mathematics

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A Time-Fractional Parabolic Inequality on a Bounded Interval

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Abstract: We study a time-fractional parabolic inequality posed on a bounded interval and involving a wight function W, where the fractional derivative is considered in the Caputo sense. We establish a general condition ensuring that the set of weak solutions is empty. Next, some particular cases of the weight function W are discussed.

Keywords: time-fractional parabolic inequality; caputo fractional derivative; weak solution; nonexistence

MSC: 34K37; 35K58; 35A01

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Citation: Alshabanat, A.; Almoalim, E.; Jleli, M.; Samet, B. A Time-Fractional Parabolic Inequality on a Bounded Interval. *Mathematics* **2023**, *11*, 4892. https://doi.org/10.3390/math11244892

Academic Editors: Osman Tunç, Vitalii Slynko and Sandra Pinelas

Received: 8 November 2023
Revised: 3 December 2023
Accepted: 5 December 2023
Published: 6 December 2023

1. Introduction

We consider problem

$$\begin{cases} u_{t^\beta} - u_{xx} - \lambda^2 u \geq W|u|^d \text{ in } \mathbb{R}_+ \times]\xi_1, \xi_2[, \\ u(0,\cdot) \geq u_0 \text{ in }]\xi_1, \xi_2[, \\ u(\cdot, \xi_2) \geq 0 \text{ in } \mathbb{R}_+, \end{cases} \tag{1}$$

where $\mathbb{R}_+ = (0,\infty)$, $u = u(t,x)$, $\lambda \neq 0$, $\xi_2 - \frac{\pi}{|\lambda|} \leq \xi_1 < \xi_2$, $0 < \beta < 1$, u_{t^β} is the partial derivative of u w.r.t. the time variable in the Caputo sense, $d > 1$ and $W > 0$ almost everywhere. Namely, we focus on nonexistence criteria for weak solutions to (1).

Existence and nonexistence theorems for parabolic equations and inequalities have attracted much attention in the literature. One of the remarkable results in this direction is due to Fujita [1], where he considered equation

$$u_t - \Delta u = u^d \text{ in } \mathbb{R}_+ \times \mathbb{R}^N, \tag{2}$$

where $d > 1$. Namely, it was proven that $d_N = 1 + \frac{2}{N}$ is critical, in the sense that, if $u(0,\cdot) > 0$ and $d < d_N$, then the set of global positive solutions to (2) is empty; if $d > d_N$ and u_0 is small, then (2) admits global positive solutions. It was proven later (see, e.g., [2]) that, if $q = q_N$, then the set of global positive solutions to (2) is empty. Problem (2) has also been studied on other domains; for instance, exterior domains [3–6] and sectorial domains [7]. In [8], by showing that any solution to equation $-\Delta u = |\nabla u|^p$, $p > 2$ in the half-space with an homogeneous Dirichlet boundary condition has to be one-dimensional, the authors provided several applications to the parabolic problem $u_t - \Delta u = |\nabla u|^p$ posed in a bounded domain (satisfying a certain regularity) with an homogeneous Dirichlet boundary condition. In [9], problem $u_t - \tau\Delta u_t = \Delta u + |u|^p + f(x)$, $t > 0$, $x \in \mathbb{R}^N$ was considered, where $\tau > 0$, $p > 1$ and f is a nontrivial continuous function satisfying $f(x) \to 0$ as $|x| \to \infty$. It was proven that this problem admits a Fujita-type critical exponent

$p_c = \infty$ if $N \in \{1,2\}$, $p_c = \frac{N}{N-2}$ if $N \geq 3$. In [10], Mitidieri and Pohozaev considered parabolic inequality

$$\frac{\partial u}{\partial t} - |x|^{\xi} \Delta u \geq |u|^d \text{ in } \mathbb{R}_+ \times \mathbb{R}^N,$$

where $0 \leq \xi < 2$. They proved that for a certain class of the initial values, the set of weak solutions is empty, provided that $1 < d \leq 1 + \frac{2-\xi}{N-\xi}$. Parabolic differential inequalities have also been studied for different kinds of nonlinearities. For instance, Filippucci and Lombardi [11] considered differential inequality

