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Enhancing Equation Solving: Extending the Applicability of Steffensen-Type Methods

Ramandeep Behl [1,*], Ioannis K. Argyros [2] and Monairah Alansari [1]

[1] Mathematical Modelling and Applied Computation Research Group (MMAC), Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia; malansari@kau.edu.sa
[2] Department of Computing and Mathematical Sciences, Cameron University, Lawton, OK 73505, USA; iargyros@cameron.edu
* Correspondence: rlal@kau.edu.sa

Abstract: Local convergence analysis is mostly carried out using the Taylor series expansion approach, which requires the utilization of high-order derivatives, not iterative methods. There are other limitations to this approach, such as the following: the analysis is limited to finite-dimensional Euclidean spaces; no a priori computable error bounds on the distance or uniqueness of the solution results are provided. The local convergence analysis in this paper positively addresses these concerns in the more general setting of a Banach space. The convergence conditions involve only the operators in the methods. The more important semi-local convergence analysis not studied before is developed by using majorizing sequences. Both types of convergence analyses are based on the concept of generalized continuity. Although we study a certain class of methods, the same approach applies to extend the applicability of other schemes along the same lines.

Keywords: Steffensen's method; convergence; divided difference; Banach space

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1. Introduction

Iterative methods are a powerful tool in numerical analysis used to solve nonlinear equations. Nonlinear equations often arise in a variety of fields, including physics, engineering, economics, and finance, and are notoriously difficult to solve analytically. Such problems can be transformed into a form like

$$G(x) = 0, \tag{1}$$

where $G : \mathbb{D} \subseteq \mathbb{E} \to \mathbb{E}$ and $\mathbb{E}$ is a Banach space.

Iterative methods are numerical techniques that start with an initial guess and iteratively refine the solution until a desired level of accuracy is achieved. This makes them particularly useful for solving complex nonlinear equations that cannot be easily solved using traditional analytical methods. The usage of iterative methods for solving nonlinear equations has revolutionized many areas of science and engineering, and continues to be an important research topic in the field of numerical analysis.

The Newton–Raphson method, one of the most significant iterative techniques, is defined as

$$x_{\sigma+1} = x_\sigma - G'(x_\sigma)^{-1} G(x_\sigma),$$

Newton's method is a classic and widely used iterative algorithm for finding the solutions to nonlinear systems. However, Newton's method has some limitations. For example, it can fail to converge if the initial guess is too far from the required solution or, if the Fréchet derivative of operator G is zero or very small, then it is hard to obtain the inverse of G. To address these issues, various extensions of Newton's method have been developed over

time, such as Steffensen's method and a higher-order version of Steffensen's method. These extensions improve the convergence rate and robustness of Newton's method, making it more effective for a wider range of problems. In this context, researchers continue to explore new ways to extend Newton's method and other numerical methods, further expanding the range of applications where they can be used effectively.

One of these fourth-order convergent methods, presented by Singh, A. [1], is defined by

$$\begin{aligned} A_\sigma &= [u_\sigma, x_\sigma; G], \\ u_\sigma &= x_\sigma + G(x_\sigma), \\ y_\sigma &= x_\sigma - A_\sigma^{-1} G(x_\sigma), \\ x_{\sigma+1} &= T(x_\sigma), \end{aligned} \tag{2}$$

where $[\cdot, \cdot; G] : \mathbb{D} \times \mathbb{D} \to \ell(\mathbb{E})$ is the space of bounded linear operators from $\mathbb{E}$ into $\mathbb{E}$ and T is any iteration function of convergence order four.

Some of the special cases of scheme (2) are given below:

Special case 1:

$$\begin{aligned} B_\sigma &= [y_\sigma, x_\sigma; G] + [u_\sigma, y_\sigma; G] - [u_\sigma, x_\sigma; G] \\ T(x_\sigma) &= y_\sigma - B_\sigma^{-1} G(y_\sigma) \end{aligned}$$

Method (2) becomes

$$\begin{aligned} y_\sigma &= x_\sigma - A_\sigma^{-1} G(y_\sigma) \\ x_{\sigma+1} &= y_\sigma - B_\sigma^{-1} G(y_\sigma) \end{aligned} \tag{3}$$

Method (3), also studied by Wang et al. [2], was the multidimensional extension to the scalar method proposed by Ren et al. [3].

Special case 2:

$$T(x_\sigma) = y_\sigma - \left(3I - A_\sigma^{-1} \Big([y_\sigma, x_\sigma; G] + [u_\sigma, y_\sigma; G] \Big) \right) A_\sigma^{-1} G(y_\sigma)$$

Then, method (2) is reduced to

$$y_\sigma = x_\sigma - A_\sigma^{-1} G(x_\sigma) x_{\sigma+1} \quad = y_\sigma - \left(3I - A_\sigma^{-1} \Big([y_\sigma, x_\sigma; G] + [u_\sigma, y_\sigma; G] \Big) \right) A_\sigma^{-1} G(y_\sigma) \tag{4}$$

Scheme (4) is presented by Sharma and Arora in [4].

Special case 3:

$$T(x_\sigma) = y_\sigma - [y_\sigma, x_\sigma; G]^{-1} [u_\sigma, x_\sigma; G] [u_\sigma, y_\sigma; G]^{-1} G(y_\sigma)$$

It follows by this choice that method (2) specializes to

$$\begin{aligned} y_\sigma &= x_\sigma - A_\sigma^{-1} G(x_\sigma) \\ x_{\sigma+1} &= y_\sigma - [y_\sigma, x_\sigma; G]^{-1} [u_\sigma, x_\sigma; G] [u_\sigma, y_\sigma; G]^{-1} G(y_\sigma) \end{aligned} \tag{5}$$

Scheme (5) is proposed by Cordero et al. [5]. Some other important cases are mentioned by Singh, A. [1].

