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General Stability for the Viscoelastic Wave Equation with Nonlinear Time-Varying Delay, Nonlinear Damping and Acoustic Boundary Conditions

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Abstract: This paper is focused on energy decay rates for the viscoelastic wave equation that includes nonlinear time-varying delay, nonlinear damping at the boundary, and acoustic boundary conditions. We derive general decay rate results without requiring the condition $a_2 > 0$ and without imposing any restrictive growth assumption on the damping term f_1, using the multiplier method and some properties of the convex functions. Here we investigate the relaxation function ψ, namely $\psi'(t) \leq -\mu(t)G(\psi(t))$, where G is a convex and increasing function near the origin, and μ is a positive nonincreasing function. Moreover, the energy decay rates depend on the functions μ and G, as well as the function F defined by f_0, which characterizes the growth behavior of f_1 at the origin.

Keywords: optimal decay; viscoelastic wave equation; nonlinear time-varying delay; nonlinear damping; acoustic boundary conditions

MSC: 35B40; 35L05; 37L45; 74D99

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1. Introduction

In this paper, we study the energy decay rates for the viscoelastic wave equation with nonlinear time-varying delay, nonlinear damping at the boundary, and acoustic boundary conditions

$$
\begin{aligned}
& u_{tt}(x,t) - \Delta u(x,t) + \int_0^t \psi(t-s)\Delta u(x,s)ds = 0, \ \text{ in } \Omega \times (0,\infty), && (1)\\
& u(x,t) = 0, \ \text{ on } \Gamma_0 \times (0,\infty), && (2)\\
& \frac{\partial u}{\partial \nu}(x,t) - \int_0^t \psi(t-s)\frac{\partial u}{\partial \nu}(x,s)ds + a_1 f_1(u_t(x,t)) + a_2 f_2(u_t(x,t-\varrho(t))) \\
& \quad = w_t(x,t), \ \text{ on } \Gamma_1 \times (0,\infty), && (3)\\
& u_t(x,t) + h(x)w_t(x,t) + m(x)w(x,t) = 0, \ \text{ on } \Gamma_1 \times (0,\infty), && (4)\\
& u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \text{ in } \Omega, && (5)\\
& u_t(x,t) = j_0(x,t), \ \text{ in } \Gamma_1 \times (-\varrho(0),0), && (6)
\end{aligned}
$$

where Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$ with smooth boundary Γ of class C^2; $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are closed and disjoint; $w(x,t)$ is the normal displacement into the domain of a point $x \in \Gamma_1$ at time t; and $h, m : \Gamma_1 \to \mathbb{R}$ are essential bounded functions that represent resistivity and spring constant per unit area, respectively. $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are given functions, and f_1 represents the nonlinear frictional damping. a_1, a_2 are real numbers with $a_1 > 0, a_2 \neq 0$. The integral term is the memory responsible for the viscoelastic damping. The functions ψ and $\varrho(t)$ represent the kernel of the memory term and the time-varying delay, respectively. ν is the outward unit normal vector to Γ. The initial data

(u_0, u_1, j_0) belong to a suitable space. Boundary conditions (3) and (4) are called acoustic boundary conditions.

In the past decades, the non-delayed wave equation with a viscoelastic term has garnered significant attention in the field of partial differential equations. Research on the energy decay rate of the solution to the viscoelastic wave equation is vital in various fields, contributing to technological advancements, safety assurance, environmental protection, energy efficiency, and academic exploration. The stability of solutions for such equations has recently been studied by many authors (see [1–3] and references therein). When $a_1 = a_2 = 0$, models (1)–(5) are pertinent to noise control and suppression in practical applications. The noise propagates through some acoustic medium, like air, in a room that is defined by a bounded domain Ω and whose floor, walls, and ceiling are determined by the boundary conditions [4,5]. Under the conditions that $\int_0^\infty \psi(s)ds < \frac{1}{2}$ and $\psi'(t) \leq -\mu(t)\psi(t)$, for $t \geq 0$, Park and Park [6] considered the general decay for problems (1)–(5). Liu [7] improved the research of [6] by achieving arbitrary rates of decay, which may not necessarily be an exponential or a polynomial one. Recently, Yoon et al. [8] generalized the work of [6,7] without the assumption condition $\int_0^\infty \psi(s)ds < \frac{1}{2}$. The assumption on relaxation function ψ has been weakened compared to the conditions assumed in previous literature [6,7].

Numerous phenomena are influenced by both the current state and the previous occurrences of the system. There has been a notable increase in the research on the equation with delay effects, which frequently arise in various physical, biological, chemical, medical, and economic problems [9–11]. However, the delay effects can generally be considered a cause of instability. In order to stabilize a system containing delay terms, additional control terms will be necessary. Kirane and Said-Houari [12] showed the global existence and asymptotic stability for the following wave equation with memory and constant delay,

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t \psi(t-s)\Delta u(x,s)ds + a_1 u_t(x,t) + a_2 u_t(x,t-\varrho) = 0,$$

where a_1, a_2, and ϱ are positive constants. They used the damping term $a_1 u_t(x,t)$ to control the delay term in obtaining the decay estimate of the energy. They proved that its energy was exponentially decaying when $a_2 \leq a_1$. Dai and Yang [13] investigated the exponential decay of an unsolved problem proposed by Kirane and Said-Houari [12], namely, the problem with $a_1 = 0$. In the case of constant weight and constant delay, the delay term typically considers the past history of strain, only up to some finite time $\varrho(t) \equiv \varrho$. Nicaise and Pignotti [14] investigated the following wave equation with internal time-varying delay instead of constant delay,

$$u_{tt}(x,t) - \Delta u(x,t) + a_1 u_t(x,t) + a_2 u_t(x,t-\varrho(t)) = 0,$$

where $\varrho(t) > 0$, a_1, and a_2 are real numbers with $a_1 > 0$. They proved the exponential stability result for the wave equation under the condition $|a_2| < \sqrt{1-\zeta_0}\, a_1$, where the constant ζ_0 satisfies $\varrho'(t) \leq \zeta_0 < 1, \forall t > 0$. Liu [15] studied the following wave equation involving memory and time-varying delay:

$$u_{tt}(x,t) - \Delta u(x,t) + \alpha(t) \int_0^t \psi(t-s)\Delta u(x,s)ds + a_1 u_t(x,t) + a_2 u_t(x,t-\varrho(t)) = 0.$$

Systems with time-varying delays have been extensively considered by many authors (see [16–22] and references therein). Recently, Zennir [23] considered the stability for solutions of plate equations with a time-varying delay and weak viscoelasticity in $\mathbb{R}^n$. Moreover, Benaissa et al. [24] proved the global existence and stability for solutions of the following wave equation with a time-varying delay in the weakly nonlinear feedback,

$$u_{tt}(x,t) - \Delta u(x,t) + a_1\sigma(t) f_1(u_t(x,t)) + a_2\sigma(t) f_2(u_t(x,t-\varrho(t))) = 0,$$

where $\varrho(t) > 0$, a_1, and a_2 are positive real numbers, and f_1, f_2 satisfy some conditions. This result extended the previous work [10,14]. Park [25] investigated the decay result of the energy for a von Karman equation with time-varying delay by dropping the restriction $a_2 > 0$ under the same conditions as ϱ, f_1, and f_2 in [24]. For the viscoelastic problem with time-varying delay in the nonlinear internal or boundary feedback, we also refer to [26,27]. As far as we know, there are few results for the viscoelastic wave equation with a nonlinear time-varying delay. Recently, Djeradi et al. [28] and Mukiawa et al. [29] showed the stability of the thermoelastic laminated beam and thermoelastic Timoshenko beam with nonlinear time-varying delay, respectively. The papers introduced so far have studied the energy decay rate of the solution for the equation with nonlinear time-varying delay in the Dirichlet boundary condition.

Motivated by these results, we study the general decay rates of the solution for problems (1)–(6) with a nonlinear time-varying delay term, nonlinear damping at the boundary, and acoustic boundary conditions. Research on the energy decay rate of solutions for the viscoelastic wave equation with nonlinear time-delay terms plays a critical role in various application areas, including stability assessment, understanding complex behaviors, advancing neuroscience, disaster preparedness, and improving energy efficiency. We consider the general assumption on the relaxation function ψ,

$$\psi'(t) \leq -\mu(t)G(\psi(t)), \tag{7}$$

where $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive nonincreasing function, and G is linear or is a strictly increasing and strictly convex function. We derive the general decay rate results without requiring the condition $a_2 > 0$ and without imposing any restrictive growth assumption on the damping term f_1. The energy decay rates depend on the functions μ and G, as well as the function F defined by f_0, which represents the growth f_1 at the origin. Our result improves upon previous work [6–8].

This paper is composed of the following. In Section 2, we prepare some notations and materials needed for our work. In Section 3, we introduce some technical lemmas to prove our stability result. In Section 4, we state and prove the general energy decay.

2. Preliminaries

In this section, we present some materials required for our results. Throughout this paper, we use the notation

$$V = \{u \in H^1(\Omega) : u = 0 \ \text{ on } \Gamma_0\}.$$

For simplicity, we denote $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^2(\Gamma_1)}$ by $\|\cdot\|$ and $\|\cdot\|_{\Gamma_1}$, respectively.

The Poincaré inequality holds in V; that is, there exist the positive constants λ_0 and λ_1 such that

$$\|u\|^2 \leq \lambda_0 \|\nabla u\|^2 \ \text{ and } \ \|u\|^2_{\Gamma_1} \leq \lambda_1 \|\nabla u\|^2 \ \text{ for all } u \in V. \tag{8}$$

As in [1,3,8,26,30], we consider the following assumptions for $\psi, f_1, f_2, \varrho, h$, and m.