$$u_t - \mathrm{div}(|\nabla u|^{d-2} \nabla u) \geq F(u, |\nabla u|) \text{ in } \mathbb{R}_+ \times \mathbb{R}^N,$$

where $F(u, |\nabla u|) = a(x) u^q - b(x) u^m |\nabla u|^s$, $0 \leq m < q$, $0 < s \leq \frac{p(q-m)}{q+1}$ and a, b are nonnegative weights. We also refer to [12], where quasilinear parabolic inequalities were studied in $\mathbb{R}_+ \times \mathbb{R}^N$ with nonlocal terms of the form $(K * u^p) u^q$, where $p, q > 0$, $K > 0$ belongs to a certain class of weight functions and $*$ is the convolution product w.r.t. the variable space. In [13], Kartsatos and Kurta studied the blow-up of solutions to parabolic inequalities of the form $u_t \geq F(u) + |u|^{p-1} u$, $t > 0$, $x \in \mathbb{R}^N$, where $p > 1$ and $F(u)$ includes several classes of quasilinear differential operators.

The study of blow-up for partial differential equations and inequalities with partial fractional derivatives w.r.t. the time variable has attracted much attention in the literature, see, e.g., [14–19]. In particular, the time-fractional version (in the Caputo sense) of (2) was studied by Zhang and Sun in [19], where it was shown that $q_N = 1 + \frac{2}{N}$ is still critical (in the Fujita sense). We notice that when $\lambda \to 0$ and $\beta \to 1^-$, (1) reduces to parabolic inequality

$$u_t - u_{xx} \geq W|u|^d \text{ in } \mathbb{R}_+ \times]\xi_1, \xi_2[.$$

We refer to [20] for the study of problems of the above type in the N-dimensional case.

The novelty of this work is the consideration of time-fractional parabolic inequalities involving modified Helmholtz operator $-\frac{\partial^2}{\partial x^2} - \lambda^2$ on bounded intervals. Namely, as far as we know, the study of nonexistence for evolution inequalities involving the above operator has not been previously considered. For some numerical studies of problems involving the modified Helmholtz operator, see, e.g., [21–23].

We refer to [24] for more details about the following notions. We let $\Gamma(\cdot)$ be the Gamma function. We let χ be a continuous function in $[0, \tau]$, $\tau > 0$. For $\varsigma > 0$, we let

$$I_0^{\varsigma} \chi(t) = [\Gamma(\varsigma)]^{-1} \int_0^t (t-s)^{\varsigma-1} \chi(s)\, ds, \ I_\tau^{\varsigma} \chi(t) = [\Gamma(\varsigma)]^{-1} \int_t^\tau (s-t)^{\varsigma-1} g(s)\, ds, \ 0 < t < \tau.$$

Lemma 1. *We let χ_i, $i = 1, 2$, be two continuous functions in $[0, \tau]$, and $\varsigma > 0$. We have*

$$\int_0^\tau I_0^{\varsigma} \chi_1(s)\, \chi_2(s)\, ds = \int_0^\tau \chi_1(s)\, I_\tau^{\varsigma} \chi_2(s)\, ds.$$

We let χ be a continuously differentiable function in $[0, \tau]$. The derivative of χ of order $0 < \varsigma < 1$ in the sense of Caputo is defined by

$${}^{C}D_0^{\varsigma} \chi(t) = I_0^{1-\varsigma} \chi'(t), \ \ 0 < t < \tau.$$

The above definition can be extended to functions χ that are absolutely continuous in $[0, \tau]$ (see [24]).

