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The Fractional Dunkl Laplacian: Definition and Harmonization via the Mellin Transform

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Abstract: In this paper, we extend the scope of the Tate and Ormerod Lemmas to the Dunkl setting, revealing a profound interconnection that intricately links the Dunkl transform and the Mellin transform. This illumination underscores the pivotal significance of the Mellin integral transform in the realm of fractional calculus associated with differential-difference operators. Our primary focus centers on the Dunkl–Laplace operator, which serves as a prototype of a differential-difference second-order operator within an unbounded domain. Following influential research by Pagnini and Runfola, we embark on an innovative exploration employing Bochner subordination approaches tailored for the fractional Dunkl Laplacian (FDL). Notably, the Mellin transform emerges as a robust and enlightening tool, particularly in its application to the FDL.

Keywords: Dunkl operators; special functions associated with root systems; fractional Dunkl Laplacian

MSC: 33C10; 35R11

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1. Introduction

In his groundbreaking work [1], Dunkl introduced an extraordinary family of first-order differential-difference operators, intricately intertwined with finite reflection groups in the Euclidean space. These operators have recently sparked substantial interest spanning a myriad of mathematical domains and have found profound applications within the realm of physics. For a comprehensive grasp of these operators, one can delve into seminal references like [2–10], along with the wealth of citations contained within.

The Dunkl–Laplace operators emerge as compelling κ-deformations of the conventional Laplacian operator $\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_d^2$. Their significance transcends mere theoretical abstraction, as they stand as pivotal instruments for extending and generalizing numerous classical results well rooted in mathematics. This elevates them to the status of a versatile framework, laying the groundwork for the development of fractional Dunkl–Laplace operators.

This paper delves into the profound impact of the Mellin integral transform within the context of the fractional Dunkl Laplacian. The primary objective is to elucidate how the Mellin transform provides a conclusive means of defining the fractional Dunkl Laplacian. While the Mellin integral transform has been sparingly employed in prior works centered around fractional calculus, its recognition has been significantly amplified by notable instances of its application. For instance, in [11], the Mellin integral transform was ingeniously harnessed to provide a supplementary equivalent definition suitable for cases where the fractional Laplacian is applied to radial functions. Additionally, Ref. [12] demonstrated the establishment of Erdélyi–Kober-type mixed operators as generators for integral transforms of the Mellin convolution type.

The significance of the Mellin integral transform was further highlighted [13,14], where it played a pivotal role in deriving Leibniz-type rules for a variety of fractional calculus operators. Moreover, this transformative technique found a natural synergy with special functions intrinsic to fractional calculus. A plethora of references, including [15–18], have validated this interpretation.

Lastly, a tribute is extended to the comprehensive compendium [15], standing as a testament to the exhaustive exploration of fractional calculus theory, where the Mellin integral transform emerges as a cornerstone.

The structure of the paper is as follows:

In Section 2, we provide an initial overview of the foundational concepts. The topics covered encompass the Dunkl operators, Dunkl transform, and Mellin transforms, collectively setting the stage for a comprehensive understanding of the subsequent content.

Section 3 delves into the effective utilization of Bochner subordination approaches, offering insights into their application within the context of the fractional Dunkl Laplacian.

Section 4 succinctly presents the primary research outcomes. Here, we summarize the significant findings that were attained through our investigation.

In Section 5, we furnish a comprehensive proof of the core results. Through meticulous derivation and thorough explanation, we establish the validity of our findings, providing readers with an in-depth grasp of the underlying mathematical foundations.

2. Preliminaries

To establish the context, we begin by presenting some fundamental aspects of Dunkl operators. Key references for this discussion are [1,5,8,9]. Let $\mathcal{R}$ denote a reduced root system in $\mathbb{R}$. For a vector $v \in \mathcal{R}$, we define the reflection σ_v as follows:

$$\sigma_v(x) = x - 2\frac{\langle x, v\rangle}{\|v\|^2}v, \quad x \in \mathbb{R}^n, \tag{1}$$

where $\langle\, .\, , .\, \rangle$ signifies the standard Euclidean inner product, and $\|x\| = \sqrt{\langle x, x\rangle}$ represents the Euclidean norm in $\mathbb{R}^n$. Consider a function $\kappa : \mathcal{R} \to [0, +\infty)$ that is invariant under the action of the group G of finite reflections associated with the root system $\mathcal{R}$. This function, known as the multiplicity function, guides our study. The Dunkl operators $\mathscr{D}_j$, where $1 \leq j \leq n$, are κ-deformations of partial derivatives ∂_j achieved through difference operators:

$$\begin{aligned}\mathscr{D}_j f(x) &= \partial_j f(x) + \frac{1}{2}\sum_{v\in\mathcal{R}} \kappa(v)\, \langle v, e_j\rangle\, \frac{f(x) - f(\sigma_v(x))}{\langle v,\, x\rangle} \\ &= \partial_j f(x) + \sum_{v\in\mathcal{R}_+} \kappa(v)\, \langle v, e_j\rangle\, \frac{f(x) - f(\sigma_v(x))}{\langle v,\, x\rangle}, \quad j = 1, 2, \ldots, d.\end{aligned} \tag{2}$$

