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Depth and Stanley Depth of the Edge Ideals of *r*-Fold Bristled Graphs of Some Graphs

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Abstract: In this paper, we find values of depth, Stanley depth, and projective dimension of the quotient rings of the edge ideals associated with r-fold bristled graphs of ladder graphs, circular ladder graphs, some king's graphs, and circular king's graphs.

Keywords: depth; Stanley depth; projective dimension; edge ideal; r-fold bristled graph; ladder graph; circular ladder graph; king's graph; circular king's graph

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1. Introduction

Let $\digamma := K[x_1, x_2, \ldots, x_v]$ be a polynomial ring over a field K with standard grading, that is, $\deg(x_i) = 1$, for all i. Let M be a finitely generated graded $\digamma$-module. Suppose that M admits the following minimal free resolution:

$$0 \longrightarrow \bigoplus_{j\in\mathbb{Z}} \digamma(-j)^{\beta_{p,j}(M)} \longrightarrow \bigoplus_{j\in\mathbb{Z}} \digamma(-j)^{\beta_{p-1,j}(M)} \longrightarrow \cdots \longrightarrow \bigoplus_{j\in\mathbb{Z}} \digamma(-j)^{\beta_{0,j}(M)} \longrightarrow M \longrightarrow 0.$$

The projective dimension of M is defined as $\operatorname{pdim}(M) = \max\{i : \beta_{i,j}(M) \neq 0\}$. The *depth* of M is defined to be the common length of all maximal M-sequences in the unique graded maximal ideal $(x_1, x_2, \ldots, x_v)$. Let M be a finitely generated $\mathbb{Z}^v$-graded $\digamma$-module. For a homogeneous element $u \in M$ and a subset $A \subset \{x_1, x_2, \ldots, x_n\}$, $uK[A]$ denotes the K-subspace of M generated by all homogeneous elements of the form uv, where v is a monomial in $K[A]$. The K-subspace, $uK[A]$, is called a *Stanley space* of dimension $|A|$ if it is a free $K[A]$-module, where $|A|$ denotes the number of indeterminates in A. A *Stanley decomposition* $\mathcal{D}$ of M is a presentation of the K-vector space M as a finite direct sum of Stanley spaces:

$$\mathcal{D} : M = \bigoplus_{i=1}^{s} a_i K[A_i].$$

The Stanley depth of decomposition $\mathcal{D}$ is defined as $\operatorname{sdepth}(\mathcal{D}) = \min\{|A_i| : i = 1, 2, \ldots, s\}$. The *Stanley depth* of M is defined as

$$\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } M\}.$$

Stanley conjectured in [1] that sdepth $(M) \geq$ depth (M); this conjecture was later disproved by Duval et al. [2] in 2016. However, it is still important to prove Stanley's inequality for some special classes of ideals. Herzog et al. gave a method in [3] to compute the Stanley depth of modules of the form I/J, where $J \subset I$ are monomial ideals. But in general, it is still too hard to compute Stanley depth even using their method. For further details, we refer the reader to [4–6].

Let $G = (V(G), E(G))$ be a graph, where $V(G) = \{x_1, x_2, \dots, x_v\}$ is the vertex set and $E(G)$ is the edge set of graph G. All graphs considered in this paper are simple and undirected. The edge ideal $I(G)$ of the graph G is the ideal generated by all monomials of the form $x_i x_j$ such that $\{x_i, x_j\} \in E(G)$. In the last decade, the study of edge ideals has gained considerable attention. Various findings on these ideals have demonstrated how combinatorial and algebraic aspects interact; see, for instance, [7,8]. The algebraic invariant depth, Stanley depth, and projective dimension have significant importance in the field of commutative algebra. Establishing the relationship of these invariants with other invariants of commutative algebra and invariants of graph theory are current trends in research.

In general, the invariant depth, Stanley depth, and projective dimension are hard to compute. There are very few classes of ideals for which the formulae of these invariants are known; see, for instance, [4,9,10]. We prove that when we consider the r-fold graph of a ladder graph, circular ladder graph, some king's graphs, and some circular king's graphs, then the value of depth, Stanley depth, and projective dimension of the quotients rings of the edge ideals of these graphs are functions of r. We also prove that Stanley's inequality also holds for these quotient rings. Furthermore, our results give strong motivation for further studies in this direction. For our main results, see Theorem 3, Corollary 4, Theorem 4, Theorem 6, Corollary 4, and Theorem 7.

2. Preliminaries

In this section, we will recall some definitions and notations from graph theory. For terminology and definitions from graph theory, we refer the reader to [11–14]. Some known results related to depth and Stanley depth are also given in this section. If I is a monomial ideal then $\mathcal{G}(I)$ denotes its unique minimal set of monomial generators. If u is a monomial of $\digamma$, then $\operatorname{supp}(u) := \{x_i : x_i | u\}$, and for a monomial ideal I, we define $\operatorname{supp}(I) := \{x_i : x_i | u,$ for some $u \in \mathcal{G}(I)\}$. The *degree* of a vertex x_i denoted by $\deg(x_i)$ is the number of edges that are incident to x_i. Let $v \geq 1$, a *path* of length $v-1$, denoted by P_v, be a graph with $V(P_v) = \{x_1, x_2, \dots, x_v\}$ and $E(P_v) = \{\{x_i, x_{i+1}\} : 1 \leq i < v\}$ (if $v = 1$, then $E(P_1) = \emptyset$). Let $v \geq 3$, a *cycle* of length v denoted by $\mathcal{C}_v$, be a graph with $V(\mathcal{C}_v) = \{x_1, x_2, \dots, x_v\}$ and $E(\mathcal{C}_v) = \{\{x_i, x_{i+1}\} : 1 \leq i < v\} \bigcup \{\{x_1, x_v\}\}$. A graph is said to be a *tree* if it is acyclic. A vertex x_i is called a *pendant* vertex if $\deg(x_i) = 1$. For $r \geq 2$, an *r-star* denoted by $\mathcal{S}_r$ is a tree with $(r-1)$ leaves and a single vertex with degree $r-1$. A *caterpillar* is a tree in which the removal of all pendants leaves a path.

Definition 1 ([15])**.** *Let G be a graph and $r \geq 1$ be an integer. The graph obtained by attaching r pendant vertices to each vertex of G is called the r-fold bristled graph of G. The r-fold bristled graph of G is denoted by $Brs_r(G)$.*

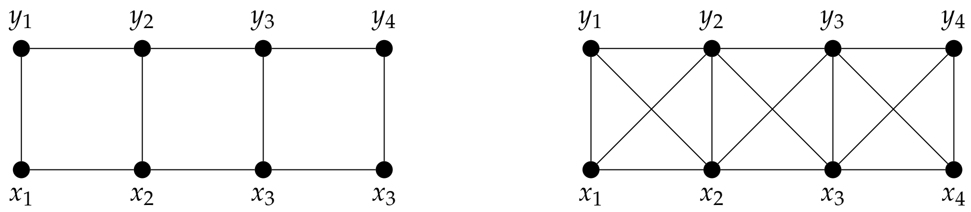
Definition 2 ([16])**.** *The Cartesian product $G_1 \square G_2$ of graphs G_1 and G_2 is a graph with vertex set $V(G_1) \times V(G_2)$ and $\{(t_1, u_1), (t_2, u_2)\} \in E(G_1 \square G_2)$, whenever*

1. $\{t_1, t_2\} \in E(G_1)$ *and* $u_1 = u_2$;
2. $t_1 = t_2$ *and* $\{u_1, u_2\} \in E(G_2)$.

Definition 3 ([16])**.** *The strong product $G_1 \boxtimes G_2$ of graphs G_1 and G_2 is a graph with vertex set $V(G_1) \times V(G_2)$ and $\{(t_1, u_1), (t_2, u_2)\} \in E(G_1 \boxtimes G_2)$, whenever*

1. $\{t_1, t_2\} \in E(G_1)$ *and* $u_1 = u_2$;
2. $t_1 = t_2$ *and* $\{u_1, u_2\} \in E(G_2)$;
3. $\{t_1, t_2\} \in E(G_1)$ *and* $\{u_1, u_2\} \in E(G_2)$.

Here we introduce some notations that will be used throughout the paper. For $v \geq 1$, let $D_v := P_v \square P_2$ and $L_v := P_v \boxtimes P_2$ be graphs. The graph D_v is known as a *ladder graph*, whereas the graph L_v is called *$(v \times 2)$-king's graph*. See Figure 1 for examples of D_v and L_v. For $v \geq 3$, let $H_v := \mathcal{C}_v \square P_2$ and $T_v := \mathcal{C}_v \boxtimes P_2$; the graph H_v is called a *circular ladder graph*. We define the graph T_v as *circular $(v \times 2)$-king's graph*.



(**a**) D_4 (**b**) L_4

Figure 1. Ladder graph and king's graph.

Now we recall some known results that are frequently used in this paper. The following lemma, which is also known as the Depth Lemma, has a crucial role in all proofs of our results concerning depth.

Lemma 1 ([17])**.** *If $0 \to U \to M \to N \to 0$ is a short exact sequence of modules over a local ring $\mathcal{F}$, or a Noetherian graded ring with $\mathcal{F}_0$ local, then*

1. $\operatorname{depth}(M) \geq \min\{\operatorname{depth}(N), \operatorname{depth}(U)\}$.
2. $\operatorname{depth}(U) \geq \min\{\operatorname{depth}(M), \operatorname{depth}(N) + 1\}$.
3. $\operatorname{depth}(N) \geq \min\{\operatorname{depth}(U) - 1, \operatorname{depth}(M)\}$.

A similar result for Stanley depth as given in the subsequent lemma is proved by Rauf.

