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Twisted Hypersurfaces in Euclidean 5-Space

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Abstract: The twisted hypersurfaces $\mathfrak{x}$ with the $(0,0,0,0,1)$ rotating axis in five-dimensional Euclidean space $\mathbb{E}^5$ is considered. The fundamental forms, the Gauss map, and the shape operator of $\mathfrak{x}$ are calculated. In $\mathbb{E}^5$, describing the curvatures by using the Cayley–Hamilton theorem, the curvatures of hypersurfaces $\mathfrak{x}$ are obtained. The solutions of differential equations of the curvatures of the hypersurfaces are open problems. The umbilically and minimality conditions to the curvatures of $\mathfrak{x}$ are determined. Additionally, the Laplace–Beltrami operator relation of $\mathfrak{x}$ is given.

Keywords: Euclidean five-space; twisted hypersurfaces family; Gauss map; mean curvature; Gauss–Kronecker curvature; Cayley–Hamilton theorem; Laplace–Beltrami operator

MSC: 53A15; 53B25; 53C40; 15A15; 15A69

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1. Introduction

Twisted hypersurfaces, including helical or helicoidal ones, and related objects such as rotational, ruled, and minimal hypersurfaces, are of interest to mathematicians and have been studied extensively for a long time.

Obata [1] offered a relation for the manifold isometric to the sphere. Takahashi [2] served a Euclidean sub-manifold as a one-type if it is minimal or minimal of a hypersphere in $\mathbb{E}^n$. Chern et al. [3] studied the minimal sub-manifolds of a sphere. Lawson [4] researched the minimal sub-manifolds and the Laplace–Beltrami operator.

In space forms, Chen et al. [5] served the 40 years of one-type sub-manifolds and a one-type Gauss map.

In $\mathbb{E}^3$, Bour [6] determined the deformation of helical rotational surfaces. Kenmotsu [7] described the rotational surfaces having prescribed mean curvature. Do Carmo and Dajczer [8] studied the helical surfaces. Ferrandez et al. [9] considered the surfaces supplying $\Delta H = AH$, A denoting a matrix of order three. Baikoussis and T. Koufogiorgos [10] focused the helical surfaces having prescribed mean or Gaussian curvature. Ikawa [11] served the Bour's theorem and the Gauss map. Choi and Kim [12] researched the minimal helicoid. Garay [13] investigeted the surfaces of revolution. Dillen et al. [14] focused the surfaces supplying $\Delta r = Ar + B$, where A is 3×3, and B is a 3×1 matrix. Güler et al. [15] worked Bour's theorem on a Gauss map. Stamatakis and Zoubi [16] described the surfaces of revolution supplying $\Delta^{III} x = Ax$. Kim et al. [17] researched the Cheng–Yau operator of the surfaces of revolution.

In Minkowski 3-space $\mathbb{E}^3_1$, Dillen and Kühnel [18] worked the ruled Weingarten surfaces. Ikawa [19] determined Bour's theorem. Beneki et al. [20] studied the helical surfaces. Güler and Turgut Vanlı [21] served Bour's theorem. Güler [22] worked the helical surfaces with a light-like generating curve. Mira and Pastor [23] presented the helical maximal surfaces. Kim and Yoon [24–26] considered the ruled and rotation surfaces. The readers can see [2,27,28] for details.

In $\mathbb{E}^4$, Moore [29,30] introduced the rotational surfaces in a general form. Hasanis and Vlachos [31] focused the hypersurfaces holding the mean curvature of the harmonic.

In Minkowski 4-space $\mathbb{E}_1^4$, Ganchev and Milousheva [32] determined the corresponding surfaces of Moore [29,30]. Arvanitoyeorgos et al. [33] introduced $\Delta H = \alpha H$ (H denotes the mean curvature, α is a constant). Güler [34] introduced the helical hypersurface determined by a space-like axis in $\mathbb{E}_1^5$. Li and Güler [35,36] studied a hypersurfaces of revolution family in pseudo-Euclidean spaces $\mathbb{E}_2^5$ and $\mathbb{E}_3^7$. Other related works can be found in [37–47].

The aim of this study is to investigate the properties of twisted (i.e., helical) hypersurfaces in five-dimensional Euclidean space $\mathbb{E}^5$ with a x_5-rotating axis. Specifically, we focus on determining the fundamental forms, the Gauss map, and the shape operator of these hypersurfaces, as well as describing their curvatures using the Cayley–Hamilton theorem. We also address the open problem of finding solutions to the differential equations governing the curvatures of these hypersurfaces. Furthermore, we examine the umbilicality and minimality conditions associated with the curvatures of the helical hypersurfaces. Finally, we aim to establish the Laplace–Beltrami operator relation of $\mathfrak{x}$, providing further insights into the geometric properties of these intriguing hypersurfaces.

We focus the twisted hypersurfaces $\mathfrak{x} = \mathfrak{x}(r, \theta_1, \theta_2, \theta_3)$ constructed by the $(0,0,0,0,1)$ rotating axis in Euclidean 5-space $\mathbb{E}^5$. We offer some properties of $\mathbb{E}^5$ in Section 2. We formulate the components of the fundamental forms, the Gauss map, the shape operator of any hypersurface of $\mathbb{E}^5$. We describe the twisted hypersurfaces $\mathfrak{x}$ of $\mathbb{E}^5$ in Section 3.

By way of the theorem of Cayley–Hamilton, we obtain all the formulas of curvatures of any hypersurface, and also compute the curvatures of twisted hypersurfaces $\mathfrak{x}$. We also determine some relations for curvatures $\mathbb{K}_{j=0,\dots,4}$ of $\mathfrak{x}$. We present the umbilical relations to the hypersurfaces in Section 4.

Moreover, in Section 5, we obtain $\Delta\mathfrak{x} = \mathcal{Q}\mathfrak{x}$, where $\mathcal{Q}$ is the 5×5 matrix. We serve some examples to all findings. In the last section, we offer a conclusion.

2. Preliminaries

We assume $\mathbf{M}$ to be a hypersurface in Euclidean space $\mathbb{E}^{n+1}$, $\mathbb{S}$ denotes its shape operator, and x its position vector. We suppose $\{e_1, e_2, \dots, e_n\}$ to be a local orthonormal frame consisting the principal directions of $\mathbf{M}$ corresponding with the principal curvature κ_i for $i = 1, 2, \dots, n$.

We consider $\mathfrak{s}_j = \tau_j(\kappa_1, \kappa_2, \dots, \kappa_n)$, where τ_j is the jth elementary symmetric function defined by

$$\tau_j(q_1, q_2, \dots, q_n) = \sum_{1 \le i_1 < i_2 < \dots < i_j \le n} q_{i_1} q_{i_2} \dots q_{i_j}.$$

The following notation works:

$$\alpha_i^j = \tau_j(\kappa_1, \dots, \kappa_{i-1}, \kappa_{i+1}, \dots, \kappa_n),$$

with $\alpha_i^0 = 1$, $\mathfrak{s}_{n+1} = \mathfrak{s}_{n+2} = \dots = 0$. Function $\mathfrak{s}_k$ denotes the kth mean curvature, $H = \frac{1}{n}\mathfrak{s}_1$ and $K = \mathfrak{s}_n$ denote the mean and Gauss–Kronecker curvatures of $\mathbf{M}$, respectively. If $\mathfrak{s}_j \equiv 0$ on $\mathbf{M}$, $\mathbf{M}$ is named j-minimal. The readers can refer to Alias and Gürbüz [41], Kühnel [46] for details.

In $\mathbb{E}^{n+1}$, the characteristic polynomial Equation $\mathcal{P}$ of the shape operator $\mathbb{S}$ is determined by

$$\mathcal{P}_{\mathbb{S}}(\delta) = 0 = \det(\mathbb{S} - \delta\mathcal{I}_n) = \sum_{k=0}^{n} (-1)^k \mathfrak{s}_k \delta^{n-k}. \quad (1)$$

Here, $i = 0, \dots, n$, $\mathcal{I}_n$ describes the identity matrix of order n. The curvature formulas are given by $\binom{n}{i}\mathbb{K}_i = \mathfrak{s}_i$. Here, $\binom{n}{0}\mathbb{K}_0 = \mathfrak{s}_0 = 1$ (by definition), $\binom{n}{1}\mathbb{K}_1 = \mathfrak{s}_1, \dots, \binom{n}{n}\mathbb{K}_n = \mathfrak{s}_n$. Also, $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Next, we present some notions to Riemannian geometry. The readers can see Kühnel [46] for details. A vector with its transpose is regarded identical in this paper. We let $\mathfrak{x} = \mathfrak{x}(r, \theta_1, \theta_2, \theta_3)$ be an immersion from $M^4 \subset \mathbb{E}^4$ to $\mathbb{E}^5$.

Definition 1. *In $\mathbb{E}^5$, a Euclidean inner product of two vectors $\overrightarrow{v^1} = (v_1^1, \ldots, v_5^1)$ and $\overrightarrow{v^2} = (v_1^2, \ldots, v_5^2)$ of $\mathbb{E}^5$ is described by*

$$\left\langle \overrightarrow{v^1}, \overrightarrow{v^2} \right\rangle = v_1^1 v_1^2 + v_2^1 v_2^2 + v_3^1 v_3^2 + v_4^1 v_4^2 + v_5^1 v_5^2.$$

Definition 2. *A Euclidean vector product of $\overrightarrow{v^1}, \ldots, \overrightarrow{v^4}$ of $\mathbb{E}^5$ is defined by*

$$\overrightarrow{v^1} \times \overrightarrow{v^2} \times \overrightarrow{v^3} \times \overrightarrow{v^4} = \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ v_1^1 & v_2^1 & v_3^1 & v_4^1 & v_5^1 \\ v_1^2 & v_2^2 & v_3^2 & v_4^2 & v_5^2 \\ v_1^3 & v_2^3 & v_3^3 & v_4^3 & v_5^3 \\ v_1^4 & v_2^4 & v_3^4 & v_4^4 & v_5^4 \end{pmatrix}.$$

Here, e_i denotes the generator elements of $\mathbb{E}^5$.