Here, $\mathcal{R}_+$ represents a fixed positive subsystem of $\mathcal{R}$, and $e_1, \ldots, e_n$ denote the standard unit vectors in $\mathbb{R}^n$. It is noteworthy that the Dunkl operators $\mathscr{D}_j$ commute pairwise and exhibit skew-symmetry concerning the G-invariant measure $w_\kappa(x)dx$. This measure incorporates the weight function w_κ, which, for $x \in \mathbb{R}^n$, assumes the form

$$w_\kappa(x) = \prod_{v\in\mathcal{R}} |\langle x, v\rangle|^{\kappa(v)} = \prod_{v\in\mathcal{R}_+} |\langle x, v\rangle|^{2\kappa(v)}.$$

For a fixed $x \in \mathbb{R}^n$, the Dunkl kernel $y \to \mathscr{E}_\kappa(x, y)$ is the unique solution to the system

$$\begin{cases} \mathscr{D}_j f = x_j f, & 1 \leq j \leq n, \\ f(0) = 1. \end{cases}$$

For a function f in the Lebesgue space $L^1(\mathbb{R}^n, w_\kappa)$ with respect to the measure $w_\kappa(x)dx$, the Dunkl transform is defined as follows:

$$\mathscr{F}_\kappa f(y) := \frac{1}{c_\kappa}\int_{\mathbb{R}^n} f(x)\, \mathscr{E}_\kappa(-iy, x) w_\kappa(x)\, dx, \tag{3}$$

where the normalized constant is given by

$$c_\kappa = \int_{\mathbb{R}^n} e^{-\frac{\|x\|^2}{2}} w_\kappa(x)\, dx. \tag{4}$$

Similarly to the Fourier transform which corresponds to the special case of $\kappa \equiv 0$, the Dunkl transform serves as a topological automorphism of the Schwartz space $S(\mathbb{R}^n)$. It can also be extended as an isometric automorphism of $L^2(\mathbb{R}^n, w_\kappa)$. Furthermore, for every $f \in L^1(\mathbb{R}^n, w_\kappa)$ satisfying $\mathscr{F}_\kappa f \in L^1(\mathbb{R}^n, w_\kappa)$, the following relation holds:

$$f(x) = \mathscr{F}_\kappa^2 f(-x), \quad x \in \mathbb{R}^n.$$

For $f \in S(\mathbb{R}^n)$, we have

$$\mathscr{F}_\kappa(\mathscr{D}_j f)(x) = i\, x_j\, \mathscr{F}_\kappa f(x), \quad x \in \mathbb{R}^n, \quad 1 \le j \le n. \tag{5}$$

Similarly to the classical case, a generalized translation operator is introduced in the Dunkl setting, specifically on $L^2(\mathbb{R}^n, w_\kappa)$, which is defined and denoted as follows [10]:

$$\tau^x f(y) = \mathscr{F}_\kappa^{-1}(\mathscr{E}_\kappa(ix, y)\mathscr{F}_\kappa f)(y), \quad y \in \mathbb{R}^n. \tag{6}$$

The Dunkl Laplacian associated with a reduced root system $\mathcal{R}$ and the multiplicity function κ is a differential-difference operator that operates on functions in C^2 as follows:

$$\Delta_\kappa := \sum_{i=1}^n \mathscr{D}_i^2.$$

Explicitly, it can be represented by

$$\Delta_\kappa f(x) = \Delta f(x) + 2 \sum_{v \in \mathcal{R}_+} \frac{\langle \nabla f(x), v\rangle}{\langle v, x\rangle} - \frac{f(x) - f(\sigma_v(x))}{\langle v, x\rangle^2},$$

where Δ signifies the conventional Laplacian operator in $\mathbb{R}^n$, given by