Lemma 2 ([18])**.** *Let $0 \to U \to V \to W \to 0$ be a short exact sequence of $\mathbb{Z}^v$-graded $\mathcal{F}$-module. Then* $\operatorname{sdepth}(V) \geq \min\{\operatorname{sdepth}(U), \operatorname{sdepth}(W)\}$.

Here is a list of some preliminary lemmas that are referred to many times in the proofs of our results.

Lemma 3 ([3])**.** *Let $I \subset \mathcal{F}$ be a monomial ideal. If $\mathcal{F}' = \mathcal{F} \otimes_K K[x_{v+1}] \cong \mathcal{F}[x_{v+1}]$, then* $\operatorname{depth}(\mathcal{F}'/I\mathcal{F}') = \operatorname{depth}(\mathcal{F}/I) + 1$ *and* $\operatorname{sdepth}(\mathcal{F}'/I\mathcal{F}') = \operatorname{sdepth}(\mathcal{F}/I) + 1$.

Lemma 4 ([19])**.** *If $I = I(\mathcal{S}_v) \subseteq \mathcal{F}$ is an edge ideal of v-star, then*

$$\operatorname{depth}(\mathcal{F}/I) = \operatorname{sdepth}(\mathcal{F}/I) = 1.$$

Lemma 5 ([20])**.** *Let $I \subset \mathcal{F}' = K[x_1, \ldots, x_r]$, $J \subset \mathcal{F}'' = K[x_{r+1}, \ldots, x_v]$ be monomial ideals, where $1 \leq r < v$ and $\mathcal{F} = \mathcal{F}' \otimes_K \mathcal{F}''$. Then*

$$\operatorname{depth}(\mathcal{F}'/I \otimes_K \mathcal{F}''/J) = \operatorname{depth}(\mathcal{F}/(I\mathcal{F} + J\mathcal{F})) = \operatorname{depth}_{\mathcal{F}'}(\mathcal{F}'/I) + \operatorname{depth}_{\mathcal{F}''}(\mathcal{F}''/J).$$

Lemma 6 ([20])**.** *Let $I \subset \mathcal{F}' = K[x_1, \ldots, x_r]$ and $J \subset \mathcal{F}' = K[x_{r+1}, \ldots, x_v]$ be monomial ideals, where $1 \leq r < v$ and $\mathcal{F} = \mathcal{F}' \otimes_K \mathcal{F}''$. Then*

$$\operatorname{sdepth}(\mathcal{F}'/I \otimes_K \mathcal{F}^{''}/J) = \operatorname{sdepth}(\mathcal{F}/(I\mathcal{F} + J\mathcal{F})) \geq \operatorname{depth}_{\mathcal{F}'}(\mathcal{F}'/I) + \operatorname{depth}_{\mathcal{F}''}(\mathcal{F}''/J).$$

The following results are useful in finding upper bounds for depth and Stanley depth.

Corollary 1 ([18])**.** *Let $I \subset \mathcal{F}$ be a monomial ideal. Then* $\operatorname{depth}(\mathcal{F}/(I : u)) \geq \operatorname{depth}(\mathcal{F}/I)$ *for all monomials $u \notin I$.*

Proposition 1 ([21]). *Let $J \subset \digamma$ be a monomial ideal. Then for all monomials $u \notin J$,*

$$\operatorname{sdepth}(\digamma/J) \leq \operatorname{sdepth}(\digamma/(J:u)).$$

Lemma 7 ([22]). *Let $I \subset \digamma$ be a squarefree monomial ideal with* $\operatorname{supp}(I) = \{x_1, x_2, \dots, x_v\}$, *let $\mu := x_{i_1}x_{i_2}\dots x_{i_q} \in \digamma/I$, such that $x_h\mu \in I$, for all $x_h \in \{x_1, x_2, \dots, x_v\} \setminus \operatorname{supp}(\mu)$. Then* $\operatorname{sdepth}(\digamma/I) \leq q$.

The following result says that once the value of depth of a module is know then one can find its projective dimension.

Theorem 1 ([17]). *(Auslander–Buchsbaum formula) If $\digamma$ is a commutative Noetherian local ring and M is a non-zero finitely generated $\digamma$-module of finite projective dimension, then*

$$\operatorname{pdim}(M) + \operatorname{depth}(M) = \operatorname{depth}(\digamma).$$

For $r \geq 1$ and $v \geq 1$, if $P_{v,r} := Brs_r(P_v)$, then clearly $P_{v,r}$ is a caterpillar and we have the following values for depth and Stanley depth.

Theorem 2 ([23]). *Let $r \geq 1$ and $v \geq 1$. Then*

$$\operatorname{depth}\left(K[V(P_{v,r})]/I(P_{v,r})\right) = \operatorname{sdepth}\left(K[V(P_{v,r})]/I(P_{v,r})\right) = \lceil\frac{v-1}{2}\rceil r + \lceil\frac{v}{2}\rceil.$$

Throughout this paper, we set $\digamma_{v,r} := K[\bigcup_{i=1}^{v}\{x_i, y_i\} \bigcup \bigcup_{j=1}^{r}\{x_{1j}, x_{2j} \dots, x_{vj}, y_{1j}, y_{2j}, \dots, y_{vj}\}]$, where r and v are positive integers. Also, $|V(\digamma_{v,r})| = 2v(1+r)$.

3. Depth and Stanley Depth of r-Fold Bristled Graph of Ladder Graph and Some King's Graph

In this section, we determine depth, projective dimension, and Stanley depth of the quotient rings associated with edge ideals of r-fold bristled graphs of graphs D_v and L_v. See Figures 2a and 3 for 2-fold bristled graph of graphs D_4 and L_4, respectively. We label the vertices of $Brs_r(D_v)$ and $Brs_r(L_v)$, as shown in Figure 2a and Figure 3, respectively. For $v, r \geq 1$, let $I_{v,r} := I(Brs_r(D_v))$ and $L_{v,r} := I(Brs_r(L_v))$. If $\mathcal{G}(I)$ denotes the minimal set of monomial generators of the monomial ideal I, using our labeling, we have

$$\mathcal{G}(I_{1,r}) = \{x_1y_1\} \bigcup \bigcup_{j=1}^{r}\{x_1x_{1j}, y_1y_{1j}\},$$

and

$$\mathcal{G}(L_{1,r}) = \{x_1y_1\} \bigcup \bigcup_{j=1}^{r}\{x_1x_{1j}, y_1y_{1j}\}.$$

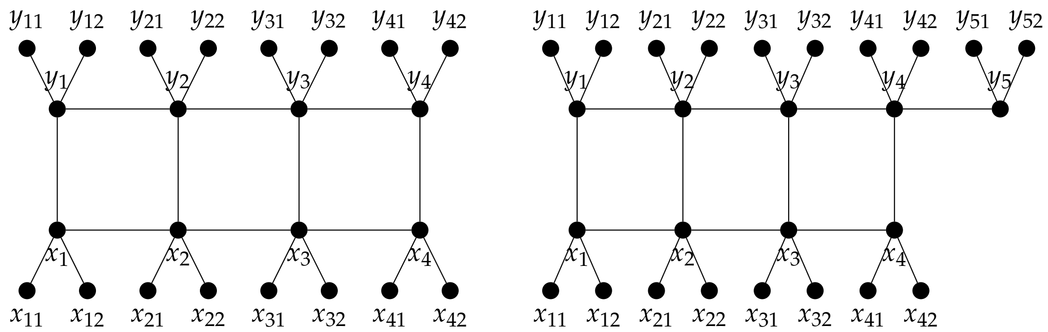
If $v \geq 2$, then we have

$$\mathcal{G}(I_{v,r}) = \bigcup_{i=1}^{v-1}\{x_ix_{i+1}, y_iy_{i+1}\} \bigcup \bigcup_{i=1}^{v}\{x_iy_i\} \bigcup \bigcup_{j=1}^{r}\{y_1y_{1j}, \dots, y_vy_{vj}, x_1x_{1j}, \dots, x_vx_{vj}\},$$

and

$$\mathcal{G}(L_{v,r}) = \bigcup_{i=1}^{v-1}\{x_ix_{i+1}, y_iy_{i+1}\} \bigcup \bigcup_{j=1}^{r}\{x_1x_{1j}, .., x_vx_{vj}, y_1y_{1j}, .., y_vy_{vj}\} \bigcup$$
$$\bigcup_{i=1}^{v}\{x_iy_i\} \bigcup \{y_1x_2, y_vx_{v-1}\} \bigcup \bigcup_{i=2}^{v-1}\{y_ix_{i-1}, y_ix_{i+1}\}.$$

Note that $Brs_r(D_1) \cong Brs_r(L_1) \cong P_{2,r}$ and $\digamma_{1,r}/I_{1,r} \cong \digamma_{1,r}/L_{1,r} \cong K[V(P_{2,r})]/I(P_{2,r})$. We also define a modified graph of $Brs_r(D_v)$ denoted by $A_{v,r}$ with the set of vertices $V(A_{v,r}) = V(Brs_r(D_v)) \bigcup \{y_{v+1}\} \bigcup \{y_{(v+1)1}, y_{(v+1)2}, \dots, y_{(v+1)r}\}$ and $E(A_{v,r}) = E(I_{v,r}) \bigcup \{\{y_v, y_{v+1}\}\} \bigcup \bigcup_{j=1}^{r} \{\{y_v, y_{(v+1)1}\}, \{y_v, y_{(v+1)2}\}, \dots \{y_v, y_{(v+1)r}\}\}$. See Figure 3b for an example of graph $A_{v,r}$ and labeling of vertices of this graph. We set $\digamma^*_{v,r} := \digamma_{v,r}[y_{v+1}, y_{(v+1)1}, y_{(v+1)1}, \dots, y_{(v+1)r}]$ and $I^* := I(A_{v,r})$. Clearly, $\mathcal{G}(I^*_{v,r}) = \mathcal{G}(I_{v,r}) \bigcup \{y_v y_{v+1}, y_v y_{(v+1)1}, y_v y_{(v+1)2}, \dots, y_v y_{(v+1)r}\}$. Note that $A_{v,r} \cong P_{3,r}$, $\digamma^*_{1,r}/I^*_{1,r} = K[V(P_{3,r})]/I(P_{3,r})$ and $|V(\digamma^*_{v,r})| = (2v+1)(1+r)$. To determine depth and Stanley depth of $\digamma_{v,r}/I_{v,r}$, we shall first determine the depth and Stanley depth of $\digamma^*_{v,r}/I^*$.