There are certain limitations with earlier works using the Taylor series expansion approach. Below is a list.

- (L_1) The local convergence analysis carried out in the case when $\mathbb{E} = \mathbb{R}^k$, where k is a natural number.
- (L_2) There are no computable error bounds on the distances $\|x^* - x_\sigma\|$. Therefore, we do not know a priori how many iterations must be carried out to reach a certain pre-decided error tolerance.
- (L_3) There is no uniqueness of the solution results.

(L_4) The existence is assumed of derivatives that are not present in the method. As an example for method (2), consider $\mathbb{D} = [-0.5, 1.5]$ and function $G : \mathbb{D} \to \mathbb{R}$ defined by

$$G(t) = \begin{cases} t^4 \log t + 2t^5 - 2t^4, & t \neq 0, \\ 0, & t = 0. \end{cases}$$

Clearly, $t^* = 1 \in \mathbb{D}$ solves the equation $G(t) = 0$. But, the fourth derivative $G^{(4)}$ is unbounded on $\mathbb{D}$, since this function is not continuous at $t = 0 \in \mathbb{D}$. Therefore, the results in [1] cannot assure the convergence of method (2) to t^*. However, method (2) converges to t^* by using the starting point $x_0 = 1.2 \in \mathbb{D}$.

(L_5) The choice of the initial point is a "shot in the dark", since no computable radius of convergence is provided.

(L_6) The more important semi-local convergence is not provided in [1] for method (2) or special cases (3)–(5).

In this paper, we address these limitations.

(L_1)′ The convergence analysis is carried out in the setting of a Banach space.

(L_2)′ A priori computable upper error bounds on the distances are provided. Hence, we know in advance the number of iterations to be carried out in order to achieve a desired error tolerance.

(L_3)′ A neighborhood is specified that contains only one solution.

(L_4)′ The convergence is established using only the operators in method (2).

(L_5)′ The radius of convergence is determined. Hence, if we choose an initial point from the ball with this radius, the convergence is assured.

(L_6)′ The semi-local convergence is developed by utilizing majorizing sequences.

It is worth noting that the convergence conditions are based on the concept of generalized continuity (see e.g., conditions (H_4)–(H_6)). Our approach can be used to extend the applicability of other methods along the same lines.

2. Convergence Analysis I: Local

Let $\mathbb{B} = [0, +\infty)$.

Assume:

(H_1) There exist functions of $g : \mathbb{B} \to \mathbb{B}, \phi_0 : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ which are increasing as well as continuous (IC) such that the equation $\phi_0\Big(t, \Delta(t)\Big) - 1 = 0$ admits a smallest positive solution (sps), denoted by ρ. Let $\mathbb{B}_0 = [0, \rho)$.

(H_2) There exists an IC function $\phi : \mathbb{B}_0 \to \mathbb{B}$ such that for $h_1 : \mathbb{B}_0 \to \mathbb{B}$ the equation $h_1(t) - 1 = 0$ has a sps $s_1 \in \mathbb{B}_0 - \{0\}$, where $h_1(t) = \dfrac{\phi\Big(t, \Delta(t)\Big)}{1 - \phi_0\Big(t, \Delta(t)\Big)}$.

(H_3) There exists an IC function $h_2 : \mathbb{B}_1 \leq \mathbb{B}_0 \to \mathbb{B}$ such that the equation $h_2(t) - 1 = 0$ admits a sps $s_2 \in \mathbb{B}_1 - \{0\}$. The set $\mathbb{B}_1$ and the function h_2 are developed later. Define the parameters by

$$s = \min\{s_1, s_2\}. \tag{6}$$

The functions ϕ_0 and ϕ are associated with the data in method (2) as follows:

(H_4) There exists an operator P such that $P^{-1} \in \ell(\mathbb{B})$ and for each $x \in \mathbb{E}, u = x + G(x)$

$$\Big\| P^{-1}([u, x; G] - P) \Big\| \leq \phi_0(\|x - x^*\|, \|u - x^*\|)$$

and

$$\|u - x^*\| \leq \Delta(\|x - x^*\|).$$

Let $\mathbb{D}_0 = \mathbb{D} \cap U(x^*, \rho_0)$.

(H_5) $\Big\| P^{-1}\Big([u, x; G] - [x, x^*; G]\Big) \Big\| \leq \phi(\|x - x^*\|, \|u - x^*\|)$

(H_6) $\|T(x) - x^*\| \leq h_2(\|x - x^*\|)\|x - x^*\|$ for each $x \in \mathbb{D}_0$
(H_7) $U[x^*, s^*] \subset \mathbb{D}$, where $s^* = \max\{s, \Delta(s^*)\}$.
(H_8) There exists a function IC $v : \mathbb{B} \to \mathbb{B}$ such that for each $x \in \mathbb{D}$

$$\left\| P^{-1}\Big([x, x^*; G] - P\Big) \right\| \leq v(\|x - x^*\|)$$

(H_9) There exists $s_3 \geq s$ such that $v(s_3) < 1$.
Let $\mathbb{D}_1 = \mathbb{D} \cap U[x^*, s_3]$. Let $\mathbb{B}_2 = [0, s)$. It follows by the definitions that for each $t \in \mathbb{B}_2$

$$0 \leq \phi_0\Big(t, \Delta(t)\Big) < 1, \tag{7}$$

$$0 \leq h_1(t) < 1 \tag{8}$$

and

$$0 \leq h_2(t) < 1. \tag{9}$$

Next, the local convergence analysis of method (2) uses the conditions (H_1)–(H_9) in combination with the preceding notation.

