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Article

Diameter Estimate in Geometric Flows

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Abstract: We prove the upper and lower bounds of the diameter of a compact manifold $(M, g(t))$ with $\dim_{\mathbb{R}} M = n (n \geq 3)$ and a family of Riemannian metrics $g(t)$ satisfying some geometric flows. Except for Ricci flow, these flows include List–Ricci flow, harmonic-Ricci flow, and Lorentzian mean curvature flow on an ambient Lorentzian manifold with non-negative sectional curvature.

Keywords: geometric flow; heat kernel; diameter bound

MSC: 35K05; 35K08; 35K10; 53C21; 53E20

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1. Introduction

The calculation and estimate of geometric quantities (e.g., volume, diameter, and curvature tensor [1]) play essential roles in the study of Riemannian geometry. It is also important and interesting to study the uniform properties of these geometric quantities under a family of Riemannian metric $g(t)$ with $t \in [0, T)$ for some $T \in (0, +\infty]$. One of the most famous examples is Ricci flow

$$\frac{\partial g_{ij}}{\partial t}(x, t) = -2R_{ij}(x, t), \tag{1}$$

introduced by Hamilton [2]. It is a nonlinear weakly parabolic equation along which the Riemannian metric is evolved, and is a powerful theoretic tool to research geometric problems such as the Poincaré conjecture (see Perelman [3–5]). Up to now, we understand the κ non-collapsing property proved by Perelman [3], the improved non-collapsing property of Ricci flow proved by Jian [6], the κ non-inflated property proved by Zhang [7] (see also Chen-Wang [8]), and the uniform (logarithmic) Sobolev inequality along the Ricci flow proved by Zhang [9] and Ye [10]. It was also proved by Perelman that the diameter is uniformly bounded along the (normalized) Kähler–Ricci flow, which is the Ricci flow with the Kähler metric as its initial metric on Fano manifolds (see Sesum-Tian [11]). Recently, Jian-Song [12] proved that if the canonical line bundle K_M of the Kähler manifold M is semi-ample, then the diameter is uniformly bounded for long-time solutions of the normalized Kähler–Ricci flow (see Jian-Song-Tian [13] for the most recent results in this direction). In [14], the author demonstrated a lower bound for the diameter along the Ricci flow with nonzero $H^1(M^n, \mathbb{R})$. For the Ricci flow (1) on a compact Riemannian manifold M with $\dim_{\mathbb{R}} M = n$, Topping [15] showed that there is a uniform constant $C = C(g_0, T)$ such that if

$$\operatorname{diam}(M, g(t)) \geq C, \quad \forall\, t \in [0, T),$$

then, it yields that

$$\operatorname{diam}(M, g(t)) \leq C \int_M |R|^{\frac{n-1}{2}} \mathrm{d}\mu(t),$$

where R denotes the scalar curvature of the Levi–Civita connection of $g(t)$. More details about the constant C can be found in [15]. Recently, Zhang [16] proved that the upper bound for the diameter of $(M, g(t))$ on compact Riemannian manifolds depends only on the $L^{\frac{n-1}{2}}$ norm of the scalar curvature of g, the volume and the Sobolev constants (see [16]). Zhang [16] also deduced the lower bound for the diameter of $(M, g(t))$, which depends only on the time t, the initial metric, and the L^∞ norm of the scalar curvature (see more details in Remark 1 and Theorem 1).

Motivated by [15,16], we investigate the geometric flow

$$\frac{\partial g_{ij}}{\partial t}(x,t) = -2\mathcal{S}_{ij}(x,t), \quad \forall\ (x,t) \in M \times [0,T), \tag{2}$$

where $T \in (0, +\infty]$ and $g(0) = g_0$ is a Riemannian metric. Here, $\mathcal{S}_{ij}(x,t)$'s denote the components of a symmetric 2-tensor $\mathcal{S}$. We deduce the bound of the diameter of a compact manifold $(M, g(t))$ with $\dim_{\mathbb{R}} M = n (n \geq 3)$ and a family of Riemannian metrics $g(t)$ satisfying the geometric flow (2) under certain assumptions. For later use, we need to define a tensor quantity $\mathcal{D}_2$ associated to the tensor $\mathcal{S}$ (see [17], Definition 1.3).

Definition 1. *Let $g(t)$ be a family of smooth Riemannian metrics satisfying the geometry flow (2) on $M \times [0,T)$. Then, we define*

$$\begin{aligned}\mathcal{D}_2(\mathcal{S}, X) :=& \frac{\partial S}{\partial t} - \Delta_{g(t)} S - 2|\mathcal{S}|^2_{g(t)} \\ &+ 4(\nabla^i \mathcal{S}_{ij})X^j - 2X^i \nabla_i S + 2R_{ij}X^iX^j - 2\mathcal{S}_{ij}X^iX^j, \quad \forall\ X \in \mathfrak{X}(M),\end{aligned} \tag{3}$$

where $S = \sum_{i,j=1}^{n} g^{ij}(t)\mathcal{S}_{ij}(t)$. Here, ∇ and R_{ij} denote the Levi–Civita connection and the Ricci curvature, respectively, of the Riemannian metric $g(t)$.