(**a**) $Brs_2(D_4)$ (**b**) $A_{4,2}$

Figure 2. 2-Fold bristled graph of a ladder graph and its modification by adding some vertices and edges.

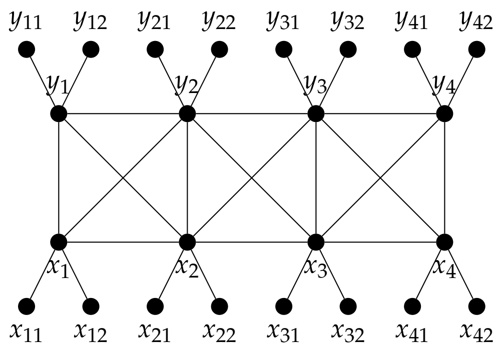


Figure 3. $Brs_2(L_4)$.

Remark 1. *Let I be a squarefree monomial ideal of $\digamma$ whose monomial generators have degrees of at most 2. We associate a graph G_I to the ideal I with $V(G_I) = \operatorname{supp}(I)$ and $E(G_I) = \{\{x_i, x_j\} : x_i x_j \in \mathcal{G}(I)\}$. Let $x_u \in \digamma$ be a variable of the polynomial ring $\digamma$ such that $x_u \notin I$. Then $(I : x_u)$ and (I, x_u) are monomial ideals of $\digamma$ such that $G_{(I:x_u)}$ and $G_{(I,x_u)}$ are subgraphs of G_I. See Figure 4a and Figure 4b for graphs $G_{(I^*_{4,2}:y_5)}$ and $G_{(L_{3,2},x_3)}$, respectively.*

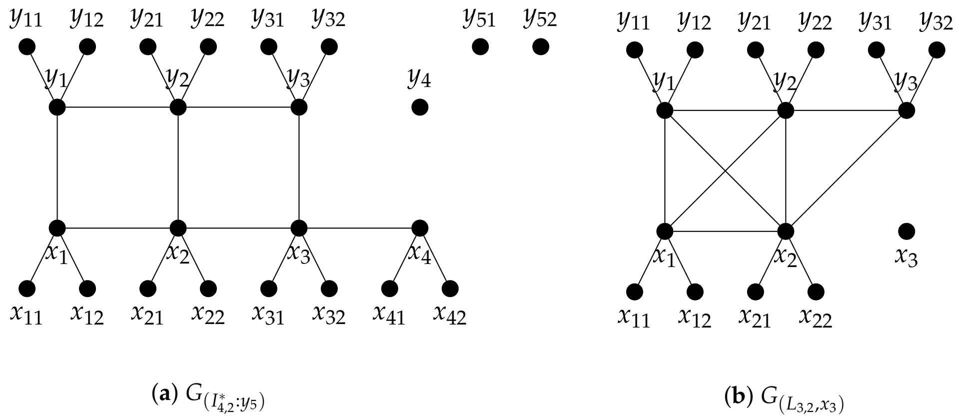


Figure 4. Graphs corresponding to ideals $(I^*_{4,2} : y_5)$ and $(L_{3,2}, x_3)$.

Remark 2. *While proving our results by induction on v, the special cases, say $F_{0,r}/L_{0,r}$ and $F^*_{0,r}/I^*_{0,r}$, that might appear in the proofs need to be addressed first. We define $F_{0,r}/L_{0,r} := K$ and $F^*_{0,r}/I^*_{0,r} := K[V(\mathcal{S}_{r+1})]/I(\mathcal{S}_{r+1})$. Thus, we have* $\operatorname{depth}(F_{0,r}/L_{0,r}) = \operatorname{sdepth}(F_{0,r}/L_{0,r}) = 0$, *and by Lemma 4, we have* $\operatorname{depth}(F^*_{0,r}/I^*_{0,r}) = \operatorname{sdepth}(F^*_{0,r}/I^*_{0,r}) = 1$.

Lemma 8. *Let $v, r \geq 1$. Then* $\operatorname{depth}(F^*_{v,r}/I^*_{v,r}) = \operatorname{sdepth}(F^*_{v,r}/I^*_{v,r}) = (r+1)v + 1$.

Proof. First we will prove the result for depth. We will prove this by induction on v. We consider the following short exact sequence:

$$0 \longrightarrow F^*_{v,r}/(I^*_{v,r} : y_v) \xrightarrow{\cdot y_v} F^*_{v,r}/I^*_{v,r} \longrightarrow F^*_{v,r}/(I^*_{v,r}, y_v) \longrightarrow 0.$$

By the Depth Lemma,

$$\operatorname{depth}(F^*_{v,r}/I^*_{v,r}) \geq \min\{\operatorname{depth}(F^*_{v,r}/(I^*_{v,r} : y_v)), \operatorname{depth}(F^*_{v,r}/(I^*_{v,r}, y_v))\}. \quad (1)$$

If $v = 1$, then by Theorem 2, $\operatorname{depth}(F^*_{1,r}/I^*_{1,r}) = \operatorname{depth}(K[V(P_{3,r})]/I(P_{3,r})) = r + 2$, as required. Let $v \geq 2$. After renumbering the variables, we have

$$F^*_{v,r}/(I^*_{v,r} : y_v) \cong F^*_{v-2,r}/I^*_{v-2,r} \bigotimes_K K[y_v, \cup_{j=1}^r \{x_{vj}, y_{(v-1)j}, y_{(v+1)j}\}].$$

Thus, by induction and Lemma 3,

$$\operatorname{depth}(F^*_{v,r}/(I^*_{v,r} : y_v)) = \operatorname{depth}(F^*_{v-2,r}/I^*_{v-2,r}) + 3r + 1 = (r+1)(v-2) + 1 + 3r + 1 = (r+1)v + r.$$

Also,

$$F^*_{v,r}/(I^*_{v,r}, y_v) \cong F^*_{v-1,r}/I^*_{v-1,r} \bigotimes_K K[V(\mathcal{S}_{r+1})]/I(\mathcal{S}_{r+1}) \bigotimes_K K[y_{v1}, y_{v2}, \ldots, y_{vr}].$$

By Lemmas 3 and 5,

$$\operatorname{depth}(F^*_{v,r}/(I^*_{v,r}, y_v)) = \operatorname{depth}(F^*_{v-1,r}/I^*_{v-1,r}) + \operatorname{depth}(K[V(\mathcal{S}_{r+1})]/I(\mathcal{S}_{r+1})) + r.$$

Using induction and Lemma 4,

$$\operatorname{depth}(F^*_{v,r}/(I^*_{v,r}, y_v)) = (r+1)(v-1) + 1 + 1 + r = (r+1)v + 1.$$

By Equation (1), we have depth $(F^*_{v,r}/(I^*_{v,r})) \geq (r+1)v+1$. Now we prove the other inequality. We have $F^*_{v,r}/(I^*_{v,r} : y_{v+1}) \cong F^*_{v-1,r}/I^*_{v-1,r} \bigotimes_K K[y_{v+1}, y_{v1}, y_{v2}, \dots, y_{vr}]$, by Lemma 3, depth $(F^*_{v,r}/(I^*_{v,r} : y_{v+1})) =$ depth $(F^*_{v-1,r}/I^*_{v-1,r}) + r + 1$. By induction, we have

$$\text{depth}\,(F^*_{v,r}/(I^*_{v,r} : y_{v+1})) = (r+1)(v-1)+1+r+1 = (r+1)v+1.$$

As $y_{v+1} \notin I^*_{v,r}$, so by Corollary 1 depth $(F^*_{v,r}/I^*_{v,r}) \leq$ depth $(F^*_{v,r}/(I^*_{v,r} : y_{v+1})) = (r+1)v+1$. This completes the proof for depth.

Now we prove the result for Stanley depth. If $v = 1$, then by Theorem 2, sdepth $(F^*_{1,r}/I^*_{1,r}) = r+2$. For $v \geq 2$, the required result follows by applying Lemma 2 instead of the Depth Lemma, Lemma 6 instead of Lemma 5, and Proposition 1 instead of Corollary 1. □

Corollary 2. *Let* $v, r \geq 1$. *Then* pdim $(F^*_{v,r}/I^*_{v,r}) = r(v+1)+v$.

Proof. The required result follows by using Lemma 8 and Theorem 1. □

Now using the previous lemma, we are able to prove one of the main results of this section.

Theorem 3. *Let* $v, r \geq 1$. *Then* depth $(F_{v,r}/I_{v,r}) =$ sdepth $(F_{v,r}/I_{v,r}) = (r+1)v$.

