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Characterization of Lie-Type Higher Derivations of von Neumann Algebras with Local Actions

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Abstract: Let m and n be fixed positive integers. Suppose that $\mathcal{A}$ is a von Neumann algebra with no central summands of type I_1, and $L_m : \mathcal{A} \to \mathcal{A}$ is a Lie-type higher derivation. In continuation of the rigorous and versatile framework for investigating the structure and properties of operators on Hilbert spaces, more facts are needed to characterize Lie-type higher derivations of von Neumann algebras with local actions. In the present paper, our main aim is to characterize Lie-type higher derivations on von Neumann algebras and prove that in cases of zero products, there exists an additive higher derivation $\phi_m : \mathcal{A} \to \mathcal{A}$ and an additive higher map $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$, which annihilates every $(n-1)^{th}$ commutator $p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)$ with $\mathfrak{S}_1\mathfrak{S}_2 = 0$ such that $L_m(\mathfrak{S}) = \phi_m(\mathfrak{S}) + \zeta_m(\mathfrak{S})$ *for all* $\mathfrak{S} \in \mathcal{A}$. We also demonstrate that the result holds true for the case of the projection product. Further, we discuss some more related results.

Keywords: Lie derivation; multiplicative Lie-type derivation; multiplicative Lie-type higher derivation; von Neumann algebra

MSC: 47B47; 16W25; 46K15

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1. Introduction

One of the mathematical disciplines called von Neumann algebras, pioneered by John von Neumann, not only plays a pivotal role in advancing pure mathematics but also finds crucial applications in quantum mechanics, functional analysis and other areas of theoretical physics, underscoring its enduring relevance and impact on diverse scientific domains. Let $\mathcal{R}$ be a commutative ring with unity, $\mathcal{A}$ be an algebra over $\mathcal{R}$ and $Z(\mathcal{A})$ be the center of $\mathcal{A}$. Recall that an $\mathcal{R}$-linear map $L : \mathcal{A} \to \mathcal{A}$ is called a derivation on $\mathcal{A}$ if for all $\mathfrak{S}, \mathfrak{T} \in \mathcal{A}$, $L(\mathfrak{S}\mathfrak{T}) = L(\mathfrak{S})\mathfrak{T} + \mathfrak{S}L(\mathfrak{T})$. An $\mathcal{R}$-linear map $L : \mathcal{A} \to \mathcal{A}$ is called a Lie derivation (resp. Lie triple derivation) on $\mathcal{A}$ if for all $\mathfrak{S}, \mathfrak{T}, \mathcal{W} \in \mathcal{A}$, $L([\mathfrak{S}, \mathfrak{T}]) = [L(\mathfrak{S}), \mathfrak{T}] + [\mathfrak{S}, L(\mathfrak{T})]\big(resp. L([[\mathfrak{S}, \mathfrak{T}], \mathcal{W}]) = [[L(\mathfrak{S}), \mathfrak{T}], \mathcal{W}] + [[\mathfrak{S}, L(\mathfrak{T})], \mathcal{W}] + [[\mathfrak{S}, \mathfrak{T}], L(\mathcal{W})]\big)$, where $[\mathfrak{S}, \mathfrak{T}] = \mathfrak{S}\mathfrak{T} - \mathfrak{T}\mathfrak{S}$ is the usual Lie product. Let $\mathbb{N}$ be the set of non-negative integers and $\mathcal{D} = \{d_m\}_{m \in \mathbb{N}}$ be a family of additive mappings $d_m : \mathcal{A} \to \mathcal{A}$ such that $d_0 = id_{\mathcal{A}}$, the identity map on $\mathcal{A}$. Then, $\mathcal{D}$ is called

(i) a *higher derivation* of $\mathcal{A}$ if for every $m \in \mathbb{N}$, $d_m(\mathfrak{S}\mathfrak{T}) = \sum\limits_{r+s=m} d_r(\mathfrak{S})d_s(\mathfrak{T})$ for all $\mathfrak{S}, \mathfrak{T} \in \mathcal{A}$.

(ii) a *Lie higher derivation* of $\mathcal{A}$ if for every $m \in \mathbb{N}$, $d_m([\mathfrak{S}, \mathfrak{T}]) = \sum\limits_{r+s=m} [d_r(\mathfrak{S}), d_s(\mathfrak{T})]$ for all $\mathfrak{S}, \mathfrak{T} \in \mathcal{A}$.

(iii) a *triple-higher derivation* of $\mathcal{A}$ if for every $m \in \mathbb{N}$, $d_m([[\mathfrak{S}, \mathfrak{T}], \mathcal{W}]) = \sum_{r+s+k=m} [[d_r(\mathfrak{S}), d_s(\mathfrak{T})], d_k(\mathcal{W})]$ for all $\mathfrak{S}, \mathfrak{T}, \mathcal{W} \in \mathcal{A}$.

Abdullaev [1] initiated the study of Lie n-derivations on von Neumann algebras. Define the sequence of polynomials: $p_1(\mathfrak{S}) = \mathfrak{S}$ and $p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n) = [p_{n-1}(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_{n-1}), \mathfrak{S}_n]$ for all $n \in \mathbb{Z}$, with $n \geq 2$. Here, $p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)$ is known as the $(n-1)^{th}$-commutator. For a fixed positive integer m, an additive (linear) map $L_m : \mathcal{A} \to \mathcal{A}$ is called a Lie n-higher derivation if

$$L_m(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)) = \sum_{l_1+l_2+\cdots+l_n=m} p_n(L_{l_1}(\mathfrak{S}_1), L_{l_2}(\mathfrak{S}_2), \cdots, L_{l_n}(\mathfrak{S}_n))$$

for all $\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n \in \mathcal{A}$. In particular, by giving different values to n, we obtain Lie higher derivation, Lie triple-higher derivation and Lie n-higher derivations. These derivations collectedly are referred to as Lie-type higher derivations. Since the last few decades, examining the various properties of derivations defined through the well-known rule given by Leibniz under the influence of various algebraic structures is a vast topic of study among the algebraists. Bresar [2] characterized an additive Lie derivation as the sum of a derivation and an additive map on a prime ring $\mathcal{R}$ with $ch(\mathcal{R}) \neq 2$, where $ch(\mathcal{R})$ denotes the characteristic of $\mathcal{R}$. Johnson [3] worked on Lie derivations on $\mathbb{C}^*$-algebras and proved that every continuous linear Lie derivation from a $\mathbb{C}^*$-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{M}$ can be written as $\tau + h$ (i.e., every continuous linear Lie derivation from a $\mathbb{C}^*$-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{M}$ is standard), where $\tau : \mathcal{A} \to \mathcal{M}$ is a derivation and $h : \mathcal{A} \to Z(\mathcal{M})$ (here, $Z(\mathcal{M})$ denotes the center of $\mathcal{M}$), vanishing at each commutator. Mathieu and Villena [4] proved that on $\mathbb{C}^*$-algebra, every linear Lie derivation is standard. Qi and Hou [5] worked on nest algebras and proved that the additive Lie derivation of nest algebras on Banach spaces is standard.

Recent questions involving finding the condition under which a linear map becomes a Lie derivation or simply a derivation influenced the observations of so many researchers (see Ashraf et al. [6], Liu [7], Ashraf et al. [8], Ji et al. [9], Qi [10], Qi et al. [11] and the references therein). The purpose of the above studies in most of cases was to obtain the restrictions under which Lie derivations or derivations can be completely determined by the action on some subsets of the algebras. There are several articles on the study of local actions of the Lie derivations of operator algebras. Lu and Jing [12] proved that for a Banach space $\mathcal{X}$ of dimensions greater than two and a linear map $L : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ such that $L([\mathfrak{S}, \mathfrak{T}]) = [L(\mathfrak{S}), \mathfrak{T}] + [\mathfrak{S}, L(\mathfrak{T})]$ for all $\mathfrak{S}, \mathfrak{T} \in \mathcal{B}(\mathcal{X})$ with $\mathfrak{S}\mathfrak{T} = 0$ (resp. $\mathfrak{S}\mathfrak{T} = P$, where $\mathfrak{Q}$ is a fixed nontrivial idempotent), then there exists an operator $\mathcal{T} \in \mathcal{B}(\mathcal{X})$ and a linear map $\phi : \mathcal{B}(\mathcal{X}) \to \mathbb{C}I$ vanishing at all the commutators $[\mathfrak{S}, \mathfrak{T}]$ with $\mathfrak{S}\mathfrak{T} = 0($ resp. $\mathfrak{S}\mathfrak{T} = P)$ such that $L(\mathfrak{S}) = \mathcal{T}\mathfrak{S} + \phi(\mathfrak{S})$ for all $\mathfrak{S} \in \mathcal{B}(\mathcal{X})$. Ji and Qi [13] proved that if $\mathcal{T}$ is a triangular algebra over a commutative ring $\mathcal{R}$, then under certain restrictions on $\mathcal{T}$, if $L : \mathcal{T} \to \mathcal{T}$ is an $\mathcal{R}$-linear map satisfying $L([\mathfrak{S}, \mathfrak{T}]) = [L(\mathfrak{S}), V] + [\mathfrak{S}, L(V)]$ for all $\mathfrak{S}, \mathfrak{T} \in \mathcal{T}$ with $\mathfrak{S}\mathfrak{T} = 0($ resp. $\mathfrak{S}\mathfrak{T} = \mathfrak{Q}$, where $\mathfrak{Q}$ is the standard idempotent of $\mathcal{T}$), then $L = d + \phi$, where $d : \mathcal{T} \to \mathcal{T}$ is a derivation and $\phi : \mathcal{T} \to Z(\mathcal{T})$ is an $\mathcal{R}$-linear map vanishing at all the commutators $[\mathfrak{S}, \mathfrak{T}]$ with $\mathfrak{S}\mathfrak{T} = 0($ resp. $\mathfrak{S}\mathfrak{T} = \mathfrak{Q})$. Qi and Hou [11] characterized Lie derivations on von Neumann algebras $\mathcal{A}$ without central summands of type I_1. Qi and Ji [14] proved the same result for $\mathfrak{S}\mathfrak{T} = \mathfrak{Q}$, where $\mathfrak{Q}$ is a core-free projection. Qi [10] characterized Lie derivations on $\mathcal{J}$-subspace lattice algebras and proved the same result due to Lu and Jing [12] on $\mathcal{J}$-subspace lattice algebra Alg$\mathcal{L}$, where $\mathcal{L}$ is a $\mathcal{J}$-subspace lattice on a Banach space $\mathcal{X}$ over the real or complex field with a dimension greater than two. Liu [15] studied the characterization of Lie triple derivations on von Neumann algebras with no central abelian projections. For further references see Bruno et al. [16], Wang [17], Wang et al. [18] and references therein. Recently, Ashraf and Jabeen [19] characterized the Lie-type derivations on von Neumann algebras with no central summands of type I_1, where they showed that every Lie-type derivation on von Neumann algebras has a standard form at zero products as well as at projection products.

The objective of this paper is to investigate Lie-type higher derivations on von Neumann algebras with no central summands of type I_1 and to prove that on a von Neumann algebra, every Lie-type higher derivation has standard form at zero products as well as at projection products. Precisely, we prove that every additive map $L_m : \mathcal{A} \to \mathcal{A}$ satisfying $L_m(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)) = \sum_{l_1+l_2+\cdots+l_n=m} p_n\Big(L_{l_1}(\mathfrak{S}_1), L_{l_2}(\mathfrak{S}_2), L_{l_3}(\mathfrak{S}_3), \cdots, L_{l_n}(\mathfrak{S}_n)\Big)$ for all $\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n \in \mathcal{A}$ with $\mathfrak{S}_1\mathfrak{S}_2 = 0$ is of the form $L_m(\mathfrak{S}) = \phi_m(\mathfrak{S}) + \zeta_m(\mathfrak{S})$ for all $\mathfrak{S} \in \mathcal{A}$, where $\phi_m : \mathcal{A} \to \mathcal{A}$ is an additive higher derivation and $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$ is an additive higher map whose range is in $Z(\mathcal{A})$. Further, we discuss some more related results.

2. Main Results

In this section, we discuss the characterization of Lie-type higher derivation on von Neumann algebras having no central summands of type I_1 at zero products.

Remark 1. *Let $\mathcal{A}$ be a von Neumann algebra with center $Z(\mathcal{A})$. For each self-adjoint operator $T \in \mathcal{A}$, we define the core of T, denoted by $\underline{T}$, to be $LUB[S \in Z(\mathcal{A})|\ S \text{ is } self-adjoint,\ S \leqq A]$. One has $T - \underline{T} \geqq 0$. Further, if $S \in Z(\mathcal{A})$ and $T - \underline{T} \geqq S \geqq 0$, then $S = 0$. If P is a projection, then $\underline{P}$ is the largest central projection $\leqq P$. We call a such projection core-free if $\underline{P} = 0$ and $\overline{P}$ is the central carrier of P.*

In proving our main results, we use the following known lemmas. Lemma 1 gives a sufficient condition for a fixed projection T of von Neumann algebra $\mathcal{A}$ to be a central element of A if it commutes with PXQ and QXP for all $X \in \mathcal{A}$.

Lemma 1 (Miers [20], Lemma 5)**.** *For projections $P, Q \in \mathcal{A}$ with $\overline{P} = \overline{Q} \neq 0$, if $T \in \mathcal{A}$ commutes with PXQ and QXP for all $X \in \mathcal{A}$, then T commutes with PXP and QXQ for all $X \in \mathcal{A}$.*

Lemma 2 (Bresar and Miers [21], Lemma 5)**.** *Let $\mathcal{A}$ be a von Neumann algebra with no central summands of type I_1. If $t \in Z(\mathcal{A})$ such that $t\mathcal{A} \subseteq Z(\mathcal{A})$, then $t = 0$.*

Lemma 3 (Miers [20], Lemma 14)**.** *Let $\mathcal{A}$ be a von Neumann algebra such that $P \in \mathcal{A}$ is a core-free projection in $\mathcal{A}$. Then, $P\mathcal{A}P \cap Z(\mathcal{A}) = 0$.*

Lemma 4 (Ashraf and Jabeen [19], Lemma 2.5)**.** *Let $\mathfrak{S}_{ii} \in \mathcal{A}_{ii}, i = 1, 2$. If $\mathfrak{S}_{11}\mathfrak{T}_{12} = \mathfrak{T}_{12}\mathfrak{S}_{11}$ for all $\mathfrak{T}_{12} \in \mathcal{A}_{12}$, then $\mathfrak{S}_{11} + \mathfrak{S}_{22} \in Z(\mathcal{A})$.*

Lemma 5 (Miers [20], Lemma 4)**.** *If $\mathcal{A}$ is a von Neumann algebra with no central summands of type I_1, then each nonzero central projection of $\mathcal{A}$ is the central carrier of a core-free projection of $\mathcal{A}$.*

The first main result of this paper is the following theorem:

Theorem 1. *Let $\mathcal{A}$ be a von Neumann algebra with no central summands of type I_1 and an additive map $L_m : \mathcal{A} \to \mathcal{A}$ satisfying*

$$L_m(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)) = \sum_{l_1+l_2+\cdots+l_n=m} p_n\Big(L_{l_1}(\mathfrak{S}_1), L_{l_2}(\mathfrak{S}_2), L_{l_3}(\mathfrak{S}_3), \cdots, L_{l_n}(\mathfrak{S}_n)\Big)$$

for all $\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n \in \mathcal{A}$ with $\mathfrak{S}_1\mathfrak{S}_2 = 0$. Then, there exists an additive higher derivation $\phi_m : \mathcal{A} \to \mathcal{A}$ and an additive higher map $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$, which annihilates every $(n-1)^{th}$-commutator $p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)$ with $\mathfrak{S}_1\mathfrak{S}_2 = 0$ such that $L_m(\mathfrak{S}) = \phi_m(\mathfrak{S}) + \zeta_m(\mathfrak{S})$ for all $\mathfrak{S} \in \mathcal{A}$.