Theorem 1. *Assume the conditions (H_1)–(H_9) are satisfied. Then, the following assertions hold provided $x_0 \in U(x_0, s) - \{x^*\}$*

$$\{x_\sigma\} \subset U(x^*, s), \tag{10}$$

$$\|y_\sigma - x^*\| \leq h_1(\|x_\sigma - x^*\|)\|x_\sigma - x^*\| \leq \|x_\sigma - x^*\| < s, \tag{11}$$

$$\|x_{\sigma+1} - x^*\| \leq h_2(\|x_\sigma - x^*\|)\|x_\sigma - x^*\| \leq \|x_\sigma - x^*\| \tag{12}$$

and

$$\lim_{\sigma \to \infty} x_\sigma = x^*. \tag{13}$$

Moreover, the point x^* is the only solution of the equation $G(x) = 0$ in the set $\mathbb{D}_1$.

Proof. The assertions (10)–(12) are validated using induction. By the conditions (H_1), (H_3), (6), and (7), and the choice of the starting point x_0 we have

$$\left\| P^{-1}(A_0 - P) \right\| \leq \phi_0(\|x_0 - x^*\|, \|u_0 - x^*\|) \leq \phi_0\Big(s, \Delta(s)\Big) < 1, \tag{14}$$

which implies $A_0^{-1} \in \ell(\mathbb{E})$ by the standard perturbation of linear operators [6–8] due to Banach and

$$\left\| A_0^{-1} P \right\| \leq \frac{1}{1 - \phi_0(\|x_0 - x^*\|, \Delta(\|x_0 - x^*\|)}. \tag{15}$$

Thus, the iterate y_0 is well defined by the first substep of method (2) if $\sigma = 0$. Moreover, we can write in turn that

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - A_0^{-1} G(x_0) \\ &= A_0^{-1}\Big(A_0 - [x_0, x^*; G]\Big)(x_0 - x^*), \end{aligned} \tag{16}$$

employing (6), (8), (H_4), (15), and (16), we obtain

$$\begin{aligned} \|y_0 - x^*\| &\leq \frac{\phi(\|x_0 - x^*\|, \|u_0 - x^*\|)\|x_0 - x^*\|}{1 - \phi_0(\|x_0 - x^*\|, \Delta(\|x_0 - x^*\|))} \\ &\leq h_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < s. \end{aligned} \tag{17}$$

Hence, the iterate $y_0 \in U(x^*, s)$ and the assertion (11) holds if $\sigma = 0$. Notice that the iterate x_1 is well defined by the second subset of method (2). Then, the application of (6), (9), and (H_6) implies

$$\|x_1 - x^*\| = \|T(x_0) - x^*\| \leq h_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|. \tag{18}$$

So, the iterate $x_1 \in U(x^*, s)$ validating (10) and also (12) if $\sigma = 0$. The induction for the assertions (10)–(13) is terminated if x_m, y_m, x_{m+1} replace, respectively, x_0, y_0, x_1 in the preceding calculations. Furthermore, the estimate

$$\|x_{m+1} - x^*\| \leq d\|x_m - x^*\| \leq \|x_m - x^*\|, \tag{19}$$

for $d = h_2(\|x_0 - x^*\|) \in [0, 1)$ leads to $\lim_{m\to\infty} x_m = x^*$ and the iterate $x_{m+1} \in U(x^*, s)$. Therefore, the assertions (10) (for $\sigma = m + 1$) and (13) are satisfied. It is left to show the uniqueness of the solution in the set $\mathbb{D}_1$. Let $y^* \in \mathbb{D}_1$ be a solution of the equation $G(x) = 0$. Then, the conditions (H_8) and (H_9) give in turn that

$$\begin{aligned} \left\| P^{-1}\Big([y^*, x^*; G] - P\Big) \right\| &\leq v(\|y^* - x^*\|), \\ &\leq v(s_3) < 1. \end{aligned}$$

Thus, the linear operator $[y^*, x^*; G]^{-1} \in \ell(\mathbb{E})$ and

$$y^* - x^* = [y^*, x^*; G]^{-1}\Big(Gy^* - Gx^*\Big) = 0, \tag{20}$$

since $G(y^*) = G(x^*) = 0$. Hence, we conclude by identity (20) that $y^* = x^*$. □

Remark 1. *The second condition in (H_3) is left uncluttered. A possible choice for the function G is motivated by the calculations:*

$$\begin{aligned} u - x^* &= x - x^* + G(x) = I + [x, x^*; G])(x - x^*) \\ &= \Big[I + PP^{-1}\Big([x, x^*; G] - P + P\Big)\Big](x - x^*) \\ &= \Big[I + P + PP^{-1}\Big([x, x^*; G] - P\Big)\Big](x - x^*) \end{aligned}$$

thus,

$$\|u - x^*\| \leq [\|I + P\| + \|P\| + v(\|x - x^*\|)]\|x - x^*\|$$

Hence, we can set

$$\Delta(t) = \Big[\|I + P\| + \|P\|v(t)\Big]t.$$

Moreover, if $\Delta(s) \leq s$ is not satisfied, then the condition (H_7) can be replaced by

$(H_7)'$ $U[x^*, s^*] \subset \mathbb{D}$, *where* $s^* = \max\{s, \Delta(s)\}$.