We let $v = v(t, x)$, $(t, x) \in [0, \tau] \times I$, where $I \subset \mathbb{R}$. We let $v_{(x,\cdot)} : [0, \tau] \ni t \to v(t, x)$, $x \in I$. For $\varsigma > 0$, by $I_0^{\varsigma} v(t, x)$ and $I_\tau^{\varsigma} v(t, x)$, we mean

$$I_0^{\varsigma} v(t, x) = I_0^{\varsigma} v_{(x,\cdot)}(t), \ \ I_\tau^{\varsigma} v(t, x) = I_\tau^{\varsigma} v_{(x,\cdot)}(t).$$

We let $0 < \varsigma < 1$. By v_{t^ς}, we mean

$$v_{t^\varsigma}(t,x) = {}^C D_0^\varsigma v_{(x,\cdot)}(t).$$

We let

$$\mathcal{A}_T = [0,T] \times]\xi_1, \xi_2], \ \ T > 0$$

and

$$\Lambda_T = \left\{ \varphi = \varphi(t,x) \in C^2(\mathcal{A}_T) : \varphi \geq 0, \mathrm{supp}(\varphi) \subset\subset \mathcal{A}_T, \varphi(\cdot, \xi_2) = 0, \varphi_x(\cdot, \xi_2) \leq 0 \right\}.$$

We now define solutions to (1).

Definition 1. *We let $\lambda \neq 0$, $\xi_2 - \frac{\pi}{|\lambda|} \leq \xi_1 < \xi_2$, $0 < \beta < 1$, $u_0 \in L^1_{\mathrm{loc}}(]\xi_1, \xi_2])$ and $W > 0$ almost everywhere. By a weak solution to (1), we mean function*

$$u \in L^d_{\mathrm{loc}}([0,\infty[\times]\xi_1,\xi_2], W\,dt\,dx) \cap L^1_{\mathrm{loc}}([0,\infty[\times]\xi_1,\xi_2]), \ d > 1$$

that satisfies

$$\int_{\mathcal{A}_T} |u|^d \varphi W\,dt\,dx + \int_{\xi_1}^{\xi_2} u_0(x) I_T^{1-\beta}\varphi(0,x)\,dx \leq -\int_{\mathcal{A}_T} u\left(\varphi_{xx} + \lambda^2\varphi + \left(I_T^{1-\beta}\varphi\right)_t\right) dt\,dx \tag{3}$$

for every $\varphi \in \Lambda_T$, $T > 0$.

The set of weak solutions to (1) is denoted by $\mathcal{S}$. Due to Lemma 1, we can show that any solution $u \in C^2([0,\infty[\times]\xi_1,\xi_2])$ to (1) belongs to $\mathcal{S}$.

The following theorem is our main result.

Theorem 1. *We let $\lambda \neq 0$, $\xi_2 - \frac{\pi}{|\lambda|} \leq \xi_1 < \xi_2$, $0 < \beta < 1$ and $d > 1$. We let $W > 0$ almost everywhere and $W^{\frac{-1}{d-1}} \in L^1_{\mathrm{loc}}([0,\infty[\times]\xi_1,\xi_2])$. We assume that*

$$u_0 \in L^1(]\xi_1,\xi_2[), \ \int_{\xi_1}^{\xi_2} u_0(x)\sin(|\lambda|(\xi_2 - x))\,dx > 0 \tag{4}$$

and there exists $j > 0$ such that

$$\liminf_{R\to\infty} F(j,R) = 0, \tag{5}$$

where

$$F(j,R) = R^{j(\beta-1)+\frac{2d}{d-1}} \int_0^{R^j} \int_{\xi_1+\frac{1}{2R}}^{\xi_1+\frac{1}{R}} W^{\frac{-1}{d-1}}\,dx\,dt + R^{\frac{j(1-d-\beta)}{d-1}} \int_0^{R^j} \int_{\xi_1+\frac{1}{2R}}^{\xi_2} W^{\frac{-1}{d-1}}\,dx\,dt.$$

Then $\mathcal{S} = \emptyset$.

We prove the above result using the test function method (see, e.g., Mitidieri and Pohozaev [10]), which requires a suitable choice of test functions. In our case, we construct a family of test functions belonging to Λ_T by taking into consideration the boundedness of the domain, the time-fractional derivative, the properties of the differential operator $-\frac{\partial^2}{\partial x^2} - \lambda^2$ and the boundary conditions.

We now investigate the case when

$$W(t,x) = (t+1)^\varrho. \tag{6}$$

From Theorem 1, we deduce the following result.