Henceforward, let $\mathcal{A}$ be a von Neumann algebra with no central summands of type I_1 and an additive map $L_m : \mathcal{A} \to \mathcal{A}$ satisfying the hypotheses of Theorem 1. For projections $\mathfrak{Q}_p, \mathfrak{Q}_q \in \mathcal{A}$, let $\mathfrak{Q}_0 = \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_q - \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_p = 0$ and let us define a map $\pi_m : \mathcal{A} \to$

$\mathcal{A}$ as an inner higher derivation $\pi_m(\mathfrak{S}) = [\mathfrak{S}, \mathfrak{Q}_0]$ for all $\mathfrak{S} \in \mathcal{A}$. Clearly, $L_m = L_m^{'} - \pi_m$ is a Lie n-higher derivation. Since

$$\begin{aligned} L_m^{'}(\mathfrak{Q}_p) &= L_m(\mathfrak{Q}_p) - [\mathfrak{Q}_p, \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_q - \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_p] \\ &= L_m(\mathfrak{Q}_p) - \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_q - \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_p \\ &= \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_p + \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_q. \end{aligned}$$

One easily obtains $\mathfrak{Q}_p L_m^{'}(\mathfrak{Q}_p)\mathfrak{Q}_q = \mathfrak{Q}_q L_m^{'}(\mathfrak{Q}_p)\mathfrak{Q}_p = 0$. Accordingly, it suffices to consider only those Lie n-higher derivations, which satisfy $\mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_q = \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_p = 0$.

We give proof of Theorem 1 in a series of lemmas. We begin with the following lemmas:

Lemma 6. *For projections $\mathfrak{Q}_p,\ \mathfrak{Q}_q \in \mathcal{A},\ L_m(\mathfrak{Q}_p)$ and $L_m(\mathfrak{Q}_q) \in Z(\mathcal{A})$.*

Proof. To prove this lemma, we use the principle of mathematical induction on m, for $m = 1$, and the result was shown to be true by Ashraf and Jabeen [19]. Assume that the result holds for all $k \leq m-1$. Then, we want to prove that it also holds for $k = m$. Since $\mathfrak{S}_{12}\mathfrak{Q}_p = \mathfrak{Q}_p \mathfrak{S} \mathfrak{Q}_q \mathfrak{Q}_p = 0$ for all $\mathfrak{S}_{12} \in \mathcal{A}_{12}$, we have

$$\begin{aligned} & L_m\Big(p_n(\mathfrak{S}_{12}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p)\Big) \\ &\quad = p_n\Big(L_m(\mathfrak{S}_{12}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) + p_n(\mathfrak{S}_{12}, L_m(\mathfrak{Q}_p), \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p) \\ &\quad + p_n\Big(\mathfrak{S}_{12}, \mathfrak{Q}_p, L_m(\mathfrak{Q}_p), \cdots, \mathfrak{Q}_p\Big) + \cdots + p_n(\mathfrak{S}_{12}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, L_m(\mathfrak{Q}_p)) \\ &\quad + \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\leq l_1, l_2, \cdots, l_n \leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{12}), L_{l_2}(\mathfrak{Q}_p), L_{l_3}(\mathfrak{Q}_p), \cdots, L_{l_n}(\mathfrak{Q}_p)\Big). \end{aligned}$$

This implies

$$\begin{aligned} L_m((-1)^{n-1}\mathfrak{S}_{12}) &= (-1)^{n-1}\mathfrak{Q}_p L_m(\mathfrak{S}_{12})\mathfrak{Q}_q + \mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p \\ &\quad + (-1)^{n-2}(n-1)[\mathfrak{S}_{12}, L_m(\mathfrak{Q}_p)]. \end{aligned} \tag{1}$$

Premultiplying by $\mathfrak{Q}_p$ to the above equation, we obtain

$$(-1)^{n-1}\mathfrak{Q}_p L_m(\mathfrak{S}_{12}) = (-1)^{n-1}\mathfrak{Q}_p L_m(\mathfrak{S}_{12})\mathfrak{Q}_q + (-1)^{n-2}(n-1)\big(\mathfrak{S}_{12}L_m(\mathfrak{Q}_p) - \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{S}_{12}\big)$$

and by postmultiplying $\mathfrak{Q}_q$ to the same equation, we obtain $\mathfrak{S}_{12}L_m(\mathfrak{Q}_p)\mathfrak{Q}_q = \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{S}_{12}$. Then, by using Lemma 4, we have $L_m(\mathfrak{Q}_p) \in Z(\mathcal{A})$. Knowing the fact that $\mathfrak{Q}_q\mathfrak{Q}_p = 0$, one can write

$$\begin{aligned} 0 &= L_m\Big(p_n(\mathfrak{Q}_q, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p)\Big) \\ &= p_n\Big(L_m(\mathfrak{Q}_q), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) + p_n\Big(\mathfrak{Q}_q, L_m(\mathfrak{Q}_p), \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\ &+ p_n\Big(\mathfrak{Q}_q, \mathfrak{Q}_p, L_m(\mathfrak{Q}_p), \cdots, \mathfrak{Q}_p\Big) + \cdots + p_n\Big(\mathfrak{Q}_q, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, L_m(\mathfrak{Q}_p)\Big) \\ &+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\leq l_1, l_2, \cdots, l_n \leq m-1}} p_n\Big(L_{l_1}(\mathfrak{Q}_q), L_{l_2}(\mathfrak{Q}_p), L_{l_3}(\mathfrak{Q}_p), \cdots, L_{l_n}(\mathfrak{Q}_p)\Big) \\ &= p_n\Big(L_m(\mathfrak{Q}_q), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\ &= (-1)^{n-1}\mathfrak{Q}_p L_m(\mathfrak{Q}_q)\mathfrak{Q}_q + \mathfrak{Q}_q L_m(\mathfrak{Q}_q)\mathfrak{Q}_p, \end{aligned}$$

which implies $\mathfrak{Q}_p L_m(\mathfrak{Q}_q)\mathfrak{Q}_q = \mathfrak{Q}_q L_m(\mathfrak{Q}_q)\mathfrak{Q}_p = 0$. Now, using $p_n(\mathfrak{Q}_q, \mathfrak{S}_{12}, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p) = 0$ and applying similar calculations as above, we obtain that $L_m(\mathfrak{Q}_q) \in Z(\mathcal{A})$. Hence, $L_m(\mathfrak{Q}_p)$ and $L_m(\mathfrak{Q}_q) \in Z(\mathcal{A})$ holds for all $m \in \mathbb{N}$. $\square$

Lemma 7. $L_m(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}$, $(1 \leq i \neq j \leq 2)$.

Proof. We show that $L_m(\mathcal{A}_{12}) \subseteq \mathcal{A}_{12}$. The other case, i.e., $L_m(\mathcal{A}_{21}) \subseteq \mathcal{A}_{21}$ can be shown similarly. For $m = 1$, it was shown to be true by Ashraf and Jabeen [19]. Now suppose that it holds for all $k \leq m-1$. We want to show that it also holds for $k = m$. Using Lemma 6 and Equation (1), we have $L_m(\mathfrak{S}_{12}) = \mathfrak{Q}_p L_m(\mathfrak{S}_{12})\mathfrak{Q}_q + (-1)^{n-1}\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p$. From this equation, one can easily obtain $\mathfrak{Q}_p L_m(\mathfrak{S}_{12})\mathfrak{Q}_p = \mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_q = 0$, and if n is even, then $2\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p = 0$. But when n is odd, then for all $\mathfrak{S}_{12}, \mathfrak{T}_{12} \in \mathcal{A}_{12}$, as $\mathfrak{S}_{12}\mathfrak{T}_{12} = 0$, one can easily see that

$$\begin{aligned}
0 &= L_m\Big(p_n(\mathfrak{S}_{12}, \mathfrak{T}_{12}, \mathcal{W}_{12}, -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{12}), \mathfrak{T}_{12}, \mathcal{W}_{12}, -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p\Big) + p_n\Big(\mathfrak{S}_{12}, L_m(\mathfrak{T}_{12}), \mathcal{W}_{12}, -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p\Big) \\
&+ p_n\Big(\mathfrak{S}_{12}, \mathfrak{T}_{12}, L_m(\mathcal{W}_{12}), -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p\Big) + \cdots + p_n\Big(\mathfrak{S}_{12}, \mathfrak{T}_{12}, \mathcal{W}_{12}, -\mathfrak{Q}_p, \cdots, L_m(-\mathfrak{Q}_p)\Big) \\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\leq l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{12}), L_{l_2}(\mathfrak{T}_{12}), L_{l_3}(\mathcal{W}_{12}), L_{l_4}(-\mathfrak{Q}_p), \cdots, L_{l_n}(-\mathfrak{Q}_p)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{12}), \mathfrak{T}_{12}, \mathcal{W}_{12}, -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p\Big) + p_n\Big(\mathfrak{S}_{12}, L_m(\mathfrak{T}_{12}), \mathcal{W}_{12}, -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p\Big).
\end{aligned}$$

which can be written as $0 = \big[[L_m(\mathfrak{S}_{12}), \mathfrak{T}_{12}], \mathcal{W}_{12}\big] + \big[[\mathfrak{S}_{12}, L_m(\mathfrak{T}_{12})], \mathcal{W}_{12}\big]$. From this equation, we obtain $[L_m(\mathfrak{S}_{12}), \mathfrak{T}_{12}] + [\mathfrak{S}_{12}, L_m(\mathfrak{T}_{12})] \in Z(\mathcal{A})$. Now put $z = [L_m(\mathfrak{S}_{12}), \mathfrak{T}_{12}] + [\mathfrak{S}_{12}, L_m(\mathfrak{T}_{12})]$. Then,

$$\begin{aligned}
[L_m(\mathfrak{S}_{12}), \mathfrak{T}_{12}] &= z - [\mathfrak{S}_{12}, L_m(\mathfrak{T}_{12})] \\
&= z - p_n\Big(\mathfrak{S}_{12}, -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p, L_m(\mathfrak{T}_{12})\Big) \\
&= z + L_m\Big(p_n(\mathfrak{S}_{12}, -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p, \mathfrak{T}_{12})\Big) - p_n\Big(L_m(\mathfrak{S}_{12}), -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p, \mathfrak{T}_{12}\Big) \\
&= z - p_n\Big(L_m(\mathfrak{S}_{12}), -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p, \mathfrak{T}_{12}\Big) \\
&= z - [\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p, \mathfrak{T}_{12}].
\end{aligned}$$

This implies that $[\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p, \mathfrak{T}_{12}] \in Z(\mathcal{A})$ and therefore $\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p\mathfrak{T}_{12} = 0$. Since $\overline{\mathfrak{Q}_q} = I$, we have $\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p = 0$. Therefore, $L_m(\mathcal{A}_{12}) \subseteq \mathcal{A}_{12}$. Hence, for all $m \in \mathbb{N}$, $L_m(\mathcal{A}_{12}) \subseteq \mathcal{A}_{12}$. $\square$

Lemma 8. *There exists maps ζ_{m_i} on $\mathcal{A}_{ii}$ such that $L_m(\mathfrak{S}_{ii}) - \zeta_{m_i}(\mathfrak{S}_{ii})I \in \mathcal{A}_{ii}$ for any $\mathfrak{S}_{ii} \in \mathcal{A}, i = 1, 2$.*

Proof. Using Lemma 6 and knowing the fact that $\mathfrak{S}_{11}\mathfrak{Q}_q = 0$, we have

$$\begin{aligned}
0 &= L_m\Big(p_n(\mathfrak{S}_{11}, \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{11}), \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}, L_m(\mathfrak{Q}_q), \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) \\
&+ p_n\Big(\mathfrak{S}_{11}, \mathfrak{Q}_q, L_m(\mathfrak{Q}_q), \cdots, \mathfrak{Q}_q\Big) + \cdots + p_n\Big(\mathfrak{S}_{11}, \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, L_m(\mathfrak{Q}_q)\Big) \\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\leq l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}), L_{l_2}(\mathfrak{Q}_q), L_{l_3}(\mathfrak{Q}_q), \cdots, L_{l_n}(\mathfrak{Q}_q)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{11}), \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) \\
&= L_m(\mathfrak{S}_{11})\mathfrak{Q}_q + (-1)^{n-1}\mathfrak{Q}_q L_m(\mathfrak{S}_{11}) \\
&= \mathfrak{Q}_p L_m(\mathfrak{S}_{11})\mathfrak{Q}_q + (-1)^{n-1}\mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_p.
\end{aligned}$$

From which we obtain $\mathfrak{Q}_p L_m(\mathfrak{S}_{11})\mathfrak{Q}_q = \mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_p = 0$. To complete the proof of the lemma, we need to show that $\mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q = 0$. For this, take any $\mathfrak{S}_{22} \in \mathcal{A}_{22}$ and $\mathfrak{T}_{12} \in \mathcal{A}_{12}$, and we have

$$\begin{aligned}
0 &= L_m\Big(p_n(\mathfrak{S}_{11}, \mathfrak{S}_{22}, \mathcal{W}_{12}, \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{11}), \mathfrak{S}_{22}, \mathcal{W}_{12}, \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}, L_m(\mathfrak{S}_{22}), \mathcal{W}_{12}, \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) \\
&+ p_n\Big(\mathfrak{S}_{11}, \mathfrak{S}_{22}, L_m(\mathcal{W}_{12}), \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + \cdots + p_n\Big(\mathfrak{S}_{11}, \mathfrak{S}_{22}, \mathcal{W}_{12}, \mathfrak{Q}_q, \cdots, L_m(\mathfrak{Q}_q)\Big) \\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\le l_1,l_2,\cdots,l_n\le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}), L_{l_2}(\mathfrak{S}_{22}), L_{l_3}(\mathfrak{T}_{12}), L_{l_4}(\mathfrak{Q}_q), L_{l_5}(\mathfrak{Q}_q), \cdots, L_{l_n}(\mathfrak{Q}_q)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{11}), \mathfrak{S}_{22}, \mathcal{W}_{12}, \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}, L_m(\mathfrak{S}_{22}), \mathcal{W}_{12}, \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) \\
&= \Big[\big[L_m(\mathfrak{S}_{11}), \mathfrak{S}_{22}\big], \mathfrak{T}_{12}\Big] + \Big[\big[\mathfrak{S}_{11}, L_m(\mathfrak{S}_{22})\big], \mathfrak{T}_{12}\Big].
\end{aligned}$$