(H1) $\psi : [0, \infty) \to \mathbb{R}^+$ is a differentiable function satisfying

$$1 - \int_0^\infty \psi(s)ds = l > 0, \tag{9}$$

and there exists a C^1 function $G : \mathbb{R}^+ \to \mathbb{R}^+$ that is linear or is a strictly convex and strictly increasing C^2 function on $(0, r_0]$, $r_0 \leq \psi(0)$ such that

$$\psi'(t) \leq -\mu(t)G(\psi(t)), \ \ \forall t \geq 0, \tag{10}$$

where $G(0) = G'(0) = 0$, and μ is a positive nonincreasing differentiable function. The function G was first introduced in [31]. These are weaker conditions on G than those introduced in [31].

(H2) $f_1 : \mathbb{R} \to \mathbb{R}$ is a nondecreasing C^0 function such that there exists a strictly increasing function $f_0 \in C^1(\mathbb{R}^+)$, with $f_0(0) = 0$, and positive constants c_0, c_1, and ε such that

$$f_0(|s|) \leq |f_1(s)| \leq f_0^{-1}(|s|) \ \text{ for all } |s| \leq \varepsilon, \tag{11}$$
$$c_0|s| \leq |f_1(s)| \leq c_1|s| \quad \text{for all } |s| \geq \varepsilon. \tag{12}$$

Moreover, we assume that the function F, defined by $F(s) = \sqrt{s} f_0(\sqrt{s})$, is a strictly convex C^2 function on $(0, r_1]$, for some $r_1 > 0$, when f_0 is nonlinear.
(H3) $f_2 : \mathbb{R} \to \mathbb{R}$ is an odd nondecreasing C^1 function such that there exist positive constants c_2, c_3, and c_4 that satisfy

$$|f_2'(s)| \leq c_2, \ \ c_3 s f_2(s) \leq F_2(s) \leq c_4 s f_1(s), \ \text{ for } \ s \in \mathbb{R}, \tag{13}$$

where $F_2(s) = \int_0^s f_2(t)dt$.
(H4) $\varrho \in W^{2,\infty}([0, T])$ is a function such that

$$0 < \varrho_1 \leq \varrho(t) \leq \varrho_2 \ \text{ and } \varrho'(t) \leq \varrho_3 < 1 \ \text{ for all } t > 0, \tag{14}$$

where T, ϱ_1, and ϱ_2 are positive constants. Moreover, the weight of dissipation and the delay satisfy

$$0 < |a_2| < \frac{c_3(1-\varrho_3)}{c_4(1-c_3\varrho_3)} a_1. \tag{15}$$

(H5) We assume that $h, m \in C(\Gamma_1)$, $h(x) > 0$, and $m(x) > 0$ for all $x \in \Gamma_1$. Then, there exist positive constants h_i and $m_i (i = 1, 2)$ such that

$$h_1 \leq h(x) \leq h_2, \ \ m_1 \leq m(x) \leq m_2 \ \text{ for all } \ x \in \Gamma_1. \tag{16}$$

Remark 1. *1. The assumption (H2) implies that $s f_1(s) > 0$, for all $s \neq 0$.*
2. The assumption (11) *of function f_1 has been weakened compared to the condition assumed in* [24,25].
3. Since f_2 is an odd nondecreasing function, F_2 is an even and convex function. Furthermore, it is satisfied that $F_2(s) = \int_0^s f_2(t)dt \leq s f_2(s)$. From (13)*, we find that $c_3 \leq 1$.*

Remark 2 ([3])**.** *1. By (H1), we obtain $\lim\limits_{t \to +\infty} \psi(t) = 0$. Then, there exists $t_0 \geq 0$ large enough that*

$$\psi(t_0) = r_0 \ \Rightarrow \ \psi(t) \leq r_0, \ \ \forall t \geq t_0. \tag{17}$$

Given ψ and μ are positive nonincreasing continuous functions, G is a positive continuous function, and for (10)*, we have, for some positive constant c_5,*

$$\psi'(t) \leq -\mu(t) G(\psi(t)) \leq -c_5 \psi(t), \ \ \forall t \in [0, t_0]. \tag{18}$$

2. If G is a strictly convex and strictly increasing C^2 function on $(0, r_0]$, with $G(0) = G'(0) = 0$, then it has an extension $\overline{G}$, which is a strictly convex and strictly increasing C^2 function on $(0, \infty)$. The same remark can be established for $\overline{F}$.

We recall the well-known Jensen inequality, which plays a pivotal role in proving our main result. If ϕ is a convex function on $[a, b]$, $p : \Omega \to [a, b]$ and k represents integrable functions on Ω such that $k(x) \geq 0$ and $\int_\Omega k(x)dx = k_0 > 0$, then Jensen's inequality holds:

$$\phi\left[\frac{1}{k_0} \int_\Omega p(x)k(x)dx\right] \leq \frac{1}{k_0} \int_\Omega \phi[p(x)]k(x)dx. \tag{19}$$

Let H^* be the conjugate of the convex function H defined by $H^*(s) = \sup\limits_{r\geq 0}(sr - H(r))$, then

$$sr \leq H^*(s) + H(r), \ \ \forall s, r \geq 0. \tag{20}$$

Moreover, due to the argument provided in [32], it holds that

$$H^*(s) = s(H')^{-1}(s) - H\big((H')^{-1}(s)\big), \ \ \forall s \geq 0. \tag{21}$$

As in [10,14], we introduce the following new function:

$$v(x,\kappa,t) = u_t(x, t - \kappa\varrho(t)), \ \ \text{for } (x,\kappa,t) \in \Gamma_1 \times (0,1) \times (0,\infty).$$

Then, problems (1)–(6) can be expressed as follows:

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t \psi(t-s)\Delta u(x,s)ds = 0, \text{ in } \Omega \times (0,\infty), \tag{22}$$
$$\varrho(t)v_t(x,\kappa,t) + (1-\kappa\varrho'(t))v_\kappa(x,\kappa,t) = 0, \text{ in } \Gamma_1 \times (0,1) \times (0,\infty), \tag{23}$$
$$u(x,t) = 0, \text{ in } \Gamma_0 \times (0,\infty), \tag{24}$$
$$\frac{\partial u}{\partial \nu}(x,t) - \int_0^t \psi(t-s)\frac{\partial u}{\partial \nu}(x,s)ds + a_1 f_1(u_t(x,t)) + a_2 f_2(v(x,1,t)) = w_t(x,t), \text{ on } \Gamma_1 \times (0,\infty), \tag{25}$$
$$u_t(x,t) + h(x)w_t(x,t) + m(x)w(x,t) = 0, \text{ on } \Gamma_1 \times (0,\infty), \tag{26}$$
$$u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), \text{ in } \Omega, \tag{27}$$
$$v(x,\kappa,0) = j_0(x, -\kappa\varrho(0)), \text{ in } \Gamma_1 \times (0,1). \tag{28}$$

We state the global existence result that can be established by the arguments of [24,33].

Theorem 1. *Let initial data $(u_0, u_1) \in (V \cap H^2(\Omega)) \times V$ and $j_0 \in L^2(\Gamma_1 \times (0,1))$. Suppose that (H1)–(H5) hold. Then, for any $T > 0$, there exists a unique pair of functions (u, w, v) that are the solution to problems (22)–(28) in the class*

$$u \in L^\infty(0,T; V \cap H^2(\Omega)), \ \ u_t \in L^\infty(0,T;V), \ \ u_{tt} \in L^\infty(0,T;L^2(\Omega)),$$
$$v \in L^\infty(0,T;L^2(\Gamma_1 \times (0,1))), \ \ w, w_t \in L^2(0,\infty;L^2(\Gamma_1)).$$

As in [6,25], we introduce the energy for problems (22)–(28),

$$\begin{aligned} E(t) = &\frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\left(1 - \int_0^t \psi(s)ds\right)\|\nabla u(t)\|^2 + \frac{1}{2}(\psi \circ \nabla u)(t) \\ &+\frac{1}{2}\int_{\Gamma_1} m(x)w^2(t)d\Gamma + \frac{\zeta\varrho(t)}{2}\int_{\Gamma_1}\int_0^1 F_2(v(x,\kappa,t))d\kappa d\Gamma, \end{aligned} \tag{29}$$

where

$$(\psi \circ \nabla u)(t) = \int_0^t \psi(t-s)\|\nabla u(t) - \nabla u(s)\|^2 ds$$

and

$$\frac{2|a_2|(1-c_3)}{c_3(1-\varrho_3)} < \zeta < \frac{2(a_1 - |a_2|c_4)}{c_4}. \tag{30}$$

Thanks to (15), this makes sense.

To show the main results of this paper, we need the following lemma.

