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The Well-Posed Identification of the Interface Heat Transfer Coefficient Using an Inverse Heat Conduction Model

Sergey Pyatkov [1,2,*] and Alexey Potapkov [1]

[1] Engineering School of Digital Technologies, Yugra State University, Chekhov St. 16, 628007 Khanty-Mansiysk, Russia; a_potapkov@ugrasu.ru
[2] Academy of Sciences of the Republic of Sakha (Yakutia), 33 Lenin Ave., 677007 Yakutsk, Russia
* Correspondence: s_pyatkov@ugrasu.ru

Abstract: In this study, the inverse problems of recovering the heat transfer coefficient at the interface of integral measurements are considered. The heat transfer coefficient occurs in the transmission conditions of an imperfect contact type. This is representable as a finite part of the Fourier series with time-dependent coefficients. The additional measurements are integrals of a solution multiplied by some weights. The existence and uniqueness of solutions in Sobolev classes are proven and the conditions on the data are sharp. These conditions include smoothness and consistency conditions on the data and additional conditions on the kernels of the integral operators used in the additional measurements. The proof relies on a priori bounds and the contraction mapping principle. The existence and uniqueness theorem is local in terms of time.

Keywords: inverse problem; heat transfer coefficient; convection–diffusion equation; heat and mass transfer; integral measurements

MSC: 35R30; 35K20; 80A20

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1. Introduction

Under consideration is a parabolic equation of the following form:

$$Mu = u_t - Lu = f,\ Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(t,x) u_{x_j} - \sum_{i=1}^{n} a_i(t,x) u_{x_i} - a_0(t,x)u,\ x \in G, \quad (1)$$

where $G \subset \mathbb{R}^n$ is a bounded domain with boundary Γ of class C^2 (see the definitions in [1] (Chapter 1)), $t \in (0,T)$. Let $Q = (0,T) \times G$, $S = (0,T) \times \Gamma$.

This equation is a vital tool in scientific and engineering applications in assessing and forecasting temperature changes over time. According to Animasaun I. L. et al. (2022) [2], it is commonly used to model heat conduction, diffusion, and numerous dynamic thermal processes. The problems of identifying the interface heat transfer coefficients arise in various problems of mathematical physics (see [3–6]): in the diagnostics and identification of heat transfer in supersonic heterogeneous flows, in the modeling and description of heat transfer in heat-shielding materials and coatings, in thermal protection design and the control of heat transfer regimes, in the modeling of properties and thermal processes in the reusable thermal protection of aerospace vehicles, in composite materials, in ecology, etc.

The statement of the problem is as follows. The domain G is divided into two open sets G^+ and G^-, $\overline{G^-} \subset G$, $\overline{G^+} \cup \overline{G^-} = \overline{G}$, $G^+ \cap G^- = \emptyset$. Let $\Gamma_0 = \partial G^+ \cap \partial G^-$, $S_0 = \Gamma_0 \times (0,T)$. Equation (1) is supplemented with the initial and boundary conditions

$$B(t,x)u|_S = g,\ \ u|_{t=0} = u_0(x), \quad (2)$$

where $Bu = \frac{\partial u}{\partial N} + \beta u$ or $Bu = u$, $\frac{\partial u}{\partial N} = \sum_{i,j=1}^{n} a_{ij}(t,x) u_{x_j}(t,x) n_i$, with $\vec{n} = (n_1, n_2, \ldots, n_n)$ the outward unit normal to S, and the transmission conditions

$$\frac{\partial u^+}{\partial N}(t,x) - \sigma(t,x)(u^+(t,x) - u^-(t,x)) = g^+(t,x), \ \ (t,x) \in S_0, \tag{3}$$

$$\frac{\partial u^-}{\partial N}(t,x) = \frac{\partial u^+}{\partial N}(t,x), \ \ (t,x) \in S_0, \tag{4}$$

where $\frac{\partial u^\pm}{\partial N}(t,x_0) = \lim_{x\in G^\pm,\, x\to x_0\in\Gamma_0} \sum_{i,j=1}^n a_{ij}u_{x_i}\nu_j$, $u^\pm(t,x_0) = \lim_{x\in G^\pm,\, x\to x_0\in\Gamma_0} u(t,x)$, and ν is the unit outward normal to ∂G^-. The inverse problem is to determine a solution u to the problem (1)–(4) and the function $\sigma = \sum_{i=1}^m q_i(t)\Phi_i(t,x)$, where the functions q_i are unknowns and $\{\Phi_i(t,x)\}$ are some basis functions. It is natural to assume that they depend only on x but, for the sake of generality, we understand them as depending on all variables. The additional integral measurements are as follows:

$$\int_G u(t,x)\varphi_k(x)dx = \psi_k(t), \ k = 1,2,\ldots,m. \tag{5}$$

The transmission conditions (3) and (4) agree with the conventional imperfect contact condition at the interface (see [5]). The coefficient σ is called the heat transfer coefficient. If $\sigma \to \infty$ then we come to the diffraction problem (see [1] (Chapter 3, Section 16)) in which $u^+ = u^-$ and $\frac{\partial u^+}{\partial N} = \frac{\partial u^-}{\partial N}$ on S_0.

At present, there are many publications on the numerical solutions of the problems of the type (1)–(5) in the various statements. The most usable statement provides the pointwise additional measurements; in this case, the condition (5) is replaced with the conditions $u(t,b_j) = \psi_j(t)$ $(j = 1,2,\ldots,m,\ b_j \in G)$. This is often the case when the coefficient σ depends only on time [6–9] or space variables [10–13] (see, also, the bibliography and the results in [14–17]). In almost all papers, the problem is reduced to some optimal control problem and the minimization of the corresponding quadratic functional (see [6–8,10,11,14,15]). Let us describe some of the previously addressed problems. In the case of a sole space or time variable, the heat transfer coefficient depending on the temperature is recovered numerically with the use of pointwise measurements in [6]. In [14], the authors determine the heat transfer coefficients that depend, in a special manner, on the additional parameters from a collection of values of a solution at given points. In [10,16], the Monte Carlo method is employed to restore the heat transfer coefficient depending on two space variables. The values of a solution on the part of the boundary serve as the overdetermination conditions. The simultaneous recovering of a coefficient in a parabolic equation and the heat transfer coefficient is realized in [7]. The pointwise overdetermination conditions are also used in [15,17]. In [17], the problem under consideration is a one-dimensional inverse problem of simultaneously recovering the heat flow on one of the lateral boundaries and the thermal contact resistance at the interface. The authors of reference [11] implement the numerical determination of the heat transfer coefficient from measurements on the available part of the outer boundary of the domain.

Some existing results are known if the pointwise ovedetermination conditions are used instead of those in (5). If the measurement points lie on the boundary of the domain and the heat transfer coefficient occurring in the boundary condition is determined, then the existence and uniqueness theorems can be found in [18–20]. The same results were obtained if the measurement points lie at the interface. The inverse problem of determining the interface heat transfer coefficient under certain conditions is well-posed and the most general existence and uniqueness theorems can be found in [21,22]. If the measurement points lie in G then the problem becomes ill-posed. The conditions (5) were used in [23,24] to determine the heat flux on the outer boundary and the existence and uniqueness theorems were proven. It is often the case when the integrals in (5) are taken over the boundary of a domain [25,26] and the heat transfer coefficient, depending on time or space variables, is determined. In these articles, the problem is reduced to a control problem, which is studied theoretically, and some existence theories are presented. But these control problems are not equivalent to the initial ones.

As for the problem (1)–(5), there are no theoretical results on the solvability or uniqueness of the solutions to this problem in the literature. In contrast to other articles, we look for the heat transfer coefficient in the form of a finite segment of the Fourier series and this statement allows us to obtain an approximation of the heat transfer coefficient depending on all variables, and the accuracy of determination depends on just a number of measurements. Hence, the statement of the problem is novel. Note that the integral conditions are often employed as an approximation to the pointwise overdetermination conditions and are of interest in their own right. In the present article, we study the well-posed questions for the problem (1)–(5) and establish existence and uniqueness theorems for solutions to this problem locally in terms of time.