$$\Delta = \sum_{i=1}^n \partial_i^2.$$

Similarly to the fractional Laplacian on $\mathbb{R}^d$, the fractional powers of $(-\Delta_\kappa)^{\alpha/2}$ are defined using the Dunkl transform (3). Indeed, the Dunkl–Laplace operator is essentially self-adjoint on $L^2_\kappa(\mathbb{R}^d)$; see for instance (Theorem 3.1 in [19]). It is a Fourier–Dunkl multiplier with symbol $|\xi|^2$,, since by (5) we have

$$\mathcal{F}_\kappa(-\Delta_\kappa f)(\xi) = |\xi|^2 \mathcal{F}_k(f)(\xi).$$

Recall that a function g defined on $\mathbb{R}^n$ is considered radial if there exists an even function g_0 defined on $\mathbb{R}$ such that $g(x) = g_0(\|x\|)$. The subspace of $S(\mathbb{R}^n)$ consisting of radial functions is denoted by $S_{\mathrm{rad}}(\mathbb{R}^n)$. Within the context of radial functions, a reduction formula for the Fourier transform can be derived as follows:

$$\mathscr{F}_\kappa g(y) = (\mathscr{H}_{\gamma+\frac{n}{2}-1} g_0)(\|y\|), \tag{7}$$

where the Fourier–Bessel transform $\mathscr{H}_{\gamma+\frac{n}{2}-1} g$ is defined as [10]

$$\mathscr{H}_{\gamma+\frac{n}{2}-1} g_0(x) := \int_0^\infty g_0(t)\, \mathscr{J}_{\gamma+\frac{n}{2}-1}(tx) \sigma_\gamma(dx). \tag{8}$$

Here,

$$\mathscr{I}_{\gamma+\frac{n}{2}-1}(x) = \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}x^2)^k}{(2\gamma+n)_k\, k!}$$

and

$$\sigma_\gamma(dx) = \frac{x^{2\gamma+n-1}}{2^{\gamma+n/2-1}\Gamma(\gamma+n/2)}\, dx. \tag{9}$$

Furthermore, the Dunkl Laplacian also has a reduced form:

$$\Delta_\kappa g(x) = \Delta f(x) + 2 \sum_{v\in\mathcal{R}_+} \frac{\langle \nabla g(x), v\rangle}{\langle v, x\rangle}.$$

In polar coordinates $x = ru$, we have

$$\Delta_\kappa g(x) = \frac{d^2 g_0(r)}{dr^2} + \frac{2\gamma+n-1}{r}\frac{dg_0(r)}{dr}. \tag{10}$$

Mellin Transform

The Mellin transform of a function $f(x)$ is defined by the integral [16]

$$\mathcal{M}\{f(x);s\} = \int_0^\infty x^{s-1} f(x)\, dx, \quad s\in\mathbb{C}.$$

For $f \in S(\mathbb{R})$, $\mathscr{M}\{f(x);s\}$ is analytic for all $\Re(s) > 0$; see [20], Lemma 1. The Mellin convolution of two functions f and g is given by

$$(f *^{\mathcal{M}} g)(x) = \int_0^\infty f\Big(\frac{x}{t}\Big) g(t)\, \frac{dt}{t}.$$

This convolution operation satisfies

$$\mathcal{M}\{(f *^{\mathcal{M}} g)(x);s\} = \mathcal{M}\{(f(x);s\}\mathcal{M}\{g(x);s\}. \tag{11}$$

3. Fractional Dunkl Laplacian

In this section, we embark on an in-depth exploration of the fractional Dunkl–Laplace operator, denoted as $(-\Delta_\kappa)^{\alpha/2}$. Our objective is to gain a comprehensive understanding of this operator's properties and implications. Notably, this operator can be likened to a Dunkl transform pseudo-differential operator, characterized by a symbol represented as $\|x\|^\alpha$. To be more precise, for $\alpha\in(0,2)$, we have the following relation:

$$\mathcal{F}_\kappa\Big((-\Delta_\kappa)^{\alpha/2} f\Big)(x) = \|x\|^\alpha \mathcal{F}_\kappa(f)(x), \quad \text{for all } f\in S(\mathbb{R}^n). \tag{12}$$

As our investigation deepens, it is crucial to recall that the Dunkl–Laplace operator $-\Delta_\kappa$ gives rise to a contractive strongly continuous semigroup $(\mathscr{G}_t)_{t\geq 0}$ in various function spaces, such as $C_0(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n, w_\kappa)$. The domain of this operator is defined as

$$D\big(-\Delta_\kappa\big) := \{f\in C_0(\mathbb{R}^n);\, \Delta_\kappa f\in C_0(\mathbb{R}^n)\}.$$