This implies that $\big[L_m(\mathfrak{S}_{11}), \mathfrak{S}_{22}\big] + \big[\mathfrak{S}_{11}, L_m(\mathfrak{S}_{22})\big] \in Z(\mathcal{A})$. By pre- and postmultiplying $\mathfrak{Q}_q$, we obtain $\big[\mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q, \mathfrak{S}_{22}\big] \in Z(\mathcal{A})\mathfrak{Q}_q$. This implies $\big[\mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q, \mathfrak{S}_{22}\big] = 0$, which means there exists some $z \in Z(\mathcal{A})$ such that $\mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q = z\mathfrak{Q}_q$ and therefore

$$\begin{aligned}
L_m(\mathfrak{S}_{11}) &= \mathfrak{Q}_p L_m(\mathfrak{S}_{11})\mathfrak{Q}_p + \mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q && (2)\\
&= \mathfrak{Q}_p L_m(\mathfrak{S}_{11})\mathfrak{Q}_p - z\mathfrak{Q}_p + z. && (3)
\end{aligned}$$

Since $z \in Z(\mathcal{A})$, we have $\mathfrak{Q}_q z \mathfrak{Q}_p = \mathfrak{Q}_p z \mathfrak{Q}_q = 0$. From the above equations, we have $z - z' = (\mathfrak{Q}_p z \mathfrak{Q}_p + \mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q) - (\mathfrak{Q}_p z' \mathfrak{Q}_p + \mathfrak{Q}_q L_m(\mathfrak{S}'_{11})\mathfrak{Q}_q)$. Then, by Lemma 3, $\mathfrak{Q}_p \mathcal{A} \mathfrak{Q}_p \cap Z(\mathcal{A}) = \{0\}$ and thus $z = z'$. One can also define a map ζ_{m_1} on $\mathcal{A}_{11}$ by $\zeta_{m_1}(\mathfrak{S}_{11}) = z \in Z(\mathcal{A})$. Then, by comparing it with Equation (3), we obtain $L_m(\mathfrak{S}_{11}) - \zeta_{m_1}(\mathfrak{S}_{11}) = \mathfrak{Q}_p L_m(\mathfrak{S}_{11})\mathfrak{Q}_p - \mathfrak{Q}_p z \mathfrak{Q}_p \in \mathcal{A}_{11}$ for all $\mathfrak{S}_{11} \in \mathcal{A}_{11}$. With the similar steps, there exists a map ζ_{m_2} on $\mathcal{A}_{22}$ such that $\zeta_{m_2}(\mathfrak{S}_{22}) = z \in Z(\mathcal{A})$ and $L_m(\mathfrak{S}_{22}) - \zeta_{m_2}(\mathfrak{S}_{22}) \in \mathcal{A}_{22}$ for all $\mathfrak{S}_{22} \in \mathcal{A}_{22}$. □

Now define two maps $\phi_m : \mathcal{A} \to \mathcal{A}$ by $\phi_m(\mathfrak{S}) = L_m(\mathfrak{S}) - \zeta_m(\mathfrak{S})$ and $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$ by $\zeta_m(\mathfrak{S}) = \zeta_{m_1}(\mathfrak{Q}_p \mathfrak{S} \mathfrak{Q}_p) + \zeta_{m_2}(\mathfrak{Q}_q \mathfrak{S} \mathfrak{Q}_q)$ for all $\mathfrak{S} \in \mathcal{A}$. Then, one can easily observe that $\phi_m(\mathfrak{Q}_i) = 0,\ \phi_m(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij};\ i,j = 1,2$ *and* $\phi_m(\mathfrak{S}_{ij}) = L_m(\mathfrak{S}_{ij})$; *for all* $\mathfrak{S}_{ij} \in \mathcal{A}_{ij};\ i,j = 1,2\ (i \neq j)$.

Lemma 9. *Let $\phi_m : \mathcal{A} \to \mathcal{A}$ be a map such that $\phi_m(\mathfrak{S}) = L_m(\mathfrak{S}) - \zeta_m(\mathfrak{S})$, where $L_m : \mathcal{A} \to \mathcal{A}$ is a Lie-type higher derivation and $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$. Then, ϕ_m is an additive map.*

Proof. As $\phi_m = L_m - \zeta_m$ *and* $\zeta_m = \zeta_{m_1} + \zeta_{m_2}$, we need to show that ζ_{m_1} *and* ζ_{m_2} are additive. For this, take any $\mathfrak{S}_{11}, \mathfrak{T}_{11} \in \mathcal{A}_{11}$, and we have

$$\zeta_{m_1}(\mathfrak{S}_{11}) = \mathfrak{Q}_p \zeta_{m_1}(\mathfrak{S}_{11})\mathfrak{Q}_p + \mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q,$$

$$\zeta_{m_1}(\mathfrak{T}_{11}) = \mathfrak{Q}_p \zeta_{m_1}(\mathfrak{T}_{11})\mathfrak{Q}_p + \mathfrak{Q}_q L_m(\mathfrak{T}_{11})\mathfrak{Q}_q$$

and

$$\zeta_{m_1}(\mathfrak{S}_{11} + \mathfrak{T}_{11}) = \mathfrak{Q}_p \zeta_{m_1}(\mathfrak{S}_{11} + \mathfrak{T}_{11})\mathfrak{Q}_p + \mathfrak{Q}_q L_m(\mathfrak{S}_{11} + \mathfrak{T}_{11})\mathfrak{Q}_q.$$

By combining all the above three expressions, one can easily find that $\zeta_{m_1}(\mathfrak{S}_{11}) + \zeta_{m_1}(\mathfrak{T}_{11}) - \zeta_{m_1}(\mathfrak{S}_{11} + \mathfrak{T}_{11}) = \mathfrak{Q}_p \zeta_{m_1}(\mathfrak{S}_{11})\mathfrak{Q}_p + \mathfrak{Q}_p \zeta_{m_1}(\mathfrak{T}_{11})\mathfrak{Q}_p - \mathfrak{Q}_p \zeta_{m_1}(\mathfrak{S}_{11} + \mathfrak{T}_{11})\mathfrak{Q}_p$. Since $\mathfrak{Q}_p \zeta_{m_1}(\mathfrak{S}_{11})\mathfrak{Q}_p + \mathfrak{Q}_p \zeta_{m_1}(\mathfrak{T}_{11})\mathfrak{Q}_p - \mathfrak{Q}_p \zeta_{m_1}(\mathfrak{S}_{11} + \mathfrak{T}_{11})\mathfrak{Q}_p \in Z(\mathcal{A})$ and $\mathfrak{Q}_p \zeta_{m_1}(\mathfrak{S}_{11})\mathfrak{Q}_p + \mathfrak{Q}_p \zeta_{m_1}(\mathfrak{T}_{11})\mathfrak{Q}_p - \mathfrak{Q}_p \zeta_{m_1}(\mathfrak{S}_{11} + \mathfrak{T}_{11})\mathfrak{Q}_p \in \mathfrak{Q}_p \mathcal{A} \mathfrak{Q}_p$, we know that $Z(\mathcal{A}) \cap \mathfrak{Q}_p \mathcal{A} \mathfrak{Q}_p = \{0\}$ by Lemma 3. We have $\mathfrak{Q}_p \zeta_{m_1}(\mathfrak{S}_{11})\mathfrak{Q}_p + \mathfrak{Q}_p \zeta_{m_1}(\mathfrak{T}_{11})\mathfrak{Q}_p - \mathfrak{Q}_p \zeta_{m_1}(\mathfrak{S}_{11} + \mathfrak{T}_{11})\mathfrak{Q}_p = 0$. Hence, $\zeta_{m_1}(\mathfrak{S}_{11} + \mathfrak{T}_{11}) = \zeta_{m_1}(\mathfrak{S}_{11}) + \zeta_{m_1}(\mathfrak{T}_{11})$. This implies that ζ_{m_1} is additive. Similarly, we can show that ζ_{m_2} is additive. Therefore, ϕ_m is an additive map. □

Our next lemma is also important to complete the proof of Theorem 1.

Lemma 10. *For any $\mathfrak{S}_{ii} \in \mathcal{A}_{ii}$, $\mathfrak{T}_{ij} \in \mathcal{A}_{ij}$, $\mathfrak{S}_{ij} \in \mathcal{A}_{ij}$ and $\mathfrak{T}_{jj} \in \mathcal{A}_{jj}$; $i, j = 1, 2, (i \neq j)$ and $\phi_m : \mathcal{A} \to \mathcal{A}$ be a map. Then,*

(*a*) $\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ij}) = \sum_{s+t=m} \phi_s(\mathfrak{S}_{ii})\phi_t(\mathfrak{T}_{ij})$,

(*b*) $\phi_m(\mathfrak{S}_{ij}\mathfrak{T}_{jj}) = \sum_{s+t=m} \phi_s(\mathfrak{S}_{ij})\phi_t(\mathfrak{T}_{jj})$.

Proof. Here, we give proof of part (a), and the second part can be proved similarly. Lets us prove the lemma with the help of mathematical induction on m. For $m = 1$, it was shown to be true by Ashraf and Jabeen [19]. Suppose it is true for all $k \leq m - 1$. Now, we show that it also holds for $k = m$. Note that $L_m : \mathcal{A} \to \mathcal{A}$ is a Lie-type higher derivation. Since for $i \neq j, \mathfrak{T}_{ij}\mathfrak{S}_{ii} = 0$, we have

$$\begin{aligned}
\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ij}) &= L_m(\mathfrak{S}_{ii}\mathfrak{T}_{ij}) \\
&= L_m\Big(p_n(\mathfrak{S}_{ii}, \mathfrak{T}_{ij}, P_j, P_j, \cdots, P_j)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{ii}), \mathfrak{T}_{ij}, P_j, P_j, \cdots, P_j\Big) + p_n\Big(\mathfrak{S}_{ii}, L_m(\mathfrak{T}_{ij}), P_j, P_j, \cdots, P_j\Big) \\
&+ p_n\Big(\mathfrak{S}_{ii}, \mathfrak{T}_{ij}, L_m(P_j), P_j, \cdots, P_j\Big) + \cdots + p_n\Big(\mathfrak{S}_{ii}, \mathfrak{T}_{ij}, P_j, \cdots, L_m(P_j)\Big) \\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\leq l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{ii}), L_{l_2}(\mathfrak{T}_{ij}), L_{l_3}(P_j), \cdots, L_{l_n}(P_j)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{ii}), \mathfrak{T}_{ij}, P_j, P_j, \cdots, P_j\Big) + p_n\Big(\mathfrak{S}_{ii}, L_m(\mathfrak{T}_{ij}), P_j, P_j, \cdots, P_j\Big) \\
&+ \sum_{\substack{s+t=m \\ 0<s,t\leq m-1}} L_s(\mathfrak{S}_{ii})L_t(\mathfrak{T}_{ij}) \\
&= \phi_m(\mathfrak{S}_{ii})\mathfrak{T}_{ij} + \mathfrak{S}_{ii}\phi_m(\mathfrak{T}_{ij}) + \sum_{\substack{s+t=m \\ 0<s,t\leq m-1}} \phi_s(\mathfrak{S}_{ii})\phi_t(\mathfrak{T}_{ij}) \\
&= \sum_{s+t=m} \phi_s(\mathfrak{S}_{ii})\phi_t(\mathfrak{T}_{ij}).
\end{aligned}$$

Similarly, one can prove $\phi_m(\mathfrak{S}_{ij}\mathfrak{T}_{jj}) = \sum_{s+t=m} \phi_s(\mathfrak{S}_{ij})\phi_t(\mathfrak{T}_{jj})$. □

Lemma 11. *For any $\mathfrak{S}_{ii}$, $\mathfrak{T}_{ii} \in \mathcal{A}_{ii}$. We have $\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii}) = \sum_{s+t=m} \phi_s(\mathfrak{S}_{ii})\phi_t(\mathfrak{T}_{ii})$; $i = 1, 2$.*

Proof. For any $\mathfrak{S}_{ii}, \mathfrak{T}_{ii} \in \mathcal{A}_{ii}$ *and* $\mathfrak{T}_{ij} \in \mathcal{A}_{ij}$, and using Lemma 10, we have

$$\begin{aligned}
&\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii}\mathfrak{T}_{ij}) \\
&= \mathfrak{S}_{ii}\mathfrak{T}_{ii}\phi_m(\mathfrak{T}_{ij}) + \phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii})\mathfrak{T}_{ij} + \sum_{\substack{r+s=m \\ 0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii}\mathfrak{T}_{ii})\phi_s(\mathfrak{T}_{ij}) \\
&= \mathfrak{S}_{ii}\mathfrak{T}_{ii}\phi_m(\mathfrak{T}_{ij}) + \phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii})\mathfrak{T}_{ij} + \sum_{\substack{r+s=m \\ 0<r,s\leq m-1}} \Bigg(\sum_{\substack{p+q=r \\ 0\leq p,q\leq r}} \phi_p(\mathfrak{S}_{ii})\phi_q(\mathfrak{T}_{ii})\Bigg)\phi_s(\mathfrak{T}_{ij}) \\
&= \mathfrak{S}_{ii}\mathfrak{T}_{ii}\phi_m(\mathfrak{T}_{ij}) + \phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii})\mathfrak{T}_{ij} + \sum_{\substack{r+s=m \\ 0<r,s\leq m-1}} \Bigg(\mathfrak{S}_{ii}\phi_r(\mathfrak{T}_{ii}) + \phi_r(\mathfrak{S}_{ii})\mathfrak{T}_{ii} \\
&+ \sum_{\substack{p+q=r \\ 0<p,q\leq r-1}} \phi_p(\mathfrak{S}_{ii})\phi_q(\mathfrak{T}_{ii})\Bigg)\phi_s(\mathfrak{T}_{ij}) \\
&= \mathfrak{S}_{ii}\mathfrak{T}_{ii}\phi_m(\mathfrak{T}_{ij}) + \phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii})\mathfrak{T}_{ij} + \sum_{\substack{r+s=m \\ 0<r,s\leq m-1}} \mathfrak{S}_{ii}\phi_r(\mathfrak{T}_{ii})\phi_s(\mathfrak{T}_{ij}) \\
&+ \sum_{\substack{r+s=m \\ 0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\phi_s(\mathfrak{T}_{ij}) + \sum_{\substack{r+s=m \\ p+q=s \\ 0<r,s\leq m-1 \\ 0<p,q\leq r-1}} \phi_p(\mathfrak{S}_{ii})\phi_q(\mathfrak{T}_{ii})\phi_r(\mathfrak{T}_{ij}).
\end{aligned} \tag{4}$$