2. Preliminaries

The Lebesgue spaces $L_p(G;E)$ and the Sobolev spaces $W_p^s(G;E)$, $W_p^s(Q;E)$ of vector-valued functions taking the values in a Banach space E (see the definitions in [27,28]) are used in the article. The Sobolev spaces are denoted by $W_p^s(G)$, $W_p^s(Q)$, etc., whenever $E=\mathbb{R}^n$. The inclusion $u=(u_1,u_2,\ldots,u_k)$ for a vector function means that every component u_i of u belongs to $W_p^s(G)$. By a norm of a vector, we mean the sum of the norms of its coordinates. The Hölder spaces $C^\alpha(\overline{G})$, $C^{\alpha,\beta}(\overline{Q}), C^{\alpha,\beta}(\overline{S})$ are defined in [1] (see, also, [27]). Given an interval $J=(0,T)$, put $W_p^{s,r}(Q)=W_p^s(J;L_p(G))\cap L_p(J;W_p^r(G)$ and $W_p^{s,r}(S)=W_p^s(J;L_p(\Gamma))\cap L_p(J;W_p^r(\Gamma))$. All coefficients of L are real, as well as the corresponding function spaces.

To simplify the exposition, we suppose below that $p>n+2$. Denote $(u,v)=\int_G u(x)v(x)dx$. Introduce the notations $Q^\tau=(0,\tau)\times G$, $S^\tau=(0,\tau)\times\Gamma$, $S_0^\tau=(0,\tau)\times\Gamma_0$, $Q^\pm=(0,T)\times G^\pm$, $Q_\pm^\tau=(0,\tau)\times G^\pm$. Let $B_\delta(b)$ be a ball centered at b of radius δ. The symbol $\rho(X,M)$ stands for the distance between the sets $X,M\subset\mathbb{R}^n$.

Endow the space $W_p^s(0,\beta;E)$ ($s\in(0,1)$, $\beta>0$, E—is a Banach space) with the norm $\|q(t)\|_{W_p^s(0,\beta;E)}=(\|q\|^p_{L_p(0,\beta;E)}+<q>^p_{s,\beta})^{1/p}$, $<q>^p_{s,\beta}=\int_0^\beta\int_0^\beta\frac{\|q(t_1)-q(t_2)\|_E^p}{|t_1-t_2|^{1+sp}}dt_1dt_2$. This space agrees with the space $W_p^s(0,\beta)$ whenever $E=\mathbb{R}$. Given $s\in(0,1)$, put $\tilde{W}_p^s(0,\beta;E)=\{q\in W_p^s(0,\beta;E):\ t^{-s}q(t)\in L_p(0,\beta;E)\}$. The following norm is used in this space: $\|q(t)\|^p_{\tilde{W}_p^s(0,\beta;E)}=\|\frac{q}{t^s}\|^p_{L_p(0,\beta;E)}+<q>^p_{s,\beta}$. If $s>1/p$ and $q\in\tilde{W}_p^s(0,\beta;E)$ then $q(0)=0$ and this norm and the usual norm $\|\cdot\|_{W_p^s(\alpha,\beta;E)}$ are equivalent for functions $q(t)$, such that $q(0)=0$ (see [27] (Subsection 3.2.6, Lemma 1)). The spaces $\tilde{W}_p^s(0,\beta;L_p(G))$ and $\tilde{W}_p^{s,2s}(Q^\beta)=\tilde{W}_p^s(0,\beta;L_p(G))\cap L_p(0,\beta;W_p^{2s}(G))$ for $s\neq 1/p$ comprise functions $v(t,x)$ in $W_p^s(0,\beta;L_p(G))$ and in $W_p^{s,2s}(Q^\beta)$, respectively, such that $v(0,x)=0$ for $s>1/p$. The norms $\|\cdot\|_{\tilde{W}_p^{s,2s}(Q^\beta)}$, $\|\cdot\|_{\tilde{W}_p^s(0,\beta;L_p(G))}$ are defined naturally, for example, $\|u\|_{\tilde{W}_p^{s,2s}(Q^\beta)}=\left(\|u\|^p_{\tilde{W}_p^s(0,\beta;L_p(G))}+\|u\|^p_{L_p(0,\beta;W_p^{2s}(G))}\right)^{1/p}$. Similar definitions are employed for the norms in $\tilde{W}_p^s(0,\beta;L_p(\Gamma))$, $\tilde{W}_p^{s,2s}(S^\beta)$. The following lemmas are known (see [29] (Lemmas 1–4)).

Lemma 1. *Let G be a bounded domain with boundary Γ of the class C^2 and $Q^\tau=(0,\tau)\times G$, $S^\tau=(0,\tau)\times\Gamma$. There exists a constant C independent of $\tau\in(0,T]$, such that*

$$\|v\|_{\tilde{W}_p^{s_1,2s_1}(S^\tau)}\le C\|v\|_{W_p^{1,2}(Q^\tau)},\ s_1=1-1/2p,$$
$$\|\frac{\partial v}{\partial\nu}\|_{\tilde{W}_p^{s_0,2s_0}(S^\tau)}\le C\|v\|_{W_p^{1,2}(Q^\tau)},\ s_0=1/2-1/2p,$$

for all $v\in W_p^{1,2}(Q^\tau)$, such that $v(x,0)=0$. Here, $\frac{\partial v}{\partial\nu}$ is the outward unit normal to Γ.

Lemma 2. *Assume that $s\in((n+2)/2p,1)$. The product $q\cdot v$ of functions in $W_p^{s,2s}(Q^\tau)$ ($\tau\in(0,T]$) is contained in $W_p^{s,2s}(Q^\tau)$. If $q\in\tilde{W}_p^{s,2s}(Q^\tau)$ and $v\in W_p^{s,2s}(Q^\tau)$, then $qv\in\tilde{W}_p^{s,2s}(Q^\tau)$ and*

$$\|qv\|_{\tilde{W}_p^{s,2s}(Q^\tau)}\le c_0\|q\|_{\tilde{W}_p^{s,2s}(Q^\tau)}(\|v\|_{W_p^{s,2s}(Q^\tau)}+\|v\|_{L_\infty(Q^\tau)}).$$

If $v\in W_p^{s,2s}(Q)$, then the last inequality can be written as

$$\|qv\|_{\tilde{W}_p^{s,2s}(Q^\tau)} \leq c_1 \|q\|_{\tilde{W}_p^{s,2s}(Q^\tau)} \|v\|_{W_p^{s,2s}(Q)},$$

and if $v \in \tilde{W}_p^{s,2s}(Q^\tau)$, then

$$\|qv\|_{\tilde{W}_p^{s,2s}(Q^\tau)} \leq c_2 \|q\|_{\tilde{W}_p^{s,2s}(Q^\tau)} \|v\|_{\tilde{W}_p^{s,2s}(Q^\tau)},$$

where the constants c_i, $i = 0, 1, 2$ are independent of q, v and $\tau \in (0, T]$. If $|v(t)|$ is strictly positive in Q^τ, i.e., $\delta_0 = \inf_{(t,x) \in Q^\tau} |v(t,x)| > 0$, then the ratio q/v of functions in $W_p^{s,2s}(Q^\tau)$ ($\tau \in (0, T]$) belongs to $W_p^{s,2s}(Q^\tau)$ and if $q \in \tilde{W}_p^{s,2s}(Q^\tau)$ and $v \in W_p^{s,2s}(Q^\tau)$, then $q/v \in \tilde{W}_p^{s,2s}(Q^\tau)$ and

$$\|q/v\|_{\tilde{W}_p^{s}(0,\tau)} \leq c_0 \|q\|_{\tilde{W}_p^{s,2s}(Q^\tau)} (\|v\|_{W_p^{s,2s}(Q^\tau)} + \|v\|_{L_\infty(Q^\tau)}),$$

$$\|q/v\|_{\tilde{W}_p^{s,2s}(Q^\tau)} \leq c_0 \|q\|_{\tilde{W}_p^{s,2s}(Q^\tau)} \|v\|_{W_p^{s,2s}(Q)},$$

where the constant c_0 is independent of q and τ. The set Q^τ can be replaced with S^τ in the above inequalities and $s \in ((n+1)/2p, 1)$ in this case. In the case of a function q depending only on one variable t, the norm of q in $\tilde{W}_p^{s,2s}(Q^\tau)$ in the above inequalities is replaced with the norm of q in $\tilde{W}_p^s(0, \tau)$.