When we consider $t>0$ and $x\in\mathbb{R}^n$, the relationship can be expressed as follows [8]:

$$\mathscr{G}_t f(x) = \mathscr{G}_\kappa(\,.\,,t) * f(x) = \int_{\mathbb{R}^n} \tau^x \mathscr{G}_\kappa(y,t) f(y) w_\kappa(y)\, dy, \tag{13}$$

where

$$\mathscr{G}_\kappa(x,t) = \frac{1}{(2t)^{\gamma+n/2}}\, e^{-\|x\|^2/4t}, \quad x\in\mathbb{R}^n,\ t>0.$$

Furthermore, it is worth highlighting that the inequality [8] remains valid:

$$\|\mathscr{G}_t f\|_{p,\kappa} \leq \|f\|_{p,\kappa}.$$

This underscores the following properties:

$$\mathscr{F}_\kappa(\mathscr{G}_\kappa(\,.\,t))(x) = e^{-t\|x\|^2}, \quad \int_{\mathbb{R}^n} \mathscr{G}_\kappa(x,t)\, w_\kappa(x)\, dx = 1, \quad t > 0. \tag{14}$$

Of particular interest is the scenario when $p \in [1,2)$ and $f \in C_0(\mathbb{R}^n) \cap L^p(\mathbb{R}^n, w_\kappa)$. In this case, the function $u(t,x) = \mathscr{G}_t f(x)$ plays a pivotal role as an infinitely smooth solution to the Cauchy problem:

$$\begin{cases} \Delta_\kappa u(x,t) = \frac{\partial u(x,t)}{\partial t} \\ u(x,0) = f(x). \end{cases}$$

Moreover, when an operator generates a strongly continuous semigroup in a Banach space, its fractional power can be defined through Bochner's subordination.

For $0 < \alpha < 2$, the function $\lambda^{\alpha/2}$ emerges as a Bernstein function and can be represented through the following integral expression:

$$\lambda^{\alpha/2} = \frac{1}{|\Gamma(-\frac{\alpha}{2})|} \int_0^\infty \left(1 - e^{-t\lambda}\right) \frac{dt}{t^{1+\frac{\alpha}{2}}}.$$

In conclusion, we arrive at the formulation

$$\big(-\Delta_\kappa\big)^{\alpha/2} f(x) = \frac{1}{|\Gamma(\frac{\alpha}{2})|} \int_0^\infty \left(f(x) - e^{t\Delta_\kappa} f(x)\right) \frac{dt}{t^{1+\frac{\alpha}{2}}}. \tag{15}$$

Theorem 1. *Let $0 < \alpha < 2$. For $f \in S(\mathbb{R}^n)$, the following pointwise formula for the Dunkl–Laplace fractional operator holds true:*

$$(-\Delta_\kappa)^{\alpha/2} f(x) = 2^{\alpha+\gamma+n} \frac{\Gamma(\gamma + \frac{n+\alpha}{2} + 1)}{|\Gamma(-\frac{\alpha}{2})|} \int_{\mathbb{R}^n} \frac{f(x) - \tau_\kappa^x f(y)}{\|y\|^{\alpha+n+2\gamma}} w_\kappa(y)\, dy.$$

Proof. Referring to the integral representation (13) and (14), we re-express (15) as follows:

$$\begin{aligned} \big(-\Delta_\kappa\big)^{\alpha/2} f(x) &= \frac{1}{|\Gamma(-\frac{\alpha}{2})|} \int_0^\infty \left(f(x) - \mathscr{G}_t f(x)\right) \frac{dt}{t^{1+\frac{\alpha}{2}}} \\ &= \frac{1}{2^{\gamma+n/2}|\Gamma(-\frac{\alpha}{2})|} \int_0^\infty \int_{\mathbb{R}^n} \left(f(x) - \tau_\kappa^x f(y)\right) \frac{e^{-\frac{\|y\|^2}{4t}}}{t^{\gamma+1+\frac{n+\alpha}{2}}} w_\kappa(y)\,(dy)\, dt. \end{aligned}$$

Intertwining the last two integrals for $f \in S(\mathbb{R}^n)$, we obtain

$$\big(-\Delta_\kappa\big)^{\alpha/2} f(x) = \frac{1}{2^{\gamma+n/2}|\Gamma(-\frac{\alpha}{2})|} \int_{\mathbb{R}^n} \left(f(x) - \tau_\kappa^x f(y)\right) \int_0^\infty \frac{e^{-\frac{\|y\|^2}{4t}}}{t^{\gamma+1+\frac{n+\alpha}{2}}} dt\, w_\kappa(y)\, dy.$$

