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Applications to Solving Variational Inequality Problems via MR-Kannan Type Interpolative Contractions

Rizwan Anjum [1,†], Andreea Fulga [2,*,†] and Muhammad Waqar Akram [1,†]

[1] Department of Mathematics, Division of Science and Technology, University of Education, Lahore 54770, Pakistan; rizwan.anjum@ue.edu.pk (R.A.); waqarakram645@gmail.com (M.W.A.)
[2] Department of Mathematics and Computer Sciences, Transilvania University of Brasov, 500123 Brasov, Romania
* Correspondence: afulga@unitbv.ro
† These authors contributed equally to this work.

Abstract: The aim of this paper is manifold. We first define the new class of operators called MR-Kannan interpolative type contractions, which includes the Kannan, enriched Kannan, interpolative Kannan type, and enriched interpolative Kannan type operators. Secondly, we prove the existence of a unique fixed point for this class of operators. Thirdly, we study Ulam-Hyers stability, well-posedness, and periodic point properties. Finally, an application of the main results to the variational inequality problem is given.

Keywords: Kannan contraction; interpolative Kannan type contraction; enriched Kannan operators; enriched interpolative Kannan type contraction; fixed point; well-posedness; variational inequality problem; periodic point property; Ulam-Hyers stability

MSC: 47H10; 47H09; 47J25; 49J40

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1. Introduction

The Banach contraction principle, detailed in [1], stands as a landmark result with a profound impact on the development of metric fixed point theory. Banach's work is highly regarded, representing an adaptable and foundational contribution to fixed point theory. This principle not only initiated significant research in the field but also spurred exploration by numerous scholars from 1922 to the present. A notable extension of the BCP was presented by Kannan [2].
We state the Kannan fixed point theorem in the context of Banach spaces.

Theorem 1 ([2])**.** *Let $(\Theta, \|\cdot\|)$ be a Banach space, and $P : \Theta \to \Theta$ be a Kannan contraction. This means that P satisfies the following condition:*

$$\|P\mathring{g} - P\mathring{s}\| \leq a\{\|\mathring{g} - P\mathring{g}\| + \|\mathring{s} - P\mathring{s}\|\}, \ \forall \mathring{g}, \mathring{s} \in \Theta, \tag{1}$$

with $0 \leq a < \frac{1}{2}$. Then, P has a unique fixed point.

In 2018, Karapinar [3] generalized Theorem 1 by introducing the concept of interpolative Kannan type contraction (IKTC).

In the framework of Banach spaces, the primary finding of [3] can be summarized as follows:

Theorem 2 ([3])**.** *Let $(\Theta, \|\cdot\|)$ be a Banach space, and $P : \Theta \to \Theta$ be an interpolative Kannan type contraction. This means that P satisfies the following condition:*

$$\|P\mathring{g} - P\mathring{s}\| \leq a(\|\mathring{g} - P\mathring{g}\|)^{\xi}(\|\mathring{s} - P\mathring{s}\|)^{1-\xi}, \ \forall \mathring{g}, \mathring{s} \in \Theta, \tag{2}$$

when $P\mathring{g} \neq \mathring{g}$, where $a \in [0,1)$ and $0 < \xi < 1$. As a consequence, it can be concluded that the operator P possesses a unique fixed point.

For further outcomes in this regard, (see [3–16]).

In 2020, Berinde and Păcurar [17] improved Theorem 1 by introducing the concept of enriched Kannan contraction.

The principal outcome highlighted in [17] is presented as follows:

Theorem 3 ([17])**.** *Let $(\Theta, \|\cdot\|)$ be a Banach space and $P : \Theta \to \Theta$ be an enriched Kannan contraction, that is an operator satisfying*

$$\|b(\mathring{g} - \mathring{s}) + P\mathring{g} - P\mathring{s}\| \leq a\{\|\mathring{g} - P\mathring{g}\| + \|\mathring{s} - P\mathring{s}\|\}, \quad \forall \mathring{g}, \mathring{s} \in \Theta, \tag{3}$$

where $0 \leq b < \infty$ and $0 \leq a < \frac{1}{2}$. Then, P has a unique fixed point.

Remark 1. *By substituting $b = 0$ into Theorem 3, we can derive Theorem 1. Therefore, Theorem 3 is a generalization of Theorem 1.*

The above Theorem 3 has been studied and generalized by many researchers (see [17–21]). Theorem 3 is generalized by Abbas et al. [4] in 2022 as follows:

Theorem 4 ([4])**.** *Let $(\Theta, \|\cdot\|)$ be a Banach space and $P : \Theta \to \Theta$ be an (a, b, ξ)-enriched IKTC, that is an operator satisfying*

$$\|b(\mathring{g} - \mathring{s}) + (P\mathring{g} - P\mathring{s})\| \leq a\big(\|\mathring{g} - P\mathring{g}\|\big)^{\xi}\big(\|\mathring{s} - P\mathring{s}\|\big)^{1-\xi}, \quad \forall \mathring{g}, \mathring{s} \in \Theta, \tag{4}$$

with $\mathring{g} \neq P\mathring{g}$, where $0 < \xi < 1$, $0 \leq a < 1$ and $0 \leq b < \infty$. Then, P has a unique fixed point.