Lemma 1. *Assume that (H3)–(H5) hold. Then, there exist positive constants γ_0 and γ_1 satisfying*

$$
\begin{aligned}
E'(t) \leq & \frac{1}{2}(\psi' \circ \nabla u)(t) - \frac{1}{2}\psi(t)\|\nabla u(t)\|^2 - h_1||w_t(t)||^2_{\Gamma_1} \\
& -\gamma_0 \int_{\Gamma_1} f_1(u_t(t))u_t(t)d\Gamma - \gamma_1 \int_{\Gamma_1} f_2(v(x,1,t))v(x,1,t)d\Gamma. \qquad (31)
\end{aligned}
$$

Proof. Multiplying by $u_t(t)$ in (22), using Green's formula, (25), and (26), we have

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left[\|u_t(t)\|^2 + \left(1 - \int_0^t \psi(s)ds\right)\|\nabla u(t)\|^2 + (\psi \circ \nabla u)(t) + \int_{\Gamma_1} m(x)w^2(t)d\Gamma\right] \\
& = \frac{1}{2}(\psi' \circ \nabla u)(t) - \frac{1}{2}\psi(t)\|\nabla u(t)\|^2 - \int_{\Gamma_1} h(x)w_t^2(t)d\Gamma \\
& -a_1 \int_{\Gamma_1} f_1(u_t(t))u_t(t)d\Gamma - a_2 \int_{\Gamma_1} f_2(v(x,1,t))u_t(t)d\Gamma, \qquad (32)
\end{aligned}
$$

where we used the relation

$$
\begin{aligned}
& -\int_\Omega \nabla u_t(t) \int_0^t \psi(t-s)\nabla u(s)dsdx \\
& = \frac{d}{dt}\left[\frac{1}{2}(\psi \circ \nabla u)(t) - \frac{1}{2}\int_0^t \psi(s)ds\|\nabla u(t)\|^2\right] - \frac{1}{2}(\psi' \circ \nabla u)(t) + \frac{1}{2}\psi(t)\|\nabla u(t)\|^2.
\end{aligned}
$$

From (29) and (32), we have

$$
\begin{aligned}
E'(t) = & \frac{1}{2}(\psi' \circ \nabla u)(t) - \frac{1}{2}\psi(t)\|\nabla u(t)\|^2 - \int_{\Gamma_1} h(x)w_t^2(t)d\Gamma \\
& -a_1 \int_{\Gamma_1} f_1(u_t(t))u_t(t)d\Gamma - a_2 \int_{\Gamma_1} f_2(v(x,1,t))u_t(t)d\Gamma \\
& +\frac{\zeta\varrho'(t)}{2}\int_{\Gamma_1}\int_0^1 F_2(v(x,\kappa,t))d\kappa d\Gamma + \frac{\zeta\varrho(t)}{2}\int_{\Gamma_1}\int_0^1 f_2(v(x,\kappa,t))v_t(x,\kappa,t)d\kappa d\Gamma, \qquad (33)
\end{aligned}
$$

where $F_2(t) = \int_0^t f_2(s)ds$. In (23), we multiply by $f_2(v(x,\kappa,t))$ and integrate over $\Gamma_1 \times (0,1)$ to obtain

$$
\begin{aligned}
& \frac{\zeta\varrho(t)}{2}\int_{\Gamma_1}\int_0^1 f_2(v(x,\kappa,t))v_t(x,\kappa,t)d\kappa d\Gamma \\
& = -\frac{\zeta}{2}\int_{\Gamma_1}\left[(1-\varrho'(t))F_2(v(x,1,t)) - F_2(v(x,0,t)) + \int_0^1 \varrho'(t)F_2(v(x,\kappa,t))d\kappa\right]d\Gamma.
\end{aligned}
$$

Applying this to (33) and noting that $v(x,0,t) = u_t(x,t)$, it follows that

$$
\begin{aligned}
E'(t) = & \frac{1}{2}(\psi' \circ \nabla u)(t) - \frac{1}{2}\psi(t)\|\nabla u(t)\|^2 - \int_{\Gamma_1} h(x)w_t^2(t)d\Gamma - a_1\int_{\Gamma_1} f_1(u_t(t))u_t(t)d\Gamma \\
& -a_2 \int_{\Gamma_1} f_2(v(x,1,t))u_t(t)d\Gamma - \frac{\zeta}{2}\int_{\Gamma_1}\left[(1-\varrho'(t))F_2(v(x,1,t)) - F_2(u_t(x,t))\right]d\Gamma. \qquad (34)
\end{aligned}
$$

From (13) and (14), we obtain

$$
\begin{aligned}
& -\frac{\zeta}{2}\int_{\Gamma_1}\left[(1-\varrho'(t))F_2(v(x,1,t)) - F_2(u_t(x,t))\right]d\Gamma \\
& \leq -\frac{\zeta c_3}{2}(1-\varrho_3)\int_{\Gamma_1} f_2(v(x,1,t))v(x,1,t)d\Gamma + \frac{\zeta c_4}{2}\int_{\Gamma_1} f_1(u_t(t))u_t(t)d\Gamma. \qquad (35)
\end{aligned}
$$

Substituting (35) into (34), we obtain

$$
\begin{aligned}
E'(t) \le &\frac{1}{2}(\psi' \circ \nabla u)(t) - \frac{1}{2}\psi(t)\|\nabla u(t)\|^2 - \int_{\Gamma_1} h(x) w_t^2(t) d\Gamma \\
&-\left(a_1 - \frac{\zeta c_4}{2}\right)\int_{\Gamma_1} f_1(u_t(t)) u_t(t) d\Gamma - \frac{\zeta c_3}{2}(1-\varrho_3)\int_{\Gamma_1} f_2(v(x,1,t)) v(x,1,t) d\Gamma \\
&-a_2 \int_{\Gamma_1} f_2(v(x,1,t)) u_t(t) d\Gamma.
\end{aligned} \tag{36}
$$

Now, we estimate the last term in the right-hand side of (36). The definition of F_2 and (21) give

$$
F_2^*(s) = s f_2^{-1}(s) - F_2(f_2^{-1}(s)), \ \text{ for } s \ge 0. \tag{37}
$$

When $f_2(v(x,1,t)) < 0$ and $u_t(t) \ge 0$, using (20) and (37) with $s = -f_2(v(x,1,t))$ and $r = u_t(t)$, we obtain (see details in [25])

$$
\begin{aligned}
&a_2 \int_{\Gamma_1} (-f_2(v(x,1,t))) u_t(t) d\Gamma \\
&\le |a_2| \int_{\Gamma_1} \Big(- f_2(v(x,1,t))(-v(x,1,t)) - F_2(-v(x,1,t)) + F_2(u_t(t))\Big) d\Gamma \\
&= |a_2| \int_{\Gamma_1} \Big(f_2(v(x,1,t)) v(x,1,t) - F_2(v(x,1,t)) + F_2(u_t(t))\Big) d\Gamma,
\end{aligned} \tag{38}
$$

where we used the fact that f_2 is odd and F_2 is even. When $f_2(v(x,1,t)) \ge 0$ and $u_t(t) < 0$, with $s = f_2(v(x,1,t))$ and $r = -u_t(t)$, we obtain

$$
\begin{aligned}
&a_2 \int_{\Gamma_1} f_2(v(x,1,t))(-u_t(t)) d\Gamma \\
&\le |a_2| \int_{\Gamma_1} \Big(f_2(v(x,1,t))(v(x,1,t)) - F_2(v(x,1,t)) + F_2(-u_t(t))\Big) d\Gamma \\
&= |a_2| \int_{\Gamma_1} \Big(f_2(v(x,1,t)) v(x,1,t) - F_2(v(x,1,t)) + F_2(u_t(t))\Big) d\Gamma.
\end{aligned} \tag{39}
$$

From (38) and (39), for the case $f_2(v(x,1,t)) u_t(t) \le 0$, we have

$$
-a_2 \int_{\Gamma_1} f_2(v(x,1,t)) u_t(t) d\Gamma \le |a_2| \int_{\Gamma_1} \Big(f_2(v(x,1,t)) v(x,1,t) - F_2(v(x,1,t)) + F_2(u_t(t))\Big) d\Gamma. \tag{40}
$$

Similarly, (40) holds when $f_2(v(x,1,t)) u_t(t) \ge 0$. Hence, using (13) and (40), we see that

$$
\begin{aligned}
&-a_2 \int_{\Gamma_1} f_2(v(x,1,t)) u_t(t) d\Gamma \\
&\le |a_2| \Big((1-c_3) \int_{\Gamma_1} f_2(v(x,1,t)) v(x,1,t) d\Gamma + c_4 \int_{\Gamma_1} f_1(u_t(t)) u_t(t) d\Gamma \Big).
\end{aligned} \tag{41}
$$

By using (16), (36), and (41), and by selecting ζ satisfying (30), we obtain the desired inequality (31) where $\gamma_0 = a_1 - \frac{\zeta c_4}{2} - |a_2| c_4 > 0$ and $\gamma_1 = \frac{\zeta c_3}{2}(1-\varrho_3) - |a_2|(1-c_3) > 0$. □

3. Technical Lemmas

In this section, we prove the following lemmas to obtain the general decay rates of the solution to problems (22)–(28).

