B}_\phi^T(s\boldsymbol{I} - \boldsymbol{A}_\phi)^{-1}\boldsymbol{\eta} \\
&= -\boldsymbol{B}_\phi^T(s\boldsymbol{I} + \boldsymbol{A}_\phi^T)^{-1}\boldsymbol{\eta}.
\end{aligned}$$

In view of Equation (10), it can be found that

$$\begin{aligned}
\boldsymbol{B}\boldsymbol{u} &= \int_{-\infty}^{0} e^{-\boldsymbol{A}\tau}\boldsymbol{B}(\mathcal{F}^{-1}\Lambda_\phi^\sim \boldsymbol{u})(\tau)d\tau, \\
&= \int_{-\infty}^{0} e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{B}_\phi^T e^{-\boldsymbol{A}_\phi^T\tau}d\tau\boldsymbol{\eta} = \boldsymbol{X}\boldsymbol{\eta}.
\end{aligned}$$

$\boldsymbol{C}\boldsymbol{\xi}$ is the orthogonal projection of $\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}$ onto S. The orthonormal basis for S is used for the following equation.

$$\begin{aligned}
\boldsymbol{C}\boldsymbol{\xi} &= \boldsymbol{C}_\phi(s\boldsymbol{I} - \boldsymbol{A}_\phi)^{-1}\int_0^\infty e^{-\boldsymbol{A}_\phi^T\tau}\boldsymbol{C}_\phi^T\boldsymbol{C}e^{-\boldsymbol{A}\tau}d\tau\boldsymbol{\eta}, \\
&= \boldsymbol{C}_\phi(s\boldsymbol{I} - \boldsymbol{A}_\phi)^{-1}Y\boldsymbol{\xi}.
\end{aligned}$$

Notice that $\boldsymbol{D}u$ is the orthogonal projection of $\Lambda_h u$ onto S. Therefore,

$$\begin{aligned}
&\frac{1}{2\pi} \int_{-\infty}^{\infty}(-j\omega\boldsymbol{I} - \boldsymbol{A}_\phi^T)^{-1}\boldsymbol{C}_\phi^T h(j\omega) \cdot \boldsymbol{C}_\phi(j\omega\boldsymbol{I} - \boldsymbol{A}_\phi)^{-1}d\omega\boldsymbol{\eta} \\
&= \frac{1}{2\pi j}\oint h(s)(s\boldsymbol{I} + \boldsymbol{A}_\phi^T)^{-1}C_\phi^T C_\phi(s\boldsymbol{I} - \boldsymbol{A}_\phi)^{-1}ds\boldsymbol{\eta}, \\
&= \frac{1}{2\pi j}\oint h(s)[(s\boldsymbol{I} + \boldsymbol{A}_\phi^T)^{-1} - (s\boldsymbol{I} - \boldsymbol{A}_\phi)^{-1}]ds\boldsymbol{\eta}, \\
&= h(-\boldsymbol{A}_\phi^T)\boldsymbol{\eta} = h^\sim(\boldsymbol{A}_\phi^T)\boldsymbol{\eta}.
\end{aligned}$$

Note that the matrices $\boldsymbol{B}$ and $\boldsymbol{D}$ are defined, mutatis mutandis, as $\tilde{\boldsymbol{B}}_1$, $\tilde{\boldsymbol{B}}_2$, $\tilde{\boldsymbol{D}}_1$, and $\tilde{\boldsymbol{D}}_2$. □

Similar to any other identification method, when the available data are not informative enough due to a low signal-to-noise ratio or insufficient input excitations, the performance of the proposed approach will deteriorate. However, the proposed approach incorporates the prior information and can, thus, diminish the deterioration of model quality.

4. Closed-Loop Subspace Identification

In this section, the restricted-least-squares method with an equality constraint is used to incorporate the prior information into the impulse response following the principal component analysis. The method closed-loop subspace identification with prior information using generalized orthonormal basis functions (CLSPI-GOBF) is proposed.

4.1. Derivation of Input–Output Algebraic Equation

In terms of the stochastic system (7), the following data can be constructed as

$$\begin{aligned}
\tilde{\boldsymbol{u}}_{k,i} &= \int_0^\infty \Lambda_\phi^k \boldsymbol{v}(t)^T\boldsymbol{u}(t + t_i)dt, \\
\tilde{\boldsymbol{w}}_{k,i} &= \int_0^\infty \Lambda_\phi^k \boldsymbol{v}(t)^T d\boldsymbol{w}(t + t_i), \\
\tilde{\boldsymbol{y}}_{k,i} &= \int_0^\infty \Lambda_\phi^k \boldsymbol{v}(t)^T d\boldsymbol{\eta}(t + t_i),
\end{aligned} \tag{12}$$

where $0 \le t_0 < t_1 < \cdots < t_i < \cdots$ is a sequence of time instances such that $t_{i+1} - t_i \ge t_{\min}$ for some $t_{\min} > 0$.