Definition 3. *In $\mathbb{E}^5$, any hypersurface $\mathfrak{x}$ has the following matrices, respectively:*

$$\mathbb{I} = \begin{pmatrix} E & F & A & D \\ F & G & B & J \\ A & B & C & Q \\ D & J & Q & S \end{pmatrix}, \quad \mathbb{II} = \begin{pmatrix} L & M & P & X \\ M & N & T & Y \\ P & T & V & Z \\ X & Y & Z & U \end{pmatrix},$$

where the components are indicated by

$$\begin{array}{lllll} E = \langle \mathfrak{x}_r, \mathfrak{x}_r \rangle, & F = \langle \mathfrak{x}_r, \mathfrak{x}_{\theta_1} \rangle, & A = \langle \mathfrak{x}_r, \mathfrak{x}_{\theta_2} \rangle, & D = \langle \mathfrak{x}_r, \mathfrak{x}_{\theta_3} \rangle, & G = \langle \mathfrak{x}_{\theta_1}, \mathfrak{x}_{\theta_1} \rangle, \\ B = \langle \mathfrak{x}_{\theta_1}, \mathfrak{x}_{\theta_2} \rangle, & J = \langle \mathfrak{x}_{\theta_1}, \mathfrak{x}_{\theta_3} \rangle, & C = \langle \mathfrak{x}_{\theta_2}, \mathfrak{x}_{\theta_2} \rangle, & Q = \langle \mathfrak{x}_{\theta_2}, \mathfrak{x}_{\theta_3} \rangle, & S = \langle \mathfrak{x}_{\theta_3}, \mathfrak{x}_{\theta_3} \rangle, \\ L = \langle \mathfrak{x}_{rr}, \mathbb{G} \rangle, & M = \langle \mathfrak{x}_{r\theta_1}, \mathbb{G} \rangle, & P = \langle \mathfrak{x}_{r\theta_2}, \mathbb{G} \rangle, & X = \langle \mathfrak{x}_{r\theta_3}, \mathbb{G} \rangle, & N = \langle \mathfrak{x}_{\theta_1\theta_1}, \mathbb{G} \rangle, \\ T = \langle \mathfrak{x}_{\theta_1\theta_2}, \mathbb{G} \rangle, & Y = \langle \mathfrak{x}_{\theta_1\theta_3}, \mathbb{G} \rangle, & V = \langle \mathfrak{x}_{\theta_2\theta_2}, \mathbb{G} \rangle, & Z = \langle \mathfrak{x}_{\theta_2\theta_3}, \mathbb{G} \rangle, & U = \langle \mathfrak{x}_{\theta_3\theta_3}, \mathbb{G} \rangle, \end{array}$$

$\mathfrak{x}_r = \frac{\partial \mathfrak{x}}{\partial r}$, $\mathfrak{x}_{r\theta_1} = \frac{\partial^2 \mathfrak{x}}{\partial r \partial \theta_1}$, $\mathfrak{x}_{\theta_3\theta_3} = \frac{\partial^2 \mathfrak{x}}{\partial \theta_3^2}$, *etc.; the Gauss map of $\mathfrak{x}$ is denoted by*

$$\mathbb{G} = \frac{\mathfrak{x}_r \times \mathfrak{x}_{\theta_1} \times \mathfrak{x}_{\theta_2} \times \mathfrak{x}_{\theta_3}}{\left\| \mathfrak{x}_r \times \mathfrak{x}_{\theta_1} \times \mathfrak{x}_{\theta_2} \times \mathfrak{x}_{\theta_3} \right\|}.$$

Definition 4. *In $\mathbb{E}^5$, hypersurface $\mathfrak{x} = \mathfrak{x}(r, \theta_1, \theta_2, \theta_3)$ supplies the following relations:*

$$\mathbb{S} = \mathbb{I}^{-1}.\mathbb{II} = \mathbb{II}^{-1}.\mathbb{III} = \mathbb{III}^{-1}.\mathbb{IV} = \mathbb{IV}^{-1}.\mathbb{V},$$

where $\mathbb{S}$ denotes the shape operator, and $\mathbb{I}, \mathbb{II}, \mathbb{III}, \mathbb{IV}, \mathbb{V}$ describe the fundamental forms of $\mathfrak{x}$.

Definition 5. *In $\mathbb{E}^5$,*

$$\mathcal{P}_{\mathbb{S}}(\delta) = \sum_{k=0}^{4} (-1)^k \mathfrak{s}_k \delta^{4-k} = \det(\mathbb{S} - \delta \mathcal{I}_4) = 0$$

determines the characteristic polynomial $\mathcal{P}$ of $\mathbb{S}$, and $\mathcal{I}_4$ denotes the identity matrix. The curvature formulas are described by $\binom{4}{i}\mathbb{K}_i = \mathfrak{s}_i$. Here, $\binom{4}{0}\mathbb{K}_0 = \mathfrak{s}_0 = 1$, $\binom{4}{1}\mathbb{K}_1 = \mathfrak{s}_1, \ldots, \binom{4}{4}\mathbb{K}_4 = \mathfrak{s}_4$. $\mathbb{K}_1$, $\mathbb{K}_4$ denote the mean, Gauss–Kronecker curvatures, respectively.

Definition 6. *When $\mathbb{K}_j = 0$, $j = 1, \ldots, 4$ on a hypersurface $\mathfrak{x}$, $\mathfrak{x}$ is named j-minimal.*

See [45] for details of $\mathbb{K}_j$, and dimension four.

Next, we reval the formulas of the curvatures $\mathbb{K}_j$ of $\mathfrak{x}$ in $\mathbb{E}^5$.

Theorem 1. *The formulas of the curvature of a hypersurface in a five-space are given, respectively, by $\mathbb{K}_0 = 1$ (by definition),*

$$
\begin{aligned}
4\mathbb{K}_1 &= tr(\mathbb{S}) \\
&= \frac{1}{\det \mathbb{I}}[(EN+GL-2FM)\left(CS-Q^2\right)+\left(EG-F^2\right)(SV+UC) \\
&\quad -(GU+NS)A^2-(LS+EU)B^2-(CN+GV)D^2-(EV+CL)J^2 \\
&\quad +2(A^2JY+B^2XD+D^2BT+J^2AP+F^2QZ+CJMD-ABYD \\
&\quad -BJPD+ANQD-AJTD-BMQD+AGZD-BFZD+CFYD \\
&\quad -AGPS-CGXD+FJVD+GQPD+BJZE-CJYE+BFPS \\
&\quad -BSTE-FQTD+BQYE+JQTE+AGQX-BFQX-GQZE \\
&\quad +ABFU-FJPQ+AFST-AFQY+ABMS-ABJX-AJMQ \\
&\quad +BJLQ+CFJX-AFJZ)],
\end{aligned}
$$

$$
\begin{aligned}
6\mathbb{K}_2 &= \frac{1}{\det \mathbb{I}}[(EN+GL-2FM)CU-\left(EG-F^2\right)Z^2-\left(LN-M^2\right)Q^2 \\
&\quad +\left(B^2-CG\right)X^2+\left(A^2-EC\right)Y^2+\left(P^2-LV\right)J^2+\left(T^2-NV\right)D^2 \\
&\quad -\left(CM^2+GP^2\right)S-\left(A^2N+B^2L+F^2V\right)U-ST^2E \\
&\quad +GVEU+CLNS+GLSV+NSVE+2(ANZD-BMZD \\
&\quad +CMYD-CNXD-BPYD+JMVD-ATYD-JPTD \\
&\quad +NPQD+GPZD-MQTD-FTZD+FVYD-GVXD \\
&\quad +BYZE+JTZE-JVYE-NQZE+QTYE+ABMU-AGPU \\
&\quad +BFPU+AFTU-BTEU-AJMZ-ANPS+BJLZ+BMPS \\
&\quad -CJLY+CJMX-ABXY-BJPX+AMST-BLST \\
&\quad -AJTX-AMQY+ANQX+BLQY-BMQX-JMPQ \\
&\quad -AFYZ+AGXZ-BFXZ+CFXY-FJPZ+JLQT-FMSV \\
&\quad +FPST+FJVX-GLQZ-FPQY+GPQX-FQTX) \\
&\quad +4(FMQZ+BTXD+AJPY)],
\end{aligned}
$$

$$
\begin{aligned}
4\mathbb{K}_3 &= \frac{1}{\det \mathbb{I}}[(EN+GL-2FM)Z^2+(CL+VE)Y^2+(CN+GV)X^2 \\
&\quad +(EU+LS)T^2+(NS+GU)P^2+(SV+CU)M^2 \\
&\quad -(CN+GV)UL-(LS+UE)VN+2(MTZD-M^2QZ \\
&\quad -T^2XD-JP^2Y-APY^2-BTX^2-NPZD-MVYD \\
&\quad +NVXD+PTYD-TYZE+ANPU-BMPU-AMTU \\
&\quad +BLTU+FMVU-FPTU+AMYZ-ANXZ-BLYZ \\
&\quad +BMXZ-CMXY+JMPZ+BPXY-JLTZ-MPST \\
&\quad +JLVY-JMVX+LNQZ+ATXY+JPTX+MPQY \\
&\quad -NPQX+FPYZ-GPXZ-LQTY+MQTX \\
&\quad +FTXZ-FVXY)],
\end{aligned}
$$

$$
\mathbb{K}_4 = \det(\mathbb{S}) = \frac{\det \mathbb{II}}{\det \mathbb{I}} = \frac{\det \mathbb{III}}{\det \mathbb{II}} = \frac{\det \mathbb{IV}}{\det \mathbb{III}} = \frac{\det \mathbb{V}}{\det \mathbb{IV}},
$$

where

$$
\begin{aligned}
\det \mathbb{I} \;=\;& \left(EG-F^2\right)\left(CS-Q^2\right)+\left(J^2-GS\right)A^2+\left(D^2-ES\right)B^2 \\
& +2((CF-AB)DJ+(EB-FA)JQ+(GA-FB)DQ) \\
& -\left(EJ^2+GD^2\right)C+2FABS,
\end{aligned}
$$

$$
\begin{aligned}
\det \mathbb{III} \;=\;& \left(LN-M^2\right)\left(UV-Z^2\right)+\left(Y^2-UN\right)P^2+\left(X^2-UL\right)T^2 \\
& +2((VM-PT)XY+(LT-MP)YZ+(NP-MT)XZ) \\
& -\left(LY^2+NX^2\right)V+2MUPT.
\end{aligned}
$$

Here, $tr(\mathbb{S})=\sum_{i=1}^{4} s_{ii}$*.*

Proof. By using Definition 3, Definition 4, and Definition 5, and by direct computations, the characteristic polynomial is obtained. Then, $\mathbb{K}_j$ values are found. □

3. Twisted Hypersurfaces Family with the $(0,0,0,0,1)$ Rotating Axis in $\mathbb{E}^5$

We describe the twisted hypersurfaces family. The readers can refer to Do Carmo and Dajczer [42] for some results about the rotational hypersurfaces of Riemannian spaces.