Lemma 2. *Under the assumption (H1), the functional Φ_1 defined by*

$$
\Phi_1(t) = \int_\Omega u(t) u_t(t) dx + \int_{\Gamma_1} u(t) w(t) d\Gamma + \frac{1}{2}\int_{\Gamma_1} h(x) w^2(t) d\Gamma
$$

satisfies

$$\Phi_1'(t) \leq \|u_t(t)\|^2 - \frac{l}{2}\|\nabla u(t)\|^2 + \frac{2C(\xi)}{l}(i \circ \nabla u)(t) + \frac{8\lambda_1}{l}\|w_t(t)\|^2_{\Gamma_1}$$
$$+ \frac{a_1 a_3}{l}\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma + \frac{|a_2|a_3}{l}\int_{\Gamma_1} f_2^2(v(x,1,t))d\Gamma - \int_{\Gamma_1} m(x)w^2(t)d\Gamma, \quad (42)$$

for any $0 < \xi < 1$, *where*

$$i(t) = \xi\psi(t) - \psi'(t) \ \text{and} \ C(\xi) = \int_0^\infty \frac{\psi^2(s)}{i(s)}ds. \quad (43)$$

Proof. Using Equations (22) and (24)–(26), and utilizing (9) and Young's inequality, we obtain

$$\Phi_1'(t) = \|u_t(t)\|^2 - \Big(1 - \int_0^t \psi(s)ds\Big)\|\nabla u(t)\|^2 + \int_0^t \psi(t-s)(\nabla u(s) - \nabla u(t), \nabla u(t))ds$$
$$-a_1\int_{\Gamma_1} f_1(u_t(t))u(t)d\Gamma - a_2\int_{\Gamma_1} f_2(v(x,1,t))u(t)d\Gamma + 2\int_{\Gamma_1} u(t)w_t(t)d\Gamma - \int_{\Gamma_1} m(x)w^2(t)d\Gamma$$
$$\leq \|u_t(t)\|^2 - \frac{7l}{8}\|\nabla u(t)\|^2 + \frac{2}{l}\int_\Omega \Big(\int_0^t \psi(t-s)|\nabla u(s) - \nabla u(t)|ds\Big)^2 dx$$
$$-a_1\int_{\Gamma_1} f_1(u_t(t))u(t)d\Gamma - a_2\int_{\Gamma_1} f_2(v(x,1,t))u(t)d\Gamma + 2\int_{\Gamma_1} u(t)w_t(t)d\Gamma - \int_{\Gamma_1} m(x)w^2(t)d\Gamma.$$

Using the Cauchy–Schwarz inequality and (43), we have (see [3,34])

$$\int_\Omega \Big(\int_0^t \psi(t-s)|\nabla u(s) - \nabla u(t)|ds\Big)^2 dx \leq \Big(\int_0^t \frac{\psi^2(s)}{i(s)}ds\Big)(i \circ \nabla u)(t) \leq C(\xi)(i \circ \nabla u)(t). \quad (44)$$

Applying Young's inequality and (8), we obtain, for $\eta > 0$,

$$\Big| -a_1\int_{\Gamma_1} f_1(u_t(t))u(t)d\Gamma\Big| \leq \eta a_1\lambda_1\|\nabla u(t)\|^2 + \frac{a_1}{4\eta}\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma, \quad (45)$$
$$\Big| -a_2\int_{\Gamma_1} f_2(v(x,1,t))u(t)d\Gamma\Big| \leq \eta|a_2|\lambda_1\|\nabla u(t)\|^2 + \frac{|a_2|}{4\eta}\int_{\Gamma_1} f_2^2(v(x,1,t))d\Gamma, \quad (46)$$

and

$$2\int_{\Gamma_1} u(t)w_t(t)d\Gamma \leq \frac{l}{8}\|\nabla u(t)\|^2 + \frac{8\lambda_1}{l}\|w_t(t)\|^2_{\Gamma_1}. \quad (47)$$

Combining estimates (44)–(47), we see that

$$\Phi_1'(t) \leq \|u_t(t)\|^2 - (\frac{3l}{4} - \eta a_1\lambda_1 - \eta|a_2|\lambda_1)\|\nabla u(t)\|^2 + \frac{2C(\xi)}{l}(i \circ \nabla u)(t) + \frac{8\lambda_1}{l}\|w_t(t)\|^2_{\Gamma_1}$$
$$+ \frac{a_1}{4\eta}\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma + \frac{|a_2|}{4\eta}\int_{\Gamma_1} f_2^2(v(x,1,t))d\Gamma - \int_{\Gamma_1} m(x)w^2(t)d\Gamma.$$

Setting $a_3 = (a_1 + |a_2|)\lambda_1$ and choosing $\eta = \frac{l}{4a_3}$ leads to (42). □

Lemma 3. *Under the assumption (H1), the functional* Φ_2 *defined by*

$$\Phi_2(t) = -\int_\Omega u_t(t)\int_0^t \psi(t-s)(u(t) - u(s))dsdx$$

satisfies

$$
\begin{aligned}
\Phi_2'(t) \le & -\Big(\int_0^t \psi(s)ds-\delta\Big)\|u_t(t)\|^2+\delta\|\nabla u(t)\|^2+\frac{C_1(1+C(\xi))}{\delta}(i\circ\nabla u)(t)\\
&+\delta\lambda_1\|w_t(t)\|^2_{\Gamma_1}+\delta a_1\lambda_1\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma+\delta|a_2|\lambda_1\int_{\Gamma_1} f_2^2(v(x,1,t))d\Gamma,
\end{aligned}
\tag{48}
$$

for any $0<\delta<1$.

Proof. Using Equations (22), (24), and (25), we obtain

$$
\begin{aligned}
\Phi_2'(t) = & \left(1-\int_0^t \psi(s)ds\right)\int_\Omega \nabla u\cdot\int_0^t \psi(t-s)(\nabla u(t)-\nabla u(s))dsdx\\
&+\int_\Omega\Big(\int_0^t \psi(t-s)(\nabla u(t)-\nabla u(s))ds\Big)^2dx-\int_{\Gamma_1} w_t(t)\int_0^t \psi(t-s)(u(t)-u(s))dsd\Gamma\\
&+a_1\int_{\Gamma_1} f_1(u_t(t))\int_0^t \psi(t-s)(u(t)-u(s))dsd\Gamma+a_2\int_{\Gamma_1} f_2(v(x,1,t))\int_0^t \psi(t-s)(u(t)-u(s))dsd\Gamma\\
&-\int_\Omega u_t(t)\int_0^t \psi'(t-s)(u(t)-u(s))dsdx-\Big(\int_0^t \psi(s)ds\Big)\|u_t(t)\|^2\\
= & \,\vartheta_1+\vartheta_2+\cdots+\vartheta_6-\Big(\int_0^t \psi(s)ds\Big)\|u_t(t)\|^2.
\end{aligned}
$$

By Young's inequality, (8), and (44), we obtain, for $\delta>0$,

$$
\begin{aligned}
&\vartheta_1 \le \delta\|\nabla u(t)\|^2+\frac{C(\xi)}{4\delta}(i\circ\nabla u)(t),\\
&\vartheta_2 \le C(\xi)(i\circ\nabla u)(t),\\
&|\vartheta_3| \le \delta\lambda_1\|w_t(t)\|^2_{\Gamma_1}+\frac{C(\xi)}{4\delta}(i\circ\nabla u)(t),\\
&|\vartheta_4| \le \delta a_1\lambda_1\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma+\frac{a_1C(\xi)}{4\delta}(i\circ\nabla u)(t),\\
&|\vartheta_5| \le \delta|a_2|\lambda_1\int_{\Gamma_1} f_2^2(v(x,1,t))d\Gamma+\frac{|a_2|C(\xi)}{4\delta}(i\circ\nabla u)(t).
\end{aligned}
$$

Using Young's inequality, (8), (9), (43), and (44), we see that

$$
\begin{aligned}
\vartheta_6 = & \int_\Omega u_t(t)\int_0^t i(t-s)(u(t)-u(s))dsdx-\xi\int_\Omega u_t(t)\int_0^t \psi(t-s)(u(t)-u(s))dsdx\\
\le & \,\delta\|u_t(t)\|^2+\frac{1}{2\delta}\int_\Omega\Big(\int_0^t i(t-s)|u(s)-u(t)|ds\Big)^2dx+\frac{\xi^2}{2\delta}\int_\Omega\Big(\int_0^t \psi(t-s)|u(t)-u(s)|ds\Big)^2dx\\
\le & \,\delta\|u_t(t)\|^2+\frac{\lambda_0(\psi(0)+\xi)}{2\delta}(i\circ\nabla u)(t)+\frac{\lambda_0\xi^2C(\xi)}{2\delta}(i\circ\nabla u)(t).
\end{aligned}
$$

Combining all above estimates and taking $C_1=\max\{\frac{\lambda_0(\psi(0)+\xi)}{2},\ \delta+\frac{1+\lambda_0\xi^2}{2}+\frac{a_1+|a_2|}{4}\}$, the desired inequality (48) is established. □

Lemma 4. *Under the assumptions (H3) and (H4), the functional* Φ_3 *defined by*

$$
\Phi_3(t)=\varrho(t)\int_{\Gamma_1}\int_0^1 e^{-\kappa\varrho(t)}F_2(v(x,\kappa,t))d\kappa d\Gamma
$$

satisfies

$$\Phi_3'(t) \leq -e^{-\varrho_2}\varrho(t)\int_{\Gamma_1}\int_0^1 F_2(v(x,\kappa,t))d\kappa d\Gamma - c_3(1-\varrho_3)e^{-\varrho_2}\int_{\Gamma_1} f_2(v(x,1,t))v(x,1,t)d\Gamma$$
$$+c_4\int_{\Gamma_1} f_1(u_t(t))u_t(t)d\Gamma. \tag{49}$$

Proof. Using Equation (23), integration by parts, (13), and (14), we obtain (see [26])

$$\begin{aligned}
\Phi_3'(t) &= \varrho'(t)\int_{\Gamma_1}\int_0^1 e^{-\kappa\varrho(t)}F_2(v(x,\kappa,t))d\kappa d\Gamma - \varrho(t)\int_{\Gamma_1}\int_0^1 \kappa\varrho'(t)e^{-\kappa\varrho(t)}F_2(v(x,\kappa,t))d\kappa d\Gamma \\
&\quad - \int_{\Gamma_1}\int_0^1 e^{-\kappa\varrho(t)}(1-\kappa\varrho'(t))\frac{d}{d\kappa}F_2(v(x,\kappa,t))d\kappa d\Gamma \\
&= -\Phi_3(t) - e^{-\varrho(t)}\int_{\Gamma_1}(1-\varrho'(t))F_2(v(x,1,t))d\Gamma + \int_{\Gamma_1}F_2(u_t(x,t))d\Gamma \\
&\leq -e^{-\varrho_2}\varrho(t)\int_{\Gamma_1}\int_0^1 F_2(v(x,\kappa,t))d\kappa d\Gamma - c_3(1-\varrho_3)e^{-\varrho_2}\int_{\Gamma_1} f_2(v(x,1,t))v(x,1,t)d\Gamma \\
&\quad + c_4\int_{\Gamma_1} f_1(u_t(t))u_t(t)d\Gamma.
\end{aligned}$$

□

Lemma 5 ([3])**.** *Under the assumption (H1), the functional Φ_4 defined by*

$$\Phi_4(t) = \int_\Omega\int_0^t G_2(t-s)|\nabla u(s)|^2 ds dx$$

satisfies

$$\Phi_4'(t) \leq 3(1-l)\|\nabla u(t)\|^2 - \frac{1}{2}(\psi\circ\nabla u)(t), \tag{50}$$

where $G_2(t) = \int_t^\infty \psi(s)ds$.