The function h_2 can be determined further provided that the operator T is specialized (see Section 4).

3. Convergence II: Semi-Local

Semi-local majorizing sequences [6,8] are employed for this type of convergence. Let $a_0 = 0$ and $b_0 \geq 0$. Define the sequence by

$$\begin{aligned} a_{\sigma+1} &= b_\sigma + \xi_\sigma, \\ \gamma_{\sigma+1} &= \psi\Big(a_\sigma, a_{\sigma+1}, \Delta(a_\sigma)\Big)(a_{\sigma+1} - a_\sigma) + \Big[1 + \psi_0\Big(\sigma, \Delta(a_\sigma)\Big)\Big](a_{\sigma+1} - b_\sigma) \\ &\text{and} \\ b_{\sigma+1} &= a_{\sigma+1} + \frac{\gamma_{\sigma+1}}{1 - \psi_0\Big(a_{\sigma+1}, \Delta(a_{\sigma+1})\Big)}. \end{aligned} \tag{21}$$

where ξ_σ is a sequence of non-negative parameters to be determined later and $g : \mathbb{B} \to \mathbb{B}, \psi_0 : \mathbb{B} \times \mathbb{B} \to \mathbb{B}, \psi : \mathbb{B} \times \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ are given IC functions. A general convergence result is needed for the sequence $\{a_\sigma\}$.

Lemma 1. *Assume:*

(C_1) $\psi_0(a_\sigma, \Delta(a_\sigma)) < 1$ *and* $a_\sigma \leq a$ *for some* $\bar{a} \geq b_0$.

Then, the following assertions hold

$$0 \leq a_\sigma \leq b_\sigma \leq a_{\sigma+1} \leq \bar{a}, \tag{22}$$

and there exists $a \in [0, a_0]$ *such that*

$$\lim_{\sigma \to \infty} a_\sigma = a. \tag{23}$$

Proof. Formula (21) and condition (C_1) imply assertion (22) by which (23) is satisfied. Limit point a is the least bound (upper) of sequence $\{a_\sigma\}$ which is unique. Sequence $\{a_\sigma\}$ is shown to be majorizing for $\{x_\sigma\}$ in Theorem 2. But, first, let us associate with the operators in method (2) as follows:

(C_2) There exists a linear operator P such that $P^{-1} \in \ell(\mathbb{E})$ and, for each $x \in \mathbb{D}$,

$$\begin{aligned} \Big\|P^{-1}\Big([u, x; G] - P\Big)\Big\| &\leq \psi_0(\|x - x_0\|, \|u - x_0\|), \\ \|u - x_0\| &\leq \Delta(\|x - x_0\|. \end{aligned}$$

It follows by the first condition that there exists $x_0 \in \mathbb{D}$ such that

$$\Big\|P^{-1}\Big([u_0, x_0; G] - P\Big)\Big\| \leq \psi_0\Big(\|x_0 - x_0\|, \Delta(\|x_0 - x_0\|)\Big) < 1,$$

thus, $A_0^{-1} \in \ell(\mathbb{E})$. Set $\Big\|A_0^{-1} G(x_0)\Big\| \leq b_0$, $\mathbb{B}_3 = [0, \bar{a})$ and $\mathbb{D}_3 = \mathbb{D} \cap U(x_0, \bar{a})$.

(C_3) $\Big\|P^{-1}\Big([y, x; G] - [u, x; G]\Big)\Big\| \leq \psi(\|x - x_0\|, \|y - x_0\|, \|u - x_0\|)$ for each $x, y, u \in \mathbb{D}_3$.

(C_4) There exists IC function $\psi_1 : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ such that

$$\Big\|P^{-1}\Big([x, y, G] - P\Big)\Big\| \leq \psi_1(\|x - x_0\|, \|y - x_0\|).$$

(C_5) $\|T(x_\sigma) - y_\sigma\| \leq a_{\sigma+1} - b$

$$y_\sigma = x_\sigma - [x_\sigma + G(x_\sigma), x_\sigma; G]^{-1} G(x_\sigma).$$

(C_6) The equation $\psi_1(t, t) - 1 = 0$ has a smallest positive solution s_1 and there exists $s_2 \geq s_1$ such that $\psi_1(s_1, s_2) < 1$.
Let $\mathbb{D}_4 = \mathbb{D} \cap U[x_0, s_2]$.

(C_7) $U[x_0, a_1^*] \subset \mathbb{D}$, where $a_1^* = \max\{a^*, \Delta(a^*)\}$.

Notice that $\bar{a}_0$ can be the smallest positive solution of the equation

$$\psi_0\Big(t, \Delta(t)\Big) - 1 = 0,$$

(if it exists). □

The semi-local convergence analysis relies on conditions (C_1)–(C_7) under the developed terminology.