Remark 2. *By substituting $b = 0$ into Theorem 4, we can derive Theorem 2. Moreover, it follows from Corollary 2.8 of [4] that Theorem 4 is a generalization of Theorem 3.*

On the other hand, in 2023, Anjum et al. [22] generalized Theorem 4 by introducing the concept of $(\urcorner, a)$-MR-Kannan type contraction.
The principal outcome highlighted in [22] is presented as follows:

Theorem 5 ([22])**.** *Let $(\Theta, \|\cdot\|)$ be a Banach space and $P : \Theta \to \Theta$ be an $(\urcorner, a)$-MR-Kannan type contraction, that is an operator satisfying*

$$\left\| \frac{\mathring{g}\urcorner(\mathring{g}) + P\mathring{g}}{1 + \urcorner(\mathring{g})} - \frac{\mathring{s}\urcorner(\mathring{s}) + P\mathring{s}}{1 + \urcorner(\mathring{s})} \right\| \leq a\left(\left| \frac{1}{1 + \urcorner(\mathring{g})} \right| \|\mathring{g} - P\mathring{g}\| + \left| \frac{1}{1 + \urcorner(\mathring{s})} \right| \|\mathring{s} - P\mathring{s}\| \right), \tag{5}$$

for all $\mathring{g}, \mathring{s} \in \Theta$, where $0 \leq a < \frac{1}{2}$ and $\urcorner \in \eta = \{\urcorner : \Theta \to \mathbb{R} : \urcorner(\mathring{g}) \neq -1, \ \ \forall \mathring{g} \in \Theta\}$. Then, P has a unique fixed point.

Remark 3. *If we take $\urcorner(\mathring{g}) = 0$ and $\urcorner(\mathring{g}) = b$, for all $\mathring{g} \in \Theta$, in Theorem 5, we obtain Theorem 1 and Theorem 3, respectively.*

Utilizing the ideas from Theorem 2 and Theorem 4, we now present the following.
Question

Under which condition can we attain an equivalent conclusion as stated in Theorem 5 by substituting the multiplication between the terms $\left|\frac{1}{1+\urcorner(\mathring{g})}\right| \|\mathring{g} - P\mathring{g}\|$ and $\left|\frac{1}{1+\urcorner(\mathring{s})}\right| \|\mathring{s} - P\mathring{s}\|$ on the right-hand side of (5)?

This paper has multiple objectives. We first define the new class of operator called MR-Kannan interpolative type contraction, which includes the contractive conditions (1)–(5). Additionally, the existence of a unique fixed point for this class of operators is proven.

Furthermore, the study encompasses Ulam-Hyers stability, well-posedness, and periodic point properties. Finally, the main results are applied to a variational inequality problem.

2. Approximating Fixed Points of MR-Kannan Type Interpolative Contractions

We introduce the following definition.

Definition 1. *Let $(\Theta, \|\cdot\|)$ be a normed space. A operator $P : \Theta \to \Theta$ is said to be MRKI type contraction, if there exist $\daleth \in \eta$, $0 \leq a < 1$ and $0 < \xi < 1$, such that for all $\mathring{g}, \mathring{s} \in \Theta$ with $P\mathring{g} \neq \mathring{g}$,*

$$\left\| \frac{\mathring{g}\daleth(\mathring{g}) + P\mathring{g}}{1 + \daleth(\mathring{g})} - \frac{\mathring{s}\daleth(\mathring{s}) + P\mathring{s}}{1 + \daleth(\mathring{s})} \right\| \leq a \left(\left\| \frac{\mathring{g} - P\mathring{g}}{1 + \daleth(\mathring{g})} \right\| \right)^{\xi} \left(\left\| \frac{\mathring{s} - P\mathring{s}}{1 + \daleth(\mathring{s})} \right\| \right)^{1-\xi}. \tag{6}$$

To emphasize the role of $a, \daleth$ and ξ in (28), we shall also call P a $(a, \daleth, \xi)$-MRKI type contraction.

Before proceeding with the proof of the main theorem of our paper, the following findings are required from [22].

Recall that we denote the set $\{\mathring{g} \in \Theta : P\mathring{g} = \mathring{g}\}$ of fixed points of P by $F(P)$.