Finally, since

$$\int_0^\infty \frac{e^{-\|y\|^2/4t}}{t^{\gamma+1+\frac{n+\alpha}{2}}}\, dt = 2^{2\gamma+n+\alpha}\Gamma(\gamma + \frac{n+\alpha}{2})\|y\|^{-(2\gamma+n+\alpha)}, \tag{16}$$

we have

$$(-\Delta_\kappa)^{\alpha/2} f(x) = \frac{2^{\alpha+\gamma+n}\Gamma(\gamma + \frac{n+\alpha}{2} + 1)}{|\Gamma(-\frac{\alpha}{2})|} \int_{\mathbb{R}^n} \frac{f(x) - \tau_\kappa^x f(y)}{\|y\|^{\alpha+n+2\gamma}} w_\kappa(y)\, dy.$$

□

4. Statement of Main Results

The following result serves as a pivotal intermediary, casting light upon the Dunkl transform via the powerful Mellin transform. This discovery extends the renowned outcome established by N. Ormerod [20] (Theorem 1), encompassing a wider spectrum of parameter values. Notably, N. Ormerod's seminal work forged a significant connection between the Fourier transform of radial functions in $\mathbb{R}^n$ and the Mellin transform.

In the Dunkl setting, we define the extended Mellin transform $\mathscr{M}\{f(x), s\}$ of a function $f \in S(\mathbb{R}^n)$ as

$$representation.\mathscr{M}\{f(x);s\} = f^*(s) = \frac{1}{2}c_\kappa\Gamma(\gamma + \frac{n}{2}) \int_{\mathbb{R}^n} f(x)\|x\|^{s-2\gamma-n}\, w_\kappa(x)\, dx, \quad s \in \mathbb{C}, \tag{17}$$

where c_κ is a constant related to the Dunkl kernel given in (4).

The following theorem represents our primary result:

Theorem 2. *For $f \in S_{rad}(\mathbb{R}^n)$, the function $\mathscr{M}\{f(x);s\}$ possesses an analytic continuation that is valid for all $s \neq 0$, and it satisfies the functional equation*

$$\mathscr{M}\{f(x);s\} = \frac{2^{s-\gamma-\frac{n}{2}}\Gamma(\frac{s}{2})}{\Gamma(\gamma + \frac{n-s}{2})}\mathscr{M}\{(\mathscr{F}_\kappa f)(x); 2\gamma + n - s\}.$$

The following theorem represents our secondary result:

Theorem 3. *Let $\alpha \in (0, 2)$. For $f \in S_{rad}(\mathbb{R}^n)$, the Mellin transform of the function $(-\Delta_\kappa)^{\alpha/2} f$ is given by*

$$\mathscr{M}\{(-\Delta_\kappa)^{\alpha/2} f(x);s\} = 2^\alpha \frac{\Gamma(\frac{s}{2})\Gamma(\gamma + \frac{n+\alpha-s-n}{2})}{\Gamma(\gamma + \frac{n-s}{2})\Gamma(\frac{s-\alpha}{2})}\mathscr{M}\{f(x);s-\alpha\}.$$

for $s \neq 0$ and $0 < \Re(s) < 2\gamma + n$.

The theorem that follows was previously established by Pagnini and Runfola [11]. However, it is important to note that the subsequent theorem is a special case where the multiplicity function κ in the above theorem is set to $\kappa = 0$.

Theorem 4. *For a radial function f in $S(\mathbb{R}^n)$, where $S(\mathbb{R}^n)$ is the Schwartz space, the fractional Laplace operator $(-\Delta)^{\alpha/2} f(x)$ with $\alpha \in (0, 2)$ is also a radial function. Furthermore,*

$$\mathscr{M}\{(-\Delta)^{\alpha/2} f(x);s\} = 2^\alpha \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{n-(s-\alpha)}{2}\right)}{\Gamma\left(\frac{n-s}{2}\right)\Gamma\left(\frac{s-\alpha}{2}\right)}\mathscr{M}\{f(x);s-\alpha\}, \quad s \in \mathbb{C}$$

for $s \neq 0$ and $0 < \Re(s) < n$.

5. Proof of Main Results

5.1. Proof of Theorem 1

In the following lemma, we provide the inversion formula for the extended Mellin transform (17).