Theorem 2. *Assume that conditions* (C_1)–(C_7) *are satisfied. Then, the following assertions hold*

$$\{x_\sigma\} \subset U(x_0, a^*), \tag{24}$$

$$\|y_\sigma - x_\sigma\| \leq b_\sigma - a_\sigma, \tag{25}$$

$$\|x_{\sigma+1} - y_\sigma\| \leq a_{\sigma+1} - b_\sigma \tag{26}$$

and there exists a solution x^* *of equation* $G(x) = 0$ *such that*

$$x^* \in U[x_0, a^*], \lim_{\sigma \to \infty} x_\sigma = x^*, F(x^*) = 0 \tag{27}$$

and

$$\|x^* - x_\sigma\| \leq a^* - a_\sigma. \tag{28}$$

Proof. As in the local case, induction is used to show assertions (24)–(26). Clearly, assertion (24) holds for $\sigma = 0$. The conditions in (C_2) and Formula (21) imply the existence of iterate y_0,

$$\|y_0 - x_0\| = \Big\| - A_0^{-1} G(x_0)\Big\| \leq b_0 - a_0 = b_0 < a^*.$$

Thus, iterate $y_0 \in U(x_0, a^*)$ and assertion (25) holds if $\sigma = 0$.

Condition (C_6) gives

$$\|x_{\sigma+1} - y_\sigma\| = \|T(x_\sigma) - y_\sigma\| \leq a_{\sigma+1} - b_\sigma \tag{29}$$

and

$$\|x_{\sigma+1} - x_0\| \leq \|x_{\sigma+1} - y_\sigma\| + \|y_\sigma - x_0\| \leq a_{\sigma+1} - b_\sigma + b_\sigma - a_0 = a_{\sigma+1} < a^*.$$

Hence, iterate $x_{\sigma+1} \in U(x_0, a^*)$ and assertion (26) holds for $\sigma = 0$. Then, by the first substep of method (2), we can write:

$$\begin{aligned} G(x_{\sigma+1}) &= G(x_{\sigma+1}) - G(x_\sigma) - A_\sigma(y_\sigma - x_\sigma) \\ &= G(x_{\sigma+1}) - G(x_\sigma) - A_\sigma(x_{\sigma+1} - x_\sigma) + A_\sigma(x_{\sigma+1} - x_\sigma) - A_\sigma(y_\sigma - x_\sigma) \\ &= \Big([x_{\sigma+1}, x_\sigma; G] - A_\sigma\Big)(x_{\sigma+1} - x_\sigma) + A_\sigma(x_{\sigma+1} - y_\sigma), \end{aligned}$$

leading by (C_2), (C_3), and the induction hypothesis to

$$\begin{aligned} \Big\|P^{-1}G(x_{\sigma+1})\Big\| &\leq \psi\Big(\|x_\sigma - x_0\|, \|x_{\sigma+1} - x_0\|, \|u_\sigma - x_0\|\Big)\|x_{\sigma+1} - x_\sigma\| \\ &\quad + (1 + \psi_0(\|x_\sigma - x_0\|, \|u_\sigma - x_0\|))\|x_{\sigma+1} - y_\sigma\| \\ &= \bar{\gamma}_{\sigma+1} \leq \psi(a_\sigma, a_{\sigma+1}, \Delta(a_\sigma)(a_{\sigma+1} - a_\sigma) \\ &\quad + (1 + \psi_0(a_\sigma, \Delta(a_\sigma))(a_{\sigma+1} - b_\sigma) \\ &= \gamma_{\sigma+1}. \end{aligned} \tag{30}$$

Thus, we obtain by the first substep of method (2) for $(\sigma+1)$ replacing σ and condition (C_2), (21), and (30)

$$\begin{aligned}\|y_{\sigma+1}-x_{\sigma+1}\| &\leq \left\|A_{\sigma+1}^{-1}P\right\|\left\|P^{-1}G(x_{\sigma+1})\right\| \\ &\leq \frac{\bar{\gamma_{\sigma+1}}}{1-\psi_0\Big(\|x_{\sigma+1}-x_0\|,\|u_{\sigma+1}-x_0\|\Big)} \\ &\leq \frac{\bar{\gamma_{\sigma+1}}}{1-\psi_0\Big(a_{\sigma+1},\Delta(a_{\sigma+1})\Big)} \\ &= b_{\sigma+1}-a_{\sigma+1}\end{aligned} \tag{31}$$

and

$$\begin{aligned}\|y_{\sigma+1}-x_0\| &\leq \|y_{\sigma+1}-x_{\sigma+1}\|+\|x_{\sigma+1}-x_0\| \\ &\leq b_{\sigma+1}-a_{\sigma+1}+a_{\sigma+1}-a_0 \\ &= b_{\sigma+1}<a^*.\end{aligned}$$

The induction is completed for relations (24)–(26). By condition (C_1) sequence $\{a_\sigma\}$ is convergent to a^*. Hence, it follows that sequence $\{a_\sigma\}$ is also fundamental. Moreover, by (29) and (31), sequence $\{a_\sigma\}$ is majorizing for $\{x_\sigma\}$. Therefore, sequence $\{x_\sigma\}$ is also fundamental in Banach space E and as such it is convergent to some $x^*\in U[x_0,a^*]$ (since $U[x_0,a^*]$ is a closed set). Furthermore, by letting $\sigma\to\infty$ in (30), $G(x^*)=0$, where we also used the continuity of the operator G. Finally, from the estimate

$$\|x_{\sigma+j}-x_\sigma\|\leq a_{\sigma+j}-a_\sigma \tag{32}$$

and by letting $j\to+\infty$, assertion (28) follows. Hence, assertions (27) and (28) are also satisfied. □

The isolation of a solution of the equation $G(x)=0$ is discussed in the following item.