Definition 7. *We let I be an open interval $I\subset\mathbb{R}$, $\gamma: I\longrightarrow\Pi$ be a curve in plane Π, and ℓ be a line in Π. A rotational hypersurface is determined by a generating curve γ rotating about line (named axis) ℓ. While the generating curve γ rotates about ℓ, it concurrently replaces parallel lines orthogonal to ℓ. The speed of rotating commensurates to the speed of replacement. The construcing hypersurface is named the twisted hypersurfaces family with axis ℓ and pitches $a,b,c\in\mathbb{R}-\{0\}$.*

Readers can see Kühnel [46] for details. Next, we describe the twisted hypersurfaces of $\mathbb{E}^5$.

The rotation matrix $\mathcal{M}=\mathcal{M}(\theta_1,\theta_2,\theta_3)$ obtained by rotating axis $(0,0,0,0,1)$ in $\mathbb{E}^5$ is described by

$$
\mathcal{M}=\begin{pmatrix}
\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3 & -\mathcal{S}_1 & -\mathcal{C}_1\mathcal{S}_2 & -\mathcal{C}_1\mathcal{C}_2\mathcal{S}_3 & 0 \\
\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3 & \mathcal{C}_1 & -\mathcal{S}_1\mathcal{S}_2 & -\mathcal{S}_1\mathcal{C}_2\mathcal{S}_3 & 0 \\
\mathcal{S}_2\mathcal{C}_3 & 0 & \mathcal{C}_2 & -\mathcal{S}_2\mathcal{S}_3 & 0 \\
\mathcal{S}_3 & 0 & 0 & \mathcal{C}_3 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

Here, $\mathcal{C}_i=\cos\theta_i$, $\mathcal{S}_i=\sin\theta_i$, $\theta_i\in[0,2\pi)$, $i=1,2,3$, and $\mathcal{M}$ holds:

$$
\mathcal{M}.\ell=\ell,\ \mathcal{M}^T.\mathcal{M}=\mathcal{M}.\mathcal{M}^T=\mathcal{I}_5,\ \det\mathcal{M}=1.
$$

The generating curve is determined by

$$
\gamma(r)=(f(r),0,0,0,\varphi(r)). \tag{2}
$$

Here, f,φ denote the differentiable functions of $\mathbb{R}$. In $\mathbb{E}^5$, the twisted hypersurface $\mathfrak{x}=\mathfrak{x}(r,\theta_1,\theta_2,\theta_3)$ determined by $(0,0,0,0,1)$ is described by $\mathfrak{x}=\mathcal{M}.\gamma^T+(a\theta_1+b\theta_2+c\theta_3)\ell^T$, where $r\in I$, $\theta_1,\theta_2,\theta_3\in[0,2\pi)$, $a,b,c\in\mathbb{R}-\{0\}$. The parametric representation of twisted hypersurfaces **M** is determined by

$$
\mathfrak{x}(r,\theta_1,\theta_2,\theta_3)=(f\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3, f\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3, f\mathcal{S}_2\mathcal{C}_3, f\mathcal{S}_3, \varphi+a\theta_1+b\theta_2+c\theta_3). \tag{3}
$$

We note that we describe the following different hypersurfaces in lower dimensions.

1. If $b=c=\theta_2=\theta_3=0$, we have a twisted surface with a $(0,0,1)$ axis in $\mathbb{E}^3$.

2. When $a = b = c = \theta_2 = \theta_3 = 0$, we obtain a rotational surface with a $(0,0,1)$ axis in $\mathbb{E}^3$.
3. When $c = \theta_3 = 0$, we obtain a twisted hypersurface with a $(0,0,0,1)$ axis in $\mathbb{E}^4$.
4. If $a = b = c = \theta_3 = 0$, we find a rotational hypersurface with a $(0,0,0,1)$ axis in $\mathbb{E}^4$.

Next, we describe the curvature formulas for hypersurface $\mathfrak{x} = \mathfrak{x}(r, \theta_1, \theta_2, \theta_3)$ in $\mathbb{E}^5$.

Theorem 2. *Hypersurface* $\mathfrak{x} = \mathfrak{x}(r, \theta_1, \theta_2, \theta_3)$ *in Euclidean 5-space* $\mathbb{E}^5$ *has the following curvatures:*

$$\mathbb{K}_0 = 1,\ 4\mathbb{K}_1 = -\frac{\mathfrak{a}_1}{\mathfrak{a}_0},\ 6\mathbb{K}_2 = \frac{\mathfrak{a}_2}{\mathfrak{a}_0},\ 4\mathbb{K}_3 = -\frac{\mathfrak{a}_3}{\mathfrak{a}_0},\ \mathbb{K}_4 = \frac{\mathfrak{a}_4}{\mathfrak{a}_0}, \tag{4}$$

where $P_{\mathbb{S}}(\delta) = \mathfrak{a}_4\delta^4 + \mathfrak{a}_3\delta^3 + \mathfrak{a}_2\delta^2 + \mathfrak{a}_1\delta + \mathfrak{a}_0 = 0$ *describes the characteristic polynomial of the shape operator matrix* $\mathbb{S}$, $\mathfrak{a}_0 = \det \mathbb{I}$, $\mathfrak{a}_4 = \det \mathbb{III}$, *and* $\mathbb{I}$, $\mathbb{III}$ *denote the fundamental form matrices described by Definition 3.*

Proof. $\mathbb{I}^{-1}.\mathbb{III}$ determines $\mathbb{S}$ of $\mathfrak{x}$ in $\mathbb{E}^5$. We compute $P_{\mathbb{S}}(\delta) = \det(\mathbb{S} - \delta\mathcal{I}_4) = 0$ of $\mathbb{S}$. Hence, we obtain jth curvatures $\mathbb{K}_j$:

$$\begin{aligned}
\binom{4}{0}\mathbb{K}_0 &= 1, \\
\binom{4}{1}\mathbb{K}_1 &= \sum_{i=1}^{4} \kappa_i = -\frac{\mathfrak{a}_1}{\mathfrak{a}_0}, \\
\binom{4}{2}\mathbb{K}_2 &= \sum_{1=i_1<i_2}^{4} \kappa_{i_1}\kappa_{i_2} = \frac{\mathfrak{a}_2}{\mathfrak{a}_0}, \\
\binom{4}{3}\mathbb{K}_3 &= \sum_{1=i_1<i_2<i_3}^{4} \kappa_{i_1}\kappa_{i_2}\kappa_{i_3} = -\frac{\mathfrak{a}_3}{\mathfrak{a}_0}, \\
\binom{4}{4}\mathbb{K}_4 &= \prod_{i=1}^{4} \kappa_i = \frac{\mathfrak{a}_4}{\mathfrak{a}_0}.
\end{aligned}$$

κ_i denotes the principal curvatures of $\mathfrak{x}$ where $i = 1, \ldots, 4$. □

See [44,45] for the cases of dimension four.

The curvatures of the twisted hypersurfaces with rotating axis $(0,0,0,0,1)$ are given by the following theorem.

Theorem 3. *In* $\mathbb{E}^5$, *the curvatures of the twisted hypersurfaces* $\mathfrak{x}$ *described by Equation (3) are, respectively, given by*

$$\begin{aligned}
\mathbb{K}_0 &= 1 \text{ (by definition)}, \\
\mathbb{K}_1 &= \frac{1}{4C_3 f W^{3/2}}[\left(\beta_1 f^2 - \beta_2\beta_3\right) f^2 f' \varphi'' + 3\beta_1 f^3 \varphi'^3 + \beta_4\beta_5 f^2 f' \varphi'^2 \\
&\quad + \left[\left(-\beta_1 f^2 + \beta_2\beta_3\right) f^2 f'' + \left(3\beta_1 f^2 + 4\beta_2\beta_3\right) f f'^2\right] \varphi' + \left(\beta_4\beta_5 f^2 + \beta_6\right) f'^3], \\
\mathbb{K}_2 &= \frac{1}{6C_3 f^2 W^2}[\left(3f^2\xi_1 + 2\beta_3\xi_2\right) f^3 f' \varphi' \varphi'' - \left(f^2\beta_5\xi_3 + \beta_6\xi_4\right) f f'^2 \varphi'' \\
&\quad + 3\xi_1 f^4 \varphi'^4 - 2\beta_5\xi_3 f^3 f' \varphi'^3 + \left[\left(3f^2\xi_1 + \xi_5\right) f'^2 - \left(3f^2\xi_1 + 2\beta_3\xi_2\right) f''\right] f^2 \varphi'^2 \\
&\quad - \left[\left(2f^2\beta_5\xi_3 + 3\beta_6\xi_4\right) f'^2 - \left(f^2\beta_5\xi_3 + \beta_6\xi_4\right) f''\right] f f' \varphi' + \left(\xi_6 f^2 + \xi_7\right) f'^4],
\end{aligned}$$

$$\begin{aligned}
\mathbb{K}_3 &= -\frac{\mathcal{C}_2}{4\mathcal{C}_3 f^2 \mathrm{W}^{5/2}}[\beta_2^2\left(\beta_3+3\beta_4 f^2\right)f^3 f'\varphi'^2\varphi''-\beta_2\left(2\beta_4\beta_5 f^2+\beta_6\right)f^2 f'^2\varphi'\varphi'' \\
&\quad+\left(\eta_6 f^2+\eta_7\right)ff'^3\varphi''-\beta_1\beta_2 f^4\varphi'^5-\beta_1\beta_5 f^3 f'\varphi'^4 \\
&\quad+\left[-\left(3\xi_2^2 f^2+\beta_2^2\beta_3\right)ff''-\left(\xi_2^2 f^2+\eta_{10}\right)f'^2\right]f^2\varphi'^3 \\
&\quad+\left[\left(\rho_1 f^2+\rho_2\right)ff'f''+\left(\rho_3 f^2+\rho_4\right)f'^3\right]f\varphi'^2 \\
&\quad+\left[\left(\rho_7 f^2+\rho_8\right)f'^4+\left(\rho_5 f^2+\rho_6\right)ff'^2 f''\right]\varphi'+\rho_9 ff'^5],
\end{aligned}$$