Lemma 1. *For $f \in S_{rad}(\mathbb{R}^n)$, the Mellin transform inversion formula $\mathscr{M}^{-1}\{f^*(x), s\}$ of the function f is given by*

$$\mathscr{M}^{-1}\{f^*(x), s\} = f(x) = \frac{1}{2i\,\pi}\int_{c-i\infty}^{c+i\infty} \mathscr{M}\{f(x);s\}\, \|x\|^{-s}\, ds.$$

Proof. Let $f \in S_{rad}(\mathbb{R}^n)$ such that $f(x) = f_0(\|x\|)$. Utilizing polar coordinates $x = ru$ and exploiting the homogeneity of w_κ, we can express the integral as follows:

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\|x\|^{s-2\gamma-n}\, w_\kappa(x)\, dx &= \int_0^\infty r^{s-2\gamma-n} \int_{\mathbb{S}^{n-1}} f(ru)\, w_\kappa(ru) d\sigma_n(u) r^{n-1}\, dr \\ &= \int_0^\infty f_0(r) r^{s-1} \int_{\mathbb{S}^{n-1}} w_\kappa(u) d\sigma_n(u)\, dr.\end{aligned}$$

Since

$$\int_{\mathbb{S}^{n-1}} w_\kappa(u) d\sigma_n(u) = \frac{2}{c_\kappa \Gamma(\gamma + \frac{n}{2})},$$

it follows that

$$\mathscr{M}\{f(x); s\} = \mathcal{M}\{f_0(r), s\} = \int_0^\infty f_0(r) r^{s-1}\, dr, \quad s \in \mathbb{C}.$$

The result follows from the well-known inversion formula for the Mellin transform. □

The subsequent lemma serves as an analogue to Tate's Lemma 2.4.2 [21] (p. 314), and its relevance is also highlighted by Ormerod (Lemma 2 in [20]).

Lemma 2. *Let f, g be functions in $S_{rad}(\mathbb{R}^n)$. The following equation holds:*

$$\mathscr{M}\{f(x); s\}\mathscr{M}\{(\mathscr{F}_\kappa g)(x); 2\gamma + n - s\} = \mathscr{M}\{(\mathscr{F}_\kappa f)(x); 2\gamma + n - s\}\mathscr{M}\{g(x); s\}, \tag{18}$$

for $0 < Re(s) < n + 2\gamma$.

Proof. Consider f and g in $S_{\mathrm{rad}}(\mathbb{R}^n)$ and f_0, g_0 such that $f = f_0(r)$, $g = g_0(r)$; thus, we have

$$\begin{aligned}&\mathscr{M}\{f(x); s\}\mathscr{M}\{(\mathscr{F}_\kappa) g(x); 2\gamma + n - s\} \\ &\quad = \mathcal{M}\{f_0(r_1); s\}\mathcal{M}\{(\mathscr{H}_{\gamma+\frac{n}{2}-1} g_0(r_2); 2\gamma + n - s\} \\ &\quad = \int_0^\infty \int_0^\infty f_0(r_1) r_1^{s-1}\, (\mathscr{F}_{\gamma+\frac{n}{2}-} g_0(r_2) r_2^{2\gamma+n-1-s}\, dr_1\, dr_2.\end{aligned} \tag{19}$$

Under the transformation $r_1 \to r_1, \quad r_2 \to r_1 r_2$, the right-hand side of (19) becomes

$$\int_0^\infty \int_0^\infty f(r_1) r_1^{2\gamma+n-1}\, \mathscr{H}_{\gamma+\frac{n}{2}-1} g(r_1 r_2) r_2^{2\gamma+n-1-s}\, dr_1\, dr_2.$$

Using (8), the above expression becomes

$$\frac{1}{2^{\gamma+n/2-1}\Gamma(\gamma+n/2)} \int_0^\infty \int_0^\infty \int_0^\infty f(r_1) g(r_3) \mathscr{J}_{\gamma+\frac{n}{2}-1}(r_1 r_2 r_3)(r_1 r_3)^{2\gamma+n-1} r_2^{2\gamma+n-1-s}\, dr_3 dr_1 dr_2.$$

It is evident that this expression is symmetric in f and g, thereby establishing the validity of Formula (18). Moreover, the Schwartz space $S(\mathbb{R}^n)$ is invariant under the Dunkl transform. Therefore, according to [20] (Lemma 1), both the right and left sides of (18) are analytic functions in the region $0 < \mathrm{Re}(s) < 2\gamma + n$. □

Now, we can prove Theorem 2.