If

$$\mathcal{D}_2(\mathcal{S}, X) \geq 0, \quad \forall\ X \in \mathfrak{X}(M)$$

on $[0,\ T)$, then we say that $\mathcal{D}_2(\mathcal{S}, \cdot)$ is non-negative.

Remark 1. *In fact, the quantity $\mathcal{D}_2$ is the difference between two differential Harnack-type quantities for the symmetric tensor $\mathcal{S}$ [17]. Hence, the non-negativity of $\mathcal{D}_2$ is equivalent to the corresponding differential Harnack-type inequality for the tensor $\mathcal{S}$ under the geometric flow.*

We note that, if $\mathcal{D}_2(\mathcal{S}, \cdot)$ and $(\mathrm{Ric} - \mathcal{S})$ are non-negative, then there also hold a κ non-collapsing property, the so-called κ non-inflated property, and the uniform (logarithmic) Sobolev inequality along the geometric flow (2) (see [17–23] and the references therein). Thanks to [24,25], the many properties of the heat kernel also hold along the geometric flow (2) under the same assumptions, similar to those of the heat kernel along the Ricci flow (1). Given these preliminaries, we can consider the diameter estimate along the geometric flow (2).

Let $g(t)$ be a family of smooth Riemannian metrics satisfying the geometry flow (2) on $M \times [0,T)$. Then, we use the notations

$$\begin{aligned} \mathrm{d}\mu(t) :=& \text{the volume element of } g(t), \text{ and sometimes we also write } \mathrm{d}\mu(g(x,t)) \\ & \text{to emphasize the space variants,} \\ \mathrm{Vol}_{g(t)}(M) :=& \text{the volume of } M \text{ with respect to } g(t), \\ (p,q,t) :=& \text{the distance between } p \text{ and } q \text{ with respect to } g(t), \\ B(p,r,t) :=& \text{the geodesic ball with center } p \text{ and radius } r \text{ with respect to } g(t), \\ \mathrm{Vol}_{g(t)}(B(p,r,t)) :=& \text{the volume of } B(p,r,t) \text{ with respect to } g(t). \end{aligned}$$

Moreover, we will write the trace S as S_t or $S(x,t)$ to emphasize the time(-space) variant(s). Now, we state our main theorem.

Theorem 1. *Let $g(t)$ be a family of smooth Riemannian metrics satisfying (2) on $M \times [0,T)$ with $\dim_{\mathbb{R}} M = n(\geq 3)$. Assume that $\mathcal{D}_2(\mathcal{S}, \cdot)$ defined in (3) and $(\mathrm{Ric} - \mathcal{S})$ are non-negative. For the upper bound on the diameter of $(M, g(t))$, we have*

$$\mathrm{diam}(M, g(t)) \leq C\left(\mathrm{Vol}_{g(t)}(M) + 1 + \int_M (S_t^+)^{\frac{n-1}{2}} \mathrm{d}\mu(t)\right), \tag{4}$$

where C denotes a constant depending only on n, A and B defined in (8). Here, $S_t^+ = \max\{0, S_t\}$. For the lower bound on the diameter of $(M, g(t))$, we have either $\mathrm{diam}(M, g(t)) \geq \sqrt{t}$ or

$$\begin{aligned} \mathrm{diam}(M, g(t)) \geq & c_2 e^{\frac{1}{n}\left[-\alpha t - \beta - t\|S^-(\cdot,0)\|_\infty\right]} e^{-\frac{2}{n}\int_0^t \|S(\cdot,s)\|_\infty \mathrm{d}s} \\ & \times \left[1 + \frac{2t}{n}\|S^-(\cdot,0)\|_\infty\right]^{-\frac{1}{2}} \left[\mathrm{Vol}_{g_0}(M)\right]^{\frac{1}{n}}. \end{aligned} \tag{5}$$

Here, $S^-(x,t) = \min\{0, S(x,t)\}$ and c_2 are constants depending only on n. The constants α and β are positive constants which depend only on the infimum of the $\mathcal{F}$ functional defined by (7) for (M, g_0) and the Sobolev constant of (M, g_0). Furthermore, if $S(\cdot, 0) \geq 0$, then we have $\alpha = 0$, which implies

$$\mathrm{diam}(M, g(t)) \geq c_2 e^{-\frac{\beta}{n}} e^{-\frac{2}{n}\int_0^t \|S(\cdot,s)\|_\infty \mathrm{d}s} \left(\mathrm{Vol}_{g_0}(M)\right)^{\frac{1}{n}}. \tag{6}$$

Remark 2. *Our theorem will reduce to Zhang's result [16] for the Ricci flow totally if the symmetric 2-tensor $\mathcal{S}$ is the Ricci curvature of $g(t)$.*