Proof. First we will prove the result for depth. Consider the following short exact sequence:

$$0 \longrightarrow F_{v,r}/(I_{v,r} : x_v) \xrightarrow{\cdot x_v} F_{v,r}/I_{v,r} \longrightarrow F_{v,r}/(I_{v,r}, x_v) \longrightarrow 0.$$

When $v = 1$, it is clear from Theorem 2 that depth $(F_{1,r}/I_{1,r}) =$ depth $(K[V(P_{2,r})]/I(P_{2,r})) = r+1$. Let $v \geq 2$. We have $F_{v,r}/(I_{v,r} : x_v) \cong (F^*_{v-2,r}/I^*_{v-2,r}) \bigotimes_K K[\{x_v\} \bigcup \cup_{j=1}^{r} \{y_{vj}, x_{(v-1)j}\}]$. By Lemma 3, we have

$$\text{depth}\,(F_{v,r}/(I_{v,r} : x_v)) = \text{depth}\,(F^*_{v-2,r}/I^*_{v-2,r}) + 2r + 1.$$

By Lemma 8, depth $(F_{v,r}/(I_{v,r} : x_v)) = (r+1)(v-2)+1+2r+1 = (r+1)v$. Now clearly $\mathcal{G}(I_{v,r}, x_v) = \{\mathcal{G}(I^*_{v-1}), x_v\}$ and $F_{v,r}/(I_{v,r}, x_v) \cong F^*_{v-1,r}/I^*_{v-1,r} \bigotimes_K K[x_{v1}, x_{v2}, \dots, x_{vr}]$, and using Lemma 3, depth $(F_{v,r}/(I_{v,r}, x_v)) =$ depth $(F^*_{v-1,r}/I^*_{v-1,r}) + r$. By Lemma 8,

$$\text{depth}\,(F_{v,r}/(I_{v,r}, x_v)) = (r+1)(v-1)+1+r = (r+1)v.$$

Applying the Depth Lemma, depth $(F_{v,r}/I_{v,r}) = (r+1)v$.

Now we prove the result for Stanley depth. If $v = 1$, then by Theorem 2, we have

$$\text{sdepth}\,(F_{1,r}/I_{1,r}) = \text{sdepth}\,(K[V(P_{2,r})]/I(P_{2,r})) = r+1.$$

Let $v \geq 2$. Applying Lemma 2 on the short exact sequence, we obtain

$$\text{sdepth}\,(F_{v,r}/I_{v,r}) \geq \min\{\text{sdepth}\,(F_{v,r}/(I_{v,r} : x_v)), \text{sdepth}\,(F_{v,r}/(I_{v,r}, x_v))\}. \tag{2}$$

Proceeding on the same lines as we did for the depth, we obtain sdepth$(F_{v,r}/(I_{v,r} : x_v)) \geq v(r+1)$ and sdepth$(F_{v,r}/(I_{v,r}, x_v)) \geq v(r+1)$ and by Equation (2), we have sdepth $(F_{v,r}/I_{v,r}) \geq v(r+1)$. For the other inequality, since $x_v \notin I_{v,r}$ and sdepth $(F_{v,r}/(I_{v,r} : x_v)) =$ sdepth $(F^*_{v-2,r}/I^*_{v-2,r}) + 2r + 1$, by Lemma 8,

$$\text{sdepth}\,(F_{v,r}/(I_{v,r} : x_v)) = (r+1)(v-2)+1+2r+1 = (r+1)v.$$

By Proposition 1, we have

$$\text{sdepth}\,(F_{v,r}/I_{v,r}) \leq \text{sdepth}\,(F_{v,r}/(I_{v,r} : x_v)) = (r+1)v.$$

This completes the proof for Stanley depth. □

Corollary 3. *Let* $v, r \geq 1$. *Then* $\operatorname{pdim}(\digamma_{v,r}/I_{v,r}) = (r+1)v$.

Proof. The required result follows by using Theorem 3 and Theorem 1. □

Now we find the depth and Stanley depth of $\digamma_{v,r}/L_{v,r}$.

Theorem 4. *Let* $v, r \geq 1$. *Then* $\operatorname{depth}(\digamma_{v,r}/L_{v,r}) = \operatorname{sdepth}(\digamma_{v,r}/L_{v,r}) = \lfloor \frac{3v}{2} \rfloor r + \lceil \frac{v}{2} \rceil$.

Proof. First we will prove the result for depth. We will prove this by induction on v. Consider the following short exact sequence:

$$0 \longrightarrow \digamma_{v,r}/(L_{v,r} : x_v) \xrightarrow{\cdot x_v} \digamma_{v,r}/L_{v,r} \longrightarrow \digamma_{v,r}/(L_{v,r}, x_v) \longrightarrow 0.$$

By the Depth Lemma,

$$\operatorname{depth}(\digamma_{v,r}/L_{v,r}) \geq \min\{\operatorname{depth}(\digamma_{v,r}/(L_{v,r} : x_v)), \operatorname{depth}(\digamma_{v,r}/(L_{v,r}, x_v))\}. \tag{3}$$

When $v = 1$, it is clear from Theorem 2 that $\operatorname{depth}(\digamma_{1,r}/L_{1,r}) = \operatorname{depth}(K[V(P_{2,r})]/I(P_{2,r})) = r+1$.

Let $v \geq 2$, $\digamma_{v,r}/(L_{v,r} : x_v) \cong \digamma_{v-2,r}/L_{v-2,r} \bigotimes_K K[\{x_v\} \bigcup \bigcup_{j=1}^{r} \{y_{vj}, x_{(v-1)j}, y_{(v-1)j}\}]$. Using Lemma 3 and induction on v, clearly

$$\begin{aligned}\operatorname{depth}(\digamma_{v,r}/(L_{v,r} : x_v)) = \operatorname{depth}(\digamma_{v-2,r}/L_{v-2,r}) + 3r + 1 &= \lfloor \frac{3(v-2)}{2} \rfloor r + \lceil \frac{v-2}{2} \rceil + 3r + 1 \\ &= \lfloor \frac{3v}{2} \rfloor r + \lceil \frac{v}{2} \rceil.\end{aligned}$$

Now let $J := (L_{v,r}, x_v)$ and $\mathcal{G}(J) = \mathcal{G}(L_{v-1,r}) \bigcup \{y_v x_{v-1}, y_v y_{v-1}, x_v\} \bigcup \{y_v y_{v1}, y_v y_{v2}, \ldots, y_v y_{vr}\}$. Consider the following short exact sequence:

$$0 \longrightarrow \digamma_{v,r}/(J : y_v) \xrightarrow{\cdot y_v} \digamma_{v,r}/J \longrightarrow \digamma_{v,r}/(J, y_v) \longrightarrow 0.$$

Again, using the Depth Lemma, we have

$$\operatorname{depth}(\digamma_{v,r}/J) \geq \min\{\operatorname{depth}(\digamma_{v,r}/(J : y_v)), \operatorname{depth}(\digamma_{v,r}/(J, y_v))\}. \tag{4}$$

Here $\digamma_{v,r}/(J : y_v) \cong \digamma_{v-2,r}/L_{v-2,r} \bigotimes_K K[\{y_v\} \bigcup \bigcup_{j=1}^{r} \{y_{(v-1)j}, x_{(v-1)j}, x_{vj}\}]$. Using Lemma 3 and induction on v, we have

$$\begin{aligned}\operatorname{depth}(\digamma_{v,r}/(J : y_v)) = \operatorname{depth}(\digamma_{v-2,r}/L_{v-2,r}) + 3r + 1 &= \lfloor \frac{3(v-2)}{2} \rfloor r + \lceil \frac{v-2}{2} \rceil + 3r + 1 \\ &= \lfloor \frac{3v}{2} \rfloor r + \lceil \frac{v}{2} \rceil.\end{aligned}$$

As $\mathcal{G}((J, y_v)) = \mathcal{G}(L_{v-1,r}) \bigcup \{x_v, y_v\}$ and $\digamma_{v,r}/(J, y_v) \cong \digamma_{v-1,r}/L_{v-1,r} \bigotimes_K K[\bigcup_{j=1}^{r} \{y_{vj}, x_{vj}\}]$. By Lemma 3 and induction on v, we obtain

$$\begin{aligned}\operatorname{depth}(\digamma_{v,r}/(J, y_v)) = \operatorname{depth}(\digamma_{v-1,r}/L_{v-1,r}) + 2r &= \lfloor \frac{3(v-1)}{2} \rfloor r + \lceil \frac{v-1}{2} \rceil + 2r \\ &= \lfloor \frac{3v+1}{2} \rfloor r + \lceil \frac{v-1}{2} \rceil.\end{aligned}$$

By Equation (4), we have

$$\text{depth}\,(\digamma_{v,r}/J) \geq \min\{\lfloor \frac{3v}{2} \rfloor r + \lceil \frac{v}{2} \rceil, \lfloor \frac{3v+1}{2} \rfloor r + \lceil \frac{v-1}{2} \rceil\} = \lfloor \frac{3v}{2} \rfloor r + \lceil \frac{v}{2} \rceil.$$

Now by using Equation (3), we obtain

$$\text{depth}\,(\digamma_{v,r}/L_{v,r}) \geq \min\{\lfloor \frac{3v}{2} \rfloor r + \lceil \frac{v}{2} \rceil, \lfloor \frac{3v}{2} \rfloor r + \lceil \frac{v}{2} \rceil\} = \lfloor \frac{3v}{2} \rfloor r + \lceil \frac{v}{2} \rceil.$$

For upper bound as $x_v \notin \digamma_{v,r}$ and $\text{depth}\,(\digamma_{v,r}/(L_{v,r} : x_v)) = \lfloor \frac{3v}{2} \rfloor r + \lceil \frac{v}{2} \rceil$. By Corollary 1, $\text{depth}\,(\digamma_{v,r}/L_{v,r}) \leq \text{depth}\,(\digamma_{v,r}/(L_{v,r} : x_v)) = \lfloor \frac{3v}{2} \rfloor r + \lceil \frac{v}{2} \rceil$. This completes the proof for depth. Now we prove the result for Stanley depth. When $v = 1$, it is clear from Theorem 2 that $\text{sdepth}\,(\digamma_{1,r}/L_{1,r}) = r + 1$. For $v \geq 2$, the required result follows by applying Lemma 2 instead of the Depth Lemma and Proposition 1 instead of Corollary 1. □

Corollary 4. *Let $v, r \geq 1$. Then* $\text{pdim}\,(\digamma_{v,r}/L_{v,r}) = \lceil \frac{v}{2} \rceil r + \lfloor \frac{3v}{2} \rfloor$.