Proposition 1. *Assume that there exists $z_1\in U(x_0,s_3)$ with $G(z_1)=0$ and some $s_3>0$; condition (C_2) holds in ball $U(x_0,\rho_3)$, and there exists $s_4\geq s_3$ such that*

$$\psi_0(s_3,s_4)<1. \tag{33}$$

Set $\mathbb{D}_4=\mathbb{D}\cap U[x_0,s_4]$. Then, z_1 is unique as a solution of the equation $G(x)=0$ in the set $\mathbb{D}_4$.

Proof. Let $z_2\in\mathbb{D}_4$ be such that $G(z_2)=0$ with $z_1\neq z_2$. It follows that the divided difference $Q=[z_1+G(z_1),z_2;G]$ is well defined. It follows by condition (C_2) and (33) that

$$\left\|P^{-1}(Q-P)\right\|\leq\psi_0\Big(\|z_1-x_0\|,\|z_2-x_0\|\Big)\leq\psi_0(s_3,s_4)<1.$$

Thus, $Q^{-1}\in\ell(\mathbb{E})$. Then, from the identity

$$z_1-z_2=Q^{-1}\Big(G(z_1)-G(z_2)\Big)=Q^{-1}(0)=0,$$

we conclude that $z_2=z_1$. □

Remark 2. *(i) The limit point a^* can be replaced by $\bar{a}$ in condition (C_7).*

(ii) It is clear that, under all conditions (C_1)–(C_7), one can choose $z_1=x^*$ and $s_3=a^*$ in Proposition 1.

(iii) Notice also that, as in the local case,

$$\Delta(t)=\Big[\|I+P\|+\|P\|\psi_2(t)\Big]t+\|G(x_0)\|,$$

provided that

$$\left\|P^{-1}\Big([x,x_0;G]-P\Big)\right\|\leq \psi_2(\|x-x_0\|)$$

for some IC function ψ_2.

4. Special Cases and Applications

The functions (h_1) and (h_2) can be specialized:

Local Case 1: Assume:

$(H_6)'$ There exists an IC function $\phi_1:\mathbb{B}_0\times\mathbb{B}_0\times\mathbb{B}_0\to\mathbb{B}$ such that, for each $x,y\in\mathbb{D}_0$,

$$\left\|P^{-1}\Big([y,x;G]-[u,x;G]\Big)\right\|\leq \phi_1\Big(\|x-x^*\|,\|y-x^*\|,\|u-x^*\|\Big).$$

We need the estimates:

$$\begin{aligned}\left\|P^{-1}\Big(\mathbb{B}_\sigma-L\Big)\right\| &= \left\|P^{-1}\left(\Big([y_\sigma,x_\sigma;G]-[u_\sigma,x_\sigma;G]\Big)+\Big([u_\sigma,y_\sigma;G]-L\Big)\right)\right\|\\ &\leq \phi_1(\|x_\sigma-x^*\|,\|y_\sigma-x^*\|,\|u_\sigma-x^*\|)+\phi_0(\|y_\sigma-x^*\|,\|u_\sigma-x^*\|)=\bar{q}_\sigma\end{aligned}$$

and from

$$\mathbb{B}_\sigma-[y_\sigma,x^*;G]=([y_\sigma,x_\sigma;G]-[u_\sigma,x_\sigma;G])+([u_\sigma,y_\sigma;G]-[y_\sigma,x^*;G])$$

$$\begin{aligned}\left\|P^{-1}\Big(\mathbb{B}_\sigma-[y_\sigma,x^*;G]\Big)\right\| &\leq \phi_1(\|x_\sigma-x^*\|,\|y_\sigma-x^*\|,\|u_\sigma-x^*\|)\\ &\quad+\phi_0(\|y_\sigma-x^*\|,\|u_\sigma-x^*\|)\\ &=\bar{p}_\sigma.\end{aligned}$$

Define the functions $q:\mathbb{B}_0\to\mathbb{B}$, $p:\mathbb{B}_0\to\mathbb{B}$ by

$$\begin{aligned}q(t)&=\phi_1\Big(t,h_1(t)t,\Delta(t)\Big)+\phi_0\Big(h_1(t)t,\Delta(t)\Big),\\ p(t)&=\phi_1\Big(t,h_1(t)t,\Delta(t)\Big)+\phi_0\Big(h_1(t)t,\Delta(t)\Big).\end{aligned}$$

Assume that the equation $q(t)-1=0$ admits a smallest positive solution $s_0\in\mathbb{B}_0$. Then, $\mathbb{B}_\sigma^{-1}\in\ell(\mathbb{E})$ and condition (H_6) can be replaced by $(H_6)'$ provided that

$$h_2(t)=\frac{p(t)}{1-q(t)}h_1(t).$$

The motivation for the definition of the function h_2 follows from the estimates

$$x_{\sigma+1}-x^*=y_\sigma-x^*-B_\sigma^{-1}G(y_\sigma)$$

and

$$\begin{aligned}\|x_{\sigma+1}-x^*\| &\leq \left\|\mathbb{B}_\sigma^{-1}L\right\|\left\|P^{-1}\Big(\mathbb{B}_\sigma-[y_\sigma,x^*;G]\Big)\right\|\|y_\sigma-x^*\|\\ &\leq \frac{p_\sigma}{1-q_\sigma}h_1(\|x_\sigma-x^*\|)\|x_\sigma-x^*\|.\end{aligned}$$

Local Case 2

Assume:

$(H_6)'''$

$$\begin{aligned}\left\|P^{-1}\Big([u,x;G]-[y,x^*;G]\Big)\right\| &\leq \phi_2\Big(\|x-x^*\|,\|y-x^*\|,\|u-x^*\|\Big),\\ \left\|P^{-1}\Big([u,x;G]-[u,y;G]\Big)\right\| &\leq \phi_3\Big(\|x-x^*\|,\|y-x^*\|,\|u-x^*\|\Big)\end{aligned}$$

and $(H_6)'$ for each $x, y \in \mathbb{D}_0$ where $\phi_2 : \mathbb{B}_0 \times \mathbb{B}_0 \times \mathbb{B}_0 \to \mathbb{B}$ and $\phi_3 : \mathbb{B}_0 \times \mathbb{B}_0 \times \mathbb{B}_0 \to \mathbb{B}$ are IC functions. This time, by the second substep of method (2), in turn we can write

$$x_{\sigma+1} - x^* = y_\sigma - x^* - A_\sigma^{-1}G(y_\sigma) - A_\sigma^{-1}\Big[\Big(A_\sigma - [y_\sigma, x_\sigma; G]\Big) + \Big(A_\sigma - [u_\sigma, y_\sigma; G]\Big)\Big]A_\sigma^{-1}G(y_\sigma).$$

But, $G(y_\sigma) = G(y_\sigma) - G(x^*) = [y_\sigma, x^*; G](y_\sigma - x^*)$, so

$$\begin{aligned}
\Big\|P^{-1}G(y_\sigma)\Big\| &= \Big\|P^{-1}\Big([y_\sigma, x^*; G] - P + P\Big)(y_\sigma - x^*)\Big\| \\
&\leq \Big[1 + \Big\|P^{-1}\Big([y_\sigma, x^*; G] - L\Big)\Big\|\Big]\|y_\sigma - x^*\| \\
&\leq (1 + v(\|y_\sigma - x^*\|))\|y_\sigma - x^*\| \\
&\leq \Big[1 + v\Big(h_1(\|x_\sigma - x^*\|)\|x_\sigma - x^*\|\Big)\Big]h_1(\|x_\sigma - x^*\|)\|x_\sigma - x^*\| \\
&= \bar{\mu}_\sigma h_1(\|x_\sigma - x^*\|)\|x_\sigma - x^*\|.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\Big\|y_\sigma - x^* - A_n^{-1}G(y_\sigma)\Big\| &\leq \frac{\phi_2(\|x_\sigma - x^*\|, \|y_\sigma - x^*\|, \|u_\sigma - x^*\|)\|y_\sigma - x^*\|}{1 - \phi_0(\|x_\sigma - x^*\|, \|u_\sigma - x^*\|)} \\
&= (\bar{\alpha_\sigma})\|y_\sigma - x^*\|, \\
\Big\|P^{-1}\Big(A_\sigma - [y_\sigma, x_\sigma; G]\Big)\Big\| &\leq \phi_1(\|x_\sigma - x^*\|, \|y_\sigma - x^*\|, \|u_\sigma - x^*\|) = \bar{\beta}_\sigma
\end{aligned}$$

and

$$\Big\|P^{-1}\Big(A_\sigma - [u_\sigma, y_\sigma; G]\Big)\Big\| \leq \phi_3(\|x_\sigma - x^*\|, \|y_\sigma - x^*\|, \|u_\sigma - x^*\|) = \bar{\gamma}_\sigma.$$

By summing up, we obtain

$$\|x_{\sigma+1} - x^*\| \leq \left[\bar{\alpha_\sigma} + \frac{(\bar{\beta}_\sigma + \bar{\gamma}_\sigma)\bar{\mu}_\sigma}{\Big[1 - \phi_0\Big(\|x_\sigma - x^*\|, \|u_\sigma - x^*\|\Big)\Big]^2}\right]\|y_\sigma - x^*\|.$$

The preceding estimate justifies the definition of the function h_2 as follows

$$\begin{aligned}
\mu(t) &= 1 + v\Big(h_1(t)t\Big) \\
\alpha(t) &= \frac{\phi_2\Big(t, h_1(t)t, \Delta(t)\Big)}{1 - \phi_0\Big(t, \Delta(t)\Big)} \\
\beta(t) &= \phi_1\Big(t, h_1(t)t, \Delta(t)\Big) \\
\gamma(t) &= \phi_3\Big(t, h_1(t)t, \Delta(t)\Big)
\end{aligned}$$

and

$$h_2(t) = \left[\alpha(t) + \frac{\Big(\beta(t) + \gamma(t)\Big)\mu(t)}{\Big(1 - \phi_0\Big(t, \Delta(t)\Big)\Big)^2}\right]h_1(t).$$

Thus, the delicate condition (H_6) can be replaced by $(H_6)''$. Similarly, we specialize the sequences $\{a_\sigma\}$ and $\{b_\sigma\}$ in the semi-local convergence analysis of method (2).