Remark 3. *Our theorem can also be applied to the following geometric flows except for the Ricci flow. The geometric and physical meanings of these geometric flows and more details of the calculation of $(\mathrm{Ric} - \mathcal{S})$ and $\mathcal{D}_2(\mathcal{S}, \cdot)$ can be found in [17,21,26].*

i. ***List–Ricci flow*** *(see List [27]).*

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t}(x,t) = -2R_{ij}(x,t) + 4(\mathrm{d}\phi \otimes \mathrm{d}\phi)(x, t), \\ \dfrac{\partial \phi}{\partial t}(x, t) = \left(\Delta_{g(x,t)}\phi\right)(x, t), \end{cases}$$

with $\phi \in C^\infty(M \times \mathbb{R}, \mathbb{R})$. In this case, it follows that

$$\mathrm{Ric} - \mathcal{S} = 2\mathrm{d}\phi \otimes \mathrm{d}\phi \geq 0$$

and

$$\mathcal{D}_2(\mathcal{S}, X) = 4|\Delta\phi - \nabla_X\phi|^2 \geq 0.$$

ii. ***Harmonic-Ricci flow*** *(see Müller [28]).*

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t}(x,t) = -2R_{ij}(x,t) + 2\alpha(t)(\nabla\psi \otimes \nabla\psi)(x,t), \\ \dfrac{\partial \psi}{\partial t}(x,t) = \left(\tau_{g(x,t)}\psi\right)(x,t), \end{cases}$$

where M and N are compact manifolds with the Riemannian metrics $g(\cdot, t)$ and h, respectively,

$$\psi(\cdot, t) : (M, g(\cdot, t)) \to (N, h)$$

are a family of smooth maps, $\tau_g\psi$ is the intrinsic Laplacian of ψ, and $\alpha(t)$ is a smooth, positive and non-increasing function defined on $\mathbb{R}$. In this case, there holds

$$\mathrm{Ric} - \mathcal{S} = \alpha \mathrm{d}\psi \otimes \mathrm{d}\psi \geq 0$$

and

$$\mathcal{D}_2(\mathcal{S}, X) = 2\alpha|\tau_g\psi - \nabla_X\psi|^2 - \dot{\alpha}|\nabla\psi|^2 \geq 0.$$

See Tadano [29] for a lower bound of the diameter for shrinking the Ricci-harmonic soliton.

iii. ***Lorentzian mean curvature flow*** *(see Holder [30] and Müller [17]). Let M be a compact space-like hyper-surface with $\dim_{\mathbb{R}} M = n$ in an ambient Lorentzian manifold L with $\dim_{\mathbb{R}} L = n+1$, and let*

$$F_0 : M \longrightarrow L$$

denote a smooth immersion from M into L. We denote by

$$F(\cdot, t) : M^n \longrightarrow L^{n+1}$$

a family of smooth immersions with $F(\cdot, 0) = F_0(\cdot)$ and

$$\frac{\partial F}{\partial t}(p, t) = H(p, t)\nu(p, t), \quad \forall\, (p, t) \in M \times [0, T),$$

where $\nu(p, t)$ and $H(p, t)$ are the future-oriented, time-like normal vector and the mean curvature of the hyper-surface $M_t = F(M^n, t)$ at the point $F(p, t)$, respectively. It follows that

$$\frac{\partial g_{ij}}{\partial t} = 2HA_{ij},$$

where $A = (A_{ij})$ denotes the second fundamental form on M_t. In this setup, one has

$$\mathcal{S}_{ij} = -HA_{ij}, \quad S = -H^2.$$

Mark the curvature tensor of L with a bar. If L has non-negative sectional curvature, then one can deduce that

$$\mathcal{D}_2(\mathcal{S}, X) = 2\overline{\mathrm{Ric}}(H\nu - X, H\nu - X) + 2\langle\overline{\mathrm{Rm}}(X, \nu)\nu, X\rangle + 2|\nabla H - A(X, \cdot)|^2 \geq 0$$

and that

$$\mathrm{Ric}(X, X) - \mathcal{S}(X, X) = \overline{\mathrm{Ric}}(X, X) + \langle\overline{\mathrm{Rm}}(X, \nu)\nu, X\rangle + X^i A_{i\ell} A_{\ell j} X^j \geq 0, \quad \forall\, X \in \mathfrak{X}(M).$$

2. Proof of Main Theorem

We need some preliminaries in order to prove Theorem 1. Let (M, g) be a compact Riemannian manifold with $\dim_{\mathbb{R}} M = n$ and the Riemannian metric g. Then, fixing a smooth function $S \in C^\infty(M, \mathbb{R})$, we can define the $\mathcal{F}$ entropy by

$$\mathcal{F}(g, h) = \int_M \left(S + |\nabla h|_g^2\right) e^{-h} \mathrm{d}\mu, \quad \forall\, h \in C^\infty(M, \mathbb{R}) \quad \text{with} \quad \int_M e^{-h} \mathrm{d}\mu = 1, \tag{7}$$

where $\mathrm{d}\mu$ is the volume element of the Riemannian metric g. When we take S to be the scalar curvature R of the Levi–Civita connection of the Riemannian metric g,, the $\mathcal{F}$ entropy defined in (7) is exactly the one defined by Perelman [3].