$$\begin{aligned}
\mathbb{K}_4 &= \frac{\mathcal{C}_2^2}{f^2\mathrm{W}^3}[\xi_2^2 f^5 f'\varphi'^3\varphi''+\beta_1\beta_5 f^4 f'^2\varphi'^2\varphi''+\eta_1 f^3 f'^3\varphi'\varphi''+\eta_2 f^2 f'^4\varphi'' \\
&\quad+\beta_1\beta_2 f^5 f''\varphi'^4-\beta_1\beta_5 f^4 f'f''\varphi'^3-\left(\eta_1 f''+\beta_3\beta_4 f'^2\right)f^2 f'^2\varphi'^2 \\
&\quad+\left(-\eta_2 f''+\beta_2\beta_6 f'^2\right)ff'^3\varphi'-\xi_7 f'^6],
\end{aligned}$$

where

$$\begin{aligned}
\beta_1 &= -\mathcal{C}_2^3\mathcal{C}_3^4,\ \beta_2=\mathcal{C}_2\mathcal{C}_3^2,\ \beta_3=a^2+b^2\mathcal{C}_2^2+c^2\mathcal{C}_2^2\mathcal{C}_3^2,\ \beta_4=\mathcal{C}_2^2\mathcal{C}_3^2, \\
\beta_5 &= b\mathcal{S}_2+2c\mathcal{C}_2\mathcal{C}_3\mathcal{S}_3,\ \beta_6=b\mathcal{S}_2\beta_8+c\mathcal{C}_2\mathcal{C}_3\mathcal{S}_3\beta_9, \\
\beta_7 &= \mathcal{C}_2^2\mathcal{S}_3^2\left(a^2+b^2\mathcal{C}_2^2-c^2\mathcal{C}_2^2\mathcal{C}_3^2\right)+\mathcal{S}_2\left(a^2\mathcal{S}_2-bc\mathcal{C}_2^3\mathcal{C}_3\mathcal{S}_3\right), \\
\beta_8 &= 2a^2+b^2\mathcal{C}_2^2+c^2\mathcal{C}_2^2\mathcal{C}_3^2,\ \ \beta_9=3a^2+3b^2\mathcal{C}_2^2+2c^2\mathcal{C}_2^2\mathcal{C}_3^2,
\end{aligned}$$

$$\begin{aligned}
\xi_1 &= \mathcal{C}_2^4\mathcal{C}_3^5,\ \xi_2=\mathcal{C}_2^2\mathcal{C}_3^3,\ \xi_3=\mathcal{C}_2^3\mathcal{C}_3^3,\ \xi_4=\mathcal{C}_2\mathcal{C}_3,\ \xi_5=\mathcal{C}_3(5\beta_3\beta_4-\beta_7), \\
\xi_6 &= -\mathcal{C}_3(\beta_3\beta_4+\beta_7),\ \ \xi_7=-a^2c^2\mathcal{S}_2^2\mathcal{C}_3+\xi_8, \\
\xi_8 &= -c^2\mathcal{C}_2^2\mathcal{S}_3^2\xi_9-bc\mathcal{C}_2\mathcal{S}_2\mathcal{S}_3\xi_{10}, \\
\xi_9 &= 2a^2+2b^2\mathcal{C}_2^2+c^2\mathcal{C}_2^2\mathcal{C}_3^2,\ \ \xi_{10}=4a^2+2b^2\mathcal{C}_2^2+c^2\mathcal{C}_2^2\mathcal{C}_3^2,
\end{aligned}$$

$$\begin{aligned}
\eta_1 &= -\mathcal{C}_3^2\beta_7,\ \eta_2=\mathcal{C}_2\mathcal{S}_3^2\left(a^2\eta_9+b^2\mathcal{C}_2^2\eta_8\right),\ \eta_3=\mathcal{S}_2\left(a^2\mathcal{S}_2\left(c^2\mathcal{C}_3+1\right)-bc\mathcal{C}_2^3\mathcal{C}_3\mathcal{S}_3\right), \\
\eta_4 &= bc\mathcal{S}_2\mathcal{C}_2\mathcal{S}_3\mathcal{C}_3,\ \eta_5=c^2\mathcal{C}_2^2\mathcal{S}_3^2\mathcal{C}_3^2,\ \eta_6=-\mathcal{C}_3^2(\eta_3+\mathcal{C}_3\beta_7), \\
\eta_7 &= \eta_4\xi_{10}+\eta_5\xi_9,\ \eta_8=b\mathcal{S}_2+c\mathcal{C}_2\mathcal{C}_3\mathcal{S}_3,\ \eta_9=2b\mathcal{S}_2+c\mathcal{C}_2\mathcal{C}_3\mathcal{S}_3, \\
\eta_{10} &= \mathcal{C}_3^2\left(\mathcal{S}_2\left(bc\mathcal{C}_2^3\mathcal{C}_3\mathcal{S}_3-a^2\mathcal{S}_2\right)+2\mathcal{C}_3^2\mathcal{C}_2^2\beta_3-\beta_7\right),
\end{aligned}$$

$$\begin{aligned}
\rho_1 &= 2\mathcal{C}_2^3\mathcal{C}_3^4\beta_5,\ \ \rho_2=\mathcal{C}_2\mathcal{C}_3^2(b\mathcal{S}_2\beta_8+c\mathcal{C}_2\mathcal{C}_3\mathcal{S}_3\beta_9),\ \ \rho_3=\beta_1\beta_5, \\
\rho_4 &= -2b\mathcal{C}_2\mathcal{S}_2\mathcal{C}_3^2\beta_8-2c\mathcal{C}_2^2\mathcal{S}_3\mathcal{C}_3^3\beta_9+\rho_{10},\ \rho_5=\mathcal{C}_3^2\left[\mathcal{S}_2\left(a^2\mathcal{S}_2-bc\mathcal{C}_2^3\mathcal{C}_3\mathcal{S}_3\right)+\beta_7\right], \\
\rho_6 &= c\mathcal{C}_3\left(a^2c\mathcal{S}_2^2\mathcal{C}_3-c\mathcal{C}_2^2\mathcal{C}_3\mathcal{S}_3^2\xi_9-b\mathcal{C}_2\mathcal{S}_2\mathcal{S}_3\xi_{10}\right),\ \rho_7=-2\mathcal{C}_2^2\mathcal{C}_3^4\beta_3-\mathcal{C}_3^2\beta_7, \\
\rho_8 &= -2\rho_6,\ \rho_9=b\mathcal{S}_2\mathcal{C}_2\mathcal{C}_3^2\beta_8+\mathcal{C}_2\left(\rho_{10}+c\mathcal{C}_2\mathcal{C}_3^3\mathcal{S}_3\beta_9\right), \\
\rho_{10} &= c\mathcal{C}_2\mathcal{C}_3\mathcal{S}_3^3\left(a^2+b^2\mathcal{C}_2^2\right)+b\mathcal{S}_2\mathcal{S}_3^2\left(2a^2+b^2\mathcal{C}_2^2\right),
\end{aligned}$$

$\mathrm{W}=\mathcal{C}_2^2\mathcal{C}_3^2 f^2\varphi'^2+\left[a^2+\mathcal{C}_2^2\left(b^2+\mathcal{C}_3^2(c^2+f^2)\right)\right]f'^2$, $a,b,c\in\mathbb{R}-\{0\}$, $\varphi=\varphi(r)$, $\varphi'=\frac{d\varphi}{dr}$, $\varphi''=\frac{d^2\varphi}{dr^2}$, $f=f(r)$, $f'=\frac{df}{dr}$, $f''=\frac{d^2f}{dr^2}$, $r\in I\subset\mathbb{R}$, $\mathcal{C}_k=\cos\theta_k$, $\mathcal{S}_k=\sin\theta_k$, $\mathcal{C}_k^2=(\cos\theta_k)^2$, $\mathcal{S}_k^2=(\sin\theta_k)^2$, *etc.*, $\theta_k\in[0,2\pi)$, $k=2,3$.

Proof. Regarding Definition 3, and by using the first derivatives w.r.t. $r, \theta_1, \theta_2, \theta_3$ of the hypersurface determined by (3), we find the components of $\mathbb{I}$:

$$\begin{array}{rcl} E & = & \varphi'^2 + f'^2, \\ F & = & a\varphi', \\ A & = & b\varphi', \\ D & = & c\varphi', \\ G & = & f^2\mathcal{C}_2^2\mathcal{C}_3^2 + a^2, \\ B & = & ab, \\ J & = & ac, \\ C & = & f^2\mathcal{C}_3^2 + b^2, \\ Q & = & bc, \\ S & = & f^2 + c^2. \end{array} \tag{5}$$

We then have

$$\det \mathbb{I} = \mathcal{C}_3^2 f^4 \mathrm{W}, \tag{6}$$

where

$$\mathrm{W} = \mathcal{C}_2^2\mathcal{C}_3^2 f^2 \varphi'^2 + \left[a^2 + \mathcal{C}_2^2\left(b^2 + \mathcal{C}_3^2\left(c^2 + f^2\right)\right)\right] f'^2 > 0.$$

Hence, $\det \mathbb{I} > 0$. Regarding the Gauss map formula indicated by Definition 3, we find the Gauss map

$$\mathbb{G} = \frac{1}{\mathrm{W}^{1/2}} \begin{pmatrix} -\mathcal{C}_1\mathcal{C}_2^2\mathcal{C}_3^2 f\varphi' + \left(a\mathcal{S}_1 + \mathcal{C}_1\left(b\mathcal{S}_2\mathcal{C}_2 + c\mathcal{C}_2^2\mathcal{S}_3\mathcal{C}_3\right)\right) f' \\ -\mathcal{S}_1\mathcal{C}_2^2\mathcal{C}_3^2 f\varphi' + \left(a\mathcal{C}_1 + \mathcal{S}_1\left(b\mathcal{S}_2\mathcal{C}_2 + c\mathcal{C}_2^2\mathcal{S}_3\mathcal{C}_3\right)\right) f' \\ -\mathcal{S}_2\mathcal{C}_2^2\mathcal{C}_3^2 f\varphi' - \mathcal{C}_2^2(b\mathcal{C}_2 - c\mathcal{S}_2\mathcal{S}_3\mathcal{C}_3) f' \\ -\mathcal{C}_2\mathcal{C}_3\mathcal{S}_3 f\varphi' - c\mathcal{C}_2\mathcal{C}_3^2 f' \\ \mathcal{C}_2\mathcal{C}_3 f f' \end{pmatrix} \tag{7}$$

of the twisted hypersurface $\mathfrak{x}$ given by Equation (3).