Corollary 1. *We let $\lambda \neq 0$, $\xi_2 - \frac{\pi}{|\lambda|} \leq \xi_1 < \xi_2$ and $0 < \beta < 1$. We let W be the function defined by (6). We assume that $u_0 \in L^1(]\xi_1, \xi_2[)$ and (4) holds. If*

$$d > 1, \ \varrho > \beta(d-1),$$

then $\mathcal{S} = \emptyset$.

We next consider the case when

$$W(t,x) = (x - \xi_1)^{\ell}. \tag{7}$$

Corollary 2. *We let $\lambda \neq 0$, $\xi_2 - \frac{\pi}{|\lambda|} \leq \xi_1 < \xi_2$ and $0 < \beta < 1$. We let W be the function defined by (7). We assume that $u_0 \in L^1(]\xi_1, \xi_2[)$ and (4) holds. If*

$$\ell < -2, \ 1 < d < -\ell - 1,$$

then $\mathcal{S} = \emptyset$.

Finally, we study the case when

$$W(t,x) = (x - \xi_1)^{\ell}(t+1)^{\varrho}. \tag{8}$$

Corollary 3. *We let $\lambda \neq 0$, $\xi_2 - \frac{\pi}{|\lambda|} \leq \xi_1 < \xi_2$ and $0 < \beta < 1$. We let W be the function defined by (8). We assume that $u_0 \in L^1(]\xi_1, \xi_2[)$ and (4) holds. If $\varrho > 0$ and*

$$1 < d < 1 + \frac{\varrho}{\beta}; \quad \text{or} \quad \beta(\ell+2) < -\varrho, \ d = 1 + \frac{\varrho}{\beta},$$

then $\mathcal{S} = \emptyset$.

Some preliminary estimates are established in Section 2. We prove our main result in Section 3.

We use the following notations:

- C_i, C: positive constants that are independent of T, R and solution u. The values of such constants are not important, and could be changed from one equation (or inequality) to another;
- $s \gg 1$ ($s \in \mathbb{R}$): $s > 0$ is sufficiently large.

2. Preliminary Estimates

We let $\lambda \neq 0$, $\xi_2 - \frac{\pi}{|\lambda|} \leq \xi_1 < \xi_2$, $d > 1$ and $0 < \beta < 1$. We let $W > 0$ almost everywhere, and $u_0 \in L^1_{\mathrm{loc}}(]\xi_1, \xi_2])$. For all $T > 0$, we let

$$\overline{\Lambda}_T = \{\varphi \in \Lambda_T : \omega_i(\varphi) < \infty, i = 1, 2\},$$

where

$$\begin{aligned} \omega_1(\varphi) &= \int_{\mathrm{supp}(\varphi_{xx} + \lambda^2\varphi)} \left|\varphi_{xx} + \lambda^2\varphi\right|^{\frac{d}{d-1}} (\varphi W)^{\frac{-1}{d-1}} \, dx \, dt, & (9) \\ \omega_2(\varphi) &= \int_{\mathrm{supp}\left(\left(I_T^{1-\beta}\varphi\right)_t\right)} \left|\left(I_T^{1-\beta}\varphi\right)_t\right|^{\frac{d}{d-1}} (\varphi W)^{\frac{-1}{q-1}} \, dx \, dt. & (10) \end{aligned}$$

Lemma 2. *We let $\varphi \in \overline{\Lambda}_T$, $T > 0$. If $u \in \mathcal{S}$, then*

$$\int_{\xi_1}^{\xi_2} u_0(x) I_T^{1-\beta}\varphi\,(0,x) \, dx \leq C \sum_{i=1}^{2} \omega_i(\varphi). \tag{11}$$

Proof. We let $u \in \mathcal{S}$. By (3), we have

$$\begin{aligned}
&\int_{\mathcal{A}_T} |u|^d \varphi W \, dx \, dt + \int_{\xi_1}^{\xi_2} u_0(x) I_T^{1-\beta} \varphi \,(0,x) \, dx \\
&\leq \int_{\mathcal{A}_T} |u| \left| \varphi_{xx} + \lambda^2 \varphi \right| dx \, dt + \int_{\mathcal{A}_T} |u| \left| \left(I_T^{1-\beta} \varphi \right)_t \right| dx \, dt := I_1 + I_2.
\end{aligned} \tag{12}$$