Next, let us define the perturbed modified energy by

$$L(t) = NE(t) + N_1\Phi_1(t) + N_2\Phi_2(t) + \Phi_3(t) + b_1 E(t), \tag{51}$$

where N, N_1, N_2, and b_1 are some positive constants.

As in [6,26], for a large enough $N > 0$, there exist positive constants β_1 and β_2 such that

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t).$$

Lemma 6. *Assume that (H1) and (H3)–(H5) hold. Then, there exist positive constants β_3, β_4, and β_5 such that*

$$L'(t) \leq -\beta_3 E(t) + \beta_4\int_{t_0}^t \psi(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dx ds + \beta_5\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma, \ \ \forall t \geq t_0, \tag{52}$$

where t_0 was introduced in (17)*.*

Proof. Let $\psi_0 = \int_0^{t_0}\psi(s)ds$. Using the fact that $i(t) = \xi\psi(t) - \psi'(t)$ and combining (31), (42), (48), (49), and (51), we obtain, for all $t \geq t_0$,

$$
\begin{aligned}
L'(t) \le{}& \frac{\xi N}{2}(\psi \circ \nabla u)(t) - \Big(\frac{lN_1}{2} - \delta N_2\Big)\|\nabla u(t)\|^2 - \Big(\psi_0 N_2 - \delta N_2 - N_1\Big)\|u_t(t)\|^2 \\
&-\Big(\frac{N}{2} - \frac{2C(\xi)N_1}{l} - \frac{C_1(1+C(\xi))N_2}{\delta}\Big)(i \circ \nabla u)(t) - N_1 \int_{\Gamma_1} m(x) w^2(t) d\Gamma + b_1 E'(t) \\
&-\Big(h_1 N - \frac{8\lambda_1 N_1}{l} - \delta\lambda_1 N_2\Big)\|w_t(t)\|^2_{\Gamma_1} - e^{-\varrho_2}\varrho(t)\int_{\Gamma_1}\int_0^1 F_2(v(x,\kappa,t))d\kappa d\Gamma \\
&-(\gamma_0 N - c_4)\int_{\Gamma_1} f_1(u_t(t))u_t(t) d\Gamma - \Big(\gamma_1 N + c_3(1-\varrho_3)e^{-\varrho_2}\Big)\int_{\Gamma_1} f_2(v(x,1,t))v(x,1,t)d\Gamma \\
&+\Big(\frac{a_1 a_3 N_1}{l} + \delta a_1 \lambda_1 N_2\Big)\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma + \Big(\frac{|a_2|a_3 N_1}{l} + \delta|a_2|\lambda_1 N_2\Big)\int_{\Gamma_1} f_2^2(v(x,1,t))d\Gamma.
\end{aligned} \tag{53}
$$

From (13), we find that

$$
\int_{\Gamma_1} f_2^2(v(x,1,t))d\Gamma \le c_2 \int_{\Gamma_1} f_2(v(x,1,t))v(x,1,t)d\Gamma. \tag{54}
$$

Applying (54) to (53) and taking $\delta = \frac{l}{4N_2}$, we obtain, for all $t \ge t_0$,

$$
\begin{aligned}
L'(t) \le{}& \frac{\xi N}{2}(\psi \circ \nabla u)(t) - \Big(\frac{lN_1}{2} - \frac{l}{4}\Big)\|\nabla u(t)\|^2 - \Big(\psi_0 N_2 - N_1 - \frac{l}{4}\Big)\|u_t(t)\|^2 \\
&-\Big(\frac{N}{2} - \frac{4C_1 N_2^2}{l} - C(\xi)\Big[\frac{2N_1}{l} + \frac{4C_1 N_2^2}{l}\Big]\Big)(i \circ \nabla u)(t) - N_1 \int_{\Gamma_1} m(x) w^2(t) d\Gamma \\
&-\Big(h_1 N - \frac{8\lambda_1 N_1}{l} - \frac{l\lambda_1}{4}\Big)\|w_t(t)\|^2_{\Gamma_1} - e^{-\varrho_2}\varrho(t)\int_{\Gamma_1}\int_0^1 F_2(v(x,\kappa,t))d\kappa d\Gamma \\
&-(\gamma_0 N - c_4)\int_{\Gamma_1} f_1(u_t(t))u_t(t) d\Gamma + \Big(\frac{a_1 a_3 N_1}{l} + \frac{a_1 l \lambda_1}{4}\Big)\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma + b_1 E'(t) \\
&-\Big(\gamma_1 N + c_3(1-\varrho_3)e^{-\varrho_2} - \frac{|a_2|a_3 c_2 N_1}{l} - \frac{|a_2|c_2 l\lambda_1}{4}\Big)\int_{\Gamma_1} f_2(v(x,1,t))v(x,1,t)d\Gamma.
\end{aligned}
$$

We choose N_1 large enough so that

$$
\frac{lN_1}{2} - \frac{l}{4} > 4(1-l),
$$

then N_2 large enough so that

$$
\psi_0 N_2 - N_1 - \frac{l}{4} > 1.
$$

Using the fact that $\frac{\xi\psi^2(s)}{i(s)} < \psi(s)$ and the Lebesgue dominated convergence theorem, we deduce that

$$
\xi C(\xi) = \int_0^\infty \frac{\xi\psi^2(s)}{i(s)} ds \to 0 \text{ as } \xi \to 0.
$$

Hence, there is $0 < \xi_0 < 1$ such that if $\xi < \xi_0$, then

$$
\xi C(\xi)\Big[\frac{2N_1}{l} + \frac{4C_1 N_2^2}{l}\Big] < \frac{1}{8}.
$$

Finally, selecting $\xi = \frac{1}{2N}$ and choosing N large enough so that

$$
N > \max\Big\{\frac{16C_1 N_2^2}{l},\ \frac{1}{h_1}\Big(\frac{8\lambda_1 N_1}{l} + \frac{l\lambda_1}{4}\Big),\ \frac{c_4}{\gamma_0},\ \frac{1}{\gamma_1}\Big(\frac{|a_2|a_3 c_2 N_1}{l} + \frac{|a_2|c_2 l\lambda_1}{4} - c_3(1-\varrho_3)e^{-\varrho_2}\Big)\Big\},
$$

we obtain

$$L'(t) \le -\|u_t(t)\|^2 - 4(1-l)\|\nabla u(t)\|^2 + \frac{1}{4}(\psi \circ \nabla u)(t) - N_1 \int_{\Gamma_1} m(x) w^2(t) d\Gamma$$
$$-e^{-\varrho_2}\varrho(t)\int_{\Gamma_1}\int_0^1 F_2(v(x,\kappa,t))d\kappa d\Gamma + \beta_5 \int_{\Gamma_1} f_1^2(u_t(t))d\Gamma + b_1 E'(t), \ \ \forall t \ge t_0, \tag{55}$$

where $\beta_5 = \frac{a_1 a_3 N_1}{l} + \frac{a_1 l \lambda_1}{4}$. Using (18) and (31), we find that, for any $t \ge t_0$,

$$\int_0^{t_0} \psi(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dx ds \le -\frac{1}{c_5}\int_0^{t_0} \psi'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dx ds \le -\frac{2}{c_5} E'(t). \tag{56}$$

Combining (29), (55), and (56) and making a suitable choice for b_1, we obtain the estimate (52). □

To evaluate the two terms on the right side of (52), we establish the following lemmas.

Lemma 7 ([1])**.** *Assume that (H2) holds and* $\max\{r_1, f_0(r_1)\} < \varepsilon$*, where* ε *was introduced in* (11)*. Then, there exist positive constants* C_2, C_3*, and* C_4 *such that*

$$\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma \le C_2 \int_{\Gamma_1} f_1(u_t(t))u_t(t)d\Gamma, \ \ \textit{if } f_0 \textit{ is linear}, \tag{57}$$

$$\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma \le C_3 F^{-1}(\chi(t)) - C_3 E'(t), \ \ \textit{if } f_0 \textit{ is nonlinear}, \tag{58}$$

where

$$\chi(t) = \frac{1}{|\Gamma_{11}|}\int_{\Gamma_{11}} f_1(u_t(t))u_t(t)d\Gamma \le -C_4 E'(t), \tag{59}$$

$\Gamma_{11} = \{x \in \Gamma_1 : |u_t(t)| \le \varepsilon_1\}$ *and* $0 < \varepsilon_1 = \min\{r_1, f_0(r_1)\}$.