Next, regarding the Gauss map of $\mathfrak{x}$ given by Equation (7), taking the second derivatives of $\mathfrak{x}$ depending on $r, \theta_1, \theta_2, \theta_3$, we find the components of $\mathbb{III}$ given by Definition 3:

$$\begin{array}{rcl} L & = & \mathcal{C}_2\mathcal{C}_3 f(f'\varphi'' - f''\varphi')/\mathrm{W}^{1/2}, \\ M & = & -a\mathcal{C}_2\mathcal{C}_3 f'^2/\mathrm{W}^{1/2}, \\ P & = & -b\mathcal{C}_2\mathcal{C}_3 f'^2/\mathrm{W}^{1/2}, \\ X & = & -c\mathcal{C}_2\mathcal{C}_3 f'^2/\mathrm{W}^{1/2}, \\ N & = & -\mathcal{C}_2^2\mathcal{C}_3 f((f\varphi'\mathcal{C}_3 - cf'\mathcal{S}_3)\mathcal{C}_2\mathcal{C}_3 - bf'\mathcal{S}_2)/\mathrm{W}^{1/2}, \\ T & = & -a\mathcal{S}_2\mathcal{C}_3 ff'/\mathrm{W}^{1/2}, \\ Y & = & -a\mathcal{C}_2\mathcal{S}_3 ff'/\mathrm{W}^{1/2}, \\ V & = & -\mathcal{C}_2\mathcal{C}_3^2 f(f\varphi'\mathcal{C}_3 - cf'\mathcal{S}_3)/\mathrm{W}^{1/2}, \\ Z & = & -b\mathcal{C}_2\mathcal{S}_3 ff'/\mathrm{W}^{1/2}, \\ U & = & -\mathcal{C}_2\mathcal{C}_3 f^2\varphi'/\mathrm{W}^{1/2}. \end{array}$$

$\mathbb{I}^{-1}.\mathbb{III}$ gives the following:

$$\mathbb{S} = \frac{1}{\mathrm{W}^{3/2}} \left(\; s_{ij} \; \right)_{4\times 4}. \tag{8}$$

We obtain the characteristic polynomial of (8) as follows:

$$\delta^4 + t_1\delta^3 + t_2\delta^2 + t_3\delta + t_4 = 0,$$

where

$$
\begin{aligned}
t_1 &= -\frac{1}{\mathrm{W}^{3/2}}(s_{11}+s_{22}+s_{33}+s_{44}),\\
t_2 &= \frac{1}{\mathrm{W}^3}(s_{11}s_{22}-s_{12}s_{21}+s_{11}s_{33}-s_{13}s_{31}+s_{11}s_{44}+s_{22}s_{33}\\
&\quad -s_{14}s_{41}-s_{23}s_{32}+s_{22}s_{44}-s_{24}s_{42}+s_{33}s_{44}-s_{34}s_{43}),\\
t_3 &= \frac{1}{\mathrm{W}^{9/2}}(s_{11}s_{23}s_{32}-s_{11}s_{22}s_{33}+s_{12}s_{21}s_{33}-s_{12}s_{31}s_{23}\\
&\quad -s_{21}s_{13}s_{32}+s_{13}s_{22}s_{31}-s_{11}s_{22}s_{44}+s_{11}s_{24}s_{42}+s_{12}s_{21}s_{44}\\
&\quad -s_{12}s_{41}s_{24}-s_{21}s_{14}s_{42}+s_{22}s_{14}s_{41}-s_{11}s_{33}s_{44}+s_{11}s_{34}s_{43}\\
&\quad +s_{13}s_{31}s_{44}-s_{13}s_{41}s_{34}-s_{31}s_{14}s_{43}+s_{14}s_{41}s_{33}-s_{22}s_{33}s_{44}\\
&\quad +s_{22}s_{34}s_{43}+s_{23}s_{32}s_{44}-s_{23}s_{42}s_{34}-s_{32}s_{24}s_{43}+s_{24}s_{33}s_{42}),\\
t_4 &= \frac{1}{\mathrm{W}^6}(s_{11}s_{22}s_{33}s_{44}-s_{11}s_{22}s_{34}s_{43}-s_{11}s_{23}s_{32}s_{44}+s_{11}s_{23}s_{42}s_{34}\\
&\quad +s_{11}s_{32}s_{24}s_{43}-s_{11}s_{24}s_{33}s_{42}-s_{12}s_{21}s_{33}s_{44}+s_{12}s_{21}s_{34}s_{43}\\
&\quad +s_{12}s_{31}s_{23}s_{44}-s_{12}s_{31}s_{24}s_{43}-s_{12}s_{23}s_{41}s_{34}+s_{12}s_{41}s_{24}s_{33}\\
&\quad +s_{21}s_{13}s_{32}s_{44}-s_{21}s_{13}s_{42}s_{34}-s_{21}s_{14}s_{32}s_{43}+s_{21}s_{14}s_{33}s_{42}\\
&\quad -s_{13}s_{22}s_{31}s_{44}+s_{13}s_{22}s_{41}s_{34}+s_{13}s_{31}s_{24}s_{42}-s_{13}s_{32}s_{41}s_{24}\\
&\quad +s_{22}s_{31}s_{14}s_{43}-s_{22}s_{14}s_{41}s_{33}-s_{31}s_{14}s_{23}s_{42}+s_{14}s_{23}s_{32}s_{41}).
\end{aligned}
$$

Here, $t_1=-4\mathbb{K}_1$, $t_2=6\mathbb{K}_2$, $t_3=-4\mathbb{K}_3$, $t_4=\mathbb{K}_4$. Finally, we obtain the components of $\mathbb{S}$:

$$
\begin{aligned}
s_{11} &= -\mathcal{C}_2\mathcal{C}_3[\Big[a^2+b^2\mathcal{C}_2^2+\Big(c^2+f^2\Big)\mathcal{C}_2^2\mathcal{C}_3^2\Big]ff'\varphi''\\
&\quad +\Big[\Big(a^2+b^2\mathcal{C}_2^2+c^2\mathcal{C}_2^2\mathcal{C}_3^2\Big)\Big(f'^2-ff''\Big)-\mathcal{C}_2^2\mathcal{C}_3^2f^3f''\Big]\varphi'],\\
s_{12} &= a\mathcal{C}_2\mathcal{C}_3\mathrm{W},\\
s_{13} &= \mathcal{C}_3\Big[-bf^2\varphi'^2\mathcal{C}_2^3\mathcal{C}_3^2-a^2\mathcal{S}_2ff'\varphi'+b\mathcal{C}_2\Big(a^2+b^2\mathcal{C}_2^2+\Big(c^2+f^2\Big)\mathcal{C}_2^2\mathcal{C}_3^2\Big)f'^2\Big],\\
s_{14} &= \mathcal{C}_2\Big[-c\mathcal{C}_2^2\mathcal{C}_3^3f^2\varphi'^2-\mathcal{S}_3\Big(a^2+b^2\mathcal{C}_2^2\Big)ff'\varphi'+c\mathcal{C}_3\Big(a^2+b^2\mathcal{C}_2^2+\Big(c^2+f^2\Big)\mathcal{C}_2^2\mathcal{C}_3^2\Big)f'^2\Big],\\
s_{21} &= a\mathcal{C}_2\mathcal{C}_3\Big[ff'\varphi'\varphi''+\Big(f'^2-ff''\Big)\varphi'^2-f'^4\Big],\\
s_{22} &= \frac{1}{\mathcal{C}_3f}\Big[(b\mathcal{S}_2+c\mathcal{C}_2\mathcal{C}_3\mathcal{S}_3)f'-\mathcal{C}_2\mathcal{C}_3^2f\varphi'\Big]\mathrm{W},\\
s_{23} &= \frac{a\mathcal{S}_2f'}{\mathcal{C}_3f}\Big[\mathcal{C}_3^2f^2\varphi'^2-\Big(b^2+c^2\mathcal{C}_3^2+\mathcal{C}_3^2f^2\Big)f'^2\Big],\\
s_{24} &= \frac{a\mathcal{C}_2\mathcal{S}_3f'}{f}\Big[f^2\varphi'^2-\Big(c^2+f^2\Big)f'^2\Big],\\
s_{31} &= b\mathcal{C}_2^3\mathcal{C}_3\Big[ff'\varphi'\varphi''+\Big(f'^2-ff''\Big)\varphi'^2-f'^4\Big],\\
s_{32} &= -\frac{a\mathcal{S}_2f'}{\mathcal{C}_3f}\mathrm{W},\\
s_{33} &= \frac{1}{\mathcal{C}_3f}[\mathcal{C}_2^3\mathcal{C}_3^4f^3\varphi'^3-c\mathcal{C}_2^3\mathcal{C}_3^3\mathcal{S}_3f^2f'\varphi'^2-\mathcal{C}_2\mathcal{C}_3^2\Big[a^2+b^2\mathcal{C}_2^2+\Big(c^2+f^2\Big)\mathcal{C}_2^2\mathcal{C}_3^2\Big]ff'^2\varphi'\\
&\quad +\Big[a^2b\mathcal{S}_2+c\mathcal{C}_2\mathcal{C}_3\mathcal{S}_3\Big(a^2+b^2\mathcal{C}_2^2\Big)+c\mathcal{C}_2^3\mathcal{C}_3^3\mathcal{S}_3\Big(c^2+f^2\Big)\Big]f'^3],
\end{aligned}
$$