Let $x_i = x(t_i)$. For fixed integers p, q and N, define

$$\boldsymbol{X}_N = [x_0 \ \ x_1 \ \ \cdots \ \ x_{N-1}], \boldsymbol{X}_{q,N} = [x_q \ \ x_{q+1} \ \ \cdots \ \ x_{q+N-1}],$$
$$\boldsymbol{U}_{p,q,N} = \begin{bmatrix} \tilde{u}_{p,0} & \tilde{u}_{p,1} & \cdots & \tilde{u}_{p,N-1} \\ \tilde{u}_{p+1,0} & \tilde{u}_{p+1,1} & \cdots & \tilde{u}_{p+1,N-1} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{u}_{p+q-1,0} & \tilde{u}_{p+q-1,1} & \cdots & \tilde{u}_{p+q-1,N-1} \end{bmatrix}, \tag{13}$$

$$\boldsymbol{W}_{p,q,N} = \begin{bmatrix} \tilde{w}_{p,0} & \tilde{w}_{p,1} & \cdots & \tilde{w}_{p,N-1} \\ \tilde{w}_{p+1,0} & \tilde{w}_{p+1,1} & \cdots & \tilde{w}_{p+1,N-1} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{w}_{p+q-1,0} & \tilde{w}_{p+q-1,1} & \cdots & \tilde{w}_{p+q-1,N-1} \end{bmatrix}, \tag{14}$$

$$\boldsymbol{Y}_{p,q,N} = \begin{bmatrix} \tilde{y}_{p,0} & \tilde{y}_{p,1} & \cdots & \tilde{y}_{p,N-1} \\ \tilde{y}_{p+1,0} & \tilde{y}_{p+1,1} & \cdots & \tilde{y}_{p+1,N-1} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{y}_{p+q-1,0} & \tilde{y}_{p+q-1,1} & \cdots & \tilde{y}_{p+q-1,N-1} \end{bmatrix}. \tag{15}$$

Let

$$\boldsymbol{\Gamma}_q = \begin{bmatrix} \tilde{\boldsymbol{C}} \\ \tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}} \\ \vdots \\ \tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{q-1} \end{bmatrix},$$
$$\boldsymbol{H}_{1,q} = \begin{bmatrix} \tilde{\boldsymbol{D}}_1 & 0 & \cdots & 0 \\ \tilde{\boldsymbol{C}}\tilde{\boldsymbol{B}}_1 & \tilde{\boldsymbol{D}}_1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{q-2}\tilde{\boldsymbol{B}}_1 & \tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{q-3}\tilde{\boldsymbol{B}}_1 & \cdots & \tilde{\boldsymbol{D}}_1 \end{bmatrix}, \tag{16}$$
$$\boldsymbol{H}_{2,q} = \begin{bmatrix} \tilde{\boldsymbol{D}}_2 & 0 & \cdots & 0 \\ \tilde{\boldsymbol{C}}\tilde{\boldsymbol{B}}_2 & \tilde{\boldsymbol{D}}_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{q-2}\tilde{\boldsymbol{B}}_2 & \tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{q-3}\tilde{\boldsymbol{B}}_2 & \cdots & \tilde{\boldsymbol{D}}_2 \end{bmatrix}.$$

From Theorem 1, the above matrices satisfy the input–output algebraic equation:

$$\boldsymbol{Y}_{p,q,N} = \boldsymbol{\Gamma}_q \boldsymbol{X}_{q,N} + \boldsymbol{H}_{1,q}\boldsymbol{W}_{p,q,N} + \boldsymbol{H}_{2,q}\boldsymbol{U}_{p,q,N}. \tag{17}$$

4.2. Consistent Estimation via Principal Component Analysis

Multiplying both sides by $(\boldsymbol{\Gamma}_q^{\perp})^T$, Equation (17) can be converted to

$$(\boldsymbol{\Gamma}_q^{\perp})^T \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{H}_{2,q} \end{bmatrix} \begin{bmatrix} \boldsymbol{Y}_{p,q,N} \\ \boldsymbol{U}_{p,q,N} \end{bmatrix} = (\boldsymbol{\Gamma}_q^{\perp})^T (\boldsymbol{H}_{1,q}\boldsymbol{W}_{p,q,N}). \tag{18}$$

Define $\boldsymbol{Z}_q = \begin{bmatrix} \boldsymbol{Y}_{p,q,N} \\ \boldsymbol{U}_{p,q,N} \end{bmatrix}$. Equation (18) becomes

$$(\boldsymbol{\Gamma}_q^{\perp})^T \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{H}_{2,q} \end{bmatrix} \boldsymbol{Z}_q = (\boldsymbol{\Gamma}_q^{\perp})^T (\boldsymbol{H}_{1,q}\boldsymbol{W}_{p,q,N}), \tag{19}$$

The instrumental variable $\boldsymbol{Z}_h = \begin{bmatrix} \boldsymbol{U}_{0,p,N} \\ \boldsymbol{Y}_{0,p,N} \end{bmatrix}$ is introduced to eliminate the estimate bias [16]. It gives

$$\lim_{N\to\infty} \frac{1}{N} \boldsymbol{H}_{1,q}\boldsymbol{W}_{p,q,N}\boldsymbol{Z}_h^T = 0, \tag{20}$$

where $\boldsymbol{Z}_h = \begin{bmatrix} \boldsymbol{U}_{0,p,N} \\ \boldsymbol{Y}_{0,p,N} \end{bmatrix}$ consisting of the input–output data are the instrumental variables.