Proof. The result follows by using Theorem 4 and Theorem 1. □

Example 1. *If $v = 9$ and $r = 4$, then by Theorem 4, we have* $\text{depth}(\digamma_{9,4}/L_{9,4}) = \text{sdepth}(\digamma_{9,4}/L_{9,4}) = \lfloor \frac{3(9)}{2} \rfloor (4) + \lceil \frac{9}{2} \rceil = 52 + 5 = 57$. *Also, by Corollary 4, we have* $\text{pdim}(\digamma_{9,4}/L_{9,4}) = \lceil \frac{9}{2} \rceil (4) + \lfloor \frac{3(9)}{2} \rfloor = 20 + 13 = 33$.

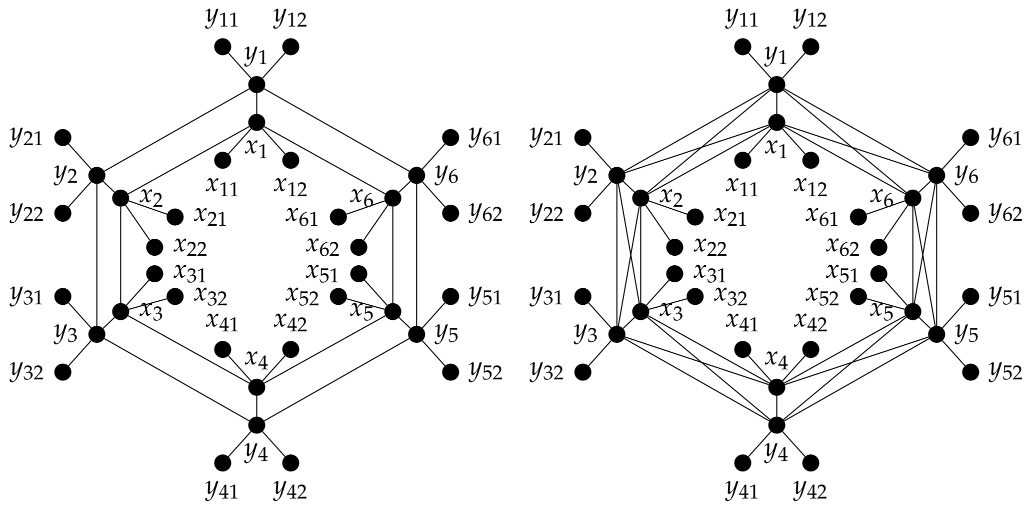
4. Depth and Stanley Depth of r-Fold Bristled Graph of Circular Ladder Graph and Some Circular King's Graph

In this section, we determine the depth and Stanley depth of the quotient rings associated with the edge ideal of r-fold bristled graph of circular ladder graph and T_v graph. Figure 5a,b are examples of 2-fold bristled graphs of a circular ladder graph and T_6 graph, respectively. For positive integers r, v such that $r \geq 1$ and $v \geq 3$, the minimal set of monomial generators of the edge ideal $\mathfrak{C}_{v,r} = I(Brs_r(H_v))$ is given as $\mathcal{G}(\mathfrak{C}_{v,r}) = \mathcal{G}(I_{v,r}) \bigcup \{x_1x_v, y_1y_v\}$. For $v \geq 1$, we also define a new graph $A'_{v,r}$ with $V(A'_{v,r}) = \bigcup_{i=1}^{v} \{x_i, y_i\} \bigcup \{y_{v+1}, y_{v+2}\} \bigcup \bigcup_{i=1}^{r} \{x_{1i}, \dots, x_{vi}, y_{1i}, \dots, y_{(v+2)i}\}$ and

$$\begin{aligned} E(A'_{v,r}) = &\bigcup_{i=1}^{v-1} \{\{x_i, x_{i+1}\}\} \bigcup \bigcup_{i=1}^{v+1} \{\{y_i, y_{i+1}\}\} \bigcup \bigcup_{i=1}^{v} \{\{x_i, y_{i+1}\}\} \bigcup \\ &\bigcup_{i=1}^{v+2} \{\{y_i, y_{i1}\}, \{y_i, y_{i2}\}, \dots, \{y_i, y_{ir}\}\} \bigcup \bigcup_{i=1}^{v} \{\{x_i, x_{i1}\}, \{x_i, x_{i2}\}, \dots \{x_i, x_{ir}\}\}. \end{aligned}$$

See Figure 6 for an example of $A'_{v,r}$ graph. We set $\digamma^{**}_{v,r} := \digamma_{v,r}[\{y_{v+1}, y_{v+2}\} \bigcup \bigcup_{j=1}^{r} \{y_{(v+1)j}, y_{(v+2)j}\}]$ and $|V(\digamma^{**}_{v,r})| = 2(v+1)(1+r)$. Let $E_{v,r} := I(A'_{v,r})$ and $C_{v,r} := I(Brs_r(T_v))$. Clearly, $\mathcal{G}(C_{v,r}) = \mathcal{G}(L_{v,r}) \bigcup \{x_1x_v, y_1y_v, x_1y_v, x_vy_1\}$.

To determine the depth and Stanley depth of the quotient rings associated with the edge ideal of the r-fold bristled graph of the circular ladder graph, we shall first determine the depth and Stanley depth of the quotient ring associated with the edge ideal of $A'_{v,r}$ graph. In Figure 7 we give examples of graphs associated to squarefree monomial ideals $(E_{1,2} : y_3)$, $(E_{1,2}, y_3)$, $(\mathfrak{C}_{6,2} : x_6)$ and $(\mathfrak{C}_{6,2}, x_6)$, as discussed in Remark 1. These examples will be helpful in understanding the proofs of our next results.



(**a**) $Brs_2(H_6)$ (**b**) $Brs_2(T_6)$

Figure 5. 2-fold bristled graphs of circular ladder and circular king's graphs.

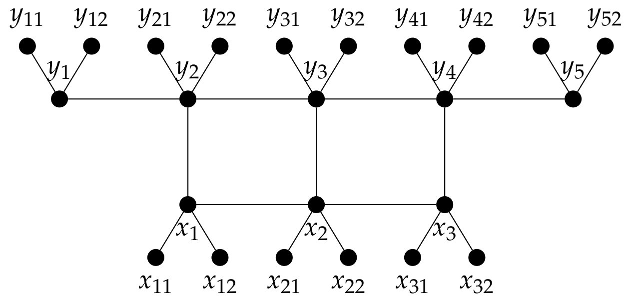


Figure 6. $A'_{3,2}$.

Remark 3. *While proving our results by induction on v, we have special case $F^{**}_{0,r}/E_{0,r}$, so we define $F^{**}_{0,r}/E_{0,r} := K[V(P_{2,r})]/I(P_{2,r})$. By using Theorem 2,* depth $(F^{**}_{0,r}/E_{0,r}) =$ sdepth $(F^{**}_{0,r}/E_{0,r}) = r+1$.

Theorem 5. *Let $r, v \geq 1$. Then*

$$\operatorname{depth}(F^{**}_{v,r}/E_{v,r}) = \operatorname{sdepth}(F^{**}_{v,r}/E_{v,r}) = \begin{cases} (v+1)(r+1), & \textit{if } v \textit{ is even;} \\ v(r+1)+2, & \textit{if } v \textit{ is odd.} \end{cases}$$

Proof. First we will prove the result for depth by using induction on v. Consider the following short exact sequence:

$$0 \longrightarrow F^{**}_{v,r}/(E_{v,r} : y_{v+2}) \xrightarrow{\cdot y_{v+2}} F^{**}_{v,r}/E_{v,r} \longrightarrow F^{**}_{v,r}/(E_{v,r}, y_{v+2}) \longrightarrow 0.$$

Let $v = 1$. We have $F^{**}_{1,r}/(E_{1,r} : y_3) \cong \bigotimes_{i=1}^{2} K[V(\mathcal{S}_{r+1})]/I(\mathcal{S}_{r+1}) \otimes_K K[y_3, y_{21}, y_{22}, \ldots, y_{2r}]$, and by Lemmas 3–5, we have

$$\operatorname{depth}(F^{**}_{1,r}/(E_{1,r} : y_3)) = 2 \cdot \operatorname{depth}(K[V(\mathcal{S}_{r+1})]/I(\mathcal{S}_{r+1})) + r + 1 = 2 + r + 1 = r + 3.$$

Also, we can see that $F^{**}_{1,r}/(E_{1,r}, y_3) \cong K[V(P_{3,r})]/I(P_{3,r}) \otimes_K K[y_{31}, y_{32}, \ldots, y_{3r}]$. By Lemma 3 and Theorem 2, we have

$$\operatorname{depth}(F^{**}_{1,r}/(E_{1,r}, y_3)) = \operatorname{depth}(K[V(P_{3,r})]/I(P_{3,r})) + r = r + 2 + r = 2r + 2.$$

Since depth $(F^{**}_{1,r}/(E_{1,r} : y_3)) \leq$ depth $(F^{**}_{1,r}/(E_{1,r}, y_3))$, then by then Depth Lemma,

$$\operatorname{depth}(F^{**}_{1,r}/E_{1,r}) = \operatorname{depth}(F^{**}_{1,r}/(E_{1,r} : y_3)) = r + 3.$$

This prove the result for $v = 1$.
Let $v \geq 2$, and $J^* := (E_{v,r} : y_{v+2})$. Now consider the following short exact sequence:

$$0 \longrightarrow F^{**}_{v,r}/(J^* : x_v) \xrightarrow{\cdot x_v} F^{**}_{v,r}/J^* \longrightarrow F^{**}_{v,r}/(J^*, x_v) \longrightarrow 0.$$