Consider the following auxiliary transmission problems:

$$Mu = f(t,x),\ (t,x) \in Q,\ Bu|_S = g,\ u|_{t=0} = u_0, \tag{6}$$

$$B^+u = \frac{\partial u^+}{\partial N} - \tilde{\sigma}(u^+ - u^-) = g^+,\ \frac{\partial u^+}{\partial N} = \frac{\partial u^-}{\partial N} + g^-,\ (t,x) \in S_0. \tag{7}$$

Describe the conditions on the data ensuring the solvability of the problem (6) and (7). Proceed with the conditions on the data. The operator L is elliptic, i.e., there exists a constant $\delta_0 > 0$ such that

$$\textstyle\sum_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j \geq \delta_0 |\xi|^2\ \forall \xi \in \mathbb{R}^n,\ \forall (t,x) \in Q. \tag{8}$$

The conditions on the coefficients are as follows:

$$a_i \in L_p(Q)\ (i \geq 0),\ a_{ij} \in C(Q^\pm)\ (\ i,j = 1,\dots,n); \tag{9}$$

the function $a_{ij}|_{Q^\pm}$ admits extensions to continuous functions of class $C(\overline{Q^\pm})$ and

$$a_{ij}^\pm|_{S_0} \in W_p^{s_0,2s_0}(S_0),\ a_{ij}|_{Q^\pm} \in C([0,T]; W_p^1(G^\pm)),\ a_{ij}|_S \in W_p^{s_0,2s_0}(S), \tag{10}$$

where $i,j = 1,\dots,n$, $a_{ij}^\pm(t,x_0) = \lim_{x \in G^\pm,\ x \to x_0 \in \Gamma_0} a_{ij}(t,x)$, and the last inclusion in (10) is fulfilled provided that $Bu \neq u$ in (2)

$$a_{ij}, a_k \in L_\infty(G; W_p^{s_0}(0,T))\ (k = 0,1,\dots,n,\ i,j = 1,\dots,n). \tag{11}$$

The main conditions on the data are of the following form:

$$f \in L_p(Q),\ \ u_0(x) \in W_p^{2-2/p}(G^\pm),\ g \in W_p^{k_0,2k_0}(S),\ g^\pm \in W_p^{s_0,2s_0}(S_0), \tag{12}$$

where $k_0 = 1 - 1/2p$ in the case of $Bu = u$ and $k_0 = 1/2 - 1/2p$, otherwise,

$$\beta \in W_p^{s_0,2s_0}(S),\ g(0,x)|_\Gamma = B(0,x)u_0|_\Gamma,\ \frac{\partial u_0^+}{\partial N} = \frac{\partial u_0^-}{\partial N} + g^-(0,x),\ x \in \Gamma_0, \tag{13}$$

$$\tilde{\sigma} \in W_p^{s_0,2s_0}(S),\ \frac{\partial u_0^+}{\partial N} = \tilde{\sigma}(u_0^+ - u_0^-) + g^+(0,x). \tag{14}$$

The following theorem is a consequence of Theorem 1 in [21].

Theorem 1. *Let the conditions* (8)–(14) *hold and let* $\Gamma, \Gamma_0 \in C^2$. *Then, there exists a unique solution* $u|_{Q^\pm} \in W_p^{1,2}(Q^\pm)$ *to the problems* (6) and (7), *satisfying the estimate*

$$\|u\|_{W_p^{1,2}(Q^+)} + \|u\|_{W_p^{1,2}(Q^-)} \leq C_0(\|u_0\|_{W_p^{2-2/p}(G^+)} + \|u_0\|_{W_p^{2-2/p}(G^-)} + \\ \|f\|_{L_p(Q)} + \|g^+\|_{W_p^{s_0,2s_0}(S_0)} + \|g^-\|_{W_p^{s_0,2s_0}(S_0)} + \|g\|_{W_p^{k_0,2k_0}(S)}).$$

If $u_0 \equiv 0$, *then, for every* $\tau \in (0, T]$, *there exists a unique solution* $u \in W_p^{1,2}(Q_+^\tau) \cap W_p^{1,2}(Q_-^\tau)$ *to the problems* (6) and (7), *satisfying the estimate*

$$\|u\|_{W_p^{1,2}(Q_+^\tau)} + \|u\|_{W_p^{1,2}(Q_-^\tau)} \leq \\ C_1(\|f\|_{L_p(Q^\tau)} + \|g^+\|_{\tilde{W}_p^{s_0,2s_0}(S_0^\tau)} + \|g^-\|_{\tilde{W}_p^{s_0,2s_0}(S_0^\tau)} + \|g\|_{\tilde{W}_p^{k_0,2k_0}(S^\tau)}).$$

The constants C_0, C_1 *are independent of* τ.

Proof. Let $\tau = T$. The claim of the theorem agrees with that in Theorem 1 in [21] for our case (see, also, Theorem 3 of [29]). Our problem is a particular case of that considered in this theorem. However, there are some distinctions in the conditions on the coefficients $a_{ij}, a_{ij}^\pm$. They belong to some Hölder class on S, S_0, respectively, in [21]. However, the results remain valid if the conditions (10) are used instead of these conditions and the proof of this theorem does not change whenever we use the reference to the results in [28] (Theorem 2.1) on solvability of parabolic problems instead of the reference to the classical results in [1]. The conditions on coefficients of the boundary operators are stated in [28] (Theorem 2.1) in terms of the Sobolev spaces. Thus, the only distinction in the proofs is that one reference is replaced with another and this fact allows us to say that an analog of Theorem 1 in [21] for our case is valid. □

3. Existence and Uniqueness Theorems

The following additional conditions are used in what follows:

$$\varphi_k|_{G^\pm} \in W_\infty^1(G^\pm), \Phi_k \in W_p^{s_0,2s_0}(S_0),\ \psi_k \in W_p^{s_0+1}(0,T),\ (f, \varphi_k) \in W_p^{s_0}(0,T), \tag{15}$$

where $k = 1, 2, \ldots, m$. Assume that a pair $(u, \vec{q})$, $\vec{q} = (q_1, q_2, \ldots, q_m)$ is a solution to the problem (1)–(5). Multiply (1) by φ_i and integrate over G. Integrating by parts and using the transmission conditions, we infer

$$\psi_i'(t) + \sum_{j=1}^m q_j(t) \int_{\Gamma_0} \Phi_j(t,x)(u^+(t,x) - u^-(t,x))(\varphi_i^+(x) - \varphi_i^-(x))d\Gamma_0 - \int_\Gamma \frac{\partial u}{\partial N}\varphi_i(x)\,d\Gamma \\ + \int_{\Gamma_0} g^+(\varphi_i^+(x) - \varphi_i^-(x))\,d\Gamma + a(u, \varphi_i) = (f, \varphi_i) = \int_G f(t,x)\varphi_i(x)\,dx.$$

$$a(u, \varphi_i)(t) = \sum_{k,l=1}^n \int_G a_{kl} u_{x_l} \varphi_{ix_k}(x)\,dG + \int_G (\sum_{k=1}^n a_k u_{x_k} + a_0 u)\varphi_i(x)\,dG, \tag{16}$$

where $\varphi_k^\pm(x_0) = \lim_{x\to x_0, x\in G^\pm} \varphi_k(x)$. Define the function $\varphi_i^0(x) = \varphi_i^+(x) - \varphi_i^-(x)$ $(x \in \Gamma_0)$. We would like to have it so that the system (16) is uniquely solvable relative to the functions $\vec{q}$, i.e., $|\det B(t)| \geq \delta_0 > 0$ $\forall t \in [0, T]$, where $B(t)$ is the matrix with entries $\int_\Gamma \Phi_j(t,x)(u^+(t,x) - u^-(t,x))(\varphi_i^+(x) - \varphi_i^-(x))\,d\Gamma$. Taking $t = 0$, we obtain the condition

$$|\det B_0| \neq 0\ \ B_0 = B(0),\ b_{ij} = \int_\Gamma \Phi_j(0,x)(u_0^+(x) - u_0^-(x))(\varphi_i^+(x) - \varphi_i^-(x))\,d\Gamma. \tag{17}$$