We have

$$\lim_{N\to\infty} \frac{1}{N}(\boldsymbol{\Gamma}_q^{\perp})^T \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{H}_{2,q} \end{bmatrix} \boldsymbol{Z}_q \boldsymbol{Z}_h^T = 0; \tag{21}$$

thus, Equation (21) implies that $(\boldsymbol{\Gamma}_q^{\perp})^T \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{H}_{2,q} \end{bmatrix}$ is in the left null space of $\lim\limits_{N\to\infty} \frac{1}{N}\boldsymbol{Z}_q\boldsymbol{Z}_h^T$.

The PCA decomposition is performed on

$$\frac{1}{N}\boldsymbol{Z}_q\boldsymbol{Z}_h^T = \boldsymbol{P}\boldsymbol{T}^T + \tilde{\boldsymbol{P}}\tilde{\boldsymbol{T}}^T, \tag{22}$$

where $\boldsymbol{P}$, $\tilde{\boldsymbol{P}}$ are the loading matrices and are mutually orthogonal. $\boldsymbol{T}$, $\tilde{\boldsymbol{T}}$ are the score matrices, and the number of principal components is selected as $\lim\limits_{N\to\infty} \tilde{\boldsymbol{T}} = 0$.

It can be found that

$$(\boldsymbol{\Gamma}_q^{\perp})^T \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{H}_{2,q} \end{bmatrix} \boldsymbol{P} = 0. \tag{23}$$

Equation (23) can be reformulated as

$$\begin{bmatrix} \boldsymbol{\Gamma}_q^{\perp} \\ -\boldsymbol{H}_{2,q}^T\boldsymbol{\Gamma}_q^{\perp} \end{bmatrix}^T \boldsymbol{P} = 0. \tag{24}$$

Since Equation (24) indicates that $\begin{bmatrix} \boldsymbol{\Gamma}_q^{\perp} \\ -\boldsymbol{H}_{2,q}^T\boldsymbol{\Gamma}_q^{\perp} \end{bmatrix}$ shares the same column space as $\tilde{\boldsymbol{P}}$, we have

$$\begin{bmatrix} \boldsymbol{\Gamma}_q^{\perp} \\ -\boldsymbol{H}_{2,q}^T\boldsymbol{\Gamma}_q^{\perp} \end{bmatrix} = \tilde{\boldsymbol{P}}\boldsymbol{M}. \tag{25}$$

where $\boldsymbol{M}$ is an arbitrary non-singular matrix.

Matrix $\tilde{\boldsymbol{P}}$ can be partitioned as

$$\begin{bmatrix} \boldsymbol{\Gamma}_q^{\perp} \\ -\boldsymbol{H}_{2,q}^T\boldsymbol{\Gamma}_q^{\perp} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{P}}_y \\ \tilde{\boldsymbol{P}}_u \end{bmatrix} \boldsymbol{M}. \tag{26}$$

We have

$$\boldsymbol{\Gamma}_q^{\perp} = \tilde{\boldsymbol{P}}_y\boldsymbol{M}, \tag{27}$$

$$-\boldsymbol{H}_{2,q}^T\boldsymbol{\Gamma}_q^{\perp} = \tilde{\boldsymbol{P}}_u\boldsymbol{M} \tag{28}$$

According to Equation (27), we select a specific $\boldsymbol{M}$.

$$\boldsymbol{\Gamma}_q = \tilde{\boldsymbol{P}}_y^{\perp}. \tag{29}$$

Substituting Equation (27) into Equation (28) gives

$$-\tilde{\boldsymbol{P}}_y^T\boldsymbol{H}_{2,q} = \tilde{\boldsymbol{P}}_u^T \tag{30}$$

Define

$$-\tilde{\boldsymbol{P}}_y^T = \Xi = \begin{bmatrix} \Xi_1 & \Xi_2 & \cdots & \Xi_i \end{bmatrix}, \tag{31}$$

and

$$\tilde{\boldsymbol{P}}_u^T = \Upsilon = \begin{bmatrix} \Upsilon_1 & \Upsilon_2 & \cdots & \Upsilon_i \end{bmatrix}, \tag{32}$$

where Ξ_i is the i-th block column of Ξ and Υ_i is the i-th block column of Υ.

Equation (30) can be described as

$$\Xi\boldsymbol{H}_{2,q} = \Upsilon, \tag{33}$$

that is

$$\begin{bmatrix} \Xi_1 & \Xi_2 & \cdots & \Xi_i \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{D}}_2 & 0 & \cdots & 0 \\ \tilde{\boldsymbol{C}}\tilde{\boldsymbol{B}}_2 & \tilde{\boldsymbol{D}}_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{q-2}\tilde{\boldsymbol{B}}_2 & \tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{q-3}\tilde{\boldsymbol{B}}_2 & \cdots & \tilde{\boldsymbol{D}}_2 \end{bmatrix} = \begin{bmatrix} \Upsilon_1 & \Upsilon_2 & \cdots & \Upsilon_i \end{bmatrix}. \tag{34}$$