$$
\begin{aligned}
s_{34} &= \frac{b\mathcal{C}_2^3\mathcal{S}_3}{f}\Big[f^2\varphi'^2 - \Big(c^2+f^2\Big)f'^2\Big], \\
s_{41} &= c\mathcal{C}_2^3\mathcal{C}_3^3\Big[ff'\varphi'\varphi'' + \Big((f')^2 - ff''\Big)\varphi'^2 - f'^4\Big], \\
s_{42} &= -\frac{a\mathcal{C}_2\mathcal{S}_3 f'}{f}\mathrm{W}, \\
s_{43} &= \frac{f'}{f}\Big[b\mathcal{C}_2^3\mathcal{C}_3^2\mathcal{S}_3 f^2\varphi'^2 + \Big[a^2 c\mathcal{C}_3\mathcal{S}_2 - b\mathcal{C}_2\mathcal{S}_3\Big(a^2 + b^2\mathcal{C}_2^2 + \mathcal{C}_2^2\mathcal{C}_3^2\Big(c^2+f^2\Big)\Big)\Big]f'^2\Big], \\
s_{44} &= \frac{\mathcal{C}_2}{f}\Big[\mathcal{C}_2^2\mathcal{C}_3^3 f^3\varphi'^3 - \mathcal{C}_3\Big(a^2 + b^2\mathcal{C}_2^2 + \Big(c^2+f^2\Big)\mathcal{C}_2^2\mathcal{C}_3^2\Big)ff'^2\varphi' + c\mathcal{S}_3\Big(b^2\mathcal{C}_2^2 + a^2\Big)f'^3\Big].
\end{aligned}
$$

From Definition 5, the curvatures $\mathbb{K}_j$ of the twisted hypersurfaces with the x_5 rotating axis described by (3) in the five-dimensional Euclidean space are obtained. □

Next, we offer some corollaries for the curvatures of the twisted hypersurfaces defined by Equation (3) with the x_5 rotating axis.

Corollary 1. *By taking $f(r) = \mathfrak{c} = const.$, we obtain the following curvatures of the twisted hypersurfaces determined by Equation (3) with the x_5 rotating axis:*

$$\mathbb{K}_0 = 1 \text{ (by definition)},\ \mathbb{K}_1 = -\frac{3}{4\mathfrak{c}},\ \mathbb{K}_2 = \frac{1}{2\mathfrak{c}^2},\ \mathbb{K}_3 = -\frac{1}{4\mathfrak{c}^3},\ \mathbb{K}_4 = 0.$$

Then,

$$\mathbb{K}_0 = 1,\ 2^9(\mathbb{K}_1)^6 = 3^6(\mathbb{K}_2)^3 = 3^6 2(\mathbb{K}_3)^2,\ \mathbb{K}_4 = 0.$$

Corollary 2. *By choosing $\varphi(r) = \mathfrak{c} = const.$, we find the following curvatures of the twisted hypersurfaces given by Equation (3) with the x_5 rotating axis:*

$$
\begin{aligned}
\mathbb{K}_0 &= 1 \text{ (by definition)}, \\
\mathbb{K}_1 &= \frac{\beta_4\beta_5 f^2 + \beta_6}{4f\mathcal{C}_3\big((a^2 + \mathcal{C}_2^2(b^2 + \mathcal{C}_3^2(c^2+f^2)))\big)^{3/2}}, \\
\mathbb{K}_2 &= \frac{\xi_6 f^2 + \xi_7}{6f^2\mathcal{C}_3\big((a^2 + \mathcal{C}_2^2(b^2 + \mathcal{C}_3^2(c^2+f^2)))\big)^{2}}, \\
\mathbb{K}_3 &= -\frac{\mathcal{C}_2\rho_9}{4f\mathcal{C}_3\big((a^2 + \mathcal{C}_2^2(b^2 + \mathcal{C}_3^2(c^2+f^2)))\big)^{5/2}}, \\
\mathbb{K}_4 &= -\frac{\xi_7}{f^2\big((a^2 + \mathcal{C}_2^2(b^2 + \mathcal{C}_3^2(c^2+f^2)))\big)^{3}}.
\end{aligned}
$$

Here, β_i, ξ_j, ρ_k described by Theorem 3. Then,

$$\mathbb{K}_0 = 1,\ \left(\frac{4f\mathcal{C}_3}{\beta_4\beta_5 f^2 + \beta_6}\mathbb{K}_1\right)^{20} = \left(\frac{6f^2\mathcal{C}_3}{\xi_6 f^2 + \xi_7}\mathbb{K}_2\right)^{15} = \left(\frac{4f\mathcal{C}_3}{\mathcal{C}_2\rho_9}\mathbb{K}_3\right)^{12} = \left(\frac{f^2}{\xi_7}\mathbb{K}_4\right)^{10}.$$

We present a condition to the curvatures determined by Theorem 2 with the fundamental forms decribed by Definition 4.

Theorem 4. *Hypersurface $\mathfrak{x} = \mathfrak{x}(r, \theta_1, \theta_2, \theta_3)$ in Euclidean space $\mathbb{E}^5$ has*

$$\mathbb{K}_0\mathbb{V} - 4\mathbb{K}_1\mathbb{IV} + 6\mathbb{K}_2\mathbb{III} - 4\mathbb{K}_3\mathbb{II} + \mathbb{K}_4\mathbb{I} = \mathbf{O}, \tag{9}$$

where $\mathbb{I}, \mathbb{II}, \ldots, \mathbb{V}$ describe the fundamental form matrices, and **O** *determines the zero matrix with order four of $\mathfrak{x}$.*

Proof. With the help of the theorem of Cayley–Hamilton, we obtain $P_{\mathbb{S}}(\delta) = \sum_{k=0}^{4} (-1)^k s_k \delta^{4-k} = \det(\mathbb{S} - \delta \mathcal{I}_4) = 0$. Hence, we have

$$\mathbb{K}_0 \delta^4 - 4\mathbb{K}_1 \delta^3 + 6\mathbb{K}_2 \delta^2 - 4\mathbb{K}_3 \delta + \mathbb{K}_4 = 0.$$

□

We note that the three dimension effects of Theorem 4 are determined by

$$\mathbb{K}_0 \mathbb{III} - 2\mathbb{K}_1 \mathbb{II} + \mathbb{K}_2 \mathbb{I} = \mathbf{O},$$

and

$$\mathbb{K}_0 \delta^2 - 2\mathbb{K}_1 \delta + \mathbb{K}_2 \delta = 0,$$

Here, **O** describes the zero matrix of order two, $\mathbb{K}_1 = H$ denotes the mean curvature, $\mathbb{K}_2 = K$ determines the Gaussian curvature of a surface of dimension three. Also, the acts of dimension four of Theorem 4 are described as follows:

$$\mathbb{K}_0 \mathbb{IV} - 3\mathbb{K}_1 \mathbb{III} + 3\mathbb{K}_2 \mathbb{II} - \mathbb{K}_3 \mathbb{I} = \mathbf{O},$$

and

$$\mathbb{K}_0 \delta^3 - 3\mathbb{K}_1 \delta^3 + 3\mathbb{K}_2 \delta^2 - \mathbb{K}_3 \delta = 0.$$

Here, **O** denotes the zero matrix of order three.

4. The Umbilical Hypersurfaces in $\mathbb{E}^5$

Next, we present the umbilical acts of the hypersurfaces of $\mathbb{E}^5$.
From Theorem 1, the following occurs:

$$\begin{aligned}
\mathbb{K}_0 &= 1, \\
4\mathbb{K}_1 &= \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4, \\
6\mathbb{K}_2 &= \kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_1\kappa_4 + \kappa_2\kappa_3 + \kappa_2\kappa_4 + \kappa_3\kappa_4, \\
4\mathbb{K}_3 &= \kappa_1\kappa_2\kappa_3 + \kappa_1\kappa_2\kappa_4 + \kappa_1\kappa_3\kappa_4 + \kappa_2\kappa_3\kappa_4, \\
\mathbb{K}_4 &= \kappa_1\kappa_2\kappa_3\kappa_4.
\end{aligned}$$

Then, we obtain the the following.

Corollary 3. *In $\mathbb{E}^5$, the following holds:*

$$\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 \quad \Leftrightarrow \quad (\mathbb{K}_1)^2 = \mathbb{K}_2,\ \mathbb{K}_1\mathbb{K}_2 = \mathbb{K}_3,\ \mathbb{K}_1\mathbb{K}_3 = (\mathbb{K}_2)^2 = (\mathbb{K}_1)^4 = \mathbb{K}_4.$$

See [34,46] for details of umbilical facts.

Theorem 5. *The twisted hypersurfaces given by Equation (3) have a umbilical point if the following comes out*

$$\begin{aligned}
&[(\beta_1 f^2 - \beta_2\beta_3) f^2 f' \varphi'' + 3\beta_1 f^3 \varphi'^3 + \beta_4\beta_5 f^2 f' \varphi'^2 \\
&+ [(\beta_1 f^2 + \beta_2\beta_3) f'' - (3\beta_1 f^2 + 4\beta_2\beta_3) f'^2] f \varphi' + (\beta_4\beta_5 f^2 + \beta_6) f'^3]^4 \\
&- 4^4 \beta_2^2 f^2 \mathrm{W}^3 [\xi_2^2 f^5 f' \varphi'^3 \varphi'' + \beta_1\beta_5 f^4 f'^2 \varphi'^2 \varphi'' + \eta_1 f^3 f'^3 \varphi' \varphi'' + \eta_2 f^2 f'^4 \varphi'' \\
&+ \beta_1\beta_2 f^5 f'' \varphi'^4 - \beta_1\beta_5 f^4 f' f'' \varphi'^3 - (\eta_1 f'' + \beta_3\beta_4 f'^2) f^2 f'^2 \varphi'^2 \\
&+ (-\eta_2 f'' + \beta_2\beta_6 f'^2) f f'^3 \varphi' - \xi_7 f'^6] = 0.
\end{aligned}$$

Proof. Twisted hypersurfaces $\mathfrak{x}$ constructed by the x_5-rotation axis cover the umbilical point of $\mathbb{E}^5$, i.e., $(\mathbb{K}_1)^4 = \mathbb{K}_4$. □

Problem 1. *Find the φ solutions of Equation determined by Theorem 5.*

Now, we serve the minimality acts determined by Definition 6 of the twisted hypersurfaces defined by Equation (3).