Let

$$\beta = \{ F : \Theta \to \mathbb{R} : F(\mathring{g}) \neq 0, \ \forall \mathring{g} \in \Theta \}. \tag{7}$$

Let $P_F : \Theta \to \Theta$ be an operator defined as

$$P_F \mathring{g} = (1 - F(\mathring{g}))\mathring{g} + F(\mathring{g})P\mathring{g}, \ \forall \mathring{g} \in \Theta, \tag{8}$$

where $F \in \beta$ is called a generalized averaged operator ([22,23]). We would like to direct the reader's attention to the fact that the term generalized averaged operator refers to a specific type of admissible perturbations [23,24]. It is worth noting that the class of generalized averaged operators includes the class of averaged operators (a term coined in [25]) . This is demonstrated by considering $\lambda \in (0,1)$ and defining $F(\mathring{g}) = \lambda$ for all $\mathring{g} \in \Theta$. Consequently, the condition (28) is reduced to

$$P_\lambda(\mathring{g}) := P_F \mathring{g} = (1 - \lambda)\mathring{g} + \lambda P\mathring{g}, \forall \mathring{g} \in \Theta. \tag{9}$$

Lemma 1 ([22])**.** *Let $P : \Theta \to \Theta$ and P_F be a generalized averaged operator as given in (8). Then, for any $F \in \beta$,*

$$F(P) = F(P_F). \tag{10}$$

Now, we present the following principal result of our paper:

Theorem 6. *Let $(\Theta, \|\cdot\|)$ be a Banach space and $P : \Theta \to \Theta$ be a $(a, \daleth, \xi)$-MRKI type contraction. Then, P has a unique fixed point.*

Proof. Let us denote $F(\mathring{g}) = \frac{1}{1+\daleth(\mathring{g})}$, for all $\mathring{g} \in \Theta$. Obviously, $F \in \beta$ and the $(a, \daleth, \xi)$-MRKI type contraction condition (28) satisfies the following

$$\left\| F(\mathring{g}) \left(\left(\frac{1}{F(\mathring{g})} - 1 \right) \mathring{g} + P\mathring{g} \right) - F(\mathring{s}) \left(\left(\frac{1}{F(\mathring{s})} - 1 \right) \mathring{s} + P\mathring{s} \right) \right\|$$
$$\leq a \left(\| F(\mathring{g})(\mathring{g} - P\mathring{g}) \| \right)^{\xi} \left(\| F(\mathring{s})(\mathring{s} - P\mathring{s}) \| \right)^{1-\xi},$$

$$\left\| F(\mathring{g}) \frac{(1 - F(\mathring{g}))\mathring{g} + F(\mathring{g})P\mathring{g}}{F(\mathring{g})} - F(\mathring{s}) \frac{(1 - F(\mathring{s})) + F(\mathring{s})P\mathring{s}}{F(\mathring{s})} \right\|$$
$$\leq a \left(\| F(\mathring{g})(\mathring{g} - P\mathring{g}) \| \right)^{\xi} \left(\| F(\mathring{s})(\mathring{s} - P\mathring{s}) \| \right)^{1-\xi},$$

we have,

$$\|P_F\mathring{g} - P_F\mathring{s}\| \leq a\|\mathring{g} - P_F\mathring{g}\|^{\xi}\|\mathring{s} - P_F\mathring{s}\|^{1-\xi}, \ \forall\, \mathring{g}, \mathring{s} \in \Theta. \tag{11}$$

We defined the Krasnoselskii iterative sequence as follows:

$$\mathring{g}_{\varsigma+1} = P_F\mathring{g}_{\varsigma}, \ \ \varsigma \geq 0. \tag{12}$$

Take $\mathring{g} := \mathring{g}_{\varsigma}$ and $\mathring{s} := \mathring{g}_{\varsigma-1}$ in (11) to get

$$\begin{aligned}\|\mathring{g}_{\varsigma+1} - \mathring{g}_{\varsigma}\| &= \|P_F\mathring{g}_{\varsigma} - P_F\mathring{g}_{\varsigma-1}\| \\ &\leq a\|\mathring{g}_{\varsigma} - P_F\mathring{g}_{\varsigma}\|^{\xi}\|\mathring{g}_{\varsigma-1} - P_F\mathring{g}_{\varsigma-1}\|^{1-\xi} \\ &\leq a\|\mathring{g}_{\varsigma} - \mathring{g}_{\varsigma+1}\|^{\xi}\|\mathring{g}_{\varsigma-1} - \mathring{g}_{\varsigma}\|^{1-\xi}\end{aligned}$$

This concludes that

$$\|\mathring{g}_{\varsigma+1} - \mathring{g}_{\varsigma}\|^{1-\xi} \leq a\|\mathring{g}_{\varsigma-1} - \mathring{g}_{\varsigma}\|^{1-\xi}. \tag{13}$$