Let

$$\boldsymbol{H}_{2,q1} = \begin{bmatrix} \tilde{\boldsymbol{D}}_2 \\ \tilde{\boldsymbol{C}}\tilde{\boldsymbol{B}}_2 \\ \vdots \\ \tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{q-2}\tilde{\boldsymbol{B}}_2 \end{bmatrix}; \tag{35}$$

we obtain

$$\begin{bmatrix} \Xi_1 & \Xi_2 & \cdots & \Xi_{i-1} & \Xi_i \\ \Xi_2 & \Xi_3 & \cdots & \Xi_i & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Xi_i & 0 & \cdots & 0 & 0 \end{bmatrix} \boldsymbol{H}_{2,q1} = \begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \\ \cdots \\ \Upsilon_i \end{bmatrix}. \tag{36}$$

4.3. Constrained-Least-Squares Approach

To enforce the structure of $\boldsymbol{H}_{2,q}$, the following relationship can be required.

$$\mathrm{vec}(\boldsymbol{ABC}) = (\boldsymbol{C}^T \otimes \boldsymbol{A})\mathrm{vec}(\boldsymbol{B}), \tag{37}$$

where vec stands for the operation of forming a long vector from a matrix by stacking its columns one under the other.

Equation (34) can be described as

$$\underbrace{\mathrm{vec}(\Upsilon)}_{y} = \underbrace{\left(\boldsymbol{I}^T \otimes \Xi\right)}_{\boldsymbol{Z}} \underbrace{\mathrm{vec}(\boldsymbol{H}_{2,q1})}_{\theta}. \tag{38}$$

Equation (38) can be solved in the least-squares sense with added equality constraints representing the prior knowledge. This leads to a new problem:

$$\min_{\theta} \|y - \boldsymbol{Z}\theta\|_2^2, \tag{39}$$

which is subject to the following equality constraints:

$$\boldsymbol{A}_{\mathrm{eq}}\theta = \boldsymbol{b}_{\mathrm{eq}}, \tag{40}$$

where $\boldsymbol{A}_{\mathrm{eq}} \in \mathbb{R}^{c\times(qn_y n_\phi(pn_y n_\phi + pn_u n_\phi) + qn_y n_\phi n_u n_\phi)}$, $\boldsymbol{b}_{\mathrm{eq}} \in \mathbb{R}^{c\times 1}$, and c is the number of constraints.

4.3.1. Known Steady-State Gain

Assume that it takes k sampling times for the system to settle and that q impulse response parameters, $g_0, g_1, \ldots, g_{q-1}$, are estimated. The constraints can be described as

$$\sum_{i=0}^{k-1} g_i = \boldsymbol{K}_{ss}, g_k = g_{k+1} = \cdots = g_{q-1} = \boldsymbol{0}_{n_y n_\phi \times n_u n_\phi}, \tag{41}$$

where $\boldsymbol{K}_{ss} \in \mathbb{R}^{n_y n_\phi \times n_u n_\phi}$ is the DC gain matrix. It can be described as

$$\boldsymbol{K}_{ss} = \begin{bmatrix} \boldsymbol{K}_{11} & \cdots & \boldsymbol{K}_{1n_u n_\phi} \\ \vdots & \vdots & \vdots \\ \boldsymbol{K}_{n_y n_\phi 1} & \cdots & \boldsymbol{K}_{n_y n_\phi n_u n_\phi} \end{bmatrix}, \tag{42}$$

and $\boldsymbol{K}_{ij}$ is the DC gain from the j-th input to the i-th output.

In view of Equation (37), applying this vec operation for both sides in each constraint in (41), we have

$$\mathbf{Y} \times \underbrace{\begin{bmatrix} \mathrm{vec} g_0 \\ \mathrm{vec} g_1 \\ \vdots \\ \mathrm{vec} g_{k-1} \\ \mathrm{vec} g_k \\ \mathrm{vec} g_{k+1} \\ \vdots \\ \mathrm{vec} g_{q-1} \end{bmatrix}}_{\theta_g} = \begin{bmatrix} \mathrm{vec}\boldsymbol{K}_{ss} \\ \mathbf{0}_{n_y n_\phi n_u n_\phi \times 1} \\ \vdots \\ \mathbf{0}_{n_y n_\phi n_u n_\phi \times 1} \end{bmatrix} \tag{43}$$

where

$$\mathbf{Y} = \begin{bmatrix} \boldsymbol{I}_{lm\times lm} & \boldsymbol{I}_{lm\times lm} & \cdots & \boldsymbol{I}_{lm\times lm} & \mathbf{0}_{lm\times lm} & \mathbf{0}_{lm\times lm} & \cdots & \mathbf{0}_{lm\times lm} \\ \mathbf{0}_{lm\times lm} & \mathbf{0}_{lm\times lm} & \cdots & \mathbf{0}_{lm\times lm} & \boldsymbol{I}_{lm\times lm} & \mathbf{0}_{lm\times lm} & \cdots & \mathbf{0}_{lm\times lm} \\ \mathbf{0}_{lm\times lm} & \mathbf{0}_{lm\times lm} & \cdots & \mathbf{0}_{lm\times lm} & \mathbf{0}_{lm\times lm} & \boldsymbol{I}_{lm\times lm} & \cdots & \mathbf{0}_{lm\times lm} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0}_{lm\times lm} & \mathbf{0}_{lm\times lm} & \cdots & \mathbf{0}_{lm\times lm} & \mathbf{0}_{lm\times lm} & \mathbf{0}_{lm\times lm} & \cdots & \boldsymbol{I}_{lm\times lm} \end{bmatrix}, \tag{44}$$