We have

$$F^{**}_{v,r}/(J^* : x_v) \cong F^{**}_{v-2,r}/E_{v-2,r} \bigotimes_K K[\{x_v, y_{v+2}\} \bigcup \bigcup_{j=1}^{r} \{y_{(v+1)j}, x_{(v-1)j}\}],$$

and

$$F^{**}_{v,r}/(J^*, x_v) \cong F^{*}_{v-1,r}/I^{*}_{v-1,r} \bigotimes_K K[\{y_{v+2}\} \bigcup \bigcup_{j=1}^{r} \{x_{vj}, y_{(v+1)j}\}].$$

Thus, by using Lemma 3, we obtain depth $(F^{**}_{v,r}/(J^* : x_v)) =$ depth $(F^{**}_{v-2,r}/E_{v-2,r}) + 2r + 2$ and depth $(F^{**}_{v,r}/(J^*, x_v)) =$ depth $(F^{*}_{v-1,r}/I^{*}_{v-1,r}) + 2r + 1$. We consider two cases:

Case 1. If v is even, then by induction on v,

$$\begin{aligned}
\operatorname{depth}(F^{**}_{v,r}/(J^* : x_v)) &= \operatorname{depth}(F^{**}_{v-2,r}/E_{v-2,r}) + 2r + 2 \\
&= (n - 2 + 1)(r + 1) + 2r + 2 \\
&= v(r + 1) - r - 1 + 2r + 2 \\
&= (v + 1)(r + 1).
\end{aligned}$$

Similarly, by induction on v, we have

$$\begin{aligned}
\operatorname{depth}(F^{**}_{v,r}/(J^*, x_v)) &= \operatorname{depth}(F^{*}_{v-1,r}/I^{*}_{v-1,r}) + 2r + 1 \\
&= (v - 1)(r + 1) + 1 + 2r + 1 \\
&= v(r + 1) - r - 1 + 1 + 2r + 1 \\
&= (v + 1)(r + 1).
\end{aligned}$$

Since depth $(F^{**}_{v,r}/(J^* : x_v)) =$ depth $(F^{**}_{v,r}/(J^*, x_v))$ Applying the Depth Lemma, we obtain

$$\operatorname{depth}(F^{**}_{v,r}/(E_{v,r} : y_{v+2})) = \operatorname{depth}(F^{**}_{v,r}/J^*) = (v + 1)(r + 1).$$

Now $F^{**}_{v,r}/(E_{v,r}, y_{v+2}) \cong F^{*}_{v,r}/I^{*}_{v,r} \otimes_K K[y_{(v+2)1}, y_{(v+2)2}, \dots, y_{(v+2)r}]$. By Lemmas 3 and 8, we have depth $(F^{**}_{v,r}/(E_{v,r}, y_{v+2})) =$ depth $(F^{*}_{v,r}/I^{*}_{v,r}) + r = (r + 1)v + 1 + r = (v + 1)(r + 1)$. Again, since depth $(F^{**}_{v,r}/(E_{v,r} : y_{v+2})) =$ depth $(F^{**}_{v,r}/(E_{v,r}, y_{v+2}))$, then by the Depth Lemma,

$$\operatorname{depth}(F^{**}_{v,r}/E_{v,r}) = (v + 1)(r + 1).$$

Case 2. If v is odd, then by induction on v,

$$\begin{aligned}
\operatorname{depth}(F^{**}_{v,r}/(J^* : x_v)) &= \operatorname{depth}(F^{**}_{v-2,r}/E_{v-2,r}) + 2r + 2 \\
&= (v - 2)(r + 1) + 2 + 2r + 2 \\
&= v(r + 1) - 2r - 2 + 2 + 2r + 2 \\
&= v(r + 1) + 2.
\end{aligned}$$

Also, by induction on v, we have

$$\begin{aligned} \text{depth}\,(F^{**}_{v,r}/(J^*, x_v)) &= \text{depth}\,(F^*_{v-1,r}/I^*_{v-1,r}) + 2r + 1 \\ &= (v-1)(r+1) + 1 + 2r + 1 \\ &= v(r+1) - r - 1 + 1 + 2r + 1 \\ &= v(r+1) + r + 1. \end{aligned}$$

By the Depth Lemma, depth $(F^{**}_{v,r}/J^*) \geq v(r+1) + 2$. It is easy to see that $F^{**}_{v,r}/(E_{v,r}, y_{v+2}) \cong F^*_{v,r}/I^*_{v,r} \bigotimes_K K[y_{(v+2)1}, y_{(v+2)2}, \ldots, y_{(v+2)r}]$. By Lemma 3, we have depth $(F^{**}_{v,r}/(E_{v,r}, y_{v+2})) = \text{depth}\,(F^*_{v,r}/I^*_{v,r}) + r = v(r+1) + 1 + r$. Using the Depth Lemma, depth $(F^{**}_{v,r}/E_{v,r}) \geq v(r+1) + 2$. For upper bound as $x_v \notin E_{v,r}$, and

$$F^{**}_{v,r}/(E_{v,r} : x_v) \cong F^{**}_{v,r}/E_{v-2,r} \bigotimes_K K[V(\mathcal{S}_{r+1})]/I(\mathcal{S}_{r+1}) \bigotimes_K K[x_v, y_{(v+1)1}, \ldots, y_{(v+1)r}, x_{(v-1)1}, \ldots, x_{(v-1)r}].$$

Thus, by Lemmas 3 and 4 and induction on v,

$$\begin{aligned} \text{depth}\,(F^{**}_{v,r}/(E_{v,r} : x_v)) &= \text{depth}\,(F^{**}_{v,r}/E_{v-2,r}) + \text{depth}\,(K[V(\mathcal{S}_{r+1})]/I(\mathcal{S}_{r+1})) + 2r + 1 \\ &= (v-2)(r+1) + 2 + 1 + 2r + 1 \\ &= v(r+1) - 2r - 2 + 2 + 2r + 2 \\ &= v(r+1) + 2. \end{aligned}$$

Using Corollary 1, depth $(F^{**}_{v,r}/E_{v,r}) \leq \text{depth}\,(F^{**}_{v,r}/(E_{v,r} : x_v)) = v(r+1) + 2$. This completes the proof for depth.

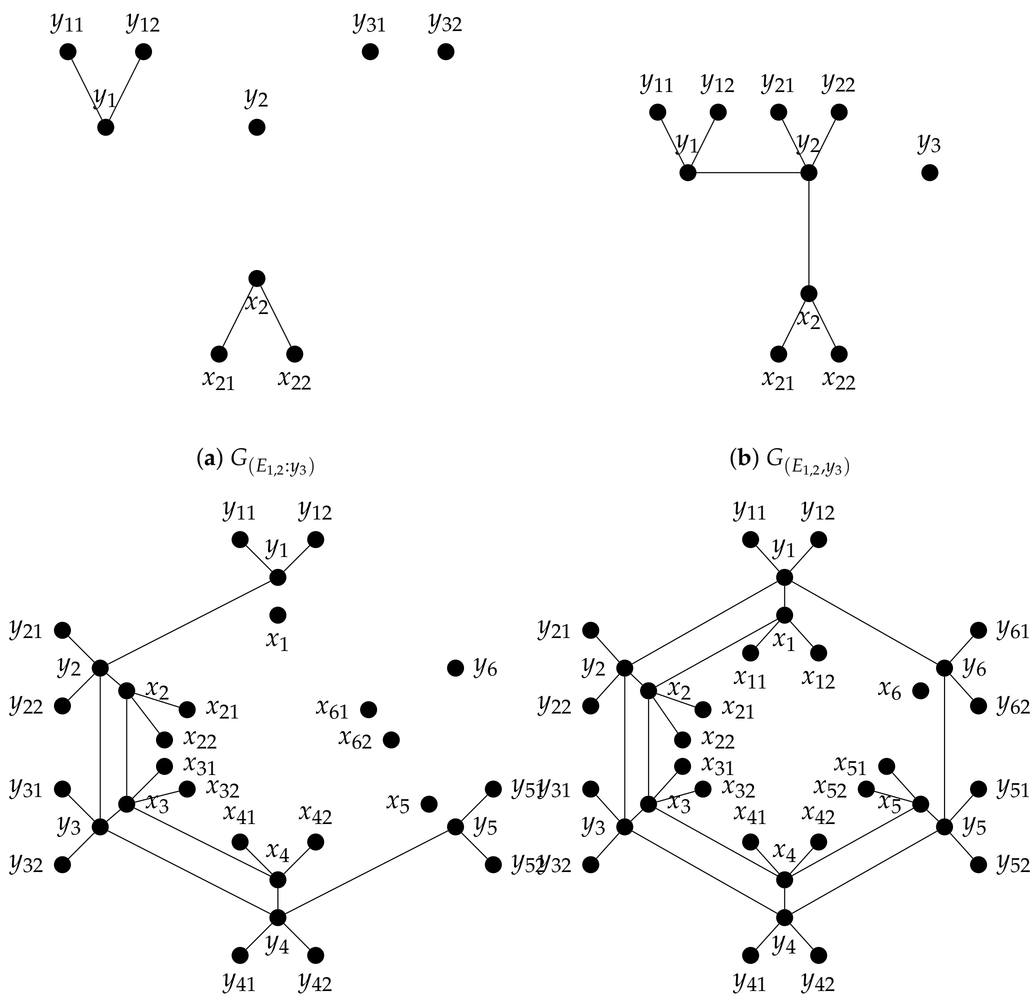
For Stanley depth, when $v = 1$, by applying Lemma 2 instead of the Depth Lemma and Lemma 6 instead of Lemma 5 on the short exact sequence, we obtain sdepth $(F^{**}_{1,r}/E_{1,r}) \geq r + 3$. For upper bound, consider $\mu = y_{21} \ldots y_{2r} y_1 y_3 x_1 \in F^{**}_{1,r}/E_{1,r}$; clearly $x\mu \in E_{1,r}$, for all $x \in \text{supp}(E_{1,r}) \setminus \text{supp}(\mu)$. Therefore, by Lemma 7, sdepth $(F^{**}_{1,r}/E_{1,r}) \leq r + 3$. For $v \geq 2$, the required result follows by applying Lemma 2 instead of the Depth Lemma, Lemma 6 instead of Lemma 5, and Proposition 1 instead of Corollary 1. If v is even, then we obtain sdepth $(F^{**}_{v,r}/E_{v,r}) \geq (v+1)(r+1)$. For upper bound, consider

$$\begin{aligned} \mu = {} & y_{11} \cdots y_{1r} \cdots y_{(v-1)1} \cdots y_{(v-1)r} y_{(v+1)1} \cdots y_{(v+1)r} x_{11} \cdots x_{1r} \cdots \\ & x_{(v-3)1} \cdots x_{(v-3)r} x_{(v-1)1} \cdots x_{(v-1)r} y_2 y_4 \cdots y_v y_{v+2} x_2 x_4 \cdots x_{v-2} x_v \in F^{**}_{v,r}/E_{v,r}. \end{aligned}$$