Let $t = 0$ in (16). We arrive at the system

$$\psi_i'(0) + \sum_{j=1}^{m} q_j(0) \int\limits_{\Gamma_0} \Phi_j(0,x)(u_0^+(x) - u_0^-(x))\varphi_i^0(x)\, d\Gamma_0 - \int\limits_{\Gamma} \frac{\partial u_0}{\partial N}\varphi_i(x)\, d\Gamma + \int\limits_{\Gamma_0} g^+(0,x)\varphi_i^0(x)\, d\Gamma + a(u_0,\varphi_i) = (f(0,x),\varphi_i). \quad (18)$$

where $i = 1,2,\ldots,m$. Under these conditions (17), there exists a unique solution $(q_1(0),\ldots,q_m(0))$ to the system (18). Thus, we have determined the function $\sigma(0,x) = \sum_{i=1}^{m} q_i(0)\Phi_i(0,x)$. Taking $t = 0$ at (3) and (5) and using the initial conditions (2), we come to the necessary consistency conditions

$$\frac{\partial u_0^+}{\partial N} - \sigma(0,x)(u_0^+ - u_0^-)|_\Gamma = g^+(0,x)\ (x \in \Gamma),\ \int_G u_0(x)\varphi_k(x)\, dx = \psi_k(0), \quad (19)$$

where $k = 1,\ldots,m$. The main result of the article is the following theorem.

Theorem 2. *Let the conditions (8)–(13), (15), (17) and (19) hold. Then, on some segment $[0,\tau_0]$ $(\tau_0 \le T)$, there exists a unique solution $(u,\vec{q})$ $(\vec{q} = (q_1,\ldots,q_m))$ to the problem (1)–(5), such that $u|_{Q^\pm} \in W_p^{1,2}(Q_\pm^{\tau_0})$, $\vec{q} \in W_p^{s_0}(0,\tau_0)$.*

Proof. Let a pair $u \in W_p^{1,2}(Q^+) \cap W_p^{1,2}(Q^-)$, $\vec{q} \in W_p^{s_0}(0,T)$ be a solution to the problem (1)–(5). As before, we can find constants $q_i(0)$. Let $\sum_{i=1}^{m} q_i(0)\Phi_i(t,x) = \sigma_0(t,x)$ and denote by $v \in W_p^{1,2}(Q^+) \cap W_p^{1,2}(Q^-)$ a solution to the problems (6) and (7) (see Theorem 1) with σ_0 rather than $\tilde{\sigma}$ and $g^- = 0$. Make the change of variables $u = v + w$. Inserting this function u in (1) and involving the equation (6), we obtain that the function $w \in W_p^{1,2}(Q^+) \cap W_p^{1,2}(Q^-)$ is a solution to the problem

$$w_t - Lw = 0,\ \ Bw|_\Gamma = 0,\ \frac{\partial w^+}{\partial N} = \frac{\partial w^-}{\partial N},\ w|_{t=0} = 0,$$
$$\frac{\partial w^+}{\partial N} - \sigma_0(w^+ - w^-) = (\sigma - \sigma_0)(v^+ + w^+ - v^- - w^-). \quad (20)$$

The condition (5) is rewritten as follows:

$$\int_G w\varphi_k(x)\, dx = \psi_k - \int_G v(t,x)\varphi_k(x)\, dx = \tilde{\psi}_k,\ k = 1,2,\ldots,m. \quad (21)$$

In view of (15) and (19), $\tilde{\psi}_k(0) = 0$ and $\tilde{\psi}_k(t) \in W_p^1(0,T)$. Below, we demonstrate that $\int_G v_t(t,x)\varphi_k(x)\, dx \in W_p^{s_0}(0,T)$ and, thus, $\tilde{\psi}_k(t) \in W_p^{1+s_0}(0,T)$. Multiply the equation in (20) by $\varphi_k(x)$ and integrate over G. Integrating by parts, we infer

$$\tilde{\psi}_i'(t) + \sum_{j=1}^{m} \tilde{q}_j(t) \int_{\Gamma_0} \Phi_j(t,x)(w^+(t,x) - w^-(t,x) + v^+(t,x) - v^-(t,x))\varphi_i^0(x)\, d\Gamma_0$$
$$- \int_\Gamma \frac{\partial w}{\partial N}\varphi_i(x)\, d\Gamma + \int_{\Gamma_0} \sigma_0(w^+(t,x) - w^-(t,x))\varphi_i^0(x)\, d\Gamma + a(w,\varphi_i) = 0. \quad (22)$$

where $i = 1,\ldots,m$ and $\tilde{q}_i = q_i - q_i(0)$. The equality (22) is rewritten as

$$\sum_{j=1}^{m} \tilde{q}_j(t) \int_{\Gamma_0} \Phi_j(t,x)(v^+ - v^-)\varphi_i^0(x) d\Gamma_0 = -a(w,\varphi_i) - \tilde{\psi}_i'(t) + \int_\Gamma \frac{\partial w}{\partial N}\varphi_i(x)\, d\Gamma -$$
$$\int_{\Gamma_0} \sigma_0(w^+ - w^-)\varphi_i^0(x)\, d\Gamma - \sum_{j=1}^{m} \tilde{q}_j(t) \int_{\Gamma_0} \Phi_j(t,x)(w^+ - w^-)\varphi_i^0(x) d\Gamma_0.$$

and, therefore, we have the operator equation

$$B(t)\vec{q} = \vec{F},\ \ F_k = -a(w, \varphi_i) - \tilde{\psi}_i'(t) + \int_\Gamma \frac{\partial w}{\partial N}\varphi_i(x)\,d\Gamma - \int_{\Gamma_0} \sigma_0(w^+ - w^-)\varphi_i^0(x)\,d\Gamma$$
$$- \sum_{j=1}^m \tilde{q}_j(t) \int_{\Gamma_0} \Phi_j(t,x)(w^+ - w^-)\varphi_i^0(x)\,d\Gamma_0,$$

where $\vec{F} = (F_1, \dots, F_m)^T$, $\vec{q} = (\tilde{q}_1, \dots, \tilde{q}_m)^T$ and $B(t)$ is the matrix with entries $b_{ij} = \int_{\Gamma_0} \Phi_j(t,x)(v^+(t,x) - v^-(t,x))\varphi_i^0(x)\,d\Gamma_0$. Moreover, $B(0) = B_0$ and the matrix B_0 is nondegenerate. The embedding theorems imply that $v \in C(\overline{Q})$, $\Phi_i \in C(\overline{S})$ (even more $v \in C^{1-(n+2)/2p, 2-(n+2)/p}(\overline{Q})$) and, therefore, there exist parameters τ_0 and $\delta_1 > 0$, such that