We now use Young's inequality to obtain

$$\begin{aligned}
I_1 &= \int_{\mathcal{A}_T} \left(|u| \varphi^{\frac{1}{d}} W^{\frac{1}{d}} \right) \left(\varphi^{\frac{-1}{d}} \left| \varphi_{xx} + \lambda^2 \varphi \right| W^{\frac{-1}{d}} \right) dx \, dt \\
&\leq \frac{1}{2} \int_{\mathcal{A}_T} |u|^d \varphi W \, dx \, dt + C \omega_1(\varphi)
\end{aligned} \tag{13}$$

and

$$I_2 \leq \frac{1}{2} \int_{\mathcal{A}_T} |u|^d \varphi W \, dx \, dt + C \omega_2(\varphi). \tag{14}$$

Finally, (11) follows from (12)–(14). □

For $m \gg 1$ and $T > 0$, we let

$$\zeta_T(s) = \left(1 - \frac{s}{T} \right)^m \tag{15}$$

for every $s \in [0,T]$. We let

$$\vartheta \in C^\infty([0,\infty[), \; 0 \leq \vartheta \leq 1, \; \vartheta|_{[0,\frac{1}{2}]} = 0, \; \vartheta|_{[1,\infty)} = 1. \tag{16}$$

For $R \gg 1$, we let

$$\vartheta_R(x) = \kappa(x) \vartheta^m(R(x - \xi_1)), \; \xi_1 \leq x \leq \xi_2, \tag{17}$$

where

$$\kappa(x) = \sin(|\lambda|(\xi_2 - x)). \tag{18}$$

We remark that

$$\kappa \geq 0, \; \kappa'' + \lambda^2 \kappa = 0, \; \kappa(\xi_2) = 0. \tag{19}$$

For $T > 0$ and $m, R \gg 1$, we let

$$\varphi(t,x) = \vartheta_R(x) \zeta_T(t) \tag{20}$$

for every $(t,x) \in \mathcal{A}_T$. We can easily check that $\varphi \in \Lambda_T$.

Lemma 3 (see [25]). *We have*

$$I_T^{1-\beta} \zeta_T\,(s) = C \left(1 - \frac{s}{T} \right)^{1-\beta+m}.$$

Lemma 4. *We let $W^{\frac{-1}{d-1}} \in L^1_{\text{loc}}(\mathcal{A}_T)$. We have*

$$\omega_1(\varphi) \leq C R^{\frac{2d}{d-1}} \int_0^T \int_{\xi_1 + \frac{1}{2R}}^{\xi_1 + \frac{1}{R}} W^{\frac{-1}{d-1}} \, dx \, dt. \tag{21}$$

Proof. By (9), (15) and (20), we have

$$\omega_1(\varphi) \leq \int_0^T \int_{\text{supp}(\vartheta_R'' + \lambda^2 \vartheta_R)} \left| \vartheta_R'' + \lambda^2 \vartheta_R \right|^{\frac{d}{d-1}} (W \vartheta_R)^{\frac{-1}{d-1}} \, dx \, dt. \tag{22}$$

Furthermore, using (17) and (19), we obtain

$$\vartheta_R''(x) + \lambda^2\vartheta_R(x) = \kappa(x)\left[\vartheta^m(Rx_{\xi_1})\right]'' + 2\kappa'(x)\left[\vartheta^m(Rx_{\xi_1})\right]', \ x_{\xi_1} = x - \xi_1, \tag{23}$$

which implies by (16) that

$$\begin{aligned}&\int_0^T \int_{\mathrm{supp}(\vartheta_R''+\lambda^2\vartheta_R)} \left|\vartheta_R'' + \lambda^2\vartheta_R\right|^{\frac{d}{d-1}} (W\vartheta_R)^{\frac{-1}{d-1}}\,dx\,dt \\ &= \int_0^T \int_{\xi_1+\frac{1}{2R}}^{\xi_1+\frac{1}{R}} \left|\vartheta_R'' + \lambda^2\vartheta_R\right|^{\frac{d}{d-1}} (W\vartheta_R)^{\frac{-1}{d-1}}\,dx\,dt := J.\end{aligned} \tag{24}$$