Clearly $x\mu \in E_{v,r}$, for all $x \in \text{supp}(E_{v,r}) \setminus \text{supp}(\mu)$; therefore, by Lemma 7, sdepth $(F^{**}_{v,r}/E_{v,r}) \leq (v+1)r + v + 1 = (v+1)(r+1)$. Hence, sdepth $(F^{**}_{v,r}/E_{v,r}) = (v+1)(r+1)$. If v is odd, then we obtain sdepth $(F^{**}_{v,r}/E_{v,r}) \geq v(r+1) + 2$. For upper bound, consider

$$\begin{aligned} \mu = {} & y_{21} \cdots y_{2r} \cdots y_{(v-1)1} \cdots y_{(v-1)r} y_{(v+1)1} \cdots y_{(v+1)r} x_{21} \cdots x_{2r} \cdots \\ & x_{(v-3)1} \cdots x_{(v-3)r} x_{(v-1)1} \cdots x_{(v-1)r} y_1 y_3 \cdots y_v y_{v+2} x_1 x_3 \cdots x_{v-2} x_v \in F^{**}_{v,r}/E_{v,r}. \end{aligned}$$

Clearly $x\mu \in E_{v,r}$, for all $x \in \text{supp}(E_{v,r}) \setminus \text{supp}(\mu)$; therefore, by Lemma 7, sdepth $(F^{**}_{v,r}/E_{v,r}) \leq vr + v + 2 = v(r+1) + 2$. This completes the proof for Stanley depth. $\square$



(**a**) $G_{(E_{1,2}:y_3)}$ (**b**) $G_{(E_{1,2},y_3)}$

(**c**) $G_{(\mathfrak{C}_{6,2}:x_6)}$ (**d**) $G_{(\mathfrak{C}_{6,2},x_6)}$

Figure 7. Graphs corresponding to ideals $(E_{1,2} : y_3)$, $(E_{1,2}, y_3)$, $(\mathfrak{C}_{6,2} : x_6)$ and $(\mathfrak{C}_{6,2}, x_6)$.

Corollary 5. *Let $r \geq 1$ and $v \geq 1$. Then*

$$\operatorname{pdim}(\digamma^{**}_{v,r}/E_{v,r}) = \begin{cases} (v+1)(r+1), & \text{if } v \text{ is even;} \\ v(r+1)+2r, & \text{if } v \text{ is odd.} \end{cases}$$

Proof. The required result can be obtained by using Theorem 5 and Theorem 1. □

Now we find depth, Stanley depth, and projective dimension of the edge ideals of the r-fold bristled graph of the circular ladder graph.

Theorem 6. *Let $v \geq 3$ and $r \geq 1$. Then*

$$\operatorname{depth}(\digamma_{v,r}/\mathfrak{C}_{v,r}) = \operatorname{sdepth}(\digamma_{v,r}/\mathfrak{C}_{v,r}) = \begin{cases} v(r+1), & \text{if } v \text{ is even;} \\ v(r+1)+r-1, & \text{if } v \text{ is odd.} \end{cases}$$

Proof. First we will prove the result for depth. Consider the following short exact sequence:

$$0 \longrightarrow \digamma_{v,r}/(\mathfrak{C}_{v,r} : x_v) \xrightarrow{\cdot x_v} \digamma_{v,r}/\mathfrak{C}_{v,r} \longrightarrow \digamma_{v,r}/(\mathfrak{C}_{v,r}, x_v) \longrightarrow 0. \quad (5)$$

After a suitable renumbering of variables, we have

$$\digamma_{v,r}/(\mathfrak{C}_{v,r} : x_v) \cong \digamma^{**}_{v-3,r}/E_{v-3,r} \bigotimes_K K[\{x_v\} \bigcup \bigcup_{j=1}^{r} \{x_{(v-2)j}, x_{(v-1)j}, y_{vj}\}].$$

By Lemma 3,

$$\text{depth}\,(\digamma_{v,r}/(\mathfrak{C}_{v,r} : x_v)) = \text{depth}\,(\digamma^{**}_{v-3,r}/E_{v-3,r}) + 3r + 1.$$

Let $A^* := (\mathfrak{C}_{v,r}, x_v)$ and $\mathcal{G}(A^*) = \mathcal{G}(I_{v-1,r}) \bigcup \{y_1 y_v, y_v y_{v-1}, x_v\} \bigcup \{y_v y_{v1}, y_v y_{v2}, \ldots, y_v y_{vr}\}$. Consider the following short exact sequence:

$$0 \longrightarrow \digamma_{v,r}/(A^* : y_v) \xrightarrow{\cdot y_v} \digamma_{v,r}/A^* \longrightarrow \digamma_{v,r}/(A^*, y_v) \longrightarrow 0.$$

After renumbering of variables, we have

$$\digamma_{v,r}/(A^* : y_v) \cong \digamma^{**}_{v-3,r}/E_{v-3,r} \bigotimes_K K[\{y_v\} \bigcup \bigcup_{j=1}^{r} \{x_{vj}, y_{(v-2)j}, y_{(v-1)j}\}],$$

and

$$\digamma_{v,r}/(A^*, y_v) \cong \digamma_{v-1,r}/I_{v-1,r} \bigotimes_K K[x_{v1}, x_{v2}, \ldots, x_{vr}, y_{v1}, y_{v2}, \ldots, y_{vr}].$$

Case 1. When v is even, using Lemma 3, depth $(\digamma_{v,r}/(A^* : y_v)) = \text{depth}\,(\digamma^{**}_{v-3,r}/E_{v-3,r}) + 3r + 1$. As v is even, so $v - 3$ will be an odd number. So by Theorem 5, we have

$$\begin{aligned} \text{depth}\,(\digamma_{v,r}/(A^* : y_v)) &= (v-3)(r+1) + 2 + 3r + 1 \\ &= v(r+1) - 3r - 3 + 3r + 3 \\ &= v(r+1). \end{aligned}$$

Similarly, by Lemma 3 and Theorem 3,

$$\begin{aligned} \text{depth}\,(\digamma_{v,r}/(A^*, y_v)) &= \text{depth}\,(\digamma_{v-1,r}/I_{v-1,r}) + 2r \\ &= (v-1)(r+1) + 2r \\ &= v(r+1) - r - 1 + 2r \\ &= v(r+1) + r - 1. \end{aligned}$$

By the Depth Lemma, depth $(\digamma_{v,r}/A^*) \geq v(r+1)$. Now by Theorem 5,

$$\begin{aligned} \text{depth}\,(\digamma_{v,r}/(\mathfrak{C}_{v,r} : x_v)) &= \text{depth}\,(\digamma^{**}_{v-3,r}/E_{v-3,r}) + 3r + 1 \\ &= (v-3)(r+1) + 2 + 3r + 1 \\ &= v(r+1) - 3r - 3 + 3r + 3 \\ &= v(r+1). \end{aligned}$$

Applying the Depth Lemma on short exact sequence 5, we obtain depth $(\digamma_{v,r}/\mathfrak{C}_{v,r}) = v(r+1)$. This completes the proof when v is even.

Case 2. If v is odd, using Lemma 3, depth $(\digamma_{v,r}/(A^* : y_v)) = \text{depth}\,(\digamma^{**}_{v-3,r}/E_{v-3,r}) + 3r + 1$. As v is odd, so $v - 3$ will be an even number. So by Theorem 5, we have

$$\begin{aligned} \text{depth}\,(\digamma_{v,r}/(A^* : y_v)) &= (n-3+1)(r+1) + 3r + 1 \\ &= v(r+1) - 2r - 2 + 3r + 1 \\ &= v(r+1) + r - 1. \end{aligned}$$

Now by Lemma 3 and Theorem 3,

$$\begin{aligned} \operatorname{depth}(\mathcal{F}_{v,r}/(A^*, y_v)) &= \operatorname{depth}(\mathcal{F}_{v-1,r}/I_{v-1,r}) + 2r \\ &= (v-1)(r+1) + 2r \\ &= v(r+1) - r - 1 + 2r \\ &= v(r+1) + r - 1. \end{aligned}$$

By the Depth Lemma, $\operatorname{depth}(\mathcal{F}_{v,r}/A^*) = v(r+1) + r - 1$. By Theorem 5,

$$\begin{aligned} \operatorname{depth}(\mathcal{F}_{v,r}/(\mathfrak{C}_{v,r} : x_v)) &= \operatorname{depth}(\mathcal{F}^{**}_{v-3,r}/E_{v-3,r}) + 3r + 1 \\ &= (v-3+1)(r+1) + 3r + 1 \\ &= v(r+1) - 2r - 2 + 3r + 1 \\ &= v(r+1) + r - 1. \end{aligned}$$

Applying the Depth Lemma on short exact sequence 5, we obtain $\operatorname{depth}(\mathcal{F}_{v,r}/\mathfrak{C}_{v,r}) = v(r+1) + r - 1$. This completes the proof for depth.