Let $v = e^{-\frac{h}{2}}$. Then, we have

$$\mathcal{F}(g, h) = \mathcal{F}^*(g, v) = \int_M \left(4|\nabla v|_g^2 + Sv^2\right) \mathrm{d}\mu, \quad \int_M v^2 \mathrm{d}\mu = 1.$$

We define

$$\begin{aligned} 4\lambda_0(g) :=& \inf\left\{\mathcal{F}^*(g, v) : \ v \in C^\infty(M, \mathbb{R}), \quad \int_M v^2 \mathrm{d}\mu = 1\right\} \\ =& \inf\left\{\mathcal{F}(g, h) : \ h \in C^\infty(M, \mathbb{R}), \quad \int_M e^{-h} \mathrm{d}\mu = 1\right\}. \end{aligned}$$

It follows from the standard theory partial differential equations that $4\lambda_0(g)$ is the first eigenvalue of the operator $-4\Delta_g + S$. Let $u_0 \in C^\infty(M, \mathbb{R})$ be a first positive eigenfunction of the operator $-4\Delta_g + S$ with

$$-4\Delta_g u_0 + Su_0 = 4\lambda_0(g)u_0.$$

Then, $h_0 = -2\log u_0$ satisfies

$$4\lambda_0(g) = \mathcal{F}(g, h_0)$$

with

$$-2\Delta_g h_0 + |\nabla h_0|^2_g - S = -4\lambda_0(g).$$

Lemma 1 (Part of Lemma 3.1 in [21])**.** *Let $g(x,t)$ be a family of Riemannian metrics along the geometric flow (2) on $M \times [0, T)$ and let $h(x,t)$ be a positive solution to the backward heat equation*

$$\frac{\partial}{\partial t}h(x,\, t) = -\Delta_{g(x,t)}h + |\nabla h|^2_{g(x,t)} - S(x,\, t).$$

Then, we have

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = \int_M \Big(2|h_{ij} + \mathcal{S}_{ij}|^2 + \mathcal{D}_2(\mathcal{S}, \nabla h)\Big)e^{-h}\mathrm{d}\mu(t)$$

where $\mathcal{D}_2(\mathcal{S}, \nabla h)$ is defined by (3).

In particular, the $\mathcal{F}$ entropy is non-decreasing in t if $\mathcal{D}_2(\mathcal{S}, \cdot)$ is non-negative for all $t \in [0, T)$, from which we can obtain that $\lambda_0(g(t))$ is non-decreasing of t.

Now, we can state the uniform Sobolev inequality along the geometric flow (2).

Lemma 2 (Theorem 1.5 in [21])**.** *Let $g(t)$ be a family of smooth Riemannian metrics satisfying (2) on $M \times [0, T)$ and assume that $\mathcal{D}_2(\mathcal{S}, \cdot)$ defined in (3) is non-negative. Then, for each $u \in W^{1,2}(M, g(t))$, there holds*

$$\left(\int_M |u|^{\frac{2n}{n-2}}\mathrm{d}\mu(t)\right)^{\frac{n-2}{n}} \leq A\int_M \Big(4|\nabla u|^2_{g(t)} + S_t u^2\Big)\mathrm{d}\mu(t) + B\int_M u^2\mathrm{d}\mu(t), \tag{8}$$

where A and B are positive constants which depend only on $M, g(0), n, t, S(\cdot, 0)$ and the Sobolev constant of $(M, g(0))$. In particular, if $\lambda_0(g(0)) > 0$, then $B = 0$ and A are independent of t.

The second ingredient of the proof can be considered as a quantified version of the non-collapsing theorem along the geometric flow (2) (see also Theorem 1.6 in [21]).

Lemma 3. *Let $g(t)$ be a family of smooth Riemannian metrics satisfying (2) on $M \times [0, T)$, and assume that $\mathcal{D}_2(\mathcal{S}, \cdot)$ defined in (3) is non-negative. Then, it yields that*

$$\mathrm{Vol}_{g(t)}(B(x, r, t)) \geq 4^{-\frac{n^2}{4}}\Big(64A + 4AM_2(x, t, S^+_t, r) + 4Br^2\Big)^{-\frac{n}{2}}r^n, \tag{9}$$

where A and B are the constants in (8) and $M_2(x, t, S^+_t, r)$ is defined by (12).

Proof. Motivated by [31], define

$$u(y) = \begin{cases} r - (x, y, t), & y \in B(x, r, t), \\ 0, & \text{otherwise.} \end{cases}$$

Substituting this u into (8), we obtain

$$
\int_M 4|\nabla u|^2_{g(t)} \mathrm{d}\mu(t) = 4\mathrm{Vol}_{g(t)}(B(x,r,t)), \tag{10}
$$

$$
\begin{aligned}
\int_M S_t u^2 \mathrm{d}\mu(t) \leq & r^2 \int_{B(x,r,t)} S_t^+ \mathrm{d}\mu(t) \\
= & \mathrm{Vol}_{g(t)}(B(x,r,t)) \frac{r^2}{\mathrm{Vol}_{g(t)}(B(x,r,t))} \int_{B(x,r,t)} S_t^+ \mathrm{d}\mu(t) \\
\leq & M_2(x,t,S_t^+,r) \mathrm{Vol}_{g(t)}(B(x,r,t)),
\end{aligned} \tag{11}
$$

where $M_2(x,t,S_t^+,r)$ is a maximal type function defined by (see [15,16])

$$
M_2(x,t,S_t^+,r) = \sup_{0<\rho\leq r} \frac{\rho^2}{\mathrm{Vol}_{g(t)}(B(x,\rho,t))} \int_{B(x,\rho,t)} S_t^+ \mathrm{d}\mu(t). \tag{12}
$$