Lemma 8. *Assume that (H1) and (H3)–(H5) hold and that* f_0 *is linear. Then, the energy functional satisfies*

$$\int_0^\infty E(s)ds < \infty. \tag{60}$$

Proof. We introduce the functional

$$\mathcal{L}(t) = L(t) + \Phi_4(t),$$

which is nonnegative. From (50) and (55), we see that, for all $t \ge t_0$,

$$\mathcal{L}'(t) \le -\|u_t(t)\|^2 - (1-l)\|\nabla u(t)\|^2 - \frac{1}{4}(\psi \circ \nabla u)(t) - N_1 \int_{\Gamma_1} m(x) w^2(t) d\Gamma$$
$$-e^{-\varrho_2}\varrho(t)\int_{\Gamma_1}\int_0^1 F_2(v(x,\kappa,t))d\kappa d\Gamma + \beta_5 \int_{\Gamma_1} f_1^2(u_t(t))d\Gamma + b_1 E'(t).$$

Applying (29), (31), and (57), we have

$$\mathcal{L}'(t) \le -d_1 E(t) + \big(b_1 - \frac{\beta_5 C_2}{\gamma_0}\big)E'(t),$$

where d_1 is some positive constant. Selecting a suitable choice for b_1, we obtain

$$\mathcal{L}'(t) \le -d_1 E(t).$$

This implies that

$$d_1 \int_{t_0}^{t} E(s)ds \leq \mathcal{L}(t_0) - \mathcal{L}(t) \leq \mathcal{L}(t_0) < \infty.$$

□

Next, we define $\Upsilon(t)$ by

$$\Upsilon(t) := -\int_{t_0}^{t} \psi'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \leq -2E'(t). \tag{61}$$

Lemma 9. *Assume that (H1) and (H2) hold and that G is nonlinear. Then, the solution to (22)–(28) satisfies the estimates*

$$\int_{t_0}^{t} \psi(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \leq \frac{1}{\theta}\, \overline{G}^{-1}\left(\frac{\theta \Upsilon(t)}{\mu(t)}\right), \ \forall t \geq t_0, \text{ if } f_0 \text{ is linear}, \tag{62}$$

$$\int_{t_0}^{t} \psi(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \leq \frac{t-t_0}{\theta}\, \overline{G}^{-1}\left(\frac{\theta \Upsilon(t)}{(t-t_0)\mu(t)}\right), \ \forall t > t_0, \text{ if } f_0 \text{ is nonlinear}, \tag{63}$$

where $\theta \in (0,1)$, and $\overline{G}$ is an extension of G such that $\overline{G}$ is a strictly convex and strictly increasing C^2 function on $(0,\infty)$.

Proof. First, we prove the estimate (62) when f_0 is linear. For $0 < \theta < 1$, we define $I(t)$ by

$$I(t) := \theta \int_{t_0}^{t} \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds.$$

By (60), θ is taken so small that, for all $t \geq t_0$,

$$I(t) < 1. \tag{64}$$

Since G is strictly convex on $(0, r_0]$, then

$$G(q\zeta) \leq qG(\zeta), \tag{65}$$

where $0 \leq q \leq 1$ and $\zeta \in (0, r_0]$. Using the fact that μ is a positive nonincreasing function and applying (10), (64), (65), and Jensen's inequality (19), we find that (see details in [1,3])

$$\begin{aligned}
\Upsilon(t) &\geq \frac{\mu(t)}{\theta I(t)} \int_{t_0}^{t} I(t) G(\psi(s)) \int_\Omega \theta |\nabla u(t) - \nabla u(t-s)|^2 dxds \\
&\geq \frac{\mu(t)}{\theta I(t)} \int_{t_0}^{t} G(I(t)\psi(s)) \int_\Omega \theta |\nabla u(t) - \nabla u(t-s)|^2 dxds \\
&\geq \frac{\mu(t)}{\theta}\, \overline{G}\left(\theta \int_{t_0}^{t} \psi(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds\right).
\end{aligned} \tag{66}$$

Since $\overline{G}$ is strictly increasing, we obtain

$$\int_{t_0}^{t} \psi(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \leq \frac{1}{\theta} \overline{G}^{-1}\left(\frac{\theta \Upsilon(t)}{\mu(t)}\right).$$

Now, we show the estimate (63) when f_0 is nonlinear. Since we cannot guarantee (60), we define the following function:

$$I_1(t) := \frac{\theta}{t-t_0} \int_{t_0}^{t} \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds, \ \forall t > t_0.$$

Using the fact that $E'(t) \leq 0$ and (29), we have

$$I_1(t) \leq \frac{2\theta}{t-t_0} \int_{t_0}^{t} (||\nabla u(t)||^2 + ||\nabla u(t-s)||^2) ds \leq \frac{8\theta E(0)}{l}.$$

Choose θ small enough so that, for all $t > t_0$,

$$I_1(t) < 1. \tag{67}$$

Similar to (67), using (10), (65), (67), and Jensen's inequality (19), we obtain

$$\begin{aligned}
\Upsilon(t) &= \frac{t-t_0}{\theta I_1(t)} \int_{t_0}^{t} I_1(t)(-\psi'(s)) \int_{\Omega} \frac{\theta}{t-t_0} |\nabla u(t) - \nabla u(t-s)|^2 dxds \\
&\geq \frac{(t-t_0)\mu(t)}{\theta I_1(t)} \int_{t_0}^{t} G(I_1(t)\psi(s)) \int_{\Omega} \frac{\theta}{t-t_0} |\nabla u(t) - \nabla u(t-s)|^2 dxds \\
&\geq \frac{(t-t_0)\mu(t)}{\theta} \overline{G}\left(\frac{\theta}{t-t_0} \int_{t_0}^{t} \psi(s) \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds \right).
\end{aligned}$$

This implies that

$$\int_{t_0}^{t} \psi(s) \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds \leq \frac{t-t_0}{\theta} \overline{G}^{-1}\left(\frac{\theta \Upsilon(t)}{(t-t_0)\mu(t)} \right).$$

□

4. General Decay of the Energy

In this section, we state and prove the main result of our work.

Theorem 2. *Assume that (H1)–(H5) hold and that f_0 is linear. Then, there exist positive constants k_1, k_2, k_3, and k_4 such that the energy functional satisfies, for all $t \geq t_0$,*

$$E(t) \leq k_2 e^{-k_1 \int_{t_0}^{t} \mu(s) ds}, \quad \text{if } G \text{ is linear}, \tag{68}$$

$$E(t) \leq k_4 G_1^{-1}\left(k_3 \int_{t_0}^{t} \mu(s) ds \right), \quad \text{if } G \text{ is nonlinear}, \tag{69}$$

where $G_1(t) = \int_t^{r_0} \frac{1}{sG'(s)} ds$ is strictly decreasing and convex on $(0, r_0]$.

Proof. Now, we consider the following two cases.
Case 1: $G(t)$ is linear. Multiplying (52) by the positive nonincreasing function $\mu(t)$ and using (10), (31), and (57), we obtain

$$\begin{aligned}
&\mu(t)L'(t) \leq -\beta_3 \mu(t) E(t) + \beta_4 \int_{t_0}^{t} \mu(s)\psi(s) \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds + \beta_5 \mu(t) \int_{\Gamma_1} f_1^2(u_t(t)) d\Gamma \\
&\leq -\beta_3 \mu(t) E(t) - \beta_4 \int_{t_0}^{t} \psi'(s) \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds + \beta_5 C_2 \mu(0) \int_{\Gamma_1} f_1(u_t(t)) u_t(t) d\Gamma \\
&\leq -\beta_3 \mu(t) E(t) - C_5 E'(t),
\end{aligned}$$

where $C_5 = 2\beta_4 + \frac{\beta_5 C_2 \mu(0)}{\gamma_0}$ is a positive constant. Since $\mu(t)$ is nonincreasing, we have

$$(\mu L + C_5 E)'(t) \leq -\beta_3 \mu(t) E(t), \quad \forall t \geq t_0.$$

Since $\mu(t)L(t) + C_5 E(t) \sim E(t)$, for some positive constants k_1 and k_2, we obtain

$$E(t) \leq k_2 e^{-k_1 \int_{t_0}^{t} \mu(s) ds}.$$

Case 2: $G(t)$ is nonlinear. This case is obtained through the ideas presented in [3] as follows. Using (31), (52), (57), and (62), we obtain

$$L'(t) \le -\beta_3 E(t) + \frac{\beta_4}{\theta}\overline{G}^{-1}\left(\frac{\theta \Upsilon(t)}{\mu(t)}\right) - \frac{\beta_5 C_2}{\gamma_0}E'(t), \quad \forall t \ge t_0. \tag{70}$$

Let $L_1(t) = L(t) + \frac{\beta_5 C_2}{\gamma_0}E(t) \sim E(t)$, and then (70) becomes

$$L_1'(t) \le -\beta_3 E(t) + \frac{\beta_4}{\theta}\overline{G}^{-1}\left(\frac{\theta \Upsilon(t)}{\mu(t)}\right), \quad \forall t \ge t_0. \tag{71}$$

For $0 < \varepsilon_0 < r_0$, using (71) and the fact that $E' \le 0, \overline{G}' > 0$ and $\overline{G}'' > 0$, we find that the functional L_2, defined by

$$L_2(t) := \overline{G}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)L_1(t) \sim E(t),$$

satisfies

$$L_2'(t) \le -\beta_3 E(t)\overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + \frac{\beta_4}{\theta}\overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\overline{G}^{-1}\left(\frac{\theta\Upsilon(t)}{\mu(t)}\right), \quad \forall t \ge t_0. \tag{72}$$