On the other hand, we have

$$
\begin{aligned}
\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii}\mathfrak{T}_{ij}) &= \mathfrak{S}_{ii}\phi_m(\mathfrak{T}_{ii}\mathfrak{T}_{ij}) + \phi_m(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\mathfrak{T}_{ij} + \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii})\phi_s(\mathfrak{T}_{ii}\mathfrak{T}_{ij})\\
&= \mathfrak{S}_{ii}\phi_m(\mathfrak{T}_{ii}\mathfrak{T}_{ij}) + \phi_m(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\mathfrak{T}_{ij} + \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii})\Bigg(\sum_{\substack{p+q=s\\0\leq p,q\leq s}} \phi_p(\mathfrak{T}_{ii})\phi_q(\mathfrak{T}_{ij})\Bigg)\\
&= \mathfrak{S}_{ii}\phi_m(\mathfrak{T}_{ii}\mathfrak{T}_{ij}) + \phi_m(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\mathfrak{T}_{ij} + \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii})\Bigg(\mathfrak{T}_{ii}\phi_s(\mathfrak{T}_{ij}) + \phi_s(\mathfrak{T}_{ii})\mathfrak{T}_{ij}\\
&+ \sum_{\substack{p+q=s\\0<p,q\leq s-1}} \phi_p(\mathfrak{T}_{ii})\phi_q(\mathfrak{T}_{ij})\Bigg)\\
&= \mathfrak{S}_{ii}\phi_m(\mathfrak{T}_{ii}\mathfrak{T}_{ij}) + \phi_m(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\mathfrak{T}_{ij} + \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\phi_s(\mathfrak{T}_{ij})\\
&+ \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii})\phi_s(\mathfrak{T}_{ii})\mathfrak{T}_{ij} + \sum_{\substack{r+s=m\\p+q=s\\0<r,s\leq m-1\\0<p,q\leq s-1}} \phi_r(\mathfrak{S}_{ii})\phi_p(\mathfrak{T}_{ii})\phi_q(\mathfrak{T}_{ij})\\
&= \mathfrak{S}_{ii}\phi_m(\mathfrak{T}_{ii})\mathfrak{T}_{ij} + \mathfrak{S}_{ii}\mathfrak{T}_{ii}\phi_m(\mathfrak{T}_{ij}) + \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \mathfrak{S}_{ii}\phi_r(\mathfrak{T}_{ii})\phi_s(\mathfrak{T}_{ij})\\
&+ \phi_m(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\mathfrak{T}_{ij} + \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\phi_s(\mathfrak{T}_{ij})\\
&+ \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii})\phi_s(\mathfrak{T}_{ii})\mathfrak{T}_{ij} + \sum_{\substack{r+s=m\\p+q=s\\0<r,s\leq m-1\\0<p,q\leq s-1}} \phi_r(\mathfrak{S}_{ii})\phi_p(\mathfrak{T}_{ii})\phi_q(\mathfrak{T}_{ij}).
\end{aligned}
\tag{5}
$$

From Equations (4) and (5), we obtain $\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii})\mathfrak{T}_{ij} = \mathfrak{S}_{ii}\phi_m(\mathfrak{T}_{ii})\mathfrak{T}_{ij} + \phi_m(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\mathfrak{T}_{ij} + \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii})\phi_s(\mathfrak{T}_{ii})\mathfrak{T}_{ij}$. Since $\overline{P_i} = I$, this follows from the fact that $\{\mathcal{A}P_i(h) : h \in \mathcal{H}\}$ is dense in $\mathcal{H}$. Hence, $\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii}) = \sum_{s+t=m} \phi_s(\mathfrak{S}_{ii})\phi_t(\mathfrak{T}_{ii})$ for all $\mathfrak{S}_{ii}, \mathfrak{T}_{ii} \in \mathcal{R}$; $i = 1, 2$. This completes the proof of the lemma. □

Lemma 12. *For any $\mathfrak{S}_{ij} \in \mathcal{A}_{ij}$ and $\mathfrak{T}_{ji} \in \mathcal{A}_{ji}$. We have $\phi_m(\mathfrak{S}_{ij}\mathfrak{T}_{ji}) = \sum_{s+t=m} \phi_s(\mathfrak{S}_{ij})\phi_t(\mathfrak{T}_{ji})$; $i, j = 1, 2, (i \neq j)$.*

Proof. To prove our lemma, we use the principle of mathematical induction. For $m = 1$ it was shown to be true by Ashraf and Jabeen [19]. Suppose it holds for all $k \leq m$. We show that it also holds for $k = m$. Since for any $\mathfrak{S}_{12} \in \mathcal{A}_{12}$, $\mathfrak{S}_{12}\mathfrak{Q}_p = 0$ and $L_m(\mathfrak{Q}_q) \in Z(\mathcal{A})$, we have

$$
\begin{aligned}
&L_m\Big(p_n(\mathfrak{S}_{12}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21})\Big)\\
&= p_n\Big(L_m(\mathfrak{S}_{12}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21}\Big) + p_n\Big(\mathfrak{S}_{12}, L_m(\mathfrak{Q}_p), \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21}\Big)\\
&+ p_n\Big(\mathfrak{S}_{12}, \mathfrak{Q}_p, L_m(\mathfrak{Q}_p), \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21}\Big) + \cdots + p_n\Big(\mathfrak{S}_{12}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, L_m(\mathfrak{T}_{21})\Big)\\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m\\0<l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{12}), L_{l_2}(\mathfrak{Q}_p), \cdots, L_{l_{n-1}}(\mathfrak{Q}_p), L_{l_n}(\mathfrak{T}_{21})\Big)\\
&= p_n\Big(L_m(\mathfrak{S}_{12}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21}\Big) + p_n\Big(\mathfrak{S}_{12}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, L_m(\mathfrak{T}_{21})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{\substack{s+t=m\\0<s,t\leq m-1}} [L_s(\mathfrak{S}_{12}), L_t(\mathfrak{T}_{21})]\\
&= p_n\Big(\phi_m(\mathfrak{S}_{12}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21}\Big) + p_n\Big(\mathfrak{S}_{12}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \phi_m(\mathfrak{T}_{21})\Big)\\
&+ \sum_{\substack{s+t=m\\0<s,t\leq m-1}} [\phi_s(\mathfrak{S}_{12}), \phi_t(\mathfrak{T}_{21})].
\end{aligned}
$$

This implies that $L_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) = \phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21} - \mathfrak{T}_{21}\phi_m(\mathfrak{S}_{12}) + \mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}) - \phi_m(\mathfrak{T}_{21})\mathfrak{S}_{12} + \sum_{\substack{s+t=m\\0<s,t\leq m-1}} [\phi_s(\mathfrak{S}_{12}), \phi_t(\mathfrak{T}_{21})]$. But, we know that for all $\mathfrak{S} \in \mathcal{A}$, $\phi_m(\mathfrak{S}) = L_m(\mathfrak{S}) - \zeta_m(\mathfrak{S})$. Therefore, one can easily arrive at

$$
\begin{aligned}
&\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) + \zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})\\
&\quad = \phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21} - \mathfrak{T}_{21}\phi_m(\mathfrak{S}_{12}) + \mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}) - \phi_m(\mathfrak{T}_{21})\mathfrak{S}_{12} + \sum_{\substack{s+t=m\\0<s,t\leq m-1}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21})\\
&\quad - \sum_{\substack{s+t=m\\0<s,t\leq m-1}} \phi_t(\mathfrak{T}_{21})\phi_s(\mathfrak{S}_{12}).
\end{aligned}
$$

Premultiplying the above equation by $\mathfrak{S}_{12}$ and using Lemma 9, we obtain

$$
\begin{aligned}
\mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}\mathfrak{S}_{12}) - \zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})\mathfrak{S}_{12} &= \mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21})\mathfrak{S}_{12} + \mathfrak{S}_{12}\mathfrak{T}_{21}\phi_m(\mathfrak{S}_{12})\\
&+ \mathfrak{S}_{12} \sum_{\substack{s+t=m\\0<s,t\leq m-1}} \phi_t(\mathfrak{T}_{21})\phi_s(\mathfrak{S}_{12})
\end{aligned}
\tag{6}
$$

and by postmultiplying the same equation by $\mathfrak{S}_{12}$, we obtain

$$
\begin{aligned}
\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21})\mathfrak{S}_{12} + \zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})\mathfrak{S}_{12} &= \phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21}\mathfrak{S}_{12} + \mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21})\mathfrak{S}_{12}\\
&+ \sum_{\substack{s+t=m\\0<s,t\leq m-1}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21})\mathfrak{S}_{12}.
\end{aligned}
\tag{7}
$$

By comparing Equations (6) and (7), we obtain

$$
\begin{aligned}
&\mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}\mathfrak{S}_{12}) - \zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})\mathfrak{S}_{12} - \mathfrak{S}_{12}\mathfrak{T}_{21}\phi_m(\mathfrak{S}_{12})\\
&\quad = \phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21})\mathfrak{S}_{12} + \zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})\mathfrak{S}_{12} - \phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21}\mathfrak{S}_{12}\\
&\quad - \sum_{\substack{s+t=m\\0\leq s,t\leq m-1}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21})\mathfrak{S}_{12}.
\end{aligned}
\tag{8}
$$

Then, through the application of Lemma 10, we obtain

$$
\begin{aligned}
\mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}\mathfrak{S}_{12}) + \phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21}\mathfrak{S}_{12} &= \phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21}\mathfrak{S}_{12})\\
&= \phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21})\mathfrak{S}_{12} + \mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}\mathfrak{S}_{12}).
\end{aligned}
$$

Now, we prove that $\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) = 0$. For any $\mathfrak{S}_{12} \in \mathcal{A}_{12}$, let $\mathfrak{S}_{12} = V|\mathfrak{S}_{12}|$ be its polar decomposition. This implies that $\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})|\mathfrak{S}_{12}| = 0$ and thus $|\mathfrak{S}_{12}|\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^* = 0$, which follows that

$$
\mathfrak{S}_{12}\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) = |\mathfrak{S}_{12}|\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^* = 0. \tag{9}
$$

On the other hand, we similarly can show that

$$
\mathfrak{T}_{21}\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) = 0. \tag{10}
$$

Then, by multiplying $\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^*$ with Equation (6) and using Equations (9) and (10), we obtain

$$\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21}-\mathfrak{T}_{21}\mathfrak{S}_{12})\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21}-\mathfrak{T}_{21}\mathfrak{S}_{12})^*+\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21}-\mathfrak{T}_{21}\mathfrak{S}_{12})\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21}-\mathfrak{T}_{21}\mathfrak{S}_{12})^*=0. \tag{11}$$

Now, by using Lemma 8 and Equations (9) and (10), one can find that

$$\begin{aligned}
&\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21})\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21}-\mathfrak{T}_{21}\mathfrak{S}_{12})^*\\
&=\phi_m\big(\mathfrak{S}_{12}\mathfrak{T}_{21}\mathfrak{Q}_p\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21}-\mathfrak{T}_{21}\mathfrak{S}_{12})^*\mathfrak{Q}_p\big)-\mathfrak{S}_{12}\mathfrak{T}_{21}\phi_m\big(\mathfrak{Q}_p\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21}-\mathfrak{T}_{21}\mathfrak{S}_{12})^*\mathfrak{Q}_p\big)\\
&=-\mathfrak{S}_{12}\mathfrak{T}_{21}\phi_m\big(\mathfrak{Q}_p(\mathfrak{S}_{12}\mathfrak{T}_{21}-\mathfrak{T}_{21}\mathfrak{S}_{12})^*\mathfrak{Q}_p\big)
\end{aligned}$$

and

$$\begin{aligned}
&\phi_m(\mathfrak{T}_{21}\mathfrak{S}_{12})\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21}-\mathfrak{T}_{21}\mathfrak{S}_{12})^*\\
&=\phi_m\big(\mathfrak{T}_{12}\mathfrak{S}_{21}\mathfrak{Q}_q\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21}-\mathfrak{T}_{21}\mathfrak{S}_{12})^*\mathfrak{Q}_q\big)-\mathfrak{T}_{21}\mathfrak{S}_{12}\phi_m\big(\mathfrak{Q}_q\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21}-\mathfrak{T}_{21}\mathfrak{S}_{12})^*\mathfrak{Q}_q\big)\\
&=-\mathfrak{T}_{21}\mathfrak{S}_{12}\phi_m\big(\mathfrak{Q}_q(\mathfrak{S}_{12}\mathfrak{T}_{21}-\mathfrak{T}_{21}\mathfrak{S}_{12})^*\mathfrak{Q}_q\big).
\end{aligned}$$

Thus, Equation (11) implies that $\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21}-\mathfrak{T}_{21}\mathfrak{S}_{12})^*\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21}-\mathfrak{T}_{21}\mathfrak{S}_{12})\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21}-\mathfrak{T}_{21}\mathfrak{S}_{12})^*=0$, and hence $\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21}-\mathfrak{T}_{21}\mathfrak{S}_{12})=0$. Therefore, from Equations (6) and (7) and using Lemma 11, we obtain $\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21})=\phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21}+\mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21})+\sum_{\substack{s+t=m\\0<s,t\leq m-1}}\phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21})$ i.e, $\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21})=\sum_{\substack{s+t=m\\0\leq s,t\leq m}}\phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21})$ and $\phi_m(\mathfrak{T}_{21}\mathfrak{S}_{12})=\sum_{\substack{s+t=m\\0\leq s,t\leq m}}\phi_s(\mathfrak{T}_{21})\phi_t(\mathfrak{S}_{12})$ for all $\mathfrak{S}_{12}\in\mathcal{A}_{12}$, $\mathfrak{T}_{21}\in\mathcal{A}_{21}$. This proves that the lemma is also true for $k=m$. Hence, the lemma is true for all $m\in\mathbb{N}$. $\square$

We have all the pieces to carry the proof of our first main result of this paper.