Note that

$$
\int_{B(x,\rho,t)} u^2 \mathrm{d}\mu(t) \leq r^2 \mathrm{Vol}_{g(t)}(B(x,\rho,t)). \tag{13}
$$

Since $u \geq \frac{r}{2}$ on $B(x, \frac{r}{2}, t)$, from (8) and the Hölder inequality, we can deduce

$$
\begin{aligned}
& \frac{r^2}{4} \mathrm{Vol}_{g(t)} \left[B\left(x, \frac{r}{2}, t\right) \right] \\
\leq & \int_{B(x,r,t)} u^2 \mathrm{d}\mu(t) \\
\leq & \Big(\mathrm{Vol}_{g(t)}(B(x,r,t)) \Big)^{\frac{2}{n}} \left(\int_{B(x,r,t)} u^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \\
\leq & \Big(\mathrm{Vol}_{g(t)}(B(x,r,t)) \Big)^{\frac{2}{n}} \left(A \int_M (4|\nabla u|^2_{g(t)} + S_t u^2) \mathrm{d}\mu(t) + B \int_M u^2 \mathrm{d}\mu(t) \right).
\end{aligned} \tag{14}
$$

Substituting (10), (11) and (13) into (14) yields

$$
\left(4A + AM_2(x,t,S_t^+,r) + Br^2 \right) \Big(\mathrm{Vol}_{g(t)}(B(x,r,t)) \Big)^{\frac{n+2}{n}} \geq \frac{r^2}{4} \mathrm{Vol}_{g(t)} \left[B\left(x, \frac{r}{2}, t\right) \right],
$$

i.e.,

$$
\begin{aligned}
\mathrm{Vol}_{g(t)}(B(x,r,t)) \geq & r^{\frac{2n}{n+2}} \left(\mathrm{Vol}_{g(t)} \left[B\left(x, \frac{r}{2}, t\right) \right] \right)^{\frac{n}{n+2}} \\
& \left(16A + 4AM_2(x,t,S_t^+,r) + 4Br^2 \right)^{-\frac{n}{n+2}}.
\end{aligned} \tag{15}
$$

Note that, for any $s \in (0, r]$, Equation (15) still holds by replacing r with s. Since

$$
M_2(x,t,S_t^+,s) \leq M_2(x,t,S_t^+,r), \quad s^2 \leq r^2,
$$

we arrive at

$$
\begin{aligned}
\mathrm{Vol}_{g(t)}(B(x,s,t)) \geq & \left(\mathrm{Vol}_{g(t)} \left[B\left(x, \frac{s}{2}, t\right) \right] \right)^{\frac{n}{n+2}} s^{\frac{2n}{n+2}} \\
& \left[16A + 4AM_2(x,t,S_t^+,s) + 4Bs^2 \right]^{-\frac{n}{n+2}}. \\
\geq & \left(\mathrm{Vol}_{g(t)} \left[B\left(x, \frac{s}{2}, t\right) \right] \right)^{\frac{n}{n+2}} s^{\frac{2n}{n+2}} \\
& \left[16A + 4AM_2(x,t,S_t^+,r) + 4Br^2 \right]^{-\frac{n}{n+2}}.
\end{aligned} \tag{16}
$$

Iterating (16) with $s = r, \frac{r}{2}, \cdots, \frac{r}{2^{k-1}}$ for positive integers k gives

$$\mathrm{Vol}_{g(t)}(B(x,r,t)) \geq 4^{-\sum\limits_{i=1}^{k}(i-1)\left(\frac{n}{n+2}\right)^{i}} \left(\mathrm{Vol}_{g(t)}\left[B\left(x,\frac{r}{2^k},t\right)\right]\right)^{\left(\frac{n}{n+2}\right)^{k}} \left[r^2\left\{16A+4AM_2(x,t,S_t^+,r)+4Br^2\right\}^{-1}\right]^{\sum\limits_{i=1}^{k}\left(\frac{n}{n+2}\right)^{i}}. \tag{17}$$

Letting $k \longrightarrow \infty$ in (17) implies (9). □

We set

$$d(t) := \text{the diameter of } M \text{ with respect to the Riemannian metric } g(t)$$
$$V(t) := \text{the volume of } M \text{ with respect to the Riemannian metric } g(t)$$

for $t \in (0,T]$. For any $x \in M$, we write

$$\kappa = \kappa(x,\, r) := \frac{\mathrm{Vol}_{g(t)}(B(x,r,t))}{r^n}.$$

Also set

$$\kappa_0 := \min\left\{4^{-\frac{n^2}{4}}(24A+4Br^2)^{-\frac{n}{2}},\, \frac{\omega_n}{2}\right\},$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. Now, we give the lower bound of $M_2(x,t,S_t^+,r)$.