Moreover, by (16) and (18), for all $\xi_1 + \frac{1}{2R} < x < \xi_1 + \frac{1}{R}$, we have

$$C_1 \le \kappa(x) \le C_2, \ |\kappa'(x)| \le C$$

and

$$\left|\left[\vartheta^m(Rx_{\xi_1})\right]'\right| \le CR\vartheta^{m-1}(Rx_{\xi_1}), \ \left|\left[\vartheta^m(Rx_{\xi_1})\right]''\right| \le CR^2\vartheta^{m-2}(Rx_{\xi_1}).$$

Hence, by (23) (and since $0 \le \vartheta \le 1$), it holds that

$$J \le CR^{\frac{2d}{d-1}} \int_0^T \int_{\xi_1+\frac{1}{2R}}^{\xi_1+\frac{1}{R}} W^{\frac{-1}{d-1}}\,dx\,dt. \tag{25}$$

From (22), (24) and (25), we obtain (21). □

Lemma 5. *We let $W^{\frac{-1}{d-1}} \in L^1_{\mathrm{loc}}(\mathcal{A}_T)$. We have*

$$\omega_2(\varphi) \le CT^{\frac{-\beta d}{d-1}} \int_0^T \int_{\xi_1+\frac{1}{2R}}^{\xi_2} W^{\frac{-1}{d-1}}\,dx\,dt. \tag{26}$$

Proof. Using (10), (16) and (20), we obtain

$$\omega_2(\varphi) = \int_0^T \int_{\xi_1+\frac{1}{2R}}^{b} \left|(I_T^{1-\beta}\zeta_T)'\right|^{\frac{d}{d-1}} \vartheta_R (W\zeta_T)^{\frac{-1}{d-1}}\,dx\,dt. \tag{27}$$

By (15) and Lemma 3, for all $0 < t < T$, we have

$$\zeta_T^{\frac{-1}{d-1}} \left|(I_T^{1-\beta}\zeta_T)'\right|^{\frac{d}{d-1}} \le CT^{\frac{-\beta d}{d-1}}. \tag{28}$$

Moreover, for all $\xi_1 + \frac{1}{2R} < x < \xi_2$, by (17) and (18) (and since $0 \le \vartheta \le 1$), we have

$$\vartheta_R(x) \le C. \tag{29}$$

Therefore, (26) follows from (27)–(29). □

3. Proofs of the Obtained Results

Proof of Theorem 1. We let $u \in \mathcal{S}$. For $m, T, R \gg 1$, we let φ be the function given by (20). Due to Lemmas 2, 4 and 5, we have $\varphi \in \overline{\Lambda}_T$ and

$$\begin{aligned}&\int_{\xi_1}^{\xi_2} u_0(x) I_T^{1-\beta}\varphi(0,x)\,dx \\ &\le C\left(\int_0^T \int_{\xi_1+\frac{1}{2R}}^{\xi_1+\frac{1}{R}} W^{\frac{-1}{d-1}}\,dx\,dt\,R^{\frac{2d}{d-1}} + \int_0^T \int_{\xi_1+\frac{1}{2R}}^{\xi_2} W^{\frac{-1}{d-1}}\,dx\,dt\,T^{\frac{-\beta d}{d-1}}\right).\end{aligned} \tag{30}$$

Using (20) and Lemma 3, we obtain

$$\begin{aligned}\int_{\xi_1}^{\xi_2} u_0(x) I_T^{1-\beta}\varphi\,(0,x)\,dx &= I_T^{1-\beta}\zeta_T(0)\int_{\xi_1}^{\xi_2} u_0(x)\vartheta_R(x)\,dx\\ &= CT^{1-\beta}\int_{\xi_1}^{\xi_2} u_0(x)\sin(|\lambda|(\xi_2-x))\vartheta^m(R(x-\xi_1))\,dx.\end{aligned} \tag{31}$$