With $s = \overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)$ and $r = \overline{G}^{-1}\left(\frac{\theta\Upsilon(t)}{\mu(t)}\right)$, using (20), (21), and (72), we obtain

$$L_2'(t) \le -\beta_3 E(t) G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + \frac{\varepsilon_0\beta_4}{\theta}\frac{E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + \frac{\beta_4\Upsilon(t)}{\mu(t)},$$

where we have used that $\varepsilon_0\frac{E(t)}{E(0)} < r_0$ and $\overline{G}' = G'$ on $(0, r_0]$. Multiplying this by $\mu(t)$ and using (61), we obtain

$$\mu(t)L_2'(t) \le -\left(\beta_3 E(0) - \frac{\varepsilon_0\beta_4}{\theta}\right)\frac{\mu(t)E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) - 2\beta_4 E'(t).$$

By defining $L_3(t) = \mu(t)L_2(t) + 2\beta_4 E(t)$, we see that, for some positive constants γ_2 and γ_3,

$$\gamma_2 L_3(t) \le E(t) \le \gamma_3 L_3(t). \tag{73}$$

With a suitable choice of ε_0, we obtain, for some positive constant d_2,

$$L_3'(t) \le -d_2\mu(t)\frac{E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) = -d_2\mu(t)G_2\left(\frac{E(t)}{E(0)}\right), \quad \forall t \ge t_0, \tag{74}$$

where $G_2(t) = tG'(\varepsilon_0 t)$. Applying the strict convexity of G on $(0, r_0]$ and $G_2'(t) = G'(\varepsilon_0 t) + \varepsilon_0 t G''(\varepsilon_0 t)$, we see that $G_2(t), G_2'(t) > 0$ on $(0,1]$. Finally, defining

$$Q(t) = \frac{\gamma_2 L_3(t)}{E(0)}$$

and using (73), we have

$$Q(t) \le \frac{E(t)}{E(0)} \le 1 \text{ and } Q(t) \sim E(t). \tag{75}$$

From (74), (75), and the fact that $G_2'(t) > 0$ on $(0,1]$, we arrive at

$$Q'(t) \le -k_3\mu(t)G_2(Q(t)), \quad \forall t \ge t_0,$$

where $k_3 = \frac{d_2\gamma_2}{E(0)}$ is a positive constant. Integrating this over (t_0, t) and using variable transformation, we find that (see details in [3])

$$\int_t^{t_0} \frac{\varepsilon_0 Q'(s)}{\varepsilon_0 Q(s) G'(\varepsilon_0 Q(s))} ds \geq k_3 \int_{t_0}^t \mu(s) ds \Longrightarrow \int_{\varepsilon_0 Q(t)}^{\varepsilon_0 Q(t_0)} \frac{1}{sG'(s)} ds \geq k_3 \int_{t_0}^t \mu(s) ds.$$

Since $\varepsilon_0 < r_0$ and $Q(t) \leq 1$, for all $t \geq t_0$, we have

$$G_1(\varepsilon_0 Q(t)) = \int_{\varepsilon_0 Q(t)}^{r_0} \frac{1}{sG'(s)} ds \geq k_3 \int_{t_0}^t \mu(s) ds \Longrightarrow Q(t) \leq \frac{1}{\varepsilon_0} G_1^{-1}\Big(k_3 \int_{t_0}^t \mu(s) ds\Big), \tag{76}$$

where $G_1(t) = \int_t^{r_0} \frac{1}{sG'(s)} ds$. Here, we have used the fact that G_1 is a strictly decreasing function on $(0, r_0]$. Therefore, using (75) and (76), the estimate (69) is established. □

Theorem 3. Assume that (H1)–(H5) hold and that f_0 is nonlinear. Then, there exist positive constants $\alpha_1, \alpha_2, \alpha_3$, and α_4 such that the energy functional satisfies

$$E(t) \leq \alpha_2 F_1^{-1}\left(\alpha_1 \int_{t_0}^t \mu(s) ds\right), \quad \forall t \geq t_0, \text{ if } G \text{ is linear}, \tag{77}$$

where $F_1(t) = \int_t^{r_1} \frac{1}{sF'(s)} ds$ and

$$E(t) \leq \alpha_4 (t - t_0) K_1^{-1}\left(\frac{\alpha_3}{(t - t_0)\int_{t_1}^t \mu(s) ds}\right), \quad \forall t \geq t_1, \text{ if } G \text{ is nonlinear}, \tag{78}$$

where $K_1(t) = tK'(\varepsilon_2 t)$, $0 < \varepsilon_2 < r_2 = \min\{r_0, r_1\}$ and $K = (\overline{G}^{-1} + \overline{F}^{-1})^{-1}$.

Proof. **Case 1**: $G(t)$ is linear. Multiplying (52) by the positive nonincreasing function $\mu(t)$ and using (10), (31), and (58), we obtain

$$\mu(t)L'(t) \leq -\beta_3 \mu(t) E(t) + \beta_5 C_3 \mu(t) F^{-1}(\chi(t)) - C_6 E'(t), \tag{79}$$

where $C_6 = 2\beta_4 + \beta_5 C_3 \mu(0)$ is a positive constant. Since $\mu(t)$ is nonincreasing, (79) becomes

$$F_3'(t) \leq -\beta_3 \mu(t) E(t) + \beta_5 C_3 \mu(t) F^{-1}(\chi(t)), \quad \forall t \geq t_0, \tag{80}$$

where $F_3(t) = \mu(t)L(t) + C_6 E(t) \sim E(t)$. For $0 < \varepsilon_1 < r_1$, using (80) and the fact that $E' \leq 0, F' > 0$ and $F'' > 0$ on $(0, r_1]$, the functional F_4, defined by

$$F_4(t) := F'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) F_3(t) \sim E(t),$$

satisfies

$$F_4'(t) \leq -\beta_3 \mu(t) E(t) F'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) + \beta_5 C_3 \mu(t) F'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) F^{-1}(\chi(t)).$$

Given (20) and (21) with $s = F'\big(\varepsilon_1 \frac{E(t)}{E(0)}\big)$ and $r = F^{-1}(\chi(t))$, using (59), we obtain that

$$\begin{aligned} F_4'(t) &\leq -\beta_3 \mu(t) E(t) F'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) + \varepsilon_1 \beta_5 C_3 \frac{\mu(t)E(t)}{E(0)} F'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) + \beta_5 C_3 \mu(0) \chi(t) \\ &\leq -(\beta_3 E(0) - \varepsilon_1 \beta_5 C_3) \frac{\mu(t)E(t)}{E(0)} F'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) - \beta_5 C_3 C_4 \mu(0) E'(t), \quad \forall t \geq t_0. \end{aligned}$$

Let $F_5(t) = F_4(t) + \beta_5 C_3 C_4 \mu(0) E(t)$; then it satisfies, for positive constants γ_4 and γ_5,

$$\gamma_4 F_5(t) \le E(t) \le \gamma_5 F_5(t). \tag{81}$$

Consequently, with a suitable choice of ε_1, we have, for some positive constant d_3,

$$F_5'(t) \le -d_3 \mu(t) \frac{E(t)}{E(0)} F'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) = -d_3 \mu(t) F_0\left(\frac{E(t)}{E(0)}\right), \ \forall t \ge t_0, \tag{82}$$

where $F_0(t) = tF'(\varepsilon_1 t)$. From the strict convexity of F on $(0, r_1]$, we obtain $F_0(t), F_0'(t) > 0$ on $(0, 1]$. Let

$$J(t) = \frac{\gamma_4 F_5(t)}{E(0)},$$

and from (81) and (82), we obtain

$$J(t) \le \frac{E(t)}{E(0)} \le 1 \ \text{ and } J'(t) \le -\alpha_1 \mu(t) F_0(J(t)), \ \forall t \ge t_0,$$

where $\alpha_1 = \frac{d_3 \gamma_4}{E(0)}$ is a positive constant. Then, similar to (76), the integration over (t_0, t) and variable transformation yield

$$J(t) \le \frac{1}{\varepsilon_1} F_1^{-1}\left(\alpha_1 \int_{t_0}^{t} \mu(s) ds\right), \tag{83}$$

where $F_1(t) = \int_t^{r_1} \frac{1}{sF'(s)} ds$, which is a strictly decreasing function on $(0, r_1]$. Combining (81) and (83), the estimate (77) is proved.