For Stanley depth, the required result follows by applying Lemma 2 instead of the Depth Lemma and Lemma 6 instead of Lemma 5. When v is even, we have $\operatorname{sdepth}(\mathcal{F}_{v,r}/\mathfrak{C}_{v,r}) \geq v(r+1)$. For upper bound as $x_v \notin \mathfrak{C}_{v,r}$ and $\operatorname{sdepth}(\mathcal{F}_{v,r}/(\mathfrak{C}_{v,r} : x_v)) = \operatorname{sdepth}(\mathcal{F}^{**}_{v-3,r}/E_{v-3,r}) + 3r + 1$, by Theorem 5 and Proposition 1 $\operatorname{sdepth}(\mathcal{F}_{v,r}/\mathfrak{C}_{v,r}) \leq \operatorname{sdepth}(\mathcal{F}_{v,r}/(\mathfrak{C}_{v,r} : x_v)) = v(r+1)$. Similarly, when v is odd, we obtain $\operatorname{sdepth}(\mathcal{F}_{v,r}/\mathfrak{C}_{v,r}) \geq v(r+1) + r - 1$. For upper bound as $x_v \notin \mathfrak{C}_{v,r}$ and $\operatorname{sdepth}(\mathcal{F}_{v,r}/(\mathfrak{C}_{v,r} : x_v)) = \operatorname{sdepth}(\mathcal{F}^{**}_{v-3,r}/E_{v-3,r}) + 3r + 1$, by Theorem 5 and Proposition 1, $\operatorname{sdepth}(\mathcal{F}_{v,r}/\mathfrak{C}_{v,r}) \leq \operatorname{sdepth}(\mathcal{F}_{v,r}/(\mathfrak{C}_{v,r} : x_v)) = v(r+1) + r - 1$. Hence,

$$\operatorname{sdepth}(\mathcal{F}_{v,r}/\mathfrak{C}_{v,r}) = v(r+1) + r - 1.$$

□

Corollary 6. *Let $v \geq 3$ and $r \geq 1$. Then*

$$\operatorname{pdim}(\mathcal{F}_{v,r}/\mathfrak{C}_{v,r}) = \begin{cases} v(r+1), & \text{if } v \text{ is even;} \\ v(r+1) - r + 1, & \text{if } v \text{ is odd.} \end{cases}$$

Proof. The required result can be obtain by using Theorem 6 and Theorem 1. □

We also have formulae for values of depth, Stanley depth, and projective dimension of the quotient rings of the edge ideals of the T_v graph, as given in the next theorem and corollary.

Theorem 7. *Let $v \geq 3$ and $r \geq 1$. Then*

$$\operatorname{depth}(\mathcal{F}_{v,r}/C_{v,r}) = \operatorname{sdepth}(\mathcal{F}_{v,r}/C_{v,r}) = \lfloor \frac{3v+1}{2} \rfloor r + \lceil \frac{v-1}{2} \rceil.$$

Proof. First we will prove the result for depth. We will prove this for $v \geq 3$. Consider the following short exact sequence:

$$0 \longrightarrow \mathcal{F}_{v,r}/(C_{v,r} : x_v) \xrightarrow{\cdot x_v} \mathcal{F}_{v,r}/C_{v,r} \longrightarrow \mathcal{F}_{v,r}/(C_{v,r}, x_v) \longrightarrow 0. \tag{6}$$

After renumbering the variables, we have

$$\mathcal{F}_{v,r}/(C_{v,r} : x_v) \cong \mathcal{F}_{v-3,r}/L_{v-3,r} \bigotimes_K K[\{x_v\} \bigcup \bigcup_{j=1}^{r} \{x_{(v-2)j}, x_{(v-1)j}, y_{(v-2)j}, y_{vj}, y_{(v-1)j}\}].$$

Using Lemma 3 and Theorem 4,

$$\begin{aligned}\operatorname{depth}(\mathcal{F}_{v,r}/(C_{v,r}:x_v)) &= \operatorname{depth}(\mathcal{F}_{v-3,r}/L_{v-3,r})+5r+1\\ &= \lfloor\frac{3(v-3)}{2}\rfloor r+\lceil\frac{v-3}{2}\rceil+5r+1\\ &= \lfloor\frac{3v+1}{2}\rfloor r+\lceil\frac{v-1}{2}\rceil.\end{aligned}$$

Let $J' := (C_{v,r}, x_v)$, where $\mathcal{G}(J') = \mathcal{G}(I_{v-1})\bigcup\{x_{v-1}y_v, y_{v-1}y_v, y_vy_1, y_vx_1, x_v\}\bigcup_{j=1}^{r}\{y_vy_{vj}\}$. Consider the following short exact sequence:

$$0 \longrightarrow \mathcal{F}_{v,r}/(J':y_v) \xrightarrow{\cdot y_v} \mathcal{F}_{v,r}/J' \longrightarrow \mathcal{F}_{v,r}/(J',y_v) \longrightarrow 0.$$

After renumbering the variables, we have

$$(\mathcal{F}_{v,r}/(J':y_v)) \cong \mathcal{F}_{v-3,r}/L_{v-3,r}\bigotimes_K K[\{y_v\}\bigcup\bigcup_{j=1}^{r}\{y_{(v-2)j}, x_{(v-2)j}, x_{vj}, x_{(v-1)j}, y_{(v-1)j}\}].$$

By Lemma 3 and Theorem 4,

$$\begin{aligned}\operatorname{depth}(\mathcal{F}_{v,r}/(J':y_v)) &= \operatorname{depth}(\mathcal{F}_{v-3,r}/L_{v-3,r})+5r+1\\ &= \lfloor\frac{3(v-3)}{2}\rfloor r+\lceil\frac{v-3}{2}\rceil+5r+1\\ &= \lfloor\frac{3v+1}{2}\rfloor r+\lceil\frac{v-1}{2}\rceil.\end{aligned}$$

Now $\mathcal{G}(J',y_v) = \mathcal{G}(L_{v-1,r})\bigcup\{y_v, x_v\}$ and $\mathcal{F}_{v,r}/(J',y_v) \cong \mathcal{F}_{v-1,r}/L_{v-1,r}\bigotimes_K K[\bigcup_{j=1}^{r}\{x_{vj}, y_{vj}\}]$. Using Lemma 3 and Theorem 4, we have $\operatorname{depth}(\mathcal{F}_{v,r}/(J',y_v)) = \operatorname{depth}(\mathcal{F}_{v-1,r}/L_{v-1,r})+2r = \lfloor\frac{3(v-1)}{2}\rfloor r+\lceil\frac{v-1}{2}\rceil+2r. = \lfloor\frac{3v+1}{2}\rfloor r+\lceil\frac{v-1}{2}\rceil$. By the Depth Lemma,

$$\operatorname{depth}(\mathcal{F}_{v,r}/J') = \operatorname{depth}(\mathcal{F}_{v,r}/(C_{v,r},x_v)) = \lfloor\frac{3v+1}{2}\rfloor r+\lceil\frac{v-1}{2}\rceil.$$

Applying the Depth Lemma on short exact sequence 6, $\operatorname{depth}(\mathcal{F}_{v,r}/C_{v,r}) = \lfloor\frac{3v+1}{2}\rfloor r+\lceil\frac{v-1}{2}\rceil$. This completes the proof for depth.

For Stanley depth, the required result follows by applying Lemma 2 instead of the Depth Lemma. We obtain $\operatorname{sdepth}(\mathcal{F}_{v,r}/C_{v,r}) \geq \lfloor\frac{3v+1}{2}\rfloor r+\lceil\frac{v-1}{2}\rceil$. For upper bound as $x_v \notin C_{v,r}$ we have

$$\operatorname{sdepth}(\mathcal{F}_{v,r}/(C_{v,r}:x_v)) = \operatorname{sdepth}(\mathcal{F}_{v-3,r}/L_{v-3,r})+5r+1,$$

by Theorem 4 and Proposition 1,

$$\operatorname{sdepth}(\mathcal{F}_{v,r}/C_{v,r}) \leq \operatorname{sdepth}(\mathcal{F}_{v,r}/(C_{v,r}:x_v)) = \lfloor\frac{3v+1}{2}\rfloor r+\lceil\frac{v-1}{2}\rceil.$$

This completes the proof. □

Corollary 7. *Let $v \geq 3$ and $r \geq 1$. Then*

$$\operatorname{pdim}(\mathcal{F}_{v,r}/C_{v,r}) = \lceil\frac{v-1}{2}\rceil r+\lfloor\frac{3v+1}{2}\rfloor.$$

Proof. The required result can be obtain by using Theorem 7 and Theorem 1. □

Example 2. *If* $v = 9$ *and* $r = 4$, *then by Theorem 7, we have* $\text{depth}(\mathcal{F}_{9,4}/C_{9,4}) = \text{sdepth}(\mathcal{F}_{9,4}/C_{9,4}) = \lfloor \frac{3(9)+1}{2} \rfloor (4) + \lceil \frac{9-1}{2} \rceil = 56 + 4 = 60$. *Also, by Corollary 4 we have* $\text{pdim}(\mathcal{F}_{9,4}/C_{9,4}) = \lceil \frac{9-1}{2} \rceil (4) + \lfloor \frac{3(9)+1}{2} \rfloor = 16 + 14 = 30$.

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