Since $u_0 \in L^1(]\xi_1,\xi_2[)$, by (16) and the dominated convergence theorem, it holds that

$$\lim_{R\to\infty}\int_{\xi_1}^{\xi_2} u_0(x)\sin(|\lambda|(\xi_2-x))\vartheta^m(R(x-\xi_1))\,dx = \int_{\xi_1}^{\xi_2} u_0(x)\sin(|\lambda|(\xi_2-x))\,dx,$$

which implies by (4) that

$$\int_{\xi_1}^{\xi_2} u_0(x)\sin(|\lambda|(\xi_2-x))\vartheta^m(R(x-\xi_1))\,dx \geq C\int_{\xi_1}^{\xi_2} u_0(x)\sin(|\lambda|(\xi_2-x))\,dx. \tag{32}$$

Then, it follows from (30)–(32) that

$$\begin{aligned}&T^{1-\beta}\int_{\xi_1}^{\xi_2} u_0(x)\sin(|\lambda|(\xi_2-x))\,dx\\ &\leq C\left(\int_0^T\int_{\xi_1+\frac{1}{2R}}^{\xi_1+\frac{1}{R}} W^{\frac{-1}{d-1}}\,dx\,dt\,R^{\frac{2d}{d-1}} + \int_0^T\int_{\xi_1+\frac{1}{2R}}^{\xi-2} W^{\frac{-1}{d-1}}\,dx\,dt\,T^{\frac{-\beta d}{d-1}}\right),\end{aligned}$$

that is,

$$\begin{aligned}&\int_{\xi_1}^{\xi_2} u_0(x)\sin(|\lambda|(\xi_2-x))\,dx\\ &\leq C\left(R^{\frac{2d}{d-1}}T^{\beta-1}\int_0^T\int_{\xi_1+\frac{1}{2R}}^{\xi_1+\frac{1}{R}} W^{\frac{-1}{d-1}}\,dx\,dt + T^{\frac{1-d-\beta}{d-1}}\int_0^T\int_{\xi_1+\frac{1}{2R}}^{\xi_2} W^{\frac{-1}{d-1}}\,dx\,dt\right).\end{aligned} \tag{33}$$

We now take $T = R^j$, where $j > 0$. In this case, (33) reduces to

$$\int_{\xi_1}^{\xi_2} u_0(x)\sin(|\lambda|(\xi_2-x))\,dx \leq F(j,R). \tag{34}$$

In particular, for $j > 0$ satisfying (5), taking the infimum limit as $R \to \infty$ in (34), it holds that

$$\int_{\xi_1}^{\xi_2} u_0(x)\sin(|\lambda|(\xi_2-x))\,dx \leq 0.$$

Then, we obtain a contradiction with (4). □

Proof of Corollary 1. For all $j > 0$ and $R \gg 1$, we have

$$\int_0^{R^j}\int_{\xi_1+\frac{1}{2R}}^{\xi_2} W^{\frac{-1}{d-1}}\,dx\,dt \leq C\left(\ln R + R^{j\left(1-\frac{\varrho}{d-1}\right)}\right),$$

which implies that

$$F(j,R) \leq C(R^{\gamma_1}\ln R + R^{\gamma_2}\ln R + R^{\gamma_3} + R^{\gamma_4}),$$

where

$$\gamma_1 = (\beta-1)j + \frac{2d}{d-1},\ \gamma_2 = -j\left(1+\frac{\beta}{d-1}\right),\ \gamma_3 = \frac{2d - j(\varrho - \beta(d-1))}{d-1}$$

and

$$\gamma_4 = \frac{-j(\varrho+\beta)}{d-1}.$$

We observe that for all $j>0$, we have $\gamma_2<0$ and $\gamma_4<0$. Moreover, taking

$$\frac{j}{2d} > \max\left\{\frac{1}{(1-\beta)(d-1)}, \frac{1}{\varrho-\beta(d-1)}\right\},$$

we obtain $\gamma_1<0$ and $\gamma_3<0$. Hence, (5) holds, and due to Theorem 1, we obtain the desired result. □