Equation (43) can be expressed as

$$\begin{bmatrix} \mathbf{1}_{1\times k} \otimes \boldsymbol{I}_{lm\times lm} & \mathbf{0}_{lm\times(q-k)lm} \\ \mathbf{0}_{(q-k)lm\times klm} & \boldsymbol{I}_{(q-k)lm\times(q-k)lm} \end{bmatrix} \theta_g = \begin{bmatrix} \mathrm{vec}(\boldsymbol{K}_{ss}) \\ \mathbf{0}_{(q-k)lm\times 1} \end{bmatrix}, \tag{45}$$

where the symbol $l = n_y n_\phi, m = n_u n_\phi$, $\mathbf{1}_{p\times q}$ is a matrix of ones of size $p \times q$ and $\mathbf{0}_{p\times q}$ is a matrix of zeros of size $p \times q$. In view of θ, Equation (45) can be described as

$$\boldsymbol{A}_{\mathrm{eq}}\theta = \boldsymbol{b}_{\mathrm{eq}}, \tag{46}$$

where

$$\begin{aligned} \boldsymbol{A}_{\mathrm{eq}} &= \begin{bmatrix} \mathbf{0}_{(q-k+1)lm\times ql(pl+pm)} & \begin{matrix} \mathbf{1}_{1\times k} \otimes \boldsymbol{I}_{lm\times lm} & \mathbf{0}_{lm\times(q-k)lm} \\ \mathbf{0}_{(q-k)lm\times klm} & \boldsymbol{I}_{(q-k)lm\times(q-k)lm} \end{matrix} \end{bmatrix}, \\ \boldsymbol{b}_{\mathrm{eq}} &= \begin{bmatrix} \mathrm{vec}(\boldsymbol{K}_{ss}) \\ \mathbf{0}_{(q-k)lm\times 1} \end{bmatrix}. \end{aligned} \tag{47}$$

4.3.2. Zero Transfer Functions

The outputs of some multiple-input multiple-output systems are unaffected by the inputs. Accordingly, some transfer functions are zero. However, by identifying such systems based on noisy data, the zero transfer functions cannot be established. The transfer functions might be pre-specified during the identification phase to address the aforementioned issue.

If the j-th input and the i-th output have a zero transfer function and all the impulse response coefficients at this channel are zeros,

$$g_0^{ij} = g_1^{ij} = \cdots = g_{q-1}^{ij} = 0, \tag{48}$$

where the q constraints are formed. The zero transfer function can be written as

$$(\mathbf{0}_{1\times ql(pl+pm)}\ \ \mathbf{0}_{1\times klm+l(j-1)+i-1}\ 1\ 0\ \dots\ 0)\theta = 0. \tag{49}$$

Therefore, $\boldsymbol{A}_{\text{eq}}$ and $\boldsymbol{b}_{\text{eq}}$ can be described as

$$\begin{aligned} \boldsymbol{A}_{\text{eq}} &= \left[\begin{array}{cccccc} & \mathbf{0}_{1\times l(j-1)+i-1} & 1 & 0 & \cdots & 0 \\ & \mathbf{0}_{1\times lm+l(j-1)+i-1} & 1 & 0 & \cdots & 0 \\ \mathbf{0}_{q\times ql(pl+pm)} & \vdots & \vdots & \vdots & \vdots & \vdots \\ & \mathbf{0}_{1\times (q-1)lm+l(j-1)+i-1} & 1 & 0 & \cdots & 0 \end{array}\right] \\ &\in \mathbb{R}^{q\times(ql(pl+pm)+qlm)}, \\ \boldsymbol{b}_{\text{eq}} &= \mathbf{0}_{q\times 1}. \end{aligned} \tag{50}$$

Noted that each zero transfer function needs q constraints.

The impulse response parameters $\hat{\theta}_g$ can be estimated, and the following Hankel matrix $\boldsymbol{T}$ is formed.

$$\boldsymbol{T} = \begin{bmatrix} \hat{g}_1 & \hat{g}_2 & \cdots & \hat{g}_{q/2} \\ \hat{g}_2 & \hat{g}_3 & \cdots & \hat{g}_{q/2+1} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{g}_{q/2} & \hat{g}_{q/2+1} & \cdots & \hat{g}_{q-1} \end{bmatrix}. \tag{51}$$

The matrix $\boldsymbol{T}$ can be factorized using Kung's realization algorithm as

$$\boldsymbol{T} = \boldsymbol{USV}^T. \tag{52}$$

The observability matrix $\boldsymbol{\Gamma}$ and controllability matrix $\boldsymbol{\Delta}$ are obtained as

$$\boldsymbol{\Gamma} = \boldsymbol{U}(:, 1:n)\boldsymbol{S}^{1/2}, \boldsymbol{\Delta} = \boldsymbol{S}^{1/2}\boldsymbol{V}(:, 1:n)^T. \tag{53}$$