Proof of Theorem 1. In view of Lemmas 10–12, one can easily see that ϕ_m is an additive higher derivation, and it can be observed that $\zeta_m(\mathfrak{S}_{jj})\in Z(\mathcal{A})$ for $j=1,2$ and $\zeta_m(\mathfrak{S}_{ji})=0$ for $j\neq i$. We now show that $\zeta_m\Big(p_n(\mathfrak{S}_1,\mathfrak{S}_2,\mathfrak{S}_3,\cdots,\mathfrak{S}_n)\Big)=0$ for all $\mathfrak{S}_i\in\mathcal{A}\,;\,1\leq i\leq n$.

$$\begin{aligned}
&\zeta_m\Big(p_n(\mathfrak{S}_1,\mathfrak{S}_2,\mathfrak{S}_3,\cdots,\mathfrak{S}_n)\Big)\\
&\qquad=L_m\Big(p_n(\mathfrak{S}_1,\mathfrak{S}_2,\mathfrak{S}_3,\cdots,\mathfrak{S}_n)\Big)-\phi_m\Big(p_n(\mathfrak{S}_1,\mathfrak{S}_2,\mathfrak{S}_3,\cdots,\mathfrak{S}_n)\Big)\\
&\qquad=p_n(L_m(\mathfrak{S}_1),\mathfrak{S}_2,\mathfrak{S}_3,\cdots,\mathfrak{S}_n)+\cdots+p_n(\mathfrak{S}_1,\mathfrak{S}_2,\mathfrak{S}_3,\cdots,L_m(\mathfrak{S}_n))\\
&\qquad+\sum_{\substack{l_1+l_2+\cdots+l_n=m\\0\leq l_1,l_2,\cdots,l_n\leq m-1}}p_n\Big(L_{l_1}(\mathfrak{S}_1,L_{l_2}(\mathfrak{S}_2),L_{l_3}(\mathfrak{S}_3),\cdots,L_{l_n}(\mathfrak{S}_n)\Big)\\
&\qquad-p_n(\phi_m(\mathfrak{S}_1),\mathfrak{S}_2,\mathfrak{S}_3,\cdots,\mathfrak{S}_n)-\cdots-p_n(\mathfrak{S}_1,\mathfrak{S}_2,\mathfrak{S}_3,\cdots,\phi_m(\mathfrak{S}_n))\\
&\qquad-\sum_{\substack{l_1+l_2+\cdots+l_n=m\\0\leq l_1,l_2,\cdots,l_n\leq m-1}}p_n\Big(\phi_{l_1}(\mathfrak{S}_1,\phi_{l_2}(\mathfrak{S}_2),\phi_{l_3}(\mathfrak{S}_3),\cdots,\phi_{l_n}(\mathfrak{S}_n)\Big)\\
&\qquad=p_n(\phi_m(\mathfrak{S}_1),\mathfrak{S}_2,\mathfrak{S}_3,\cdots,\mathfrak{S}_n)+\cdots+p_n(\mathfrak{S}_1,\mathfrak{S}_2,\mathfrak{S}_3,\cdots,\phi_m(\mathfrak{S}_n))\\
&\qquad+\sum_{\substack{l_1+l_2+\cdots+l_n=m\\0\leq l_1,l_2,\cdots,l_n\leq m-1}}p_n\Big(\phi_{l_1}(\mathfrak{S}_1,\phi_{l_2}(\mathfrak{S}_2),\phi_{l_3}(\mathfrak{S}_3),\cdots,\phi_{l_n}(\mathfrak{S}_n)\Big)\\
&\qquad-p_n(\phi_m(\mathfrak{S}_1),\mathfrak{S}_2,\mathfrak{S}_3,\cdots,\mathfrak{S}_n)-\cdots-p_n(\mathfrak{S}_1,\mathfrak{S}_2,\mathfrak{S}_3,\cdots,\phi_m(\mathfrak{S}_n))\\
&\qquad-\sum_{\substack{l_1+l_2+\cdots+l_n=m\\0\leq l_1,l_2,\cdots,l_n\leq m-1}}p_n\Big(\phi_{l_1}(\mathfrak{S}_1,\phi_{l_2}(\mathfrak{S}_2),\phi_{l_3}(\mathfrak{S}_3),\cdots,\phi_{l_n}(\mathfrak{S}_n)\Big)\\
&\qquad=0.
\end{aligned}$$

We can now conclude from the above observations that if $L_m:\mathcal{A}\to\mathcal{A}$ is an additive Lie n-higher derivation, then there exists an additive higher derivation ϕ_m of $\mathcal{A}$ and a map $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$ that vanishes at $p_n(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \cdots, \mathfrak{S}_n)$ with $\mathfrak{S}_1\mathfrak{S}_2 = 0$ for all $\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \cdots, \mathfrak{S}_n \in \mathcal{A}$, such that $L_m = \phi_m + \zeta_m$. $\square$

Note that every additive derivation $d : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is an inner derivation Semrl [22]. Nowicki [23] proved that if every additive(linear) derivation of $\mathcal{A}$ is inner, then every additive (linear) higher derivation of $\mathcal{A}$ is inner (see Wei and Xiao [24] for details). Hence, by Theorem 1, the following corollaries are immediate:

Corollary 1. *Let $\mathcal{A}$ be a von Neumann algebra with no central summands of type I_1 and a linear map $L_m : \mathcal{A} \to \mathcal{A}$ satisfying*

$$L_m\big(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)\big) = \sum_{l_1+l_2+\cdots+l_n=m} p_n\big(L_{l_1}(\mathfrak{S}_1), L_{l_2}(\mathfrak{S}_2), \cdots, L_{l_n}(\mathfrak{S}_n)\big)$$

for all $\mathfrak{S}_i \in \mathcal{A}\,; 1 \le i \le n$, with $\mathfrak{S}_1\mathfrak{S}_2 = 0$. Then, there exists an operator $\mathcal{T} \in \mathcal{A}$ and a linear map $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$, which annihilates every $(n-1)^{th}$-commutator $p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)$ with $\mathfrak{S}_1\mathfrak{S}_2 = 0$ such that $L_m(\mathfrak{S}) = \mathfrak{S}\mathcal{T} - \mathcal{T}\mathfrak{S} + \zeta_m(\mathfrak{S})$ for all $\mathfrak{S} \in \mathcal{A}$.

Corollary 2. *Let $\mathcal{A}$ be a von Neumann algebra with no central summands of type I_1 and a linear map $L_m : \mathcal{A} \to \mathcal{A}$ satisfying*

$$L_m\big(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)\big) = \sum_{l_1+l_2+\cdots+l_n=m} p_n\big(L_{l_1}(\mathfrak{S}_1), L_{l_2}(\mathfrak{S}_2), \cdots, L_{l_n}(\mathfrak{S}_n)\big)$$

for all $\mathfrak{S}_i \in \mathcal{A}\,; 1 \le i \le n$. Then, L_m is an additive Lie higher derivation if and only if there exists an additive higher derivation $\phi_m : \mathcal{A} \to \mathcal{A}$ and an additive map $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$, which annihilates every $(n-1)^{th}$-commutator $p_n(\mathfrak{S}_1, \mathfrak{S}_1, \cdots, \mathfrak{S}_1)$ such that $L_m(\mathfrak{S}) = \phi_m(\mathfrak{S}) + \zeta_m(\mathfrak{S})$ for all $\mathfrak{S} \in \mathcal{A}$.

In the next segment, we study the characterization of Lie derivations on general von Neumann algebras having no central summands of type I_1 by taking action at the projection products. Now, we state and prove the second main result of this paper.

Theorem 2. *Let $\mathcal{A}$ be a von Neumann algebra with no central summands of type I_1 and an additive higher map $L_m : \mathcal{A} \to \mathcal{A}$ satisfying $L_m(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)) = \sum_{l_1+l_2+\cdots+l_n=m} p_n$ $\Big(p_n(L_{l_1}(\mathfrak{S}_1), L_{l_2}(\mathfrak{S}_2), L_{l_3}(\mathfrak{S}_3), \cdots, L_{l_n}(\mathfrak{S}_n))\Big)$ for all $\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n \in \mathcal{A}$ with $\mathfrak{S}_1\mathfrak{S}_2 = P$, where P is a core-free projection with the central carrier I. Then, there exists an additive higher derivation $\phi_m : \mathcal{A} \to \mathcal{A}$ and an additive higher map $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$ that annihilates every $(n-1)^{th}$-commutator $p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)$ with $\mathfrak{S}_1\mathfrak{S}_2 = P$, such that $L_m(\mathfrak{S}) = \phi_m(\mathfrak{S}) + \zeta_m(\mathfrak{S})$ for all $\mathfrak{S} \in \mathcal{A}$.*

Let $\mathfrak{Q}_0 = \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_q - \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_p$ and let us define a map $\pi_m : \mathcal{A} \to \mathcal{A}$ as an inner higher derivation $\pi_m(\mathfrak{S}) = [\mathfrak{S}, \mathfrak{Q}_0]$ for all $\mathfrak{S} \in \mathcal{A}$. Clearly, $L_m = L'_m - \pi_m$ is also a Lie n-higher derivation. Since

$$\begin{aligned} L'_m(\mathfrak{Q}_p) &= L_m(\mathfrak{Q}_p) - [\mathfrak{Q}_p, \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_q - \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_p] \\ &= L_m(\mathfrak{Q}_p) - \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_q - \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_p \\ &= \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_p + \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_q. \end{aligned}$$

one easily obtains $\mathfrak{Q}_p L'_m(\mathfrak{Q}_p)\mathfrak{Q}_q = \mathfrak{Q}_q L'_m(\mathfrak{Q}_p)\mathfrak{Q}_p = 0$. Accordingly, it suffices to consider only those Lie n-higher derivations $L_m : \mathcal{A} \to \mathcal{A}$ which satisfy $\mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_q = \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_p = 0$.

Lemma 13. $L_m(\mathfrak{Q}_p), L_m(\mathfrak{Q}_q) \in Z(\mathcal{A})$.

Proof. For $k=1$, it was shown to be true by Ashraf and Jabeen [19]. Suppose that it holds for all $k \leq m-1$. We show that it also holds for $k=m$. Since $(\mathfrak{S}_{12}+\mathfrak{Q}_p)\mathfrak{Q}_p=\mathfrak{Q}_p$ for all $\mathfrak{S}_{12}\in\mathcal{A}_{12}$, we can write

$$\begin{aligned}
L_m\Big(p_n(\mathfrak{S}_{12}+\mathfrak{Q}_p),\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p\Big)
&= p_n\Big(L_m(\mathfrak{S}_{12}+\mathfrak{Q}_p),\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p\Big)+p_n\Big(\mathfrak{S}_{12}+\mathfrak{Q}_p,L_m(\mathfrak{Q}_p),\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p\Big)\\
&+p_n\Big(\mathfrak{S}_{12}+\mathfrak{Q}_p,\mathfrak{Q}_p,L_m(\mathfrak{Q}_p),\cdots,\mathfrak{Q}_p\Big)+\cdots+p_n\Big(\mathfrak{S}_{12}+\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,L_m(\mathfrak{Q}_p)\Big)\\
&+\sum_{\substack{l_1+l_2+\cdots+l_n=m\\0<l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{12}+\mathfrak{Q}_p),L_{l_2}(\mathfrak{Q}_p),L_{l_3}(\mathfrak{Q}_p),\cdots,L_{l_n}(\mathfrak{Q}_p)\Big)\\
&=(-1)^{n-1}\mathfrak{Q}_pL_m(\mathfrak{S}_{12})\mathfrak{Q}_q+\mathfrak{Q}_qL_m(\mathfrak{S}_{12})\mathfrak{Q}_p\\
&+(-1)^{n-2}(n-1)[\mathfrak{S}_{12},L_m(\mathfrak{Q}_p)].
\end{aligned}$$

From which we obtain

$$\begin{aligned}
L_m((-1)^{n-1}\mathfrak{S}_{12})=(-1)^{n-1}\mathfrak{Q}_pL_m(\mathfrak{S}_{12})\mathfrak{Q}_q+\mathfrak{Q}_qL_m(\mathfrak{S}_{12})\mathfrak{Q}_p\\
+(-1)^{n-2}(n-1)[\mathfrak{S}_{12},L_m(\mathfrak{Q}_p)].
\end{aligned}\tag{12}$$

Now, by premultiplying $\mathfrak{Q}_p$ and postmultiplying $\mathfrak{Q}_q$ to the above equation, one finds that

$$\mathfrak{Q}_pL_m(\mathfrak{Q}_p)\mathfrak{S}_{12}=\mathfrak{S}_{12}L_m(\mathfrak{Q}_p)\mathfrak{Q}_q.$$

Since $\mathfrak{Q}_pL_m(\mathfrak{Q}_p)\mathfrak{Q}_q=\mathfrak{Q}_qL_m(\mathfrak{Q}_p)\mathfrak{Q}_p=0$, by using the above equation and Lemma 4, we obtain $L_m(\mathfrak{Q}_p)\in Z(\mathcal{A})$. Now, by using $(\mathfrak{Q}_q+\mathfrak{Q}_p)\mathfrak{Q}_p=\mathfrak{Q}_p$, it follows that

$$\begin{aligned}
0&=L_m\Big(p_n(\mathfrak{Q}_q+\mathfrak{Q}_p,\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p)\Big)\\
&=p_n\Big(L_m(\mathfrak{Q}_q+\mathfrak{Q}_p),\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p\Big)+p_n\Big(\mathfrak{Q}_q+\mathfrak{Q}_p,L_m(\mathfrak{Q}_p),\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p\Big)\\
&+\cdots+p_n\Big(\mathfrak{Q}_q+\mathfrak{Q}_p,\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,L_m(\mathfrak{Q}_p)\Big)+\sum_{\substack{l_1+l_2+\cdots+l_n=m\\0<l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{Q}_q+\mathfrak{Q}_p),\\
&L_{l_2}(\mathfrak{Q}_p),L_{l_3}(\mathfrak{Q}_p),\cdots,L_{l_n}(\mathfrak{Q}_p)\Big).
\end{aligned}$$

which gives $0=(-1)^{n-1}\mathfrak{Q}_pL_m(\mathfrak{Q}_q)\mathfrak{Q}_q+\mathfrak{Q}_qL_m(\mathfrak{Q}_q)\mathfrak{Q}_p$. It follows that $\mathfrak{Q}_pL_m(\mathfrak{Q}_q)\mathfrak{Q}_q=\mathfrak{Q}_qL_m(\mathfrak{Q}_q)\mathfrak{Q}_p=0$. On the other hand, by using $p_n(\mathfrak{Q}_p+\mathfrak{S}_{12},\mathfrak{Q}_q+\mathfrak{Q}_p-\mathfrak{S}_{12},\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p)=0$ and making similar calculations as above, one obtains that $L_m(\mathfrak{Q}_q)\in Z(\mathcal{A})$. Hence, $L_m(\mathfrak{Q}_p),L_m(\mathfrak{Q}_q)\in Z(\mathcal{A})$ for all $m\in\mathbb{N}$. □

Lemma 14. $L_m(A_{ij})\subseteq\mathcal{A}_{ij}$, $1\leq i\neq j\leq 2$.