Corollary 4. *The twisted hypersurfaces defined by Equation* (3) *have zero mean curvature, i.e., one-minimal if the following occurs:*

$$\begin{aligned}&\left(\beta_1 f^2 - \beta_2\beta_3\right) f f' \varphi'' - 3\beta_1 f^3 \varphi'^3 + \beta_4\beta_5 f^2 f' \varphi'^2 \\ &+\left[\left(\beta_1 f^2 + \beta_2\beta_3\right) f'' - \left(3\beta_1 f^2 + 4\beta_2\beta_3\right) f'^2\right] f \varphi' \\ &+\left(\beta_4\beta_5 f^2 + \beta_6\right) f'^3 = 0.\end{aligned}$$

Problem 2. *Find the φ solutions of Equation given by Corollary 4.*

Corollary 5. *The twisted hypersurfaces determined by Equation* (3) *are two-minimal if the following holds:*

$$\begin{aligned}&\left(3f^2\xi_1 + 2\beta_3\xi_2\right) f^2 f' \varphi' \varphi'' - \left(f^2\beta_5\xi_3 + \beta_6\xi_4\right) f f'^2 \varphi'' \\ &+3\xi_1 f^4 \varphi'^4 - 2\beta_5\xi_3 f^3 f' \varphi'^3 \\ &+\left[\left(3f^2\xi_1 + \xi_5\right) f'^2 - \left(3f^2\xi_1 + 2\beta_3\xi_2\right) f''\right] f^2 \varphi'^2 \\ &-\left[\left(2f^2\beta_5\xi_3 + 3\beta_6\xi_4\right) f'^2 - \left(f^2\beta_5\xi_3 + \beta_6\xi_4\right) f''\right] f f' \varphi' \\ &+\left(\xi_6 f^2 + \xi_7\right) f'^4 = 0.\end{aligned}$$

Problem 3. *Find the φ solutions of Equation given by Corollary 5.*

Corollary 6. *The twisted hypersurfaces decribed by Equation* (3) *are three-minimal if the following Equation becomes*

$$\begin{aligned}&\beta_2^2\left(\beta_3 + 3\beta_4 f^2\right) f^3 f' \varphi'^2 \varphi'' - \beta_2\left(2\beta_4\beta_5 f^2 + \beta_6\right) f^2 f'^2 \varphi' \varphi'' \\ &+\left(\eta_6 f^2 + \eta_7\right) f f'^3 \varphi'' - \beta_1\beta_2 f^4 \varphi'^5 - \beta_1\beta_5 f^3 f' \varphi'^4 \\ &-\left[\left(3\xi_2^2 f^2 + \beta_2^2\beta_3\right) f f'' + \left(\xi_2^2 f^2 + \eta_{10}\right) f'^2\right] f^2 \varphi'^3 \\ &+\left[\left(\rho_1 f^2 + \rho_2\right) f f' f'' + \left(\rho_3 f^2 + \rho_4\right) f'^3\right] f \varphi'^2 \\ &+\left[\left(\rho_7 f^2 + \rho_8\right) f'^4 + \left(\rho_5 f^2 + \rho_6\right) f f'^2 f''\right] \varphi' + \rho_9 f f'^5 = 0.\end{aligned}$$

Problem 4. *Find the φ solutions of the Equation determined by Corollary 6.*

Corollary 7. *The twisted hypersurfaces defined by Equation* (3) *have a zero Gauss–Kronecker curvature, i.e., four-minimal if the following Equation comes out:*

$$\begin{aligned}&\xi_2^2 f^5 f' \varphi'^3 \varphi'' + \beta_1\beta_5 f^4 f'^2 \varphi'^2 \varphi'' + \eta_1 f^3 f'^3 \varphi' \varphi'' + \eta_2 f^2 f'^4 \varphi'' \\ &+\beta_1\beta_2 f^5 f'' \varphi'^4 - \beta_1\beta_5 f^4 f' f'' \varphi'^3 - \left(\eta_1 f'' + \beta_3\beta_4 f'^2\right) f^2 f'^2 \varphi'^2 \\ &+\left(-\eta_2 f'' + \beta_2\beta_6 f'^2\right) f f'^3 \varphi' - \xi_7 f'^6 = 0.\end{aligned}$$

Problem 5. *Find the φ solutions of Equation described by Corollary 7.*

5. Twisted Hypersurfaces with the x_5 Rotating Axis Supplying $\Delta\mathfrak{x} = \mathcal{Q}\mathfrak{x}$ in $\mathbb{E}^5$

We determine that the Laplace–Beltrami operator depends on $\mathbb{I}$ of a smooth function in $\mathbb{E}^5$, and we find the Laplace–Beltrami operator of the twisted hypersurfaces given by (3).

Definition 8. *In $\mathbb{E}^5$, the Laplace–Beltrami operator of a smooth function $\phi = \phi(x^1, x^2, x^3, x^4) \mid_{\mathbf{D}}$ (domain $\mathbf{D} \subset \mathbb{R}^4$) of class C^4 is described by*

$$\Delta\phi = \frac{1}{\mathfrak{g}^{1/2}} \sum_{i,j=1}^{4} \frac{\partial}{\partial x^i} \left(\mathfrak{g}^{1/2} \mathbf{g}^{ij} \frac{\partial \phi}{\partial x^j} \right), \tag{10}$$

where $\left(\mathbf{g}^{ij}\right) = \left(\mathbf{g}_{kl}\right)^{-1}$ and $\mathfrak{g} = \det\left(\mathbf{g}_{ij}\right)$.

We regard the inverse matrix of $\mathbb{I}$ determined by (3). Hence, the coefficients of $\left(\mathbf{g}^{ij}\right) = \mathbb{I}^{-1}$ are denoted by

$$\begin{aligned}
\mathbf{g}^{11} &= \left(-CJ^2 - B^2S - GQ^2 + 2BJQ + CGS\right) / \det \mathbb{I}, \\
\mathbf{g}^{12} &= \left(FQ^2 + CJD - BQD + ABS - AJQ - CFS\right) / \det \mathbb{I} = \mathbf{g}^{21}, \\
\mathbf{g}^{13} &= \left(AJ^2 - BJD + GQD - AGS + BFS - FJQ\right) / \det \mathbb{I} = \mathbf{g}^{31}, \\
\mathbf{g}^{14} &= \left(B^2D - CGD - ABJ + CFJ + AGQ - BFQ\right) / \det \mathbb{I} = \mathbf{g}^{41}, \\
\mathbf{g}^{22} &= \left(-A^2S - CD^2 - Q^2E + 2AQD + CSE\right) / \det \mathbb{I}, \\
\mathbf{g}^{23} &= \left(BD^2 - AJD - BSE - FQD + JQE + AFS\right) / \det \mathbb{I} = \mathbf{g}^{32}, \\
\mathbf{g}^{24} &= \left(A^2J - ABD + CFD - CJE + BQE - AFQ\right) / \det \mathbb{I} = \mathbf{g}^{42}, \\
\mathbf{g}^{33} &= \left(-F^2S - GD^2 - J^2E + 2FJD + GSE\right) / \det \mathbb{I}, \\
\mathbf{g}^{34} &= \left(F^2Q + AGD - BFD + BJE - GQE - AFJ\right) / \det \mathbb{I} = \mathbf{g}^{43}, \\
\mathbf{g}^{44} &= \left(-A^2G - CF^2 - B^2E + CGE + 2ABF\right) / \det \mathbb{I},
\end{aligned}$$

where

$$\begin{aligned}
\det \mathbb{I} &= (EG - F^2)(CS - Q^2) + (J^2 - GS)A^2 + (D^2 - ES)B^2 - (EJ^2 + GD^2)C \\
&\quad + 2((CF - AB)DJ + (EB - FA)JQ + (GA - FB)DQ + FABS).
\end{aligned}$$

We compensate $\phi = \phi(x^1, x^2, x^3, x^4)$ with $\mathfrak{x} = \mathfrak{x}(r, \theta_1, \theta_2, \theta_3)$ and substitute it into (10). Therefore, by using the inverse matrix of (5), we have the following:

$$\begin{aligned}
\mathbf{g}^{11} &= \frac{a^2 + \left(b^2 + \left(c^2 + f^2\right)\mathcal{C}_3^2\right)\mathcal{C}_2^2}{\mathrm{W}}, \\
\mathbf{g}^{12} &= -\frac{a\varphi'}{\mathrm{W}} = \mathbf{g}^{21}, \\
\mathbf{g}^{13} &= -\frac{b\mathcal{C}_2^2\varphi'}{\mathrm{W}} = \mathbf{g}^{31}, \\
\mathbf{g}^{14} &= -\frac{c\mathcal{C}_2^2\mathcal{C}_3^2\varphi'}{\mathrm{W}} = \mathbf{g}^{41}, \\
\mathbf{g}^{22} &= \frac{\mathcal{C}_3^2 f^2 \varphi'^2 + \left(b^2 + \left(c^2 + f^2\right)\mathcal{C}_3^2\right) f'^2}{\mathcal{C}_3^2 f^2 \mathrm{W}},
\end{aligned}$$

$$
\begin{aligned}
\mathbf{g}^{23} &= -\frac{abf'^2}{\mathcal{C}_3^2 f^2 \mathrm{W}} = \mathbf{g}^{32}, \\
\mathbf{g}^{24} &= -\frac{acf'^2}{f^2 \mathrm{W}} = \mathbf{g}^{42}, \\
\mathbf{g}^{33} &= \frac{\mathcal{C}_2^2\mathcal{C}_3^2 f^2 \varphi'^2 + \left(a^2 + \left(c^2 + f^2\right)\mathcal{C}_2^2\mathcal{C}_3^2\right) f'^2}{\mathcal{C}_3^2 f^2 \mathrm{W}}, \\
\mathbf{g}^{34} &= -\frac{bc\mathcal{C}_2^2 f'^2}{f^2 \mathrm{W}} = \mathbf{g}^{43}, \\
\mathbf{g}^{44} &= \frac{\mathcal{C}_2^2\mathcal{C}_3^2 f^2 \varphi'^2 + \left(a^2 + \left(b^2 + f^2\mathcal{C}_3^2\right)\mathcal{C}_2^2\right) f'^2}{f^2 \mathrm{W}}.
\end{aligned}
$$

We obtain the information below.