Proof. Consider the function $g(x) = e^{-\|x\|^2/2} \in S_{\mathrm{rad}}(\mathbb{R}^n)$. Applying the formula provided above (14), we obtain

$$(\mathscr{F}_\kappa g)(x) = g(x), \quad x \in \mathbb{R}^n. \tag{20}$$

It is clear that

$$\mathscr{M}\{g(x);s\} = \mathscr{M}\{(\mathscr{F}_\kappa g)(x);s\} = 2^{\frac{s}{2}-1}\Gamma(\tfrac{s}{2}).$$

By Lemma 2, for $0 < \mathrm{Re}(s) < 2\gamma + n$, we have:

$$\mathscr{M}\{f(x);s\} = \frac{\mathscr{M}\{g(x);s\}}{\mathscr{M}\{(\mathscr{F}_\kappa g)(x);2\gamma+n-s\}}\mathscr{M}\{(\mathscr{F}_\kappa f)(x);2\gamma+n-s\}.$$

Substituting the derived expressions, we obtain

$$\mathscr{M}\{f(x);s\} = \frac{2^{s-\gamma-\frac{n}{2}}\Gamma(\frac{s}{2})}{\Gamma(\gamma+\frac{n-s}{2})}\mathscr{M}\{(\mathscr{F}_\kappa f)(x);2\gamma+n-s\}.$$

This completes the proof. □

5.2. *Proof of Theorem 2*

We shall now establish Theorem 2.

Proof. We begin with Theorem 1, leading to the following expression:

$$\mathscr{M}\{(-\Delta_\kappa)^{\alpha/2}f(x);s\} = \frac{2^{s-\gamma-\frac{n}{2}}\Gamma(\frac{s}{2})}{\Gamma(\gamma+\frac{n-s}{2})}\mathscr{M}\{(\mathscr{F}_\kappa\Big((-\Delta_\kappa)^{\alpha/2}f\Big)(x);2\gamma+n-s\}. \tag{21}$$

Since we have the relation $\mathscr{F}_\kappa\Big((-\Delta_\kappa)^{\alpha/2}f\Big)(x) = \|x\|^\alpha(\mathscr{F}_\kappa f)(x)$, we can simplify the expression further:

$$\mathscr{M}\{\mathscr{F}_\kappa\Big((-\Delta_\kappa)^{\alpha/2}f\Big)(x);2\gamma+n-s\} = \mathscr{M}\{(\mathscr{F}_\kappa f)(x);2\gamma+n-(s-\alpha)\}.$$

Now, utilizing Theorem 1 again, we arrive at

$$\mathscr{M}\{(\mathscr{F}_\kappa f)(x);2\gamma+n-(s-\alpha)\} = 2^{\gamma+\frac{n}{2}+\alpha-s}\,\frac{\Gamma(\gamma+\frac{n+\alpha-s}{2})}{\Gamma(\frac{s-\alpha}{2})}\mathscr{M}\{f(x);s-\alpha\}. \tag{22}$$

Combining Equations (21) and (22), we obtain the final result:

$$\mathscr{M}\{(-\Delta_\kappa)^{\alpha/2}f(x);s\} = 2^\alpha\,\frac{\Gamma(\frac{s}{2})\Gamma(\gamma+\frac{n+\alpha-s-n}{2})}{\Gamma(\gamma+\frac{n-s}{2})\Gamma(\frac{s-\alpha}{2})}\mathscr{M}\{f(x);s-\alpha\}.$$

This concludes the proof. □

6. The Rank-One Case

Consider the rank-one case, where the root system $\mathcal{R}$ consists of $\{\pm\sqrt{2}\}$, $G = \mathbb{Z}_2$, and $w_\kappa(x) = |x|^{2\kappa}$. In this context, we define the Dunkl operator $\mathscr{D}_\kappa$, associated with the multiplicity parameter $\kappa \geq 0$, as follows:

$$\mathscr{D}_\kappa := \partial_x + \frac{\kappa}{x}(1-s).$$

The corresponding Dunkl–Laplace operator Δ_κ is given by

$$\Delta_\kappa := \mathscr{D}_\kappa^2 = \partial_x^2 + \frac{2\kappa}{x}\partial_x - \frac{\kappa}{x^2}(1-s), \tag{23}$$

where s represents the reflection operator, which acts on a function $f(x)$ of a real variable as

$$(sf)(x) := f(-x).$$

Now, let us consider the so-called nonsymmetric Bessel function, also known as the Dunkl-type Bessel function, in the rank-one case (see [5] §4):

$$\mathscr{E}_\kappa(x) := \mathscr{J}_{\kappa-1/2}(ix) + \frac{x}{2\kappa+1}\,\mathscr{J}_{\kappa+1/2}(ix).$$

Then, we have the eigenvalue equations:

$$\mathscr{D}_\kappa(\mathscr{E}_\kappa(i\lambda x)) = i\,\lambda\,\mathscr{E}_\kappa(i\lambda x), \quad \Delta_\kappa(\mathscr{E}_\kappa(i\lambda x)) = -\lambda^2\,\mathscr{E}_k(i\lambda x).$$