Lemma 4. *Let $g(t)$ be a family of smooth Riemannian metrics satisfying (2) on $M \times [0,T)$ and assume that $\mathcal{D}_2(\mathcal{S},\cdot)$ defined in (3) is non-negative. If $r \leq \frac{d(t)}{2}$ and $\kappa \leq \kappa_0$, then we have*

$$M_2(x,t,S_t^+,r) \geq 2. \tag{18}$$

In particular, if $d(t) \geq 2$ and $r \leq 1$, then κ_0 can be taken as

$$\kappa_0 := \min\left\{4^{-\frac{n^2}{4}}(24A+4B)^{-\frac{n}{2}},\, \frac{\omega_n}{2}\right\}. \tag{19}$$

Proof. From (9), we have

$$16A+4AM_2(x,t,S_t^+,r)+4Br^2 \geq 4^{-\frac{n}{2}}\kappa^{-\frac{2}{n}}. \tag{20}$$

This means, for $r \leq \frac{d(t)}{2}$,

$$M_2(x,t,S_t^+,r) \geq \frac{4^{-\frac{n}{2}}\kappa^{-\frac{2}{n}} - 4Br^2 - 16A}{4A}. \tag{21}$$

Finally, Equation (18) follows from the definition of κ_0 and $\kappa \leq \kappa_0$. □

Based on the preliminaries as above, we can prove Theorem 1.

Proof of Theorem 1. For the upper bound of the diameter, let

$$N := \max\left\{n \in \mathbb{N} : n \leq \frac{d(t)}{4}\right\}.$$

We choose two points $a, b \in M$ with $(a,\, b,\, t) = d(t)$. Denote by γ a minimal geodesic connecting a and b. Choose $p \in \gamma$ such that

$$(a,\, p,\, t) = (p,\, b,\, t) = \frac{d(t)}{2}.$$

We claim that, if

$$d(t) \geq \frac{V(t)4^{n+3}}{\kappa_0}, \tag{22}$$

then, for at least N many positive integers i_j in $\{1, 2, \cdots, 2N\}$, we have

$$\left| \mathrm{Vol}_{g(t)}(B(p, i_j, t)) - \mathrm{Vol}_{g(t)}(B(p, i_j - 1, t)) \right| \leq \kappa_0 4^{-n}, \quad j = 1, \cdots, N.$$

Indeed, if the claim is not right, then there exist at least $(N+1)$ integers k_j in $\{1, 2, \cdots, 2N\}$ such that

$$\left| \mathrm{Vol}_{g(t)}(B(p, k_j, t)) - \mathrm{Vol}_{g(t)}(B(p, k_j - 1, t)) \right| \leq \kappa_0 4^{-n}, \quad j = 1, \cdots, N+1,$$

which implies

$$d(t)\kappa_0 4^{-n-2} \leq (N+1)\kappa_0 4^{-n} \leq \sum_{i=1}^{2N} \left| \mathrm{Vol}_{g(t)}(B(p, i, t)) - \mathrm{Vol}_{g(t)}(B(p, i-1, t)) \right| \leq V(t).$$

We can conclude that

$$d(t) \leq \frac{V(t)4^{n+2}}{\kappa_0},$$

which contradicts (22).

From now on, without loss of generality, assume that $d(t) \geq 2$ and (22) hold. We pick N integers $i_1, i_2, \cdots, i_N$ with

$$\{i_1, i_2, \cdots, i_N\} \subset \{1, 2, \cdots, 2N\}$$

and satisfying

$$\left| \mathrm{Vol}_{g(t)}(B(p, i_k, t)) - \mathrm{Vol}_{g(t)}(B(p, i_k - 1, t)) \right| \leq 4^{-n}\kappa_0, \quad k = 1, \cdots, N. \tag{23}$$

For $i \in \{i_1, i_2, \cdots, i_N\}$, we set $\gamma_i := \gamma \cap [B(p, i, t) \setminus B(p, i-1, t)]$, and the middle point of γ_i is denoted by p_i. Then, there holds

$$B\left(p_i, \frac{1}{2}, t\right) \subset [B(p, i, t) \setminus B(p, i-1, t)],$$

which implies that, for any $x \in B\left(p_i, \frac{1}{4}, t\right)$, we have

$$B\left(x, \frac{1}{4}, t\right) \subset B\left(p_i, \frac{1}{2}, t\right) \subset [B(p, i, t) \setminus B(p, i-1, t)]. \tag{24}$$