The system matrices are given using $\boldsymbol{\Gamma}$ and $\boldsymbol{\Delta}$ as

$$\begin{aligned} \hat{\boldsymbol{A}} &= \underline{\boldsymbol{\Gamma}}^{\dagger}\overline{\boldsymbol{\Gamma}}, \underline{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma}(1:l(q-1),:), \overline{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma}(l+1;ql,:). \\ \hat{\boldsymbol{B}} &= \boldsymbol{\Delta}(:,1:m), \hat{\boldsymbol{C}} = \boldsymbol{\Gamma}(1:l,:), \hat{\boldsymbol{D}} = \hat{g}_0. \end{aligned} \tag{54}$$

Remark 1. *The proposed approach's performance will decrease, just like any other identification method, if the input–output data are insufficiently informative because of a low signal-to-noise ratio or inadequate input excitations. The generalized orthonormal basis functions are constructed and effected by the rational inner function. To solve this problem, prior information is integrated through the use of constrained least squares.*

Remark 2. *It should be noted that a large number of traditional subspace identification approaches are inapplicable to closed-loop data. This issue stems from an identification procedure step that projects onto the future horizon and necessitates that the input from the future horizon be uncorrelated with the noise from the past. The proposed approach avoids this projection. Consequently, the cases of closed-loop data can be valuably applied to the proposed approach.*

4.4. Summary of Subspace System Identification Algorithm

The proposed CLSPI-GOBF method is summarized in Table 1.

Table 1. Summary of the CLSPI-GOBF method.

step 1:
(1) Construct the data matrices $\boldsymbol{Y}_{p,q,N}, \boldsymbol{U}_{p,q,N}, \boldsymbol{Y}_{0,p,N}, \boldsymbol{U}_{0,p,N}$. Form $\boldsymbol{Z}, y$ in Equation (38).
(2) Transform the prior information into $\boldsymbol{A}_{\text{eq}}$ and $\boldsymbol{b}_{\text{eq}}$. Utilizing Equation (40) as a constraint, the least-squares optimization problem is solved in Equation (39).
step 2:
(3) Estimate the impulse response parameters $\hat{\theta}_g$ by Equation (51).
(4) Factorize $\boldsymbol{T}$ using Kung's realization algorithm in Equation (52).
step 3:
(5) Estimate the observability matrix $\boldsymbol{\Gamma}$ and controllability matrix $\boldsymbol{\Delta}$ using Equation (53).
(6) Extract the system matrices by Equation (54).

5. Numerical Simulation

The effectiveness of the proposed CLSPI-GOBF method was evaluated by the following examples.

5.1. Example 1: Known Steady-State Gain

Consider a second-order system with a first-order controller [29]. For the transfer functions of the process, the controller is

$$G(s) = \frac{b_1 s + b_2}{s^2 + a_2 s + a_1}, C(s) = \frac{10s + 15}{s}, \tag{55}$$

and the first-order inner function is used:

$$\phi(s) = \frac{s - p}{s + p}. \tag{56}$$

The reference signal $r(t)$ was chosen as pseudo-random binary sequence signals; $d(t)$ was set to zero; $v(t)$ is a white noise. The signal-to-noise ratio was equal to 10 dB. The sampling period was chosen to be 0.01 s.

We set $a_2 = 0.5, a_1 = 1, b_1 = b_2 = 1$, and the number of each of the future and past block rows $p = q = 10$. The experiments were conducted by 100 Monte Carlo experiments. The output response for Example 1 is depicted in Figure 1. The closed-loop subspace identification using generalized orthonormal basis functions (CLS-GOBF) and bias-eliminated least-squares method (BELSM) in [30] are compared in Figure 1. It indicates that the line of the step response from CLSPI-GOBF is closer to the step response of the true system.

The Bode diagram of the CLSPI-GOBF, CLS-GOBF, and BELSM is shown in Figure 2. The CLSPI-GOBF appeared to have a higher identification effectiveness than the other approaches, as both identified curves were able to converge to the true ones. This suggests a higher identification performance for the proposed CLSPI-GOBF approach.

For clarity, the estimated values for the CLSPI-GOBF, CLS-GOBF, and BELSM are described in Table 2.

Table 2. The values of parameter estimation.

Parameters	True	CLSPI-GOBF		CLS-GOBF		BELSM	
		Mean	Variance	Mean	Variance	Mean	Variance
a_1	1	0.9957	0.0004	0.9578	0.0052	0.9347	0.0082
a_2	0.5	0.4914	0.0005	0.4905	0.0041	0.4571	0.0021
b_1	1	0.9685	0.0007	0.9458	0.0009	0.9120	0.0062
b_2	1	0.9741	0.0005	0.9647	0.0010	0.9249	0.0073