Given that $\xi \in (0,1)$,

$$\|\mathring{g}_{\varsigma+1} - \mathring{g}_{\varsigma}\| \leq a\|\mathring{g}_{\varsigma-1} - \mathring{g}_{\varsigma}\|. \tag{14}$$

Inductively,

$$\|\mathring{g}_{\varsigma+1} - \mathring{g}_{\varsigma}\| \leq a^{\varsigma}\|\mathring{g}_0 - \mathring{g}_1\|. \tag{15}$$

Using the Equation (15) and the triangular inequality, we can conclude that

$$\|\mathring{g}_{\varsigma} - \mathring{g}_{\varsigma+r}\| \leq \frac{a^{\varsigma}}{1-a}\|\mathring{g}_0 - \mathring{g}_1\|, \ \ r \in \mathbb{N}, \ \varsigma \geq 1, \tag{16}$$

This concludes that, a Cauchy sequence $\{\mathring{g}_{\varsigma}\}_{\varsigma=0}^{\infty}$ is converges to Θ. This can be denoted as follows:

$$\mathring{g}^* = \lim_{\varsigma\to\infty}\mathring{g}_{\varsigma}. \tag{17}$$

Note that

$$\begin{aligned}\|\mathring{g}^* - P_F\mathring{g}^*\| &\leq \|\mathring{g}^* - \mathring{g}_{\varsigma+1}\| + \|\mathring{g}_{\varsigma+1} - P_F\mathring{g}^*\| \\ &\leq \|\mathring{g}^* - \mathring{g}_{\varsigma+1}\| + \|P_F\mathring{g}_{\varsigma} - P_F\mathring{g}^*\| \\ &\leq \|\mathring{g}^* - \mathring{g}_{\varsigma+1}\| + a\|\mathring{g}_{\varsigma} - P_F\mathring{g}_{\varsigma}\|^{\xi}\|\mathring{g}^* - P_F\mathring{g}^*\|^{1-\xi} \\ &\leq \|\mathring{g}^* - \mathring{g}_{\varsigma+1}\| + a\|\mathring{g}_{\varsigma} - \mathring{g}_{\varsigma+1}\|^{\xi}\|\mathring{g}^* - P_F\mathring{g}^*\|^{1-\xi}.\end{aligned}$$

By taking the limit ($\varsigma \to \infty$) on both sides of the aforementioned inequality, we get $\mathring{g}^* = P_F\mathring{g}^*$.

Let $\mathring{s}^*$ be another fixed point of P. Next, as shown by (11), we possess

$$\begin{aligned}\|\mathring{g}^* - \mathring{s}^*\| = \|P_F\mathring{g}^* - P_F\mathring{s}^*\| &\leq a\|\mathring{g}^* - P_F\mathring{g}^*\|^{\xi}\|\mathring{s}^* - P_F\mathring{s}^*\|^{1-\xi} \\ &\leq a\|\mathring{g}^* - \mathring{g}^*\|^{\xi}\|\mathring{s}^* - \mathring{s}^*\|^{1-\xi},\end{aligned}$$

which, gives $\mathring{g}^* = \mathring{s}^*$. □

We obtain Theorem 5 as a corollary of our main result.

Corollary 1 ([22])**.** *Let $P : \Theta \to \Theta$ be a $(\urcorner, a)$-MR-Kannan type contraction. Then, P possesses a unique fixed point.*

Proof. Take $F(\mathring{g}) = \frac{1}{\urcorner(\mathring{g})+1}$, for all $\mathring{g} \in \Theta$. Then, condition (5) becomes

$$\begin{aligned} &\left\| \frac{\mathring{g}\left(\frac{1}{F(\mathring{g})} - 1\right) + P\mathring{g}}{1 + \left(\frac{1}{F(\mathring{g})} - 1\right)} - \frac{\mathring{s}\left(\frac{1}{F(\mathring{s})} - 1\right) + P\mathring{s}}{1 + \left(\frac{1}{F(\mathring{s})} - 1\right)} \right\| \\ \leq \quad & a\left(\left\| \frac{\mathring{g} - P\mathring{g}}{1 + \left(\frac{1}{F(\mathring{g})} - 1\right)} \right\| + \left\| \frac{\mathring{s} - P\mathring{s}}{1 + \left(\frac{1}{F(\mathring{s})} - 1\right)} \right\| \right), \ \forall\, \mathring{g}, \mathring{s} \in \Theta, \end{aligned}$$

which can be written in an equivalent form as follows;

$$\|P_F\mathring{g} - P_F\mathring{s}\| \leq a\left\{\|\mathring{g} - P_F\mathring{g}\| + \|\mathring{s} - P_F\mathring{s}\|\right\}, \ \forall\, \mathring{g}, \mathring{s} \in \Theta. \tag{18}$$

By (18), P_F is a Kannan contraction. P_F satisfies condition (18) and condition (11). Since, $F(\mathring{g}) = \frac{1}{\urcorner(\mathring{g})+1}$, for all $\mathring{g} \in \Theta$, the inequality (11) is same as the condition (28).
As a result, Theorem 6 refers to the conclusion. □

We obtain Theorem 4 as a consequence of our main result.