The Dunkl transform is defined as

$$(\mathscr{F}_\kappa f)(\lambda) := \frac{1}{2^{\kappa+\frac{1}{2}}\Gamma(\kappa+\frac{1}{2})}\int_{\mathbb{R}} f(x)\,\mathscr{E}_\kappa(-i\lambda x)\,|x|^{2\kappa}dx.$$

In [22], Rösler introduced the following generalized translation τ^x:

$$\begin{aligned}\tau^x f(y) := &\frac{1}{2}\int_{-1}^{1} f(\sqrt{x^2+y^2-2xyt})(1+\frac{x-y}{\sqrt{x^2+y^2-2xyt}})h_k(t)dt\\ &+\frac{1}{2}\int_{-1}^{1} f(-\sqrt{x^2+y^2-2xyt})(1-\frac{x-y}{\sqrt{x^2+y^2-2xyt}})\,h_k(t)\,dt,\end{aligned}$$

where

$$h_\kappa(t) = \frac{\Gamma(\kappa+1/2)}{2^{2\kappa}\sqrt{\pi}\Gamma(\kappa)}(1+t)(1-t^2)^{\kappa-1}.$$

The proof of the corollary at hand is readily derived from the implications of Theorem 1.

Corollary 1. *Let $f \in S(\mathbb{R})$. Then, we have*

$$(-\Delta_\kappa)^{\alpha/2} f(x) = \frac{2^{\alpha+1}\Gamma(\kappa+\frac{\alpha+1}{2})}{\Gamma(\kappa+\frac{1}{2})|\Gamma(-\frac{\alpha}{2})|}\int_{\mathbb{R}}\frac{f(x)-\tau^x f(y)}{|y|^{\alpha+\kappa+1}}dy.$$

It is important to note that when considering even functions in the Schwartz space $S(\mathbb{R})$, the Dunkl Laplacian defined in (23) simplifies to the Bessel operator $\mathscr{B}_\kappa$, which is given by

$$\mathscr{B}_\kappa = \frac{d^2}{dx^2} + \frac{2\kappa}{x}\frac{d}{dx}.$$

The proof of the following corollary is a straightforward consequence of the implications presented in Theorem 3.

Corollary 2. *Let $\kappa \geq 0$ and $\alpha \in (0,2)$. For $f \in S_{rad}(\mathbb{R})$, the Mellin transform of the function $(-\mathscr{B}_\kappa)^{\alpha/2} f$ is given by*

$$\mathscr{M}\{(-\mathscr{B}_\kappa)^{\alpha/2} f(x); s\} = 2^\alpha \frac{\Gamma(\frac{s}{2})\Gamma(\kappa-\frac{s-\alpha-1}{2})}{\Gamma(\kappa-\frac{s-1}{2})\Gamma(\frac{s-\alpha}{2})}\mathscr{M}\{f(x); s-\alpha\}$$

for $s \neq 0$ and $0 < \Re(s) < 2\kappa+1$.

A direct result of Corollary 2 in the scenario where $\kappa = 0$ yields the following corollary, which is applicable to the second-order fractional derivative.

Corollary 3. *Let $\alpha \in (0,2)$. For $f \in S_{rad}(\mathbb{R})$, the Mellin transform of the function $(-\frac{d^2}{dx^2})^{\alpha/2} f$ is given by*

$$\mathscr{M}\{(-\frac{d^2}{dx^2})^{\alpha/2} f; s\} = \frac{\Gamma(s)\cos(\frac{\pi}{2}s)}{\Gamma(s-\alpha)\cos[\frac{\pi}{2}(s-\alpha)]}\mathscr{M}\{f(x); s-\alpha\}, \quad s \in \mathbb{C}$$

for $s \neq 0$ *and* $0 < \Re(s) < 1$.

Concluding Remarks

In summary, this paper delved into the intricate interconnections that exist between the Mellin integral transform and the fractional Dunkl Laplacian. Through the strategic application of the Mellin transform, we successfully introduced a fresh perspective on the fractional Dunkl–Laplace operator, shedding light on its intricate relationships with other fractional calculus operators. The outcomes of this study not only contribute to a profound comprehension of the interplay between diverse mathematical concepts but also set the stage for further explorations into the expansive applications of the Mellin transform within the domain of fractional calculus. The adept deployment of the Mellin transform in this particular context elegantly demonstrates its versatility and its pivotal role in unearthing novel insights in the realm of mathematical analysis. This work not only extends the frontiers of existing knowledge but also beckons forth promising avenues for prospective research within this dynamic and evolving field.

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