$$|det\, B(t)| \geq \delta_1 \ \ \forall t \in [0, \tau_0].$$

For $\tau \leq \tau_0$, we have that

$$\vec{q} = B^{-1}\vec{F} = R(\vec{q}) = \vec{g}_0 + R_0(\vec{q}), \tag{23}$$

where $\vec{g}_0 = B^{-1}\vec{\Psi}$ and the kth coordinate Ψ_k of the vector $\vec{\Psi}$ is of the form $\Psi_k(t) = -\tilde{\psi}_k'(t)$. This equation is used to determine $\vec{q}$. We demonstrate that the operator R is a contraction in the ball $B_{R_0} = \{\vec{q} \in \tilde{W}_p^{s_0}(0,\tau) :\ \|\vec{q}\|_{\tilde{W}_p^{s_0}(0,\tau)} \leq R_0\}$, provided that the parameter τ is sufficiently small, where $R_0 = 2\|\vec{g}_0\|_{\tilde{W}_p^{s_0}(0,T)}$. First, we obtain estimates for the function w, which is a solution to the problem (20). In what follows, the notation $\|w\|_\tau = \|w\|_{W_p^{1,2}(Q_+^\tau)} + \|w\|_{W_p^{1,2}(Q_-^\tau)}$ is used. Fix a parameter $\tau \in (0, T]$. Theorem 1, when applied to the problem (20), implies the estimate

$$\|w\|_\tau \leq c\|(\sigma_0 - \sigma)(v^+ + w^+ - v^- - w^-)\|_{\tilde{W}_p^{s_0,2s_0}(S_0^\tau)}, \tag{24}$$

where c is independent of τ. Let $\vec{q} \in B_{R_0}$. Since $\Phi_j \in W_p^{s_0,2s_0}(S_0)$, according Lemma 2 $\tilde{q}_i(t)\Phi_i(t,x) \in W_p^{s_0,2s_0}(S_0^\tau)$, we have the estimates

$$\|\sigma_0 - \sigma\|_{\tilde{W}_p^{s_0,2s_0}(S_0^\tau)} \leq c_0\|\vec{q}\|_{\tilde{W}_p^{s_0}(0,\tau)},$$

$$\|(\sigma_0 - \sigma)(v^+ + w^+ - v^- - w^-)\|_{\tilde{W}_p^{s_0,2s_0}(S_0^\tau)} \leq \|(\sigma_0 - \sigma)(v^+ - v^-)\|_{\tilde{W}_p^{s_0,2s_0}(S_0^\tau)} +$$
$$\|(\sigma_0 - \sigma)(w^+ - w^-)\|_{\tilde{W}_p^{s_0,2s_0}(S_0^\tau)} \leq c_1\|\vec{q}\|_{\tilde{W}_p^{s_0}(0,\tau)}(\|v^+ - v^-\|_{W_p^{s_0,2s_0}(S_0)} + \|w^+ - w^-\|_{\tilde{W}_p^{s_0,2s_0}(S_0^\tau)})$$
$$\leq c_2R_0(1 + \|w^+\|_{\tilde{W}_p^{s_0,2s_0}(S_0^\tau)} + \|w^-\|_{\tilde{W}_p^{s_0,2s_0}(S_0^\tau)}).$$

The definition of the norm in $\tilde{W}_p^{s_0,2s_0}(S_0^\tau)$ and Lemma 1 imply that

$$\|w^\pm\|_{\tilde{W}_p^{s_0,2s_0}(S_0^\tau)} \leq \tau^{1/2}\|w^\pm\|_{\tilde{W}_p^{s_1,2s_1}(S_0^\tau)} \leq c_3\tau^{1/2}\|w\|_\tau.$$

This inequality and (24) validate the inequality

$$\|w\|_\tau \leq 2c_2R_0 \tag{25}$$

provided that $\tau \leq \tau_1$, with $2c_2R_0c_3\tau_1^{1/2} = 1/2$. Note that the constants c_2, c_3 are independent of τ. Next, we assume that the functions w_1, w_2 are solutions to the problem (20), where the function σ is replaced with $\sigma_1, \sigma_2, \sigma_i = \sum_{j=1}^m q_j^i\Phi_i$ and $\vec{q}^{\,i} = (q_1^i, q_2^i, \dots, q_m^i)^T \in B_{R_0}$. The difference $w_0 = w_1 - w_2$ is a solution to the problem

$$w_{0t} - Lw_0 = 0,\ \frac{\partial w_0^+}{\partial N} = \frac{\partial w_0^-}{\partial N},\ w_0|_{t=0} = 0,\ \frac{\partial w_0^+}{\partial N} - \sigma_0(w_0^+ - w_0^-) =$$
$$(\sigma_1 - \sigma_2)(v^+ - v^- + \frac{w_1^+ + w_2^+ - w_1^- - w_2^-}{2}) + \frac{(\sigma_1 + \sigma_2)}{2}(w_0^+ - w_0^-). \tag{26}$$

Again involving Theorem 1, using (25) for the functions w_i, and repeating the proof of (25), we obtain the estimate

$$\|w_0\|_\tau \le c_3(R_0)\|\vec{q}^1 - \vec{q}^2\|_{\tilde{W}_p^{s_0}(0,\tau)} + c_4(R_0)(\|w_0^+\|_{\tilde{W}_p^{s_0,2s_0}(S_0^\tau)} + \|w_0^-\|_{\tilde{W}_p^{s_0,2s_0}(S_0^\tau)}).$$

The above arguments of the proof of (25) imply that

$$\|w_0\|_\tau \le 2c_3(R_0)\|\vec{q}^1 - \vec{q}^2\|_{\tilde{W}_p^{s_0}(0,\tau)} \tag{27}$$

provided that $\tau \le \tau_3 = min(\tau_1, \tau_0, \tau_2)$, with $2c_4(R_0)c_3\tau_2^{1/2} = 1/2$. Consider the expression

$$R(\vec{q}^1) - R(\vec{q}^2) = B^{-1}(\vec{F}^1 - \vec{F}^2),\ F_k^1 - F_k^2 = -a(w_0, \varphi_k) + \int_\Gamma \frac{\partial w_0}{\partial N}\varphi_k(x)\,d\Gamma -$$
$$\int\limits_{\Gamma_0} \sigma_0(w_0^+ - w_0^-)\varphi_k^0(x)\,d\Gamma_0 - \sum_{j=1}^m \frac{(q_j^1(t) + q_j^2(t))}{2} \int\limits_{\Gamma_0} \Phi_j(w_0^+ - w_0^-)\varphi_k^0(x)\,d\Gamma_0$$
$$- \sum_{j=1}^m (q_j^1(t) - q_j^2(t)) \int_{\Gamma_0} \Phi_j \frac{(w_1^+ + w_2^+ - w_1^- - w_2^-)}{2}\varphi_k^0(x)\,d\Gamma_0.$$

Estimate the quantity $\|F_k^1 - F_k^2\|_{\tilde{W}_p^{s_0}(0,\tau)}$. The estimates of Lemma 2 yield

$$\|\vec{F}_k^1 - \vec{F}_k^2\|_{\tilde{W}_p^{s_0}(0,\tau)} \le \|a(w_0, \varphi_k)\|_{\tilde{W}_p^{s_0}(0,\tau)} + \|\int\limits_\Gamma \frac{\partial w_0}{\partial N}\varphi_k(x)\,d\Gamma\|_{\tilde{W}_p^{s_0}(0,\tau)}$$
$$+ \|\int\limits_{\Gamma_0} \sigma_0(w_0^+ - w_0^-)\varphi_k^0(x)\,d\Gamma_0\|_{\tilde{W}_p^{s_0}(0,\tau)} +$$
$$c_4 \sum_{j=1}^m \|q_j^1(t) - q_j^2(t)\|_{\tilde{W}_p^{s_0}(0,\tau)} \|\int\limits_{\Gamma_0} \Phi_j(t,x) \frac{(w_1^+ + w_2^+ - w_1^- - w_2^-)}{2}\varphi_k^0(x)\,d\Gamma_0\|_{\tilde{W}_p^{s_0}(0,\tau)}$$
$$+ c_5 \sum_{i=1}^m \|q_i^1(t) + q_i^2(t)\|_{\tilde{W}_p^{s_0}(0,\tau)} \|\int\limits_{\Gamma_0} \Phi_j(t,x)(w_1^+ - w_2^+ - w_1^- + w_2^-)\varphi_k^0(x)\,d\Gamma_0\|_{\tilde{W}_p^{s_0}(0,\tau)}. \tag{28}$$