Proof. We prove the lemma with the help of the principle of mathematical induction. For $m=1$, it was shown to be true by Ashraf and Jabeen [19]. Suppose that the lemma holds for all $k\leq m-1$. We will prove that it is also true for $k=m$. First, consider the case for $i=1$ *and* $j=2$; the other case $i=2$ *and* $j=1$ will be proved in a similar way. By using equation (12) and $L_m(\mathfrak{Q}_p)\in Z(\mathcal{A})$, we have $L_m(\mathfrak{S}_{12})=\mathfrak{Q}_pL_m(\mathfrak{S}_{12})\mathfrak{Q}_q+(-1)^{n-1}\mathfrak{Q}_qL_m(\mathfrak{S}_{12})\mathfrak{Q}_p$. By pre- and postmultiplying $\mathfrak{Q}_p$ and $\mathfrak{Q}_q$ to the above equation, we obtain $\mathfrak{Q}_pL_m(\mathfrak{S}_{12})\mathfrak{Q}_p=0$ *and* $\mathfrak{Q}_qL_m(\mathfrak{S}_{12})\mathfrak{Q}_q=0$, respectively. Hence, $\mathfrak{Q}_pL_m(\mathfrak{S}_{12})\mathfrak{Q}_p=\mathfrak{Q}_qL_m(\mathfrak{S}_{12})\mathfrak{Q}_q=0$. Since $(\mathfrak{Q}_p+\mathfrak{S}_{12})\mathfrak{Q}_p=\mathfrak{Q}_p$, we can write

$$
\begin{aligned}
0 &= L_m\Big(p_n(\mathfrak{Q}_p+\mathfrak{S}_{12},\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p,\mathfrak{T}_{12})\Big)\\
&= p_n\Big(L_m(\mathfrak{Q}_p+\mathfrak{S}_{12}),\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p,\mathfrak{T}_{12}\Big)+p_n\Big(\mathfrak{Q}_p+\mathfrak{S}_{12},L_m(\mathfrak{Q}_p),\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p,\mathfrak{T}_{12}\Big)\\
&+\cdots+\Big(\mathfrak{Q}_p+\mathfrak{S}_{12},\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,L_m(\mathfrak{Q}_p),\mathfrak{T}_{12}\Big)+p_n\Big(\mathfrak{Q}_p+\mathfrak{T}_{12},\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p,L_m(\mathfrak{T}_{12})\Big)\\
&+\sum_{\substack{l_1+l_2+\cdots+l_n=m\\ 0\le l_1,l_2,\cdots,l_n\le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{12}),L_{l_2}(\mathfrak{Q}_p),L_{l_3}(\mathfrak{Q}_p),\cdots,L_{l_{n-1}}(\mathfrak{Q}_p),L_{l_n}(\mathfrak{T}_{12})\Big)\\
&= p_n\Big(L_m(\mathfrak{Q}_p+\mathfrak{S}_{12}),\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p,\mathfrak{T}_{12}\Big)+p_n\Big(\mathfrak{Q}_p+\mathfrak{T}_{12},\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p,L_m(\mathfrak{T}_{12})\Big)\\
&+\sum_{\substack{s+t=m\\ 0<s,t\le m-1}} L_s(\mathfrak{S}_{12})L_t(\mathfrak{T}_{12}).
\end{aligned}
$$

This follows that

$$0=\mathfrak{Q}_qL_m(\mathfrak{S}_{12})\mathfrak{T}_{12}-\mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\mathfrak{Q}_p+(-1)^{n-1}L_m(\mathfrak{T}_{12})\mathfrak{S}_{12}-(-1)^{n-1}\mathfrak{S}_{12}L_m(\mathfrak{T}_{12}).$$

Then, by multiplying $\mathfrak{Q}_q$ on both sides to the above equation, one obtains $\mathfrak{Q}_qL_m(\mathfrak{S}_{12})\mathfrak{T}_{12}=0$, and by multiplying $\mathfrak{T}_{12}$ from the right-hand side and using $\mathfrak{Q}_qL_m(\mathfrak{S}_{12})\mathfrak{T}_{12}=0$, we find that $\mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\mathfrak{T}_{12}=0$. Then, by linearizing, we obtain $\mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\mathcal{T}_{12}+\mathcal{T}_{12}L_m(\mathfrak{S}_{12})\mathfrak{T}_{12}=0$ for all $\mathcal{T}_{12},\mathfrak{T}_{12}\in\mathcal{A}_{12}$. Now it can be easily observed that

$$
\begin{aligned}
&\mathfrak{Q}_qL_m(\mathfrak{S}_{12})\mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\Big[\mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\mathcal{T}_{12}\Big]L_m(\mathfrak{S}_{12})\mathfrak{Q}_p\\
&\quad+\mathfrak{Q}_qL_m(\mathfrak{S}_{12})\mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\Big[\mathcal{T}_{12}L_m(\mathfrak{S}_{12})\mathfrak{T}_{12}\Big]L_m(\mathfrak{S}_{12})\mathfrak{Q}_p=0.
\end{aligned}
$$

which implies

$$\mathfrak{Q}_qL_m(\mathfrak{S}_{12})\mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\mathcal{T}_{12}L_m(\mathfrak{S}_{12})\mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\mathfrak{Q}_p=0.$$

As $\mathcal{A}$ is semiprime, one can easily see that $\mathfrak{Q}_qL_m(\mathfrak{S}_{12})\mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\mathfrak{Q}_p=0$ and therefore $\mathfrak{Q}_qL_m(\mathfrak{S}_{12})\mathfrak{Q}_p=0$. Hence, $L_m(\mathcal{A}_{12})\subseteq\mathcal{A}_{12}$, which shows that the lemma also holds for $k=m$. Therefore, $L_m(A_{12})\subseteq\mathcal{A}_{12}$ holds for all $m\in\mathbb{N}$. Similarly, we can easily prove that $L_m(\mathcal{A}_{21})\subseteq\mathcal{A}_{21}$. □

Lemma 15. *There exists maps ζ_{m_i} on $\mathcal{A}_{ii}$ such that $L_m(\mathfrak{S}_{ii})-\zeta_{m_i}(\mathfrak{S}_{ii})\in\mathcal{A}_{ii}$ and $L_m(\mathfrak{S}_{ii})\subseteq\mathcal{A}_{ii}+\mathcal{A}_{jj}$, for any $\mathfrak{S}_{ii}\in\mathcal{A}_{ii}$; $i=1,2$ and $i\neq j$.*

Proof. We will prove the lemma with the help of the principle of mathematical induction. For $m=1$, it was shown to be true by Ashraf and Jabeen [19]. Suppose that the lemma holds for all $k\le m-1$. We will show that it also holds for $k=m$. Here, we give the proof for the case $i=1$, and the proof for the case $i=2$ follows similar steps. Suppose $\mathfrak{S}_{11}\in\mathcal{A}_{11}$ is invertible; this implies that there exists $\mathfrak{S}_{11}^{-1}\in\mathcal{A}_{11}$, such that $\mathfrak{S}_{11}\mathfrak{S}_{11}^{-1}=\mathfrak{S}_{11}^{-1}\mathfrak{S}_{11}=\mathfrak{Q}_p$. Therefore, we can write

$$
\begin{aligned}
0 &= L_m\Big(p_n(\mathfrak{S}_{11}^{-1},\mathfrak{S}_{11},\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p)\Big)\\
&= p_n\Big(L_m(\mathfrak{S}_{11}^{-1}),\mathfrak{S}_{11},\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p\Big)+p_n\Big(\mathfrak{S}_{11}^{-1},L_m(\mathfrak{S}_{11}),\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p\Big)\\
&+p_n\Big(\mathfrak{S}_{11}^{-1},\mathfrak{S}_{11},L_m(\mathfrak{Q}_p),\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p\Big)+\cdots+p_n\Big(\mathfrak{S}_{11}^{-1},\mathfrak{S}_{11},\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,L_m(\mathfrak{Q}_p)\Big)\\
&+\sum_{\substack{l_1+l_2+\cdots+l_n=m\\ 0\le l_1,l_2,\cdots,l_n\le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1}),L_{l_2}(\mathfrak{S}_{11}),L_{l_3}(\mathfrak{Q}_p),L_{l_4}(\mathfrak{Q}_p),\cdots,L_{l_n}(\mathfrak{Q}_p)\Big).
\end{aligned}
$$

Also, since $(\mathfrak{S}_{11}^{-1}+\mathfrak{Q}_q)\mathfrak{S}_{11}=\mathfrak{Q}_p$ and by using Lemma 13, we have

$$
\begin{aligned}
0 &= L_m\Big(p_n(\mathfrak{S}_{11}^{-1}+\mathfrak{Q}_q,\mathfrak{S}_{11},\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p)\Big)\\
&= p_n\Big(L_m(\mathfrak{S}_{11}^{-1}+\mathfrak{Q}_q),\mathfrak{S}_{11},\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p\Big)+p_n\Big(\mathfrak{S}_{11}^{-1}+\mathfrak{Q}_q,L_m(\mathfrak{S}_{11}),\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p\Big)\\
&+p_n\Big(\mathfrak{S}_{11}^{-1}+\mathfrak{Q}_q,\mathfrak{S}_{11},L_m(\mathfrak{Q}_p),\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p\Big)+\cdots+p_n\Big(\mathfrak{S}_{11}^{-1}+\mathfrak{Q}_q,\mathfrak{S}_{11},\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,L_m(\mathfrak{Q}_p)\Big)\\
&+\sum_{\substack{l_1+l_2+\cdots+l_n=m\\0\le l_1,l_2,\cdots,l_n\le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1}+\mathfrak{Q}_q),L_{l_2}(\mathfrak{S}_{11}),L_{l_3}(\mathfrak{Q}_p),L_{l_4}(\mathfrak{Q}_p),\cdots,L_{l_n}(\mathfrak{Q}_p)\Big)\\
&= p_n\Big(L_m(\mathfrak{S}_{11}^{-1})+L_m(\mathfrak{Q}_q),\mathfrak{S}_{11},\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p\Big)+p_n\Big(\mathfrak{S}_{11}^{-1}+\mathfrak{Q}_q,L_m(\mathfrak{S}_{11}),\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p\Big)\\
&+\sum_{l=3}^{n} p_n\Big(\mathfrak{S}_{11}^{-1}+\mathfrak{Q}_q,\mathfrak{S}_{11},\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,L_m(\mathfrak{Q}_p)_l,\cdots,\mathfrak{Q}_p\Big)\\
&+\sum_{\substack{l_1+l_2+\cdots+l_n=m\\0\le l_1,l_2,\cdots,l_n\le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1})+L_{l_1}(\mathfrak{Q}_q),L_{l_2}(\mathfrak{S}_{11}),L_{l_3}(\mathfrak{Q}_p),L_{l_4}(\mathfrak{Q}_p),\cdots,L_{l_n}(\mathfrak{Q}_p)\Big).
\end{aligned}
$$

Upon comparing the above two equations, we have $0=p_n\Big(\mathfrak{Q}_q,L_m(\mathfrak{S}_{11}),\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p\Big)=\mathfrak{Q}_qL_m(\mathfrak{S}_{11})\mathfrak{Q}_p+(-1)^{n-1}\mathfrak{Q}_pL_m(\mathfrak{S}_{11})\mathfrak{Q}_q$. It can be easily observed from the above equation that $\mathfrak{Q}_qL_m(\mathfrak{S}_{11})\mathfrak{Q}_p=\mathfrak{Q}_pL_m(\mathfrak{S}_{11})\mathfrak{Q}_q=0$, from which we obtain $L_m(\mathfrak{S}_{ii})\subseteq\mathcal{A}_{ii}+\mathcal{A}_{jj}$ as $(\mathfrak{S}_{11}^{-1}+\mathfrak{T}_{22})\mathfrak{S}_{11}=\mathfrak{Q}_p$ and $(\mathfrak{S}_{11}^{-1})\mathfrak{S}_{11}=\mathfrak{Q}_p$. Then, for any $\mathfrak{T}_{22}\in\mathcal{A}_{22}$ and $\mathcal{W}_{12}\in\mathcal{A}_{12}$, it can be easily seen that