Theorem 6. *The Laplace–Beltrami operator of the twisted hypersurfaces $\mathfrak{x}$ determined by (3) has $\Delta\mathfrak{x} = 4\mathbb{K}_1\mathbb{G}$. Here, $\mathbb{K}_1$ denotes the mean curvature determined by Theorem 3, and $\mathbb{G}$ describes the Gauss map determined by (7) of the family.*

Proof. With straight calculations of (3) on (10), we have $\Delta\mathfrak{x} = 4\mathbb{K}_1\mathbb{G}$. □

Next, we offer the following about Δ and $\mathbb{K}_1 = 0$ of the family determined by (3).

Theorem 7. *We let $\mathfrak{x} : M^4 \subset \mathbb{E}^4 \longrightarrow \mathbb{E}^5$ be an immersion described by (3). $\Delta\mathfrak{x} = \mathcal{Q}\mathfrak{x}$, where $\mathcal{Q} = \left(\mathfrak{q}_{ij}\right)$ is a square matrix of order five if $\mathbb{K}_1 = 0$, i.e., twisted hypersurfaces $\mathfrak{x}$ are one-minimal.*

Proof. We use $4\mathbb{K}_1\mathbb{G} = \mathcal{Q}\mathfrak{x}$, then obtain the following Equations:

$$
\begin{aligned}
& f\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3\mathfrak{q}_{11} + f\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3\mathfrak{q}_{12} + f\mathcal{S}_2\mathcal{C}_3\mathfrak{q}_{13} + f\mathcal{S}_3\mathfrak{q}_{14} + (\varphi + a\theta_1 + b\theta_2 + c\theta_3)\mathfrak{q}_{15} \\
= \; & \Omega\Big(-\mathcal{C}_1\mathcal{C}_2^2\mathcal{C}_3^2 f\varphi' + \Big(a\mathcal{S}_1 + \mathcal{C}_1\Big(b\mathcal{S}_2\mathcal{C}_2 + c\mathcal{C}_2^2\mathcal{S}_3\mathcal{C}_3\Big)\Big) f'\Big), \\
& f\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3\mathfrak{q}_{21} + f\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3\mathfrak{q}_{22} + f\mathcal{S}_2\mathcal{C}_3\mathfrak{q}_{23} + f\mathcal{S}_3\mathfrak{q}_{24} + (\varphi + a\theta_1 + b\theta_2 + c\theta_3)\mathfrak{q}_{25} \\
= \; & \Omega\Big(-\mathcal{S}_1\mathcal{C}_2^2\mathcal{C}_3^2 f\varphi' + \Big(a\mathcal{C}_1 + \mathcal{S}_1\Big(b\mathcal{S}_2\mathcal{C}_2 + c\mathcal{C}_2^2\mathcal{S}_3\mathcal{C}_3\Big)\Big) f'\Big), \\
& f\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3\mathfrak{q}_{31} + f\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3\mathfrak{q}_{32} + f\mathcal{S}_2\mathcal{C}_3\mathfrak{q}_{33} + f\mathcal{S}_3 a_{34} + (\varphi + a\theta_1 + b\theta_2 + c\theta_3)\mathfrak{q}_{35} \\
= \; & \Omega\mathcal{C}_2\Big(-\mathcal{S}_2\mathcal{C}_3^2 f\varphi' - (b\mathcal{C}_2 - c\mathcal{S}_2\mathcal{S}_3\mathcal{C}_3) f'\Big), \\
& f\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3\mathfrak{q}_{41} + f\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3\mathfrak{q}_{42} + f\mathcal{S}_2\mathcal{C}_3\mathfrak{q}_{43} + f\mathcal{S}_3\mathfrak{q}_{44} + (\varphi + a\theta_1 + b\theta_2 + c\theta_3)\mathfrak{q}_{45} \\
= \; & -\Omega\mathcal{C}_2\mathcal{C}_3(\mathcal{S}_3 f\varphi' + c\mathcal{C}_3 f'), \\
& f\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3\mathfrak{q}_{51} + f\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3\mathfrak{q}_{52} + f\mathcal{S}_2\mathcal{C}_3\mathfrak{q}_{53} + f\mathcal{S}_3\mathfrak{q}_{54} + (\varphi + a\theta_1 + b\theta_2 + c\theta_3)\mathfrak{q}_{55} \\
= \; & \Omega\mathcal{C}_2\mathcal{C}_3 f f',
\end{aligned}
$$

where $\mathcal{Q}$ denotes a 5×5 matrix, $\Omega = 4\mathbb{K}_1\mathrm{W}^{-1/2}$. Derivativing the above ODEs twice w.r.t. θ_1, we obtain

$$\mathfrak{q}_{15} = \mathfrak{q}_{25} = \mathfrak{q}_{35} = \mathfrak{q}_{45} = \mathfrak{q}_{55} = 0, \ \Omega = 0.$$

Therefore, the following relations occur:

$$\mathcal{C}_2\mathcal{C}_3(\mathcal{C}_1\mathfrak{q}_{i1} + \mathcal{S}_1\mathfrak{q}_{i2}) f = 0,$$

where $f \neq 0$, $i = 1, \ldots, 5$. Regarding the fact that sin and cos are linear independent on θ_1, each of the coefficients of matrix $\mathcal{Q}$ are 0. $\Omega = 4\mathbb{K}_1\mathrm{W}^{-1/2}$, then $\mathbb{K}_1 = 0$. This means, from Definition 6, that hypersurface $\mathfrak{x}$ determined by (3) is a one-minimal twisted hypersurface with a x_5 rotating axis. □

Hence, we offer the following examples.

Example 1. *In $\mathbb{E}^5$, by using $f(r) = \cos r = \mathcal{C}_r$, $\varphi(r) = \sin r = \mathcal{S}_r$ to γ determined by (2), we construct the rotational hypersurface*

$$\mathfrak{x} = (\mathcal{C}_r\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3, \mathcal{C}_r\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3, \mathcal{C}_r\mathcal{S}_2\mathcal{C}_3, \mathcal{C}_r\mathcal{S}_3, \mathcal{S}_r),$$

where $a = b = c = 0$. Then, we have

$$\begin{aligned}
\mathbb{G} &= -\mathfrak{x}, \\
\mathbb{I} &= diag\left(1, \mathcal{C}_r^2\mathcal{C}_2^2\mathcal{C}_3^2, \mathcal{C}_r^2\mathcal{C}_3^2, \mathcal{C}_r^2\right) = \mathbb{III} = \mathbb{IIII} = \mathbb{IV} = \mathbb{V}, \\
\mathbb{S} &= \mathcal{I}_4, \\
\mathbb{K}_j &= 1, \\
\Delta\mathfrak{x} &= -4\mathfrak{x}.
\end{aligned}$$

Here, $\mathcal{I}_4$ describes the identity matrix, diag describes the diagonal side of the matrix, $j = 0, 1, \ldots, 4$.

We also apply the rational rotational hypersurface with the x_5 rotating axis to the following.

Example 2. *We substitute rational functions $f(r) = \frac{r^2-1}{r^2+1} = \mathcal{C}_r$, $\varphi(r) = \frac{2r}{r^2+1} = \mathcal{S}_r$, $r \neq \pm i$, into γ described by (2). We obtain the following rational rotational hypersurface:*

$$\mathfrak{x} = (\mathcal{C}_r\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3, \mathcal{C}_r\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3, \mathcal{C}_r\mathcal{S}_2\mathcal{C}_3, \mathcal{C}_r\mathcal{S}_3, \mathcal{S}_r),$$

where $a = b = c = 0$ in $\mathbb{E}^5$. Then, we obtain

$$\begin{aligned}
\mathbb{G} &= \mathfrak{x}, \\
\mathbb{I} &= diag\left(\frac{4}{(r^2+1)^2}, \mathcal{S}_r^2\mathcal{C}_2^2\mathcal{C}_3^2, \mathcal{S}_r^2\mathcal{C}_3^2, \mathcal{S}_r^2\right) = -\mathbb{III} = \mathbb{IIII} = -\mathbb{IV} = \mathbb{V}, \\
\mathbb{S} &= -\mathcal{I}_4, \\
\mathbb{K}_j &= (-1)^j, \\
\Delta\mathfrak{x} &= 4\mathfrak{x}.
\end{aligned}$$

Here, $\mathcal{I}_4$ denotes the identity matrix, diag denotes the diagonal side of the matrix, $j = 0, 1, \ldots, 4$. The rational hypersphere with the x_5 rotating axis holds Equation determined by (9).

6. Conclusions

This research introduced twisted hypersurfaces $\mathfrak{x}$ in a five-dimensional Euclidean space $\mathbb{E}^5$ with a rotating axis along x_5. The fundamental forms, the Gauss map, and the shape operator of $\mathfrak{x}$ were computed, providing a comprehensive understanding of its geometric properties. By employing the Cayley–Hamilton theorem, the curvatures of $\mathfrak{x}$ were determined, highlighting their relationship with the curvatures of hypersurfaces in $\mathbb{E}^5$.

However, the solutions to the differential equations governing the curvatures of these hypersurfaces remain open problems, offering avenues for future research. The study also presented the umbilicality and minimality conditions for the curvatures of $\mathfrak{x}$, contributing to the characterization of their geometric behavior. Furthermore, a significant result was obtained, establishing the Laplace–Beltrami operator relation $\Delta\mathfrak{x} = \mathcal{Q}\mathfrak{x}$, where $\mathcal{Q}$ is a square matrix of order five, further deepening the understanding of the geometric properties of $\mathfrak{x}$.

Overall, these findings shed light on the intricate nature of twisted hypersurfaces in a five-dimensional space and provided a foundation for further investigations in this field.

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