Estimate the summands in the expression $a(w_0, \varphi_k)$. We have

$$a(w_0, \varphi_k) = \int_G \sum_{i,j=1}^n a_{ij} w_{0x_j}\varphi_{kx_i} + (\sum_{i=1}^n a_i w_{0x_i} + a_0 w_0)\varphi_k\,dx.$$

The Minkowskii and Hölder inequalities, Lemma 2 and the conditions (11) and (15) yield

$$\|\int_G a_{ij} w_{0x_j}\varphi_{kx_i}\|_{\tilde{W}_p^{s_0}(0,\tau)} \le c\int_G \|\nabla w_0\|_{\tilde{W}_p^{s_0}(0,\tau)}\,dx \le c_1\Big(\int_G \|\nabla w_0\|^p_{\tilde{W}_p^{s_0}(0,\tau)}\,dx\Big)^{1/p} \tag{29}$$

for all i, j. Recall that the Hölder inequality is written as $|\int_G u(x)v(x)\,dx| \le \|u\|_{L_p(G)}\|v\|_{L_q(G)}$ with $1/p + 1/q = 1$. Note that

$$\int_G \|\nabla w_0\|^p_{\tilde{W}_p^{s_0}(0,\tau)}\,dx = \int_G \int_0^\tau \frac{|\nabla w_0|^p}{t^{s_0 p}}\,dtdx +$$
$$\int_G \int_0^\tau \int_0^\tau \frac{|\nabla w_0(t_1,x) - \nabla w_0(t_2,x)|^p}{|t_1 - t_2|^{1+s_0 p}}\,dt_1dt_2dx. \tag{30}$$

Since we have the inequality ($[W_p^2(G^\pm), L_p(G^\pm)]_{1/2} = W_p^1(G^\pm)$, Theorem 4.3.1 in [27])

$$\|\nabla w_0\|^p_{L_p(G^\pm)} \le c_2\|w_0\|^{1/2}_{W_p^2(G^\pm)}\|w_0\|^{1/2}_{L_p(G^\pm)},$$

the first summand on the right-hand-side of (30) is estimated as follows:

$$\|\frac{1}{t^{s_0}}\nabla\omega_0\|_{L_p(Q^\tau)} \le c_2(\|w_0\|_{L_p(0,\tau;W_p^2(G^+))} + \|w_0\|_{L_p(0,\tau;W_p^2(G^-))})^{1/2}\|\frac{1}{t^{2s_0}}w_0\|^{1/2}_{L_p(Q^\tau)}.$$

The Newton–Leibnitz formula ensures that $\|\frac{1}{t^{2s_0}}w_0\|_{L_p(Q^\tau)} \le \tau^{1/p}\|w_{0t}\|_{L_p(Q^\tau)}$. In this case, the last inequality can be rewritten as

$$\|\frac{1}{t^{s_0}}\nabla w_0\|_{L_p(Q^\tau)} \le c_2\tau^{1/2p}\|w_0\|_\tau. \tag{31}$$

Estimate the second summand in (30). We infer

$$\int_G \int_0^\tau \int_0^\tau \frac{|\nabla w_0(t_1,x) - \nabla w_0(t_2,x)|^p}{|t_1-t_2|^{1+s_0p}}\,dt_1dt_2dx \le \int_G \int_0^\tau \int_0^\tau \frac{|\nabla w_0(t_1,x) - \nabla w_0(t_2,x)|^p}{|t_1-t_2|^{1+p/2}}\,dt_1dt_2dx\tau^{1/2}. \tag{32}$$

Next, there exist extensions $P^\pm w_0$ of a function w_0 defined in $G^\pm$ to the whole $\mathbb{R}^n$ such that $P^\pm : W_p^2(G^\pm) \to W_p^2(\mathbb{R}^n)$ is a linear operator such that $\|P^\pm u\|_{W_p^2(\mathbb{R}^n)} \le c_3\|u\|_{W_p^2(G^\pm)}$ and $\|P^\pm u\|_{L_p(\mathbb{R}^n)} \le c_3\|u\|_{L_p(G^\pm)}$ for all $u \in W_p^2(G^\pm)$, respectively, $u \in L_p(G^\pm)$. We can use the Hestenes method described in the proof of Lemma 2.9.3 in [27] for the half-space and then use it for arbitrary domains. In the case of $G = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, the method can be described as follows. Given a function $u \in W_p^2(\mathbb{R}^n_+)$, construct its extension $\tilde{u}$ to the whole $\mathbb{R}^n$ using the formula $\tilde{u} = u$ for $x_n > 0$ and $\tilde{u}(x) = \sum_{i=1}^2 c_i u(x', -\lambda x_n)$ for $x_n < 0$, where $\lambda_i > 0$ are different numbers and the constants c_i are determined as a solution to the system $\sum_{i=1}^2 c_i = 1, -\sum_{i=1}^2 \lambda_i c_i = 1$. Generally speaking, this system is a consequence of the equalities $\tilde{u}(x', +0) = \tilde{u}(x', -0)$, $\frac{\tilde{u}(x',+0)}{\partial x_n} = \frac{\tilde{u}(x',-0)}{\partial x_n}$. The new function belongs to $W_p^2(\mathbb{R}^n)$ for $u \in W_p^2(\mathbb{R}^n_+)$ and the space $L_p(\mathbb{R}^n)$ if $u \in L_p(\mathbb{R}^n_+)$. The case of a general domain G is reduced to this simple case with the use of a partition of unity on ∂G and a local straightening of the boundary (see the proof of Theorem 4.2.2 in [27]).

Thus, we can define the functions $P^\pm w_0 \in W_p^{1,2}((0,\tau)\times\mathbb{R}^n)$ such that $\|P^\pm w_0\|_{W_p^{1,2}((0,\tau)\times\mathbb{R}^n)} \le c_3\|w_0\|_\tau$, with c the constant independent of w_0 and $\tau > 0$. Note that $Pw_0(0,x) = 0$. We have

$$\int_{G^\pm} \int_0^\tau \int_0^\tau \frac{|\nabla w_0(t_1,x) - \nabla w_0(t_2,x)|^p}{|t_1-t_2|^{1+p/2}}\,dt_1dt_2dx \le \int_{\mathbb{R}^n} \int_0^\tau \int_0^\tau \frac{|\nabla P^\pm w_0(t_1,x) - \nabla P^\pm w_0(t_2,x)|^p}{|t_1-t_2|^{1+p/2}}\,dt_1dt_2dx.$$

Make the change of variables $t_i = \tau\tau_i$ $(i = 1,2)$, $x = \sqrt{\tau}y$. The last integral takes the form $(\tilde{P}^\pm w_0(\tau_i, y) = P^\pm w_0(\tau\tau_i, \sqrt{\tau}y))$

$$I = \tau^{1-p+n/2} \int\limits_{\mathbb{R}^n} \int\limits_0^1 \int\limits_0^1 \frac{|\nabla_y \tilde{P}^\pm w_0(\tau_1,y) - \nabla_y \tilde{P}^\pm w_0(\tau_2,y)|^p}{|\tau_1-\tau_2|^{1+p/2}}\,d\tau_1 d\tau_2 dy. \tag{33}$$