Corollary 2 ([4])**.** *Let $P : \Theta \to \Theta$ be a (a, b, ξ)-enriched IKTC on a Banach space $(\Theta, \|\cdot\|)$. Then, P has a unique fixed point.*

Proof. Let $\urcorner(\mathring{g}) = b$, for all $\mathring{g} \in \Theta$. Clearly, $\urcorner \in \eta$. In this scenario, the contraction condition (28) becomes (4).
Indeed,

$$\begin{aligned} \left\| \frac{b\mathring{g} + P\mathring{g}}{1+b} - \frac{b\mathring{s} + P\mathring{s}}{1+b} \right\| &\leq a\left(\left\| \frac{\mathring{g} - P\mathring{g}}{1+b} \right\| \right)^{\xi} \left(\left\| \frac{\mathring{s} - P\mathring{s}}{1+b} \right\| \right)^{1-\xi} \\ \frac{\|b\mathring{g} + P\mathring{g} - (by + P\mathring{s})\|}{1+b} &\leq a\left(\frac{1}{1+b}\right)^{\xi+1-\xi} \left(\|\mathring{g} - P\mathring{g}\| \right)^{\xi} \left(\|\mathring{s} - P\mathring{s}\| \right)^{1-\xi} \\ \|b\mathring{g} + P\mathring{g} - (b\mathring{s} + P\mathring{s})\| &\leq a\left(\|\mathring{g} - P\mathring{g}\| \right)^{\xi} \left(\|\mathring{s} - P\mathring{s}\| \right)^{1-\xi}. \end{aligned}$$

This can be expressed equivalently as:

$$\|b(\mathring{g} - \mathring{s}) + P\mathring{g} - P\mathring{s}\| \leq a(\|\mathring{g} - P\mathring{g}\|)^{\xi}(\|\mathring{s} - P\mathring{s}\|)^{1-\xi}. \tag{19}$$

Moreover, generalized Krasnoselskii iterative method [25] related to P reduces to Krasnoselskii iterative method [26]. Hence, the conclusion follows from Theorem 6. □

Corollary 3 ([3])**.** *Let $P : \Theta \to \Theta$ be IKTC on a Banach space $(\Theta, \|\cdot\|)$. Then, P possesses a unique fixed point.*

Proof. Let $\urcorner(\mathring{g}) = 0$, for all $\mathring{g} \in \Theta$. Clearly, $\urcorner \in \eta$. In this case, the contraction condition (28) reduces to (1).
Hence, the conclusion follows from Theorem 6. □

3. Well-Posedness, Perodic Point Property and Ulam-Hyers Stability

We start this section with the following definition:

3.1. Well-Posedness

Recall that the goal of solving the fixed point problem of the operator P, represented by $FPP(P)$, is to demonstrate the nonemptiness of $F(P)$.

Definition 2 ([27]). *Consider a normed space $(\Theta, \|\cdot\|)$ and an operator $P : \Theta \to \Theta$. Then, $FPP(P)$ is claimed to be well-posed if:*

1. $F(P) = \{\mathring{g}^*\}$;
2. *$\exists$ a sequence $\{\mathring{g}_\varsigma\}$ in Θ such that $\lim_{\varsigma\to\infty}\|P\mathring{g}_\varsigma - \mathring{g}_\varsigma\| = 0$, we can conclude that $\lim_{\varsigma\to\infty}\mathring{g}_\varsigma = \mathring{g}^*$.*

Theorem 7. *Let P be an operator as defined in Theorem 6. Then, $FPP(P)$ is well-posed.*

Proof. Because $F(P) = F(P_F)$, we may derive that operator P is well-posed if and only if operator P_F is well-posed.
As Theorem 6 states that $F(P) = \{\mathring{g}^*\}$. Assume that

$$\lim_{\varsigma\to\infty}\|P_F\mathring{g}_\varsigma - \mathring{g}_\varsigma\| = 0.$$

By utilizing (11), we get

$$\|\mathring{g}_\varsigma - \mathring{g}^*\| \le \|\mathring{g}_\varsigma - P_F\mathring{g}_\varsigma\|. \tag{20}$$

By applying limit $\varsigma \to \infty$ in (20), we conclude that

$$\lim_{\varsigma\to\infty}\mathring{g}_\varsigma = \mathring{g}^*.$$

□

3.2. Perodic Point Result

Obviously, a fixed point $\mathring{g}^*$ of the operator P satisfies $F(P^\varsigma) = \{\mathring{g}^*\}$ for all $\varsigma \in \mathbb{N}$; however, the reverse assertion does not hold. An operator P possesses a periodic point property ([28]) if it satisfies the condition $F(P) = F(P^\varsigma)$ for every $\varsigma \in \mathbb{N}$.