Combining (23) and (24), we derive

$$\frac{\mathrm{Vol}_{g(t)}(B(x, 1/4, t))}{(1/4)^n} \leq \kappa_0. \tag{25}$$

Note that

$$\lim_{\rho \searrow 0} \frac{\mathrm{Vol}_{g(t)}(B(x, \rho, t))}{\rho^n} = \omega_n \geq 2\kappa_0, \tag{26}$$

where we use the definition of κ_0 in (19). From (25) and (26), there exists a positive number $s = s(x) \in (0, 1/4]$ such that

$$\frac{\mathrm{Vol}_{g(t)}(B(x,s(x),t))}{[s(x)]^n}=\kappa_0 \tag{27}$$

and for $0<\rho\le s(x)$,

$$\frac{\mathrm{Vol}_{g(t)}(B(x,\rho,t))}{\rho^n}\ge\kappa_0. \tag{28}$$

From (18) and (27), we arrive at

$$M_2(x,t,S_t^+,s(x))\ge 2,$$

which implies that there exists $s_1(x)\in(0,\,s(x)]$ such that

$$\frac{[s_1(x)]^2}{\mathrm{Vol}_{g(t)}(B(x,s_1(x),t))}\int_{B(x,s_1(x),t)}S_t^+\mathrm{d}\mu(t)\ge 1. \tag{29}$$

From (28), we also have

$$\frac{\mathrm{Vol}_{g(t)}(B(x,s_1(x),t))}{[s_1(x)]^n}\ge\kappa_0. \tag{30}$$

Denote by σ the disjoint curves

$$\bigcup_{i=i_1,\cdots,i_N}\left[\gamma_i\bigcap B\left(p_i,\frac{1}{4},t\right)\right].$$

Since $N\ge\frac{d(t)}{8}$, we know that

$$\mathrm{length}_{g(t)}(\sigma)\ge\frac{d(t)}{16}.$$

A family of balls $\{B(x,s_1(x),t):\ x\in\sigma\}$ covers σ. By Lemma 5.2 in [15], there exist a sequence of points $\{x_\ell:\ \ell=1,2,\cdots\}\subset\sigma$ such that each of the balls $B(x_\ell,s(x_\ell),t)$ is disjoint from each other and these balls cover at least $\frac{1}{3}$ of σ. Therefore, we can deduce

$$d(t)\le 16\mathrm{length}_{g(t)}(\sigma)\le 96\sum_\ell|s_1(x_\ell)|. \tag{31}$$

By (29) and the Hölder inequality, we obtain

$$\begin{aligned}&\frac{\mathrm{Vol}_{g(t)}(B(x_\ell,s_1(x_\ell),t))}{[s_1(x_\ell)]^2}\\ \le&\int_{B(x_\ell,s_1(x_\ell),t)}S_t^+\mathrm{d}\mu(t)\\ \le&\left(\int_{B(x_\ell,s_1(x_\ell),t)}(S_t^+)^{\frac{n-1}{2}}\mathrm{d}\mu(t)\right)^{\frac{2}{n-1}}\left[\mathrm{Vol}_{g(t)}(B(x_\ell,s_1(x_\ell),t))\right]^{\frac{n-3}{n-1}},\end{aligned}$$

which, combining (30), leads to

$$\kappa_0 s_1(x_\ell)\le\frac{\mathrm{Vol}_{g(t)}(B(x_\ell,s_1(x_\ell),t))}{[s_1(x_\ell)]^{n-1}}\le\int_{B(x,s_1(x_\ell),t)}(S_t^+)^{\frac{n-1}{2}}\mathrm{d}\mu(t). \tag{32}$$

Thanks to (31) and (32), we obtain

$$d(t)\le 96\kappa_0^{-1}\int_M(S_t^+)^{\frac{n-1}{2}}\mathrm{d}\mu(t).$$

This inequality implies the desired upper bound (4) together with the assumptions $d(t) \geq 2$ and (22).

For the lower bound for the diameter of $(M, g(t))$, denote by $G(p, s; x, t)$ $(0 \leq s < t)$ the fundamental solution to the conjugate heat equation

$$\partial_s f(p, s) + \Delta_{g(p,s)} f(p, s) - S(p, s) f(p, s) = 0,$$

where $G(p, s; x, t)$ is seen as a function of (p, s).

From Lemma 6.5 in [21], we know that

$$G(p, s; x, t) \geq \frac{c_1 J(t)}{(t - s)^{\frac{n}{2}}} e^{-\frac{2(p,x,t)}{t-s}} e^{-\frac{1}{\sqrt{t-s}} \int_s^t \sqrt{t-v} S(z,v) \mathrm{d}v}, \tag{33}$$

where c_1 depends only on n. Here,

$$J(t) = \exp[-\alpha t - \beta + t \inf_{y \in M} S^-(y, 0)], \tag{34}$$

where α and β are positive constants which depend only on the infimum of the $\mathcal{F}$ functional for (M, g_0) and the Sobolev constant of (M, g_0). Furthermore, if $S(\cdot, 0) \geq 0$, then we have $\alpha = 0$. From (33), we can deduce

$$G(p, s; x, t) \geq \frac{c_1 J(t)}{(t - s)^{\frac{n}{2}}} e^{-\frac{2(p,x,t)}{t-s}} e^{-\int_s^t \|S(\cdot,v)\|_\infty \mathrm{d}v}. \tag{35}$$