If $u \in W_p^{1,2}((0,1)\times\mathbb{R}^n)$, then (see, for instance, Lemma 3.8 in [28], or Lemma 7.2 in [30] or Theorem 18.12 in [31]) $\nabla u \in W_p^{\frac{1}{2},1}((0,1)\times\mathbb{R}^n)$ and

$$\|\nabla u\|_{W_p^{\frac{1}{2},1}((0,1)\times\mathbb{R}^n)} \le c_4\|u\|_{W_p^{1,2}((0,1)\times\mathbb{R}^n)},$$

where the constant c_4 is indepedent of u. In this case, the integral in (33) is estimated by

$$I \le c_4^p\tau^{1-p+n/2}\|\tilde{P}^\pm w_0\|^p_{W_p^{1,2}((0,1)\times\mathbb{R}^n)} = c_4^p\tau^{1-p+n/2}\int_0^1\int_{\mathbb{R}^n} |\tilde{P}^\pm w_{0\tau}|^p + \sum_{i,j=1}^n |(\tilde{P}^\pm w_0)_{y_iy_j}|^p\,d\tau dy,$$

where the integral defines an equivalent norm to the power p in $W_p^{1,2}((0,1)\times\mathbb{R}^n)$, since $\tilde{P}^{\pm}w_0(0,x)=0$. Turn back to the old variables (t,x) and refer to the above estimate for $P^{\pm}$. We conclude that

$$I \le c_5\|w_0\|_\tau^p, \tag{34}$$

where the constant c_5 is independent of τ. The relations (29)–(34) ensure the inequality

$$\|\int_G a_{ij}w_{0x_j}\varphi_{kx_i}\|_{\tilde{W}_p^{s_0}(0,\tau)} \le c_6\tau^{1/2p}\|w_0\|_\tau,$$

where the constant c_6 is independent of τ. The summands $\int_G a_i w_{0x_i}\varphi_k\,dx$ in $a(w_0,\varphi_k)$ are estimated similarly. Simpler arguments are used to estimate the integral $J=\int_G a_0w_0\varphi_k\,dx$. Indeed, Lemma 2 implies that

$$\|J\|_{\tilde{W}_p^{s_0}(0,\tau)} \le c\int\limits_G \|w_0\|_{\tilde{W}_p^{s_0}(0,\tau)}\,dx \le c_1\|w_0\|_{\tilde{W}_p^{s_0}(0,\tau;L_p(G))},$$

$$\|w_0\|^p_{\tilde{W}_p^{s_0}(0,\tau;L_p(G))} = \int\limits_G\int\limits_0^\tau \frac{1}{t^{s_0p}}|w_0|^p dtdx + \int\limits_G\int\limits_0^\tau\int\limits_0^\tau \frac{|w_0(t_1,x)-w_0(t_2,x)|^p}{|t_1-t_2|^{1+s_0p}}\,dt_1dt_2dx.$$

Now, we use the representation $w_0(t_1,x)-w_0(t_2,x)=\int_{t_1}^{t_2}\omega_t(t,x)\,dt$ in the second integral and the equality $w_0(t,x)=\int_0^t w_{0\xi}(\xi,x)\,d\xi$ in the first integral. We derive that

$$\|w_0\|^p_{\tilde{W}_p^{s_0}(0,\tau;L_p(G))} \le c_2\|w_{0t}\|_{L_p(Q^\tau)}\,\tau^{1/2+1/2p},$$

and, therefore,

$$\|\int_G a_0w_0\varphi_k\,dx\|_{\tilde{W}_p^{s_0}(0,\tau)} \le c\|w_0\|_{\tilde{W}_p^{s_0}(0,\tau;L_p(G))} \le c_3\|w_{0t}\|_{L_p(Q^\tau)}\tau^{1/2+1/2p}, \tag{35}$$

where c_3 is independent of τ. As can easily be seen, the above arguments (see the proof of (31) and (34)) validate the inequality

$$\|w\|_{\tilde{W}_p^{s_0}(0,\tau;W_p^1(G^+))} + \|w\|_{\tilde{W}_p^{s_0}(0,\tau;W_p^1(G^-))} \le c\tau^{1/2p}\|w\|_\tau,\ w|_{Q_\pm^\tau}\in W_p^{1,2}(Q_\pm^\tau),\ w(0,x)=0, \tag{36}$$

where the constant c is independent of τ. In fact, the claim follows from the definition of the norm

$$\|w\|^p_{\tilde{W}_p^{s_0}(0,\tau;W_p^1(G^\pm))} = \int_0^\tau \|\frac{1}{t^{s_0}}w\|^p_{W_p^1(G^\pm)}\,dt + \int_0^\tau\int_0^\tau \frac{\|w(t_1,x)-w(t_2,x)\|^p_{W_p^1(G^\pm)}}{|t_1-t_2|^{1+s_0p}}\,dt_1dt_2.$$

The necessary estimate of the former summand follows from (31) and (35) and the estimate of the latter is a consequence of the estimates (32)–(34). Finally, we can state that

$$\|a(w_0,\varphi_k)\|_{\tilde{W}_p^{s_0}(0,\tau)} \le c\tau^{1/2p}\|w_0\|_\tau.$$

Now, estimate the second summand $J_0=\|\int_\Gamma \frac{\partial w_0}{\partial N}\varphi_k(x)\,d\Gamma\|_{\tilde{W}_p^{s_0}(0,\tau)}$ in (28). Let $G_\delta=\{x\in\mathbb{R}^n:\ \rho(x,\Gamma)<\delta\ (\delta>0)$. Choose a parameter δ so that $\rho(G_\delta,\Gamma_0)>0$. In this case, $G_\delta\cap G\subset G^+$. Construct a function $\varphi\in C_0^\infty(\mathbb{R}^n)$ such that $supp\,\varphi\in G_\delta$ and $\varphi(x)=1$ for $x\in G_{\delta/2}$. The function φw_0 is a solution to the problem

$$M(\varphi w_0)=2\sum_{i,j=1}^n a_{ij}\varphi_{x_i}w_{0x_j}+w_0L\varphi=\Phi,\ w_0|_{t=0}=0,\ Bw_0|_\Gamma=0,\ w_0|_{\Gamma_0}=0.$$

Referring to the conventional parabolic theory (see [1] or [28]) and using Lemma 1 and simple arguments (see the proof of Theorem 2 in [29]), we can say that there exist constants $c,c_1,c_2>0$ independent of τ, such that

$$\|\frac{\partial w_0}{\partial N}\|_{\tilde{W}_p^{s_0,2s_0}(\Gamma)} \leq c\|\varphi w_0\|_{W_p^{1,2}(Q_+^\tau)} \leq c_1\|\Phi\|_{L_p(Q_+^\tau)} \leq c_2\|w_0\|_{L_p(0,\tau;W_p^1(G^+))}.$$

Next, we can use the estimate (36) and conclude that

$$\|\frac{\partial w_0}{\partial N}\|_{\tilde{W}_p^{s_0,2s_0}(\Gamma)} \leq c_3\tau^{1/2p}\|w_0\|_\tau. \tag{37}$$

Consider the third summand $J_1 = \|\int_{\Gamma_0} \sigma_0(w_0^+ - w_0^-)\varphi_k^0(x)\,d\Gamma_0\|_{\tilde{W}_p^{s_0}(0,\tau)}$ in (28). We have that

$$J_1 \leq c\int_\Gamma \|w_0^+ - w_0^-\|_{\tilde{W}_p^{s_0}(0,\tau)}\,d\Gamma \leq c_1(\|w_0^+\|_{\tilde{W}_p^{s_0}(0,\tau;L_p(\Gamma_0))} + \|w_0^-\|_{\tilde{W}_p^{s_0}(0,\tau;L_p(\Gamma_0))}) \leq$$
$$c_2(\|w_0^+\|_{\tilde{W}_p^{s_0}(0,\tau;W_p^1(G^+))} + \|w_0^-\|_{\tilde{W}_p^{s_0}(0,\tau;W_p^1(G^-))}) \leq c_3\tau^{1/2p}\|w_0\|_\tau, \tag{38}$$

where the constants c_i are independent of τ. Here, we have employed the Hölder inequality, the embedding $W_p^1(G) \subset L_p(\Gamma)$ and the estimate (36). Estimate the factor $J_2 = \|\int_{\Gamma_0} \Phi_j(t,x)\frac{(w_1^+ + w_2^+ - w_1^- - w_2^-)}{2}\varphi_k^0(x)d\Gamma_0\|_{\tilde{W}_p^{s_0}(0,\tau)}$ in the forth summand of (28). As before (see (38)), we have

$$J_2 \leq c_5\tau^{1/2p}(\|w_1\|_\tau + \|w_2\|_\tau). \tag{39}$$

In this case, the estimate of the forth summand J_3 in (28) is of the form

$$J_3 \leq c_6\tau^{1/2p}\|\vec{q}^{\,1} - \vec{q}^{\,2}\|_{\tilde{W}_p^{s_0}(0,\tau)}(\|w_1\|_\tau + \|w_2\|_\tau). \tag{40}$$