Theorem 8. *Let P be an operator defined in Theorem 6. Then, P possesses a preodic point property.*

Proof. Because $F(P) = F(P_F)$, we may derive that

$$\text{P has preodic point property} \Leftrightarrow P_F \text{ has preodic point property.}$$

Since P possesses a unique fixed point, then $\mathring{s}^* \in F(P^\varsigma)$. Now referring to Equation (11), we have

$$\begin{aligned}
\|\mathring{s}^* - P_F\mathring{s}^*\| &= \left\|P_F^\varsigma\mathring{s}^* - P_F\left(P_F^\varsigma\mathring{s}^*\right)\right\| \\
&= \left\|P_F\left(P_F^{\varsigma-1}\mathring{s}^*\right) - P_F\left(P_F^\varsigma\mathring{s}^*\right)\right\| \\
&\le a\left\|P_F^{\varsigma-1}\mathring{s}^* - P_F^\varsigma\mathring{s}^*\right\|^\xi \left\|P_F^\varsigma\mathring{s}^* - P_F^{\varsigma+1}\mathring{s}^*\right\|^{1-\xi},
\end{aligned}$$

that is,

$$\left\|P_F^\varsigma\mathring{s}^* - P_F^{\varsigma+1}\mathring{s}^*\right\|^\xi \le a\left\|P_F^{\varsigma-1}\mathring{s}^* - P_F^\varsigma\mathring{s}^*\right\|^\xi. \tag{21}$$

Since $\xi \in (0,1)$, then (21) becomes

$$\begin{aligned}
\|\mathring{s}^* - P_F\mathring{s}^*\| = \left\|P_F^\varsigma\mathring{s}^* - P_F^{\varsigma+1}\mathring{s}^*\right\| \le a\left\|P_F^{\varsigma-1}\mathring{s}^* - P_F^\varsigma\mathring{s}^*\right\| \le \\
\le a^2\left\|P_F^{\varsigma-2}\mathring{s}^* - P_F^{\varsigma-1}\mathring{s}^*\right\| \le \ldots \le a^\varsigma\|\mathring{s}^* - P_F\mathring{s}^*\|.
\end{aligned}$$

Now, $0 \le a < 1$ implies that $\|\mathring{s}^* - P_F\mathring{s}^*\| = 0$ and hence $\mathring{s}^* = P\mathring{s}^*$. □

3.3. Ulam-Hyers Stability

Before presenting the definition, let's establish the following concept from [29].

Let $(\Theta, \|\cdot\|)$ be a normed space and $P : \Theta \to \Theta$ be an operator such that a point $s^* \in \Theta$ as an ς-solution to the $FPP(P)$, if it satisfies the inequality

$$\|s^* - Ps^*\| \leq \varsigma,$$

where $\varsigma > 0$.
Let
$\Im = \{\chi : [0, \infty) \to [0, \infty),\ \chi(0) = 0,\ \ \chi$ is an increasing and continuous function$\}$.

Let us begin with definition.

Definition 3 ([29])**.** *The $FPP(P)$ exhibits generalized Ulam-Hyers stability if there exist $\chi \in \Im$ such that for every ς-solution $s^* \in \Theta$ and there is also a solution $\mathring{g}^* \in \Theta$ of $P(\mathring{g}) = \mathring{g}$ in Θ such that*

$$\|\mathring{g}^* - s^*\| \leq \chi(\varsigma).$$

where $\varsigma > 0$.

Remark 4 ([29])**.** *The fixed point equation $P\mathring{g} = \mathring{g}$ is considered to be Ulam-Hyers stable if the function χ is defined as $\chi(\varsigma) = m\varsigma$ for all $\varsigma \geq 0$, where $m > 0$.*

Theorem 9. *Let P be an operator as in the Theorem 6. Then, $FPP(P)$ possesses a Ulam-Hyers stability.*

Proof. Because $F(P) = F(P_F)$, we may derive that

P has Ulam-Hyers stability $\Leftrightarrow$ the operator $F(P_F)$ has Ulam-Hyers stability.