Semi-local Case 1
We have

$$\|x_{\sigma+1}-y_\sigma\| = \left\| -B_\sigma^{-1}G(y_\sigma)\right\| \leq \frac{\psi\Big(a_\sigma,b_\sigma,\Delta(a_\sigma)\Big)(b_\sigma-a_\sigma)}{1-c_\sigma} = a_{\sigma+1}-b_\sigma,$$

with $c_\sigma = \psi(a_\sigma,b_\sigma,\Delta(a_\sigma)) + \psi_0(b_\sigma,\Delta(a_\sigma))$ where the following estimates are used by the definition of method (2)

$$\begin{aligned} G(y_\sigma) &= G(y_\sigma)-G(x_\sigma)-A_\sigma(y_\sigma-x_\sigma)\\ &= \Big([y_\sigma,x_\sigma;G]-[u_\sigma,x_\sigma;G]\Big)(y_\sigma-x_\sigma)\\ \left\|P^{-1}G(y_\sigma)\right\| &\leq \psi\Big(\|x_\sigma-x^*\|,\|y_\sigma-x_0\|,\|u_\sigma-x_0\|\Big)\|y_\sigma-x_\sigma\|\\ &\leq \psi\Big(a_\sigma,b_\sigma,\Delta(a_\sigma)\Big)(b_\sigma-a_\sigma)\end{aligned}$$

and

$$\left\|B_\sigma^{-1}P\right\| \leq \frac{1}{1-c_\sigma}.$$

Thus, condition (C_5) is dropped provided that it is replaced by $(C_5)'$

$$a_{\sigma+1} = b_\sigma + \frac{\psi\Big(a_\sigma,b_\sigma,\Delta(a_\sigma)\Big)(b_\sigma-a_\sigma)}{1-c_\sigma}$$

and $c_\sigma < 1$ for each $\sigma = 0,1,2,....$
Semi-local Case 2:
Assume:
$(C_5)''$ $\left\|P^{-1}\Big([u,x;G]-[u,y;G]\Big)\right\| \leq \psi_1(\|x-x_0\|,\|y-x_0\|,\|u-x_0\|)$ for each $x,y \in \mathbb{D}_0$ and some IC function $\psi_1 : \mathbb{B}_0\times\mathbb{B}_0\times\mathbb{B}_0 \to \mathbb{B}$.

We can write as before

$$x_{\sigma+1}-y_\sigma = -A_\sigma^{-1}G(y_\sigma) - A_\sigma^{-1}\left[\Big(A_\sigma-[y_\sigma,x_\sigma;G]\Big)+\Big(A_\sigma-[u_\sigma,y_\sigma;G]\Big)\right]A_\sigma^{-1}G(y_\sigma),$$

so

$$\begin{aligned}\|x_{\sigma+1}-y_{(\sigma)}\| &\leq \left[1+\frac{\psi(a_\sigma,b_\sigma,\Delta(a_\sigma))+\psi_1(a_\sigma,b_\sigma,\Delta(a_\sigma))}{1-\psi_0(a_\sigma,\Delta(a_\sigma))}\right]\frac{\psi(a_\sigma,b_\sigma,\Delta(a_\sigma))(b_\sigma-a_\sigma)}{1-\psi_0(a_\sigma,\Delta(a_\sigma))}\\ &= a_{\sigma+1}-b_\sigma.\end{aligned}$$

Thus, again, condition (C_5) can be replaced by $(C_5)''$ in Theorem 2.

Remark 3. *Concerning the choice of the linear operators we can suggest two interesting cases:*

(1) $P = G'(x^*)$, *if the operator G is differentiable in the local convergence case.*
(2) $P = [x_0,x_{-1};G], x_0,x_{-1} \in \mathbb{D}$, *if the operator is not necessarily differentiable or* $P = G'(x^*)$ *if the operator is differentiable in the semi-local convergence case.*

Other possible choices are given in [6,7,9].

5. Numerical Applications

To evaluate the effectiveness of our mathematical methods, we test them on a variety of problems, including systems of differential equations, first kind Hammerstein integrals [7,8,10], steering motion problems, and boundary value problems. In addition to this, we also chose a nonlinear, nondifferentiable function; these functions arise in a wide range

of applications and are particularly challenging to solve. By solving these problems using our methods, we can determine their accuracy, efficiency, and suitability for various applications.

First of all, we obtained the radii of convergence for iterative solver (2), so that we have an idea of how much closeness of the initial approximation is required for convergence to the exact solution. We used the iterative approach to perform the computation and found the computational order of convergence after choosing a suitable beginning approximation. This made it easier for us to see how quickly the iterative solver was reaching the precise solution. For computing computational order of convergence (COC), the following formulas are used:

$$\kappa = \frac{\ln \frac{\|x_{\rho+1}-x_*\|}{\|x_\rho-x_*\|}}{\ln \frac{\|x_\rho-x_*\|}{\|x_{\rho-1}-x_*\|}}, \quad \text{for } \rho = 1, 2, \ldots \tag{34}$$

or approximated computational order of convergence $(ACOC)$ [5,10] by:

$$\kappa^* = \frac{\ln \frac{\|x_{\rho+1}-x_\rho\|}{\|x_\rho-x_{\rho-1}\|}}{\ln \frac{\|x_\rho-x_{\rho-1}\|}{\|x_{\rho-1}-x_{\rho-2}\|}}, \quad \text{for } \rho = 2, 3, \ldots \tag{35}$$

To make sure the approach was effective, we additionally recorded the CPU timing for the computation. Finally, we determined how many iterations would be necessary to achieve the specified accuracy as well as the residual error.

Iterative solvers use stopping criteria to decide when to stop iterating and accept the current approximation as the solution. Depending on the type of problem being solved and the method being used, a variety of stopping criteria can be used. We opt for the following standard criteria:

(i) $\|x_{k+1} - x_k\| < \epsilon$, and
(ii) $\|G(x_k)\| < \epsilon$,

where $\epsilon = 10^{-100}$ is the error tolerance. The computations and multiple precision arithmetic are performed with the help of Mathematica-11.

Example 1. *Consider the following system of differential