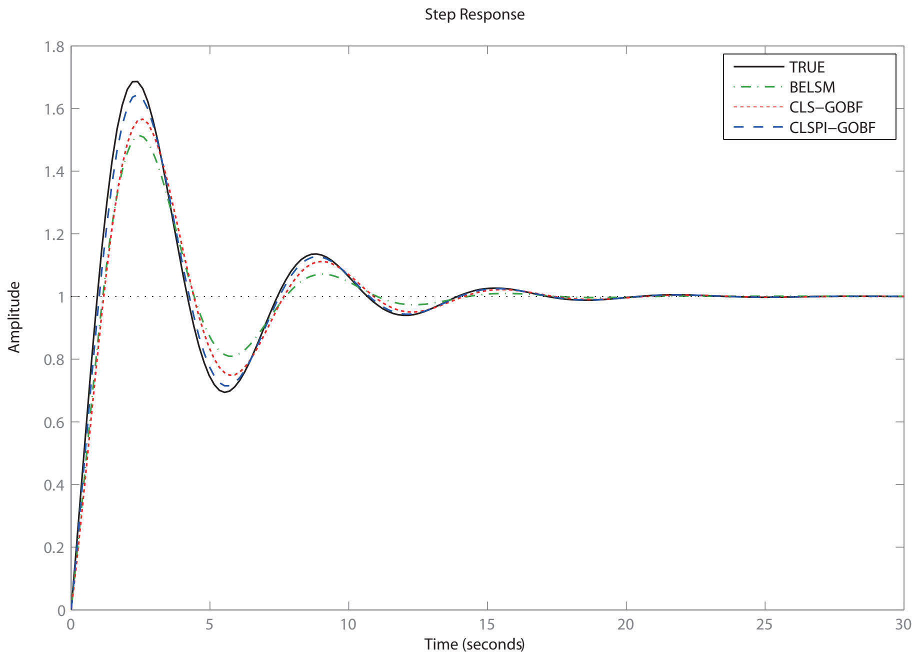

Figure 1. Comparison of step responses for simulation Example 1.

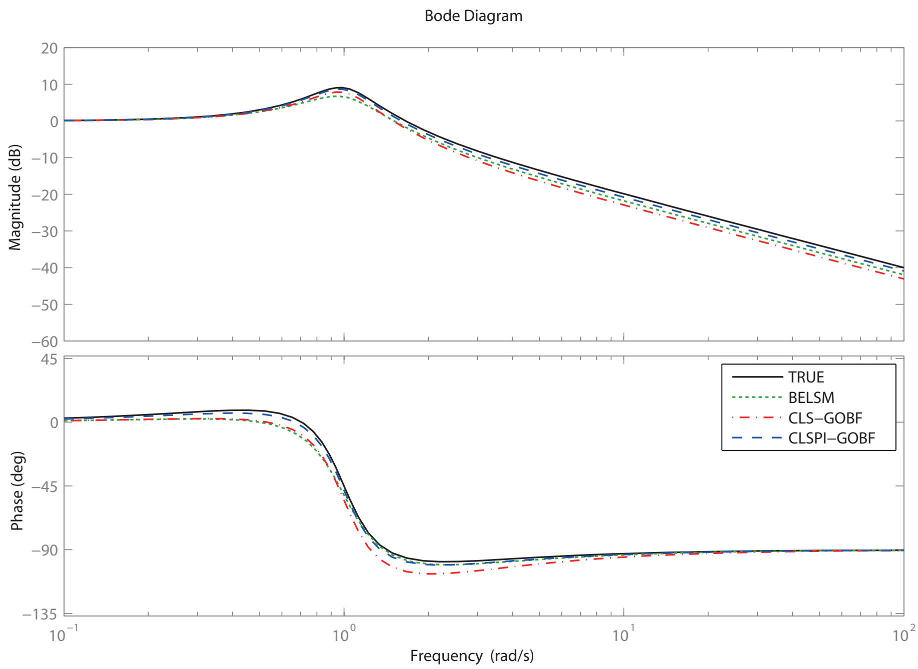

Figure 2. Comparison of Bode diagram for Example 1.

The mean values of the CLSPI-GOBF approach were more accurate than those obtained using the other methods, as indicated by the results displayed in Table 2. The corresponding values of the CLSPI-GOBF were smaller than the other methods' values in terms of variance estimations, indicating that the CLSPI-GOBF fluctuations were more mild. The results showed that the CLSPI-GOBF, the proposed method, performed better in the identification process.

5.2. Example 2: Zero Transfer Functions

Consider the following continuous-time system [29]:

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G(s)_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{2}{(s+2)} & 0 \\ 0 & \dfrac{2}{(s+2)} \end{bmatrix}, \tag{57}$$

with the following feedback controller:

$$u(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y(t) + r(t). \tag{58}$$

The first-order inner function is used:

$$\phi(s) = \frac{s-p}{s+p}. \tag{59}$$

which has two zero transfer functions G_{12} and G_{21}, while G_{11} and G_{22} have unity gains. The system was excited using the zero-mean white Gaussian process with unit variance. One hundred Monte Carlo simulations were performed.

The Hankel matrices had $N = 81$ columns, and the horizons for the three methods were all set to $p = q = 10$. We set the sampling time to 0.01 s. The CLSPI-GOBF was compared with the CLS-GOBF and BELSM in order to verify the identification algorithm's superiority. The average step responses from each of the three methods with white noise are displayed in Figure 3. The step response from the CLSPI-GOBF is shown in Figure 3 as a line of small circles, which is more consistent with the step response of the true system. Figure 4 describes the Bode diagram of the CLSPI-GOBF, CLS-GOBF, and BELSM. It can clearly be seen that the identified curves of the CLSPI-GOBF were closer than the other identified methods. Therefore, it had a better identification performance.