Taking s^* as an ς-solution to the $FPP(P)$, we can infer the following:

$$\|s^* - P_F s^*\| \leq \varsigma. \tag{22}$$

Utilizing (11) and (22), we obtain:

$$\begin{aligned}\|\mathring{g}^* - s^*\| &= \|P_F \mathring{g}^* - s^*\| \leq \|P_F \mathring{g}^* - P_F s^*\| + \|P_F s^* - s^*\| \\ &\leq a\|\mathring{g}^* - P_F \mathring{g}^*\|^{\xi} \|s^* - P_F s^*\|^{1-\xi} + \varsigma \\ &= \varsigma.\end{aligned}$$

□

4. Application to Variational Inequality Problems

The theory of variational inequalities, independently demonstrated by Stampacchia [30] and Ficchera [31], has evolved into a captivating branch of applied mathematics. Its diverse applications span across industry, social sciences, economics, finance and both pure and applied sciences. The Variational Inequality Problem, as discussed [32–42] a, has been and remains a focal point in nonlinear analysis.

Let H be a Hilbert space with the inner product denoted by $\langle \cdot, \cdot \rangle$, and consider a nonempty, closed, and convex subset C of H. This article is dedicated to exploring the classical variational inequality, seeking the presence of a point $\mathring{g}^*$ within C that satisfies

$$\langle S(\mathring{g}^*),\ \mathring{g} - \mathring{g}^* \rangle \geq 0, \quad \forall \mathring{g} \in C, \tag{23}$$

where $S : H \to H$ represents an operator. We denote $VIP(S, C)$ as the variational inequality problem associated with S and C. According to [33], it is well known that when Υ is a positive number, then $\mathring{g}^* \in C$ is a solution to $VIP(S,C)$ if and only if $\mathring{g}^*$ satisfies the fixed-point problem:

$$\mathring{g} = Pc(I - \Upsilon G)\mathring{g}, \tag{24}$$

Here, the closest point projection onto C is indicated by Pc.

We choose an alternative approach by investigating $VIP(S,C)$ with $(a, ㄱ, \xi)$-MRKI contraction operators, which can exhibit discontinuity, unlike nonexpansive operators, that are inherently continuous. According to the next theorem, we expect that $VIP(S,C)$ will have a unique solution in this situation. In addition, we anticipate substantial convergence of the algorithm outlined in (25) towards the $VIP(S,C)$ solution.

Theorem 10. *Let* Υ *be a positive value and* $P : C \to C$ *represent a* $(a, ㄱ, \xi)$*-MRKI type operator satisfying*

$$P\mathring{g} = (2 + \|\mathring{g}\|)P_C(I - \Upsilon S)(\mathring{g}) - \mathring{g} - \mathring{g}\|\mathring{g}\|, \ \ \forall \mathring{g} \in C, \tag{25}$$

Then, the iterative sequence $\{\mathring{g}_\varsigma\}_{\varsigma=0}^{\infty}$ *is given by*

$$\mathring{g}_{\varsigma+1} = (1 - F(\mathring{g}_\varsigma))\mathring{g}_\varsigma + F(\mathring{g}_\varsigma)P\mathring{g}_\varsigma, \ \forall \varsigma \geq 0 \tag{26}$$

where $F \in \beta$*, exhibits strong convergence towards the unique solution* $\mathring{g}^*$ *of the* $VIP(S,C)$*, for any* $\mathring{g}_0 \in C$*.*

Proof. As C is a closed set, let $\Theta = C$ and employ the definition of P as given in (24). Subsequently, we apply Theorem 6. Consequently, there exists an element $\mathring{g}^* \in C$ such that

$$\begin{aligned}
(2 + \|\mathring{g}^*\|)P_C(I - \Upsilon S)(\mathring{g}^*) - \mathring{g}^* - \mathring{g}^*\|\mathring{g}^*\| &= \mathring{g}^* \\
(2 + \|\mathring{g}^*\|)P_C(I - \Upsilon S)(\mathring{g}^*) &= 2\mathring{g}^* + \mathring{g}^*\|\mathring{g}^*\| \\
P_C(I - \Upsilon S)(\mathring{g}^*) &= \mathring{g}^*.
\end{aligned}$$

□

Example 1. *Let* $\Theta = \mathbb{R}^2$ *and the inner product for any* $\mathring{g} = (\mathring{g}_1, \mathring{g}_2)$ *and* $\mathring{s} = (\mathring{s}_1, \mathring{s}_2)$ *in* Θ*, is defined as follows:*

$$\langle \mathring{g}, \mathring{s} \rangle = \mathring{g}_1\mathring{s}_1 + \mathring{g}_2\mathring{s}_2.$$