$$
\begin{aligned}
0 &= L_m\Big(p_n(\mathfrak{S}_{11}^{-1},\mathfrak{S}_{11},\mathcal{W}_{12},\mathfrak{Q}_q,\cdots,\mathfrak{Q}_q)\Big)\\
&= p_n\Big(L_m(\mathfrak{S}_{11}^{-1}),\mathfrak{S}_{11},\mathcal{W}_{12},\mathfrak{Q}_q,\cdots,\mathfrak{Q}_q\Big)+p_n\Big(\mathfrak{S}_{11}^{-1},L_m(\mathfrak{S}_{11}),\mathcal{W}_{12},\mathfrak{Q}_q,\cdots,\mathfrak{Q}_q\Big)\\
&+p_n\Big(\mathfrak{S}_{11}^{-1},\mathfrak{S}_{11},L_m(\mathcal{W}_{12}),\mathfrak{Q}_q,\cdots,\mathfrak{Q}_q\Big)+\cdots+p_n\Big(\mathfrak{S}_{11}^{-1},\mathfrak{S}_{11},\mathcal{W}_{12},\mathfrak{Q}_q,\cdots,L_m(\mathfrak{Q}_q)\Big)\\
&+\sum_{\substack{l_1+l_2+\cdots+l_n=m\\0\le l_1,l_2,\cdots,l_n\le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1}),L_{l_2}(\mathfrak{S}_{11}),L_{l_3}(\mathcal{W}_{12}),L_{l_4}(\mathfrak{Q}_q),\cdots,L_{l_n}(\mathfrak{Q}_q)\Big).
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
0 &= L_m\Big(p_n(\mathfrak{S}_{11}^{-1}+\mathfrak{T}_{22},\mathfrak{S}_{11},\mathcal{W}_{12},\mathfrak{Q}_q,\cdots,\mathfrak{Q}_q)\Big)\\
&= p_n\Big(L_m(\mathfrak{S}_{11}^{-1}+\mathfrak{T}_{22}),\mathfrak{S}_{11},\mathcal{W}_{12},\mathfrak{Q}_q,\cdots,\mathfrak{Q}_q\Big)+p_n\Big(\mathfrak{S}_{11}^{-1}+\mathfrak{T}_{22},L_m(\mathfrak{S}_{11}),\mathcal{W}_{12},\mathfrak{Q}_q,\cdots,\mathfrak{Q}_q\Big)\\
&+p_n\Big(\mathfrak{S}_{11}^{-1}+\mathfrak{T}_{22},\mathfrak{S}_{11},L_m(\mathcal{W}_{12}),\mathfrak{Q}_q,\cdots,\mathfrak{Q}_q\Big)+\cdots+p_n\Big(\mathfrak{S}_{11}^{-1}+\mathfrak{T}_{22},\mathfrak{S}_{11},\mathcal{W}_{12},\mathfrak{Q}_q,\cdots,L_m(\mathfrak{Q}_q)\Big)\\
&+\sum_{\substack{l_1+l_2+\cdots+l_n=m\\0\le l_1,l_2,\cdots,l_n\le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1}+\mathfrak{T}_{22}),L_{l_2}(\mathfrak{S}_{11}),L_{l_3}(\mathcal{W}_{12}),L_{l_4}(\mathfrak{Q}_q),\cdots,L_{l_n}(\mathfrak{Q}_q)\Big)\\
&= p_n\Big(L_m(\mathfrak{S}_{11}^{-1})+L_m(\mathfrak{T}_{22}),\mathfrak{S}_{11},\mathcal{W}_{12},\mathfrak{Q}_q,\cdots,\mathfrak{Q}_q\Big)+p_n\Big(\mathfrak{S}_{11}^{-1}+\mathfrak{T}_{22},L_m(\mathfrak{S}_{11}),\mathcal{W}_{12},\mathfrak{Q}_q,\cdots,\mathfrak{Q}_q\Big)\\
&+p_n\Big(\mathfrak{S}_{11}^{-1}+\mathfrak{T}_{22},\mathfrak{S}_{11},L_m(\mathcal{W}_{12}),\mathfrak{Q}_q,\cdots,\mathfrak{Q}_q\Big)+\cdots+p_n\Big(\mathfrak{S}_{11}^{-1}+\mathfrak{T}_{22},\mathfrak{S}_{11},\mathcal{W}_{12},\mathfrak{Q}_q,\cdots,L_m(\mathfrak{Q}_q)\Big)\\
&+\sum_{\substack{l_1+l_2+\cdots+l_n=m\\0\le l_1,l_2,\cdots,l_n\le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1})+L_{l_1}(\mathfrak{T}_{22}),L_{l_2}(\mathfrak{S}_{11}),L_{l_3}(\mathcal{W}_{12}),L_{l_4}(\mathfrak{Q}_q),\cdots,L_{l_n}(\mathfrak{Q}_q)\Big)
\end{aligned}
\tag{14}
$$

Comparing Equations (13) and (14), we obtain

$$
\begin{aligned}
0 &= p_n\Big(L_m(\mathfrak{T}_{22}),\mathfrak{S}_{11},\mathcal{W}_{12},\mathfrak{Q}_q,\cdots,\mathfrak{Q}_q\Big)+p_n\Big(\mathfrak{T}_{22},L_m(\mathfrak{S}_{11}),\mathcal{W}_{12},\mathfrak{Q}_q,\cdots,\mathfrak{Q}_q\Big)\\
&= p_{n-1}\Big(\big[L_m(\mathfrak{T}_{22}),\mathfrak{S}_{11}\big],\mathcal{W}_{12},\mathfrak{Q}_q,\cdots,\mathfrak{Q}_q\Big)+p_{n-1}\Big(\big[\mathfrak{T}_{22},L_m(\mathfrak{S}_{11})\big],\mathcal{W}_{12},\mathfrak{Q}_q,\cdots,\mathfrak{Q}_q\Big)\\
&= \Big[\big[L_m(\mathfrak{T}_{22}),\mathfrak{S}_{11}\big],\mathcal{W}_{12}\Big]+\Big[\big[\mathfrak{T}_{22},L_m(\mathfrak{S}_{11})\big],\mathcal{W}_{12}\Big].
\end{aligned}
$$

which leads to $[L_m(\mathfrak{T}_{22}), \mathfrak{S}_{11}] + [\mathfrak{T}_{22}, L_m(\mathfrak{S}_{11})] \in Z(\mathcal{A})$. Then, multiplying both sides of the above equation by $\mathfrak{Q}_q$, one arrives at $[\mathfrak{T}_{22}, \mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q] \in Z(\mathcal{A})\mathfrak{Q}_q$ and therefore $[\mathfrak{T}_{22}, \mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q] = 0$. This implies that there exists some $z \in Z(\mathcal{A})$ such that $\mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q = z\mathfrak{Q}_q$. If $\mathfrak{S}_{11}$ is not invertible in $\mathcal{A}_{11}$, then one can find a sufficiently large number, say r, in a way such that $r\mathfrak{Q}_p - \mathfrak{S}_{11}$ is invertible in $\mathcal{A}_{11}$ following the preceding cases $\mathfrak{Q}_p L_m(r\mathfrak{Q}_p - \mathfrak{S}_{11})\mathfrak{Q}_q + \mathfrak{Q}_q L_m(r\mathfrak{Q}_p - \mathfrak{S}_{11})\mathfrak{Q}_p = 0$ and $\mathfrak{Q}_q L_m(r\mathfrak{Q}_p - \mathfrak{S}_{11})\mathfrak{Q}_q = z\mathfrak{Q}_q$. As $L_m(\mathfrak{Q}_p) \in Z(\mathcal{A})$, we have $\mathfrak{Q}_p L_m(\mathfrak{S}_{11})\mathfrak{Q}_q + \mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_p = 0$ and $\mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q \in Z(\mathcal{A})\mathfrak{Q}_q$. Without a loss of generality, we denote $\mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q = z\mathfrak{Q}_q$. Therefore, for any $\mathfrak{S}_{11} \in \mathcal{A}_{11}$, we have

$$\begin{aligned} L_m(\mathfrak{S}_{11}) &= \mathfrak{Q}_p L_m(\mathfrak{S}_{11})\mathfrak{Q}_p + \mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q \\ &= \mathfrak{Q}_p L_m(\mathfrak{S}_{11})\mathfrak{Q}_p - z\mathfrak{Q}_p + z. \end{aligned}$$

We define a map, say ζ_{m_1}, on $\mathcal{A}_{11}$ by $\zeta_{m_1}(\mathfrak{S}_{11}) = z$, and then by combining it with the above equation, we obtain $L_m(\mathfrak{S}_{11}) - \zeta_{m_1}(\mathfrak{S}_{11}) = \mathfrak{Q}_p L_m(\mathfrak{S}_{11})\mathfrak{Q}_p - \mathfrak{Q}_p\zeta_{m_1}(\mathfrak{S}_{11})\mathfrak{Q}_p \in \mathcal{A}_{11}$ for any $\mathfrak{S}_{11} \in \mathcal{A}_{11}$. Hence, the lemma is true for $k = m$. Therefore, the lemma is true for all $m \in \mathbb{N}$. For the case when $i = 2$, we take $(\mathfrak{Q}_p + \mathfrak{T}_{22})\mathfrak{Q}_p = \mathfrak{Q}_p$ to obtain $\mathfrak{Q}_q L_m(\mathfrak{T}_{22})\mathfrak{Q}_p + (-1)^{n-1}\mathfrak{Q}_p L_m(\mathfrak{T}_{22})\mathfrak{Q}_q = 0$, and then following similar steps as that for $i = 1$, we find that

$$L_m(\mathfrak{T}_{22}) - \zeta_{m_2}(\mathfrak{T}_{22}) = \mathfrak{Q}_q L_m(\mathfrak{T}_{22})\mathfrak{Q}_q - \mathfrak{Q}_q\zeta_{m_1}(\mathfrak{T}_{22})\mathfrak{Q}_q \in \mathcal{A}_{22}$$

for any $\mathfrak{T}_{22} \in \mathcal{A}_{22}$, which completes the proof of the lemma. □

We now define maps $\phi_m : \mathcal{A} \to \mathcal{A}$ and $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$ by $\phi_m\mathfrak{S} = L_m\mathfrak{S} - \zeta_m\mathfrak{S}$ and $\zeta_m\mathfrak{S} = \zeta_{m_1}(\mathfrak{Q}_p\mathfrak{S}\mathfrak{Q}_p) + \zeta_{m_2}(\mathfrak{Q}_q\mathfrak{S}\mathfrak{Q}_q)$ for all $\mathfrak{S} \in \mathcal{A}$. One can easily observe that $\phi_m(P_i) = 0$, $\phi_m(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}$, $i, j = 1, 2$ *and* $\phi_m(\mathfrak{S}_{ij}) = L_m(\mathfrak{S}_{ij})$ *for all* $\mathfrak{S}_{ij} \in \mathcal{A}_{ij}, 1 \le i \ne j \le 2$.

Lemma 16. *ϕ_m is an additive map.*

Proof. The proof is similar to that of Lemma 9. □

Lemma 17. *For any $\mathfrak{S}_{ii} \in \mathcal{A}_{ii}$, $\mathfrak{S}_{ij}, \mathfrak{T}_{ij} \in \mathcal{A}_{ij}$ and $\mathfrak{T}_{jj} \in \mathcal{A}_{jj}$, $i, j = 1, 2$, $(i \ne j)$, we have,*

(a) $\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ij}) = \sum_{\substack{s+t=m \\ 0 \le s,t \le m}} \phi_s(\mathfrak{S}_{ii})\phi_t(\mathfrak{T}_{ij})$,

(b) $\phi_m(\mathfrak{S}_{ij}\mathfrak{T}_{jj}) = \sum_{\substack{s+t=m \\ 0 \le s,t \le m}} \phi_s(\mathfrak{S}_{ij})\phi_t(\mathfrak{T}_{jj})$.

Proof. (a) We prove it with the help of the principle of mathematical induction. For $m = 1$, it was shown to be true by Ashraf and Jabeen [19]. Suppose that it holds for all $k \le m - 1$. We prove that it is also true for $k = m$. We take the case for $i = 1$ *and* $j = 2$. If $\mathfrak{S}_{11} \in \mathcal{A}_{11}$ is invertible, then for any $\mathcal{W}_{12} \in \mathcal{A}_{12}$, we have $(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1})\mathfrak{S}_{11} = \mathfrak{Q}_p$. Therefore, we have

$$\begin{aligned} &\phi_m(\mathcal{W}_{12}) = L_m\Big(p_n(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q)\Big) \\ &= p_n\Big(L_m(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1}), \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1}, L_m(\mathfrak{S}_{11}), \\ &\mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, L_m(\mathfrak{Q}_p), \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + \cdots \\ &+ p_n\Big(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, L_m(\mathfrak{Q}_q)\Big) + \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0 \le l_1, l_2, \cdots, l_n \le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1}), \\ &L_{l_2}(\mathfrak{S}_{11}), L_{l_3}(\mathfrak{Q}_p), L_{l_4}(\mathfrak{Q}_q), \cdots, L_{l_n}(\mathfrak{Q}_q)\Big) \end{aligned}$$

$$
\begin{aligned}
&= p_n\Big(L_m(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}) + L_m(\mathfrak{S}_{11}^{-1}), \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1}, L_m(\mathfrak{S}_{11}), \mathfrak{Q}_p,\\
&\mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + \cdots + p_n\Big(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, L_m(\mathfrak{Q}_q)\Big)\\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m\\ 0\leq l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}) + L_{l_1}(\mathfrak{S}_{11}^{-1}), L_{l_2}(\mathfrak{S}_{11}), L_{l_3}(\mathfrak{Q}_p), L_{l_4}(\mathfrak{Q}_q), \cdots, L_{l_n}(\mathfrak{Q}_q)\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= p_n\Big(L_m(\mathfrak{S}_{11}^{-1}), \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}^{-1}, L_m(\mathfrak{S}_{11}), \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big)\\
&+ p_n\Big(\mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, L_m(\mathfrak{Q}_p), \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + \cdots + p_n\Big(\mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, L_m(\mathfrak{Q}_q)\Big)\\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m\\ 0\leq l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1}), L_{l_2}(\mathfrak{S}_{11}), L_{l_3}(\mathfrak{Q}_p), L_{l_4}(\mathfrak{Q}_q), \cdots, L_{l_n}(\mathfrak{Q}_q)\Big).
\end{aligned}
$$

since $L_m(\mathfrak{Q}_p), L_m(\mathfrak{Q}_q) \in Z(\mathcal{A})$ and ϕ_m is additive. From the above two equations, we obtain

$$
\begin{aligned}
&\phi_m(\mathcal{W}_{12})\\
&= p_n\Big(L_m(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}), \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}, L_m(\mathfrak{S}_{11}), \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big)\\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m\\ 0\leq l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}), L_{l_2}(\mathfrak{S}_{11}), L_{l_3}(\mathfrak{Q}_p), L_{l_4}(\mathfrak{Q}_q), \cdots, L_{l_n}(\mathfrak{Q}_q)\Big)\\
&= p_n\Big(\phi_m(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}), \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}, \phi_m(\mathfrak{S}_{11}), \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big)\\
&+ \sum_{\substack{s+t=m\\ 0<s,t\leq m-1}} \phi_t(\mathfrak{S}_{11})\phi_s(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12})\\
&= \phi_m(\mathfrak{S}_{11})\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}\phi_m(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}) + \sum_{\substack{s+t=m\\ 0<s,t\leq m-1}} \phi_t(\mathfrak{S}_{11})\phi_s(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}).
\end{aligned}
$$

By replacing $\mathcal{W}_{12}$ with $\mathfrak{S}_{11}\mathfrak{T}_{12}$ in the above equation, we obtain

$$
\begin{aligned}
\phi_m(\mathfrak{S}_{11}\mathfrak{T}_{12}) &= \phi_m(\mathfrak{S}_{11})\mathfrak{T}_{12} + \mathfrak{S}_{11}\phi_m(\mathfrak{T}_{12}) + \sum_{\substack{s+t=m\\ 0<s,t\leq m-1}} \phi_s(\mathfrak{T}_{12})\phi_t(\mathfrak{S}_{11})\\
&= \sum_{\substack{s+t=m\\ 0\leq s,t\leq m}} \phi_t(\mathfrak{S}_{11})\phi_s(\mathfrak{T}_{12}).
\end{aligned}
$$