Fix a time $t_0 > 0$ and a point $x_0 \in M$. Denote by r the diameter of $(M, g(t_0))$. Without loss of generality, we assume $r < \sqrt{t_0}$. Choosing $p = x_0,\ t = t_0,\ s = t_0 - r^2$ in (35), for any $x \in M$, we have $(x_0, x, t_0) \leq r$ and arrive at

$$G(x_0, t_0 - r^2; x, t_0) \geq \frac{c_1 J(t_0)}{r^n} e^{-1 - \int_0^{t_0} \|S(\cdot,v)\|_\infty \mathrm{d}v}. \tag{36}$$

From (3.20) in [21], we know that

$$\|S^-(\cdot, t)\|_\infty \leq \frac{1}{\frac{1}{\|S^-(\cdot,0)\|_\infty} + \frac{2t}{n}}. \tag{37}$$

It follows from the adjoint property of the fundamental solution (see [32]) that, for fixed p and s, $G(p, s; x, t)$ is a function of (x, t), and is the fundamental solution of the heat equation

$$\partial_t G(p, s; x, t) - \Delta_{g(x,t)} G(p, s; x, t) = 0.$$

This yields that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_M G(p, s; x, t) \mathrm{d}\mu(g(x, t)) \\
&= \int_M \Delta_{g(x,t)} G(p, s; x, t) \mathrm{d}\mu(g(x, t)) - \int_M S(x, t) G(p, s; x, t) \mathrm{d}\mu(g(x, t)) \\
&\leq \|S^-(\cdot, t)\|_\infty \int_M G(p, s; x, t) \mathrm{d}\mu(g(x, t)),
\end{aligned}$$

which, combining (37), yields

$$\int_M G(p, s; x, t) \mathrm{d}\mu(g(x, t)) \leq \left[1 + \frac{2}{n} \|S^-(\cdot, 0)\|_\infty (t - s)\right]^{\frac{n}{2}}. \tag{38}$$

It follows from (36) and (38) that

$$
\begin{aligned}
&\left(\frac{2r^2}{n}\|S^-(\cdot,0)\|_\infty+1\right)^{\frac{n}{2}} \\
\geq &\int_M G(x_0,t_0-r^2;y,t_0)\mathrm{d}\mu(g(y,t_0)) \\
\geq &\int_{(x_0,y,t_0)\leq r} G(x_0,t_0-r^2;y,t_0)\mathrm{d}\mu(g(y,t_0)) \\
\geq &\frac{c_1 J(t_0)}{r^n}e^{-1-\int_0^{t_0}\|S(\cdot,v)\|_\infty \mathrm{d}v}\int_{(x_0,y,t_0)\leq r}\mathrm{d}\mu(g(y,t_0)).
\end{aligned}
\tag{39}
$$

Noting that $r=\mathrm{diam}(M,g(t_0))$, we know

$$
\int_{(x_0,x,t_0)\leq r}\mathrm{d}\mu(g(x,t_0))=\mathrm{Vol}_{g(t_0)}(M). \tag{40}
$$

In addition, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}_{g(t)}(M)=-\int_M S(x,t)\mathrm{d}\mu(g(x,\,t))\geq -\|S(\cdot,t)\|_\infty\mathrm{Vol}_{g(t)}(M),
$$

which implies

$$
\mathrm{Vol}_{g(t_0)}(M)\geq e^{-\int_0^{t_0}\|S(\cdot,t)\|_\infty \mathrm{d}t}\mathrm{Vol}_{g_0}(M). \tag{41}
$$

Combining (39)–(41) leads to

$$
\left[1+\frac{2}{n}\|S^-(\cdot,0)\|_\infty r^2\right]^{\frac{n}{2}}r^n\geq c_1J(t_0)e^{-1-2\int_0^{t_0}\|S(\cdot,s)\|_\infty \mathrm{d}s}\mathrm{Vol}_{g_0}(M). \tag{42}
$$

Since $r=\mathrm{diam}(M,g(t_0))$ and $r<\sqrt{t_0}$, by assumption, we can obtain

$$
\begin{aligned}
\mathrm{diam}(M,g(t_0))\geq &c_2e^{\frac{1}{n}[-\alpha t_0-\beta-t_0\|S^-(\cdot,0)\|_\infty]}e^{-\frac{2}{n}\int_0^{t_0}\|S(\cdot,s)\|_\infty \mathrm{d}s} \\
&\times\left[1+\frac{2t_0}{n}\|S^-(\cdot,0)\|_\infty\right]^{-\frac{1}{2}}\left[\mathrm{Vol}_{g_0}(M)\right]^{\frac{1}{n}}.
\end{aligned}
$$

If $S(\cdot,0)\geq 0$, then we have $\alpha=0$ and $S^-(\cdot,0)=0$, which implies that

$$
\mathrm{diam}(M,g(t_0))\geq c_2e^{-\frac{\beta}{n}}e^{-\frac{2}{n}\int_0^{t_0}\|S(\cdot,s)\|_\infty \mathrm{d}s}\left[\mathrm{Vol}_{g_0}(M)\right]^{\frac{1}{n}}.
$$

Since t_0 is arbitrary, we can deduce (5) and (6). $\square$

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