With this definition, Θ *becomes a Hilbert space. The associated norm is given by:*

$$\|\mathring{g}\| = (<\mathring{g}, \mathring{s}>)^{1/2}.$$

Let's define the operator $S : \Theta \to \Theta$ *as follows:*

$$S(\mathring{g}) = \frac{(1,0) + \mathring{g}}{\Upsilon} \ \forall \, \mathring{g} \in \Theta,$$

where Υ *is a fixed positive real number.*

Next, consider the operator $P_C : \Theta \to C$ *defined by*

$$P_C(\mathring{g}) = \begin{cases} \frac{\mathring{g}}{\|\mathring{g}\|} & ; \mathring{g} \notin C \\ \mathring{g} & ; \mathring{g} \in C, \end{cases}$$

where $C = \{\mathring{g} \in \Theta : \|\mathring{g}\| \leq 1\}$, *The operator* P *defined by (25) is* $(a, \daleth, \xi)$*-MRKI type operator. Certainly, when* $\daleth(\mathring{g}) = 1 + \|\mathring{g}\|$, $\forall \mathring{g} \in \Theta$, *the left-hand side of (28) transforms to,*

$$
\begin{aligned}
&= \left\| \frac{\mathring{g}(1+\|\mathring{g}\|) + P\mathring{g}}{2+\|\mathring{g}\|} - \frac{\mathring{s}(1+\|\mathring{s}\|) + P\mathring{s}}{2+\|\mathring{s}\|} \right\| \\
&= \left\| \frac{1}{2+\|\mathring{g}\|}\left(\mathring{g}(1+\|\mathring{g}\|) + P\mathring{g}\right) - \frac{1}{2+\|\mathring{s}\|}\left(\mathring{s}(1+\|\mathring{s}\|) + P\mathring{s}\right) \right\| \\
&= \Bigg\| \left(\frac{(\mathring{g} + \mathring{g}\|\mathring{g}\| + (2+\|\mathring{g}\|)P_C(I - \Upsilon S)(\mathring{g}) - \mathring{g} - \mathring{g}\|\mathring{g}\|)}{2+\|\mathring{g}\|} \right) - \\
&\qquad \left(\frac{(\mathring{s} + \mathring{s}\|\mathring{s}\| + (2+\|\mathring{g}\|)P_C(I - \Upsilon S)(\mathring{s}) - \mathring{s} - \mathring{s}\|\mathring{s}\|)}{2+\|\mathring{g}\|} \right) \Bigg\| \\
&= \|P_C(\mathring{g} - \Upsilon S(\mathring{g})) - P_C(\mathring{s} - \Upsilon S(\mathring{s}))\| \\
&= \|P_C(\mathring{g} - (1,0) + \mathring{g}) - P_C(\mathring{s} - (1,0) + \mathring{s})\| \\
&= \|(-(1,0) + (1,0)\| = 0.
\end{aligned}
$$

Therefore, we obtain that

$$
\left\| \frac{\mathring{g}(1+\|\mathring{g}\|) + P\mathring{g}}{2+\|\mathring{g}\|} - \frac{\mathring{s}(1+\|\mathring{s}\|) + P\mathring{s}}{2+\|\mathring{s}\|} \right\| = 0. \tag{27}
$$

It follows from (27) the condition in (28) satisfy for $\daleth(\mathring{g}) = 1 + \|\mathring{g}\|$, $\forall \mathring{g} \in \Theta$. Hence, $F(P)$ is a singleton set, which becomes a solution for $\mathrm{VIP}(S, C)$.

5. Conclusions

We provide a broad class of contractive operators called contractions of the MR-Kannan interpolative kind. Interpolative Kannan type, enriched interpolative Kannan type, Kannan, and enhanced Kannan are among the operators included in this class. A Krasnoselskii-type technique has been developed by us to estimate fixed points of MR-Kannan interpolative type operators. Our exploration involves the analysis of the set of fixed points (see Theorem 6). Furthermore, we have derived Theorems 7–9, which address well-posedness, periodic points, and Ulam-Hyers stability for the fixed-point problem of MR-Kannan interpolative type operators, respectively. Moreover, leveraging our primary findings (see Corollary 10), we have introduced Krasnoselskii projection-type algorithms to solve variational inequality problems within the class of MR-Kannan interpolative type operators.

Here, we now present an open problem.

Open Problem: Following the approach proposed in [9] for the interpolation technique, we present a new problem. Suppose we have positive numbers a and b, where $a + b < 1$, and consider the following condition instead of (28):

$$
\left\| \frac{\mathring{g}\daleth(\mathring{g}) + P\mathring{g}}{1+\daleth(\mathring{g})} - \frac{\mathring{s}\daleth(\mathring{s}) + P\mathring{s}}{1+\daleth(\mathring{s})} \right\| \leq a \left(\left\| \frac{\mathring{g} - P\mathring{g}}{1+\daleth(\mathring{g})} \right\| \right)^a \left(\left\| \frac{\mathring{s} - P\mathring{s}}{1+\daleth(\mathring{s})} \right\| \right)^b, \tag{28}
$$

The open problem is whether the conclusion of Theorem 6 still holds under this new condition.

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