Now, if $\mathfrak{S}_{11}$ is not invertible in $\mathcal{A}_{11}$, we can find a sufficiently large number, say r, such that $r\mathfrak{Q}_p - \mathfrak{S}_{11}$ is invertible in $\mathcal{A}_{11}$. Then, $\phi_m\big((r\mathfrak{Q}_p - \mathfrak{S}_{11})\mathfrak{T}_{12}\big) = (r\mathfrak{Q}_p - \mathfrak{S}_{11})\phi_m(\mathfrak{T}_{12}) + \phi_m(r\mathfrak{Q}_p - \mathfrak{S}_{11})\mathfrak{T}_{12}$. Since $\mathfrak{Q}_p$ is invertible in $\mathcal{A}_{11}$, from the above equation, we obtain

$$
\phi_m(\mathfrak{S}_{11}\mathfrak{T}_{12}) = \sum_{\substack{s+t=m\\ 0\leq s,t\leq m-1}} \phi_t(\mathfrak{S}_{11})\phi_s(\mathfrak{T}_{12}).
$$

For $i = 2$ *and* $j = 1$, since $(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21})(\mathfrak{Q}_p + \mathfrak{T}_{21}) = \mathfrak{Q}_p$, we have

$$\begin{aligned}
-\phi_m(\mathfrak{T}_{21}) &= L_m\Big(p_n(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}, \mathfrak{Q}_p + \mathfrak{T}_{21}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p)\Big) \\
&= p_n\Big(L_m(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}), \mathfrak{Q}_p + \mathfrak{T}_{21}, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\
&+ p_n\Big(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}, L_m(\mathfrak{Q}_p + \mathfrak{T}_{21}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\
&+ \cdots + p_n\Big(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}, \mathfrak{Q}_p + \mathfrak{T}_{21}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, L_m(\mathfrak{Q}_p)\Big) \\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\le l_1,l_2,\cdots,l_n\le m-1}} p_n\Big(L_{l_1}(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}), L_{l_2}(\mathfrak{Q}_p + \mathfrak{T}_{21}), L_{l_3}(\mathfrak{Q}_p), L_{l_4}(\mathfrak{Q}_p), \\
&\cdots, L_{l_n}(\mathfrak{Q}_p)\Big) \\
&= p_n\Big(L_m(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}), \mathfrak{Q}_p + \mathfrak{T}_{21}, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\
&+ p_n\Big(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}, L_m(\mathfrak{Q}_p + \mathfrak{T}_{21}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\
&+ \sum_{\substack{s+t=m \\ 0<s,t\le m-1}} p_n\Big(L_s(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}), L_t(\mathfrak{Q}_p + \mathfrak{T}_{21}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\
&= p_n\Big(\phi_m(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}), \mathfrak{Q}_p + \mathfrak{T}_{21}, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\
&+ p_n\Big(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}, \phi_m(\mathfrak{Q}_p + \mathfrak{T}_{21}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\
&+ \sum_{\substack{s+t=m \\ 0<s,t\le m-1}} p_n\Big(\phi_s(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}), \phi_t(\mathfrak{Q}_p + \mathfrak{T}_{21}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big).
\end{aligned}$$

Since ϕ_m is additive, the above equation gives

$$\begin{aligned}
&-\phi_m(\mathfrak{T}_{21}) = p_n\Big(\phi_m(\mathfrak{Q}_p) + \phi_m(\mathfrak{S}_{22}) - \phi_m(\mathfrak{S}_{22}\mathfrak{T}_{21}), \mathfrak{Q}_p + \mathfrak{T}_{21}, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\
&+ p_n\Big(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}, \phi_m(\mathfrak{Q}_p) + \phi_m(\mathfrak{T}_{21}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\
&+ \sum_{\substack{s+t=m \\ 0<s,t\le m-1}} p_n\Big(\phi_s(\mathfrak{Q}_p) + \phi_s(\mathfrak{S}_{22}) - \phi_s(\mathfrak{S}_{22}\mathfrak{T}_{21}), \phi_t(\mathfrak{Q}_p) + \phi_t(\mathfrak{T}_{21}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\
&= -\phi_m(\mathfrak{S}_{22}\mathfrak{T}_{21}) + \phi_m(\mathfrak{S}_{22})\mathfrak{T}_{21} + \mathfrak{S}_{22}\phi_m(\mathfrak{T}_{21}) - \phi_m(\mathfrak{T}_{21}) + \sum_{\substack{s+t=m \\ 0<s,t\le m-1}} \phi_s(\mathfrak{S}_{22})\phi_t(\mathfrak{T}_{21}).
\end{aligned}$$

Which follows that

$$\phi_m(\mathfrak{S}_{22}\mathfrak{T}_{21}) = \phi_m(\mathfrak{S}_{22})\mathfrak{T}_{21} + \mathfrak{S}_{22}\phi_m(\mathfrak{T}_{21}) + \sum_{\substack{s+t=m \\ 0<s,t\le m-1}} \phi_s(\mathfrak{S}_{22})\phi_t(\mathfrak{T}_{21})$$

i.e, $\phi_m(\mathfrak{S}_{22}\mathfrak{T}_{21}) = \sum_{\substack{s+t=m \\ 0\le s,t\le m}} \phi_s(\mathfrak{S}_{22})\phi_t(\mathfrak{T}_{21})$ for all $\mathfrak{S}_{22} \in \mathcal{A}_{22}$ and $\mathfrak{T}_{21} \in \mathcal{A}_{21}$.

(b) For $i = 1,\ j = 2$, by considering $(\mathfrak{Q}_p + \mathfrak{S}_{12})(\mathfrak{Q}_p - \mathfrak{T}_{22} + \mathfrak{S}_{12}\mathfrak{T}_{22}) = \mathfrak{Q}_p$ and using the same approach as above, one can easily obtain $\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{22}) = \sum_{\substack{s+t=m \\ 0\le s,t\le m}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{22})$ for all $\mathfrak{S}_{12} \in \mathcal{A}_{12}$ and $\mathfrak{T}_{22} \in \mathcal{A}_{22}$, and for the case when $i = 2,\ j = 1$, by considering $\mathfrak{S}_{11}(\mathcal{W}_{21}\mathfrak{S}_{11}^{-1} + \mathfrak{S}_{11}^{-1}) = \mathfrak{Q}_p$, we can easily prove that $\phi_m(\mathfrak{S}_{21}\mathfrak{T}_{11}) = \sum_{\substack{s+t=m \\ 0\le s,t\le m}} \phi_s(\mathfrak{S}_{21})\phi_t(\mathfrak{T}_{11})$ for all $\mathfrak{S}_{21} \in \mathcal{A}_{21}$ *and* $\mathfrak{T}_{11} \in \mathcal{A}_{11}$. □

Lemma 18. *For any* $\mathfrak{S}_{ii}$, $\mathfrak{T}_{ii} \in \mathcal{A}_{ii}$, *we have* $\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii}) = \phi_m(\mathfrak{S}_{ii})\mathfrak{T}_{ii} + \mathfrak{S}_{ii}\phi_m(\mathfrak{T}_{ii}),\ i = 1, 2$.

Proof. The proof of this lemma is same as that of Lemma 11. □

Lemma 19. *For any $\mathfrak{S}_{ij} \in \mathcal{A}_{ij}$, $\mathfrak{T}_{ji} \in \mathcal{A}_{ji}$, we have $\phi_m(\mathfrak{S}_{ij}\mathfrak{T}_{ji}) = \phi_m(\mathfrak{S}_{ij})\mathfrak{T}_{ji} + \mathfrak{S}_{ij}\phi_m(\mathfrak{T}_{ji}); 1 \leq i \neq j \leq 2$.*

Proof. We prove the lemma with the help of the principle of mathematical induction. For $m = 1$, it was shown to be true by Ashraf and Jabeen [19]. Suppose that the lemma holds for all $k \leq m-1$. We prove that it is true for $k = m$. Take any $\mathfrak{S}_{12} \in \mathcal{A}_{12}$ since $(\mathfrak{S}_{12} + \mathfrak{Q}_p)\mathfrak{Q}_p = \mathfrak{Q}_p$. Then,

$$\begin{aligned}
& L_m\Big(p_n(\mathfrak{S}_{12} + \mathfrak{Q}_p, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21})\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{12} + \mathfrak{Q}_p), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21}\Big) + p_n\Big(\mathfrak{S}_{12} + \mathfrak{Q}_p, L_m(\mathfrak{Q}_p), \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21}\Big) \\
&+ \cdots + p_n\Big(\mathfrak{S}_{12} + \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, L_m(\mathfrak{Q}_p), \mathfrak{T}_{21}\Big) + p_n\Big(\mathfrak{S}_{12} + \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, L_m(\mathfrak{T}_{21})\Big) \\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\leq l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{12} + \mathfrak{Q}_p), L_{l_2}(\mathfrak{Q}_p), L_{l_3}(\mathfrak{Q}_p), \cdots, L_{l_{n-1}}(\mathfrak{Q}_p), L_{l_n}(\mathfrak{T}_{21})\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{12} + \mathfrak{Q}_p), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21}\Big) + p_n\Big(\mathfrak{S}_{12} + \mathfrak{Q}_p, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, L_m(\mathfrak{T}_{21})\Big) \\
&+ \sum_{\substack{s+t=m \\ 0<s,t\leq m-1}} L_s(\mathfrak{S}_{12} + \mathfrak{Q}_p) L_t(\mathfrak{T}_{21}) \\
&= p_n\Big(\phi_m(\mathfrak{S}_{12} + \mathfrak{Q}_p), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21}\Big) + p_n\Big(\mathfrak{S}_{12} + \mathfrak{Q}_p, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \phi_m(\mathfrak{T}_{21})\Big) \\
&+ \sum_{\substack{s+t=m \\ 0<s,t\leq m-1}} \phi_s(\mathfrak{S}_{12} + \mathfrak{Q}_p) \phi_t(\mathfrak{T}_{21}) \\
&= p_n\Big(\phi_m(\mathfrak{S}_{12}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21}\Big) + p_n\Big(\mathfrak{S}_{12}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \phi_m(\mathfrak{T}_{21})\Big) \\
&+ \sum_{\substack{s+t=m \\ 0<s,t\leq m-1}} \phi_s(\mathfrak{S}_{12}) \phi_t(\mathfrak{T}_{21}).
\end{aligned}$$

From this, we obtain $L_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) = \phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21} + \mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}) - \phi_m(\mathfrak{T}_{21})\mathfrak{S}_{12} - \mathfrak{T}_{21}\phi_m(\mathfrak{S}_{12}) + \sum_{\substack{s+t=m \\ 0<s,t\leq m-1}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21})$ since $\phi_m(\mathfrak{S}) = L_m(\mathfrak{S}) - \zeta_{m_1}(\mathfrak{Q}_p\mathfrak{S}\mathfrak{Q}_p) - \zeta_{m_2}(\mathfrak{Q}_q\mathfrak{S}\mathfrak{Q}_q)$ for all $\mathfrak{S} \in \mathcal{A}$. We have

$$\begin{aligned}
& \phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) + \zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) \\
&= \phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21} + \mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}) - \phi_m(\mathfrak{T}_{21})\mathfrak{S}_{12} - \mathfrak{T}_{21}\phi_m(\mathfrak{S}_{12}) \\
&+ \sum_{\substack{s+t=m \\ 0<s,t\leq m-1}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21}).
\end{aligned}$$

by using Lemma 12 and applying similar steps to obtain $\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) = 0$. Then, from the above relation, we obtain

$$\begin{aligned}
\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21}) &= \phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21} + \mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}) + \sum_{\substack{s+t=m \\ 0<s,t\leq m-1}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21}) \\
&= \sum_{\substack{s+t=m \\ 0\leq s,t\leq m}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21})
\end{aligned}$$

and

$$\begin{aligned}
\phi_m(\mathfrak{T}_{21}\mathfrak{S}_{12}) &= \phi_m(\mathfrak{T}_{21})\mathfrak{S}_{12} + \mathfrak{T}_{21}\phi_m(\mathfrak{S}_{12}) + \sum_{\substack{s+t=m \\ 0<s,t\leq m-1}} \phi_s(\mathfrak{T}_{21})\phi_t(\mathfrak{S}_{12}) \\
&= \sum_{\substack{s+t=m \\ 0\leq s,t\leq m}} \phi_s(\mathfrak{T}_{21})\phi_t(\mathfrak{S}_{12})
\end{aligned}$$

for all $\mathfrak{S}_{12} \in \mathcal{A}_{12}$, $\mathfrak{T}_{21} \in \mathcal{A}_{12}$. This shows that the lemma is true for all $m \in \mathbb{N}$. $\square$

Proof of Theorem 2. The proof is similar to that of Theorem 1 and by using Lemmas 13–19 instead of Lemmas 10–12. $\square$

As a direct consequence of Theorem 2, we have the following corollary:

Corollary 3. *Let $\mathcal{A}$ be a von Neumann algebra with no central summands of type I_1 and a linear map $L_m : \mathcal{A} \to \mathcal{A}$ satisfying*

$$L_m\big(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)\big) = \sum_{l_1+l_2+\cdots+l_n=m} p_n\big(L_{l_1}(\mathfrak{S}_1), L_{l_2}(\mathfrak{S}_2), \cdots, L_{l_n}(\mathfrak{S}_n)\big)$$

for all $\mathfrak{S}_i \in \mathcal{A}$; $1 \leq i \leq n$, with $\mathfrak{S}_1\mathfrak{S}_2 = \mathfrak{Q}_p$, where $\mathfrak{Q}_p$ is a core-free projection with the central carrier I. Then, there exists an operator $\mathcal{T} \in \mathcal{A}$ and a linear map $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$, which annihilates every $(n-1)^{th}$-commutator $p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)$ with $\mathfrak{S}_1\mathfrak{S}_2 = \mathfrak{Q}_p$ such that $L_m(\mathfrak{S}) = \mathfrak{S}\mathcal{T} - \mathcal{T}\mathfrak{S} + \zeta_m(\mathfrak{S})$ for all $\mathfrak{S} \in \mathcal{A}$.

3. Conclusions

In the present paper, firstly, we studied the action of Lie-type higher derivations of von Neumann algebras and described their structures. Precisely, we established that every additive Lie-type higher derivation of von Neumann algebras has a standard form at zero products as well as the projection products; that is, every additive map L_m from von Neumann algebra $\mathcal{A}$ into itself can be written as $L_m(\mathfrak{S}) = \phi_m(\mathfrak{S}) + \zeta_m(\mathfrak{S})$ *for all* $\mathfrak{S} \in \mathcal{A}$, where $\phi_m : \mathcal{A} \to \mathcal{A}$ is an additive higher derivation and $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$ is an additive higher map, which annihilates every $(n-1)^{th}$-commutator $p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)$ with $\mathfrak{S}_1\mathfrak{S}_2 = 0$. Secondly, we characterized Lie derivations on general von Neumann algebras with no central summands of type I_1 by using the actions at projection products. Finally, we described the structures of linear maps via core-free projections with the central carrier.

In view of Ferreira et al. [16], these results are still open for alternative rings, where one can study and characterize the Lie-type higher derivations of alternative rings.

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