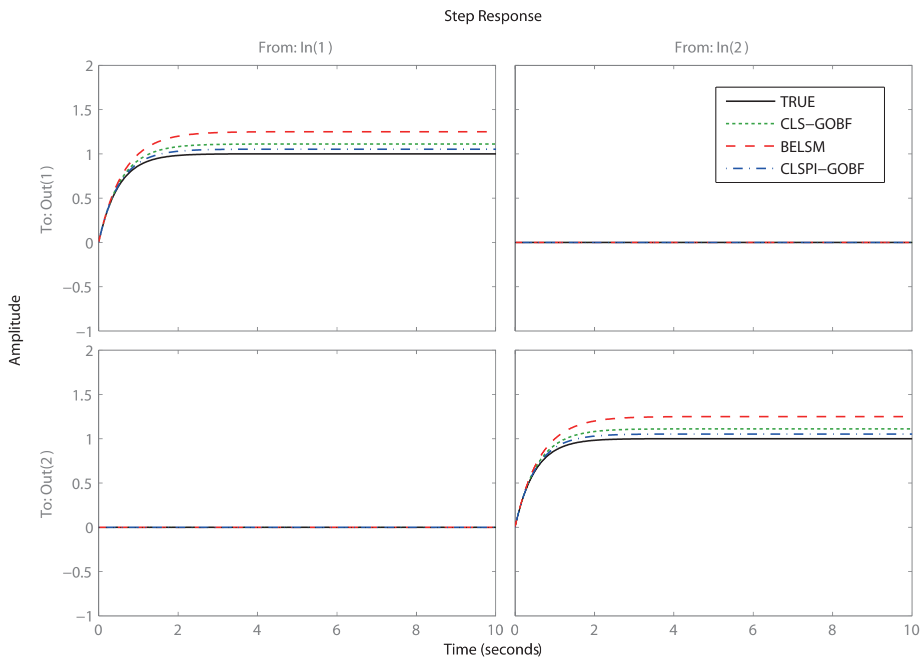

Figure 3. Comparison of step responses for Example 2.

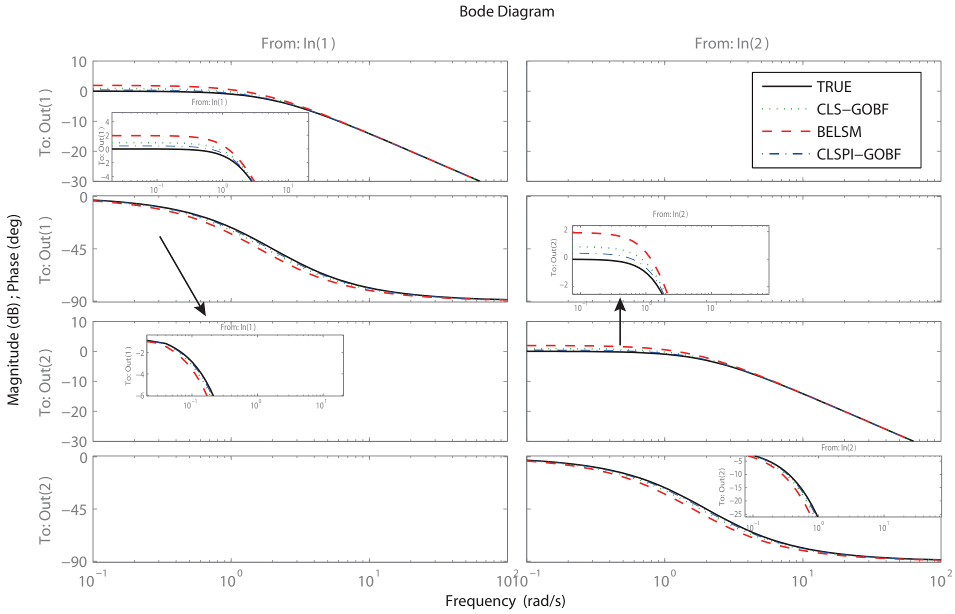

Figure 4. Comparison of Bode diagram for Example 2.

For clarity, the mean and variance of the pole estimations are depicted in Table 3.

Table 3. The values of pole estimations for Example 2.

Approach	Mean of p_1	Variance of p_1	Mean of p_2	Variance of p_2
CLSPI-GOBF	−1.9759	0.0025	−1.9654	0.0086
CLS-GOBF	−1.9524	0.0457	−1.9204	0.0527
BELSM	−1.9140	0.1547	−1.8580	0.8426

Table 3 describes that the estimated values by the CLSPI-GOBF were superior to those generated by the other methods. Comparing the CLSPI-GOBF to the other methods, the former yielded more-accurate and -consistent results.

6. Conclusions

A closed-loop continuous-time subspace identification with prior information was proposed. The transformation of the continuous-time stochastic system into a discrete-time stochastic system was the foundation of the method. Then, the continuous-time system model was obtained by inversely transforming the deterministic portion. Based on the principal component analysis method, the consistent estimate of the closed-loop systems can be obtained. Prior information was introduced in terms of equality constraints based on the constrained least squares. Kung's realization approach can be used to extract the system matrices and increase the accuracy of the impulse response parameters with the prior information. The simulation results showed that the proposed method increased the accuracy of the identified model. It is important to extend the proposed method to inaccurate or incomplete prior information, which will be the future work.

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