Case 2: $G(t)$ is nonlinear. This case is obtained by the arguments presented in [1] as follows. Using (52), (58), and (63), we obtain

$$L'(t) \le -\beta_3 E(t) + \frac{\beta_4 (t - t_0)}{\theta} \overline{G}^{-1}\left(\frac{\theta \Upsilon(t)}{(t - t_0)\mu(t)}\right) + \beta_5 C_3 F^{-1}(\chi(t)) - \beta_5 C_3 E'(t), \ \forall t > t_0. \tag{84}$$

Since $\lim\limits_{t \to \infty} \dfrac{1}{t - t_0} = 0$, there exists $t_1 > t_0$ such that

$$\frac{1}{t - t_0} < 1, \ \forall t \ge t_1. \tag{85}$$

Using the strictly convex and strictly increasing function of $\overline{F}$ and (65) with $q = \frac{1}{t - t_0}$, we see that

$$\overline{F}^{-1}(\chi(t)) \le (t - t_0) \overline{F}^{-1}\left(\frac{\chi(t)}{t - t_0}\right), \ \forall t \ge t_1. \tag{86}$$

Combining (84) and (86), we arrive at

$$R_1'(t) \le -\beta_3 E(t) + \frac{\beta_4 (t - t_0)}{\theta} \overline{G}^{-1}\left(\frac{\theta \Upsilon(t)}{(t - t_0)\mu(t)}\right) + \beta_5 C_3 (t - t_0) \overline{F}^{-1}\left(\frac{\chi(t)}{t - t_0}\right), \ \forall t \ge t_1, \tag{87}$$

where $R_1(t) = L(t) + \beta_5 C_3 E(t) \sim E(t)$. Let

$$r_2 = \min\{r_0, r_1\}, \ \varphi(t) = \max\left\{\frac{\theta \Upsilon(t)}{(t - t_0)\mu(t)}, \frac{\chi(t)}{t - t_0}\right\} \text{ and } K = \left(\overline{G}^{-1} + \overline{F}^{-1}\right)^{-1}, \ \forall t \ge t_1. \tag{88}$$

Therefore, (87) reduces to

$$R_1'(t) \le -\beta_3 E(t) + C_7 (t - t_0) K^{-1}(\varphi(t)), \ \forall t \ge t_1, \tag{89}$$

where $C_7 = \max\{\frac{\beta_4}{\theta}, \beta_5 C_3\}$. The strictly increasing and strictly convex properties of $\overline{G}$ and $\overline{F}$ imply that

$$K' = \frac{\overline{G}'\overline{F}'}{\overline{G}' + \overline{F}'} > 0 \text{ and } K'' = \frac{\overline{G}''(\overline{F}')^2 + (\overline{G}')^2\overline{F}''}{(\overline{G}' + \overline{F}')^2} > 0, \tag{90}$$

on $(0, r_2]$.

Now, for $0 < \varepsilon_2 < r_2$, using (85), we see that $\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)} < r_2$. Defining

$$R_2(t) = K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right)R_1(t), \quad \forall t \geq t_1,$$

and using (89) and (90), we find that

$$\begin{aligned}
R_2'(t) &= \left(-\frac{\varepsilon_2}{(t-t_0)^2}\frac{E(t)}{E(0)} + \frac{\varepsilon_2}{t-t_0}\frac{E'(t)}{E(0)}\right)K''\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right)R_1(t) + K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right)R_1'(t) \\
&\leq -\beta_3 E(t)K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right) + C_7(t-t_0)K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right)K^{-1}(\varphi(t)), \quad \forall t \geq t_1.
\end{aligned} \tag{91}$$

Using (20) and (21) with $s = K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right)$ and $r = K^{-1}(\varphi(t))$ and applying (91), we obtain

$$R_2'(t) \leq -\beta_3 E(t)K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right) + \varepsilon_2 C_7 \frac{E(t)}{E(0)}K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right) + C_7(t-t_0)\varphi(t). \tag{92}$$

From (59), (61), and (88), we obtain

$$(t-t_0)\mu(t)\varphi(t) \leq -C_8 E'(t), \tag{93}$$

where $C_8 = \min\{2\theta, C_4\mu(0)\}$. Multiplying (92) by the positive nonincreasing function $\mu(t)$ and using (93), we have

$$R_3'(t) \leq -\left(\beta_3 E(0) - \varepsilon_2 C_7\right)\frac{\mu(t)E(t)}{E(0)}K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right), \quad \forall t \geq t_1,$$

where $R_3(t) = \mu(t)R_2(t) + C_7 C_8 E(t) \sim E(t)$. For a suitable choice of ε_2, we find that

$$R_3'(t) \leq -d_4\frac{\mu(t)E(t)}{E(0)}K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right), \quad \forall t \geq t_1, \tag{94}$$

where d_4 is a positive constant. An integration of (94) yields

$$\frac{d_4}{E(0)}\int_{t_1}^{t} E(s)K'\left(\frac{\varepsilon_2}{s-t_0}\frac{E(s)}{E(0)}\right)\mu(s)ds \leq \int_{t}^{t_1} R_3'(s)ds \leq R_3(t_1).$$

Using (90) and the non-increasing property of E, we see that the map $t \to E(t)K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right)$ is non-increasing and, consequently, we obtain

$$d_4\frac{E(t)}{E(0)}K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right)\int_{t_1}^{t}\mu(s)ds \leq R_3(t_1), \quad \forall t \geq t_1. \tag{95}$$

Multiplying (95) by $\frac{1}{t-t_0}$, we obtain

$$d_4 K_1\left(\frac{1}{t-t_0}\frac{E(t)}{E(0)}\right)\int_{t_1}^{t}\mu(s)ds \leq \frac{R_3(t_1)}{t-t_0}, \quad \forall t \geq t_1,$$

where $K_1(s) = sK'(\varepsilon_2 s)$, which is strictly increasing. Therefore, we deduce that

$$E(t) \leq \alpha_4 (t-t_0) K_1^{-1}\left(\frac{\alpha_3}{(t-t_0)\int_{t_1}^{t}\mu(s)ds}\right), \ \forall t \geq t_1,$$

where α_3 and α_4 are positive constants. This completes the proof. □

Examples. We provide examples to explain the decay of energy (see [1]).
1. Case: f_0 and G are linear.

Let $\psi(t) = ae^{-b(1+t)}, \mu(t) = b$, and $G(t) = t$, where $b > 0$, and $a > 0$ is small enough. Assume that $f_0(t) = ct$ and $F(t) = \sqrt{t} f_0(\sqrt{t}) = ct$. Then, we can obtain

$$E(t) \leq k_2 e^{-k_1 t}, \text{ for all } t \geq t_0.$$

2. Case: f_0 is linear and G is nonlinear.

Let $\psi(t) = ae^{-t^p}, \mu(t) = 1$, and $G(t) = \frac{p^t}{(\ln(\frac{a}{t}))^{\frac{1}{p}-1}}$, where $0 < p < 1$, and $a > 0$ is small enough. Assume that $f_0(t) = ct$ and $F(t) = \sqrt{t} f_0(\sqrt{t}) = ct$. Then, G satisfies the condition (H1) on $(0, r_0]$ for any $0 < r_0 < a$.

$$G_1(t) = \int_t^{r_0} \frac{1}{sG'(s)} ds = \int_t^{r_0} \frac{[\ln \frac{a}{s}]^{\frac{1}{p}}}{s[1-p+p\ln\frac{a}{s}]} ds = \int_{\ln\frac{a}{r_0}}^{\ln\frac{a}{t}} \frac{u^{\frac{1}{p}}}{1-p+pu} du \leq \left(\ln\frac{a}{t}\right)^{\frac{1}{p}}.$$

Then, we can have

$$E(t) \leq k_4 e^{-k_3 t^p}, \text{ for all } t \geq t_0.$$

3. Case: f_0 is nonlinear and G is linear.

Let $\psi(t) = ae^{-b(1+t)}, \mu(t) = b$, and $G(t) = t$, where $b > 0$, and $a > 0$ is small enough. Assume that $f_0(t) = ct^p$, where $p > 1$ and $F(t) = \sqrt{t} f_0(\sqrt{t}) = ct^{\frac{p+1}{2}}$. Then,

$$F_1(t) = \int_t^{r_1} \frac{1}{sF'(s)} ds = \int_t^{r_1} \frac{2}{c(p+1)} s^{-\frac{p+1}{2}} ds = -\alpha_0\left(r_1^{-\frac{p-1}{2}} - t^{-\frac{p-1}{2}}\right)$$

and

$$F_1^{-1}(t) = \left(r_1^{-\frac{p-1}{2}} + \frac{1}{\alpha_0} t\right)^{-\frac{2}{p-1}},$$

where $\alpha_0 = \frac{4}{c(p+1)(p-1)}$. Therefore, we find that

$$E(t) \leq (\alpha_1 t + \alpha_2)^{-\frac{2}{p-1}}, \text{ for all } t \geq t_0.$$

4. Case: f_0 is nonlinear and G is nonlinear.

Let $\psi(t) = \frac{a}{(1+t)^2}, \mu(t) = b$, and $G(t) = t^{\frac{3}{2}}$, where $b > 0$, and $a > 0$ is taken so that (9) remains valid. Assume that $f_0(t) = t^5$ and $F(t) = t^3$. Then,

$$K(s) = (G^{-1} + F^{-1})^{-1}(s) = \left(\frac{-1+\sqrt{1+4s}}{2}\right)^3.$$

Therefore, we see that

$$E(t) \leq \frac{\alpha_3}{(t-t_0)^{\frac{1}{3}}}, \text{ for all } t \geq t_1,$$

where $t_1 > t_0$.

5. Conclusions

Numerous phenomena are influenced by both the current state and the previous occurrences of the system. There has been a notable increase in the research on the equation with delay effects, which frequently arise in various physical, biological, chemical, medical, and economic problems. In this paper, we study the energy decay rates for the viscoelastic wave equation with nonlinear time-varying delay, nonlinear damping at the boundary, and acoustic boundary conditions. We consider the relaxation function ψ, namely $\psi'(t) \leq -\mu(t)G(\psi(t))$, where G is an increasing and convex function near the origin, and μ is a positive nonincreasing function. We establish general decay rate results without the need for the condition $a_2 > 0$ and without imposing any limiting growth assumption on the damping term f_1, using the multiplier method and some properties of the convex functions. Moreover, the energy decay rates depend on the functions μ and G, as well as the function F defined by f_0, which characterizes the growth behavior of f_1 at the origin.

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