Proof of Corollary 2. For all $j>0$ and $R \gg 1$, we have

$$\int_0^{R^j}\int_{\xi_1+\frac{1}{2R}}^{\xi_1+\frac{1}{R}} W^{\frac{-1}{d-1}}\,dx\,dt \le CR^{j+\frac{\ell}{d-1}-1}, \quad \int_0^{R^j}\int_{\xi_1+\frac{1}{2R}}^{\xi_2} W^{\frac{-1}{q-1}}\,dx\,dt \le CR^j\left(\ln R + R^{\frac{\ell}{d-1}-1}\right),$$

which implies that

$$F(j,R) \le C(R^{\mu_1}+R^{\mu_2}+R^{\mu_3}\ln R),$$

where

$$\mu_1 = j\beta + \frac{d+\ell+1}{d-1},\ \mu_2 = \frac{-\beta j+\ell-d+1}{d-1},\ \mu_3 = \frac{-\beta j}{d-1}.$$

We observe that for all $j>0$, we have $\mu_2<0$ and $\mu_3<0$. Moreover, for

$$0 < \beta(d-1)j < -(d+\ell+1),$$

we obtain $\mu_1<0$. Then, (5) holds. Therefore, Theorem 1 yields the following result. □

Proof of Corollary 3. For all $j>0$ and $R \gg 1$, it holds that

$$F(j,R) \le C\left(R^{\beta_1}\ln R + R^{\beta_2}\ln R + R^{\beta_3}\ln R + R^{\beta_4}(\ln R)^2 + R^{\beta_5} + R^{\beta_6}\right),$$

where

$$\beta_1 = j(\beta-1) + \frac{\ell+1+d}{d-1},\ \beta_2 = \frac{-j(\varrho+\beta)}{d-1},\ \beta_3 = \frac{\ell+1-d+j(1-\beta-d)}{d-1}$$

and

$$\beta_4 = \frac{j(-\beta+1-d)}{d-1},\ \beta_5 = j\left(\beta - \frac{\varrho}{d-1}\right) + \frac{\ell+1+d}{d-1},\ \beta_6 = \frac{\ell+1-d-j(\beta+\varrho)}{d-1}.$$

We remark that $\beta_4<0$ for every $j>0$. Moreover, when

$$j > \max\left\{0, \frac{d+\ell+1}{(1-\beta)(d-1)}, \frac{\ell-d+1}{d+\beta-1}, \frac{\ell-d+1}{\beta+\varrho}\right\}, \tag{35}$$

we obtain $\beta_1<0$, $\beta_3<0$ and $\beta_6<0$. We also have $\beta_2<0$ if $\varrho>-\beta$. We next consider two cases.
Case 1: If

$$\varrho > 0,\ 1 < d < 1 + \frac{\varrho}{\beta},$$

then, in addition to (35), taking

$$j > \frac{d+\ell+1}{\varrho-\beta(d-1)},$$

we obtain $\beta_5<0$.
Case 2: If

$$\varrho > 0,\ d = \frac{\varrho}{\beta}+1 < -\ell-1,$$

then
$$\beta_5 = \frac{d+\ell+1}{d-1} < 0.$$
Therefore, (5) holds in each case. Theorem 1 yields the desired result. □

4. Conclusions

A general nonexistence result is obtained for Problem (1) (see Theorem 1). Namely, we proved that, if the initial value satisfies (4) and (5) holds for some $j > 0$, then the set of weak solutions is empty. Next, we discussed some examples of the potential function W (see Corollaries 1–3). In this paper, we only considered the one-dimensional case. It will be also interesting to investigate the higher-dimensional case, namely problem

$$\begin{cases} u_{t^\beta} - \Delta u - \lambda^2 u \geq W|u|^d \text{ in } \mathbb{R}_+ \times \Omega, \\ u(0,\cdot) \geq u_0 \text{ in } \Omega, \\ u \geq 0 \text{ in } \mathbb{R}_+ \times \Sigma, \end{cases}$$

where $0 < \beta < 1$, $d > 1$, $\Omega = \{x \in \mathbb{R}^N : \xi_1 < |x| < \xi_2\}$, $0 < \xi_1 < \xi_2$ and $\Sigma = \{x \in \mathbb{R}^N : |x| = \xi_2\}$.

Author Contributions: All authors contributed equally in this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research project was funded by the Deanship of Scientific Research, Princess Nourah bint Abdulrahman University, through the Program of Research Project Funding After Publication, grant No (43- PRFA-P-68).

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare that there is no conflict of interest.

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