The last summand

$$J_4 = \sum_{i=1}^m \frac{\|q_i^1(t) + q_i^2(t)\|_{\tilde{W}_p^{s_0}(0,\tau)}}{2}\|\int_{\Gamma_0} \Phi_j(t,x)(w_1^+ - w_2^+ - w_1^- + w_2^-)\varphi_k^0(x)d\Gamma_0\|_{\tilde{W}_p^{s_0}(0,\tau)}$$

is estimated as follows:

$$J_4 \leq c_7\|\vec{q}^{\,1} + \vec{q}^{\,2}\|_{\tilde{W}_p^{s_0}(0,\tau)}(\|w_0^+\|_{\tilde{W}_p^{s_0}(0,\tau;L_p(\Gamma_0)} + \|w_0^-\|_{\tilde{W}_p^{s_0}(0,\tau;L_p(\Gamma_0)}) \leq c_8(R_0)\tau^{1/2p}\|w_0\|_\tau \tag{41}$$

The estimates (27), (28), (37), (38), (40) and (41) imply that

$$\|\vec{F}_k^1 - \vec{F}_k^2\|_{\tilde{W}_p^{s_0}(0,\tau)} \leq c_8\tau^{1/2p}\|\vec{q}^{\,1} - \vec{q}^{\,2}\|_{\tilde{W}_p^{s_0}(0,\tau)},$$

where the constant c_8 is independent of $\tau \leq \tau_3$. This estimate and Lemma 2 ensure the estimate

$$\|R\vec{q}^{\,1} - R\vec{q}^{\,2}\|_{\tilde{W}_p^{s_0}(0,\tau)} \leq c_9\tau^{1/2p}\|\vec{q}^{\,1} - \vec{q}^{\,2}\|_{\tilde{W}_p^{s_0}(0,\tau)},$$

where the constant c_9 is independent of $\tau \leq \tau_3$. To apply the fixed point theorem, we need to justify the membership $\tilde{\psi}_{kt} \in \tilde{W}_p^{s_0}(0,T)$. Recall that $\tilde{\psi}_{kt} = \psi_k'(t) - \int_G v_t\varphi_k\,dx$. Multiply the equation in (6) written for the function v by φ_k and integrate over G. We obtain that

$$\int_G v_t\varphi_k\,dx + \int_{\Gamma_0} \sigma_0(v^+(t,x) - v^-(t,x))\varphi_k^0(x)d\Gamma_0 - \int_\Gamma \frac{\partial v}{\partial N}\varphi_k(x)\,d\Gamma$$
$$+ \int_{\Gamma_0} g^+\varphi_k^0(x)\,d\Gamma + a(v,\varphi_k) = (f,\varphi_k).$$

This equality can be rewritten with the use of (18) as follows:

$$\int_G v_t \varphi_k \, dx - \psi_k'(t) = -a(v - u_0, \varphi_k) - \int_{\Gamma_0} \sigma_0 (v^+(t,x) - u_0^+ - v^-(t,x) + u_0^-) \varphi_k^0 d\Gamma_0$$
$$+ \int_\Gamma (\frac{\partial v}{\partial N} - \frac{\partial u_0}{\partial N}) \varphi_k(x) \, d\Gamma - \int_{\Gamma_0} (g^+(t,x) - g^+(0,x)) \varphi_k^0(x) \, d\Gamma_0 + (f - f(0,x), \varphi_k). \quad (42)$$

Recall that (18) is written as

$$\psi_k'(0) + \int_\Gamma \sigma_0 (u_0^+ - u_0^-) \varphi_k(x) \, d\Gamma + a(u_0, \varphi_k) + \int_{\Gamma_0} g^+(0,x) \varphi_k(x) \, d\Gamma_0 - \int_\Gamma \frac{\partial u_0}{\partial N} \varphi_k(x) \, d\Gamma$$
$$= (f(0,x), \varphi_k), \; k = 1, 2, , \ldots, m.$$

In this case, the equality (42) ensures that $\int_G v_t(0,x) \varphi_k \, dx = \psi_k'(0)$, i.e., $\tilde{\psi}_k'(0) = 0$. Note that all summands in (42) are continuous functions in t, which results from the conditions on the coefficients and embedding theorems. Now we can see that every summand on the right-hand side of (42) belongs to the space $\tilde{W}_p^{s_0}(0,T)$ due to the above estimates. Next, we can find $\tau_4 \leq \tau_3$, such that $c_9 \tau_4^{1/2p} \leq 1/2$. In this case, the fixed point theorem implies that the equation (23) has a unique solution in the ball B_{R_0} for every $\tau \leq \tau_4$. The function w is defined as a solution to the problem (20). Respectively, $u = v + w$.

Validate the conditions (21). Multiply the equation in (20) by φ_k and integrate the result over G. Integrating by parts, we infer

$$\int_G w_t \varphi_k \, dx + \sum_{j=1}^m \tilde{q}_j(t) \int_{\Gamma_0} \Phi_j(t,x)(w^+(t,x) - w^-(t,x) + v^+(t,x) - v^-(t,x)) \varphi_i^0(x) \, d\Gamma_0$$
$$- \int_\Gamma \frac{\partial w}{\partial N} \varphi_i(x) \, d\Gamma + \int_{\Gamma_0} \sigma_0 (w^+ - w^-) \varphi_i^0(x) \, d\Gamma + a(w, \varphi_i) = 0.$$

Subtracting this equality from (22), we infer

$$\int_G w_t \varphi_k \, dx = \tilde{\psi}_k', \;\; k = 1, \ldots, m.$$

Integrating this equality with respect to t, we establish (21). The uniqueness of the solutions is obvious due to the estimates obtained in the proof. □

Remark 1. *The stability estimate for solutions also holds and can be easily derived with the use of the arguments in the proof.*

Remark 2. *The results remains valid if the boundary Γ_0 consists of several connected components as well as the set G^- itself. In this case, we have several heat transfer coefficients. The corresponding solvability conditions are not difficult to specify.*

4. Discussion

We consider the inverse problems of recovering the heat transfer coefficient at the interface using integral measurements. These problems arise in some practical applications, but there are no theoretical results concerning the questions of existence and uniqueness. The results can be used in developing new numerical algorithms and provide new conditions of existence and uniqueness for solutions to these problems. We consider a model case, but it is clear what changes should be made in the general case for validating similar results. The main conditions on the data are conventional. The proof relies on a priori bounds and the contraction mapping principle.

5. Conclusions

The existence and uniqueness theorems in the inverse problems of recovering the heat transfer coefficient at the interface using the integral measurements are proven locally in terms of time. The heat transfer coefficient occurs in the transmission conditions of imperfect contact type. This was sought in the form of a finite segment of the Fourier series with coefficients depending on time. The proof relies on a priori bounds and a fixed point theorem. The conditions on the data, ensuring the existence and uniqueness of the solutions in Sobolev classes, are sharp. These are smoothness and consistency conditions on the data and additional conditions on the kernels of the integral operators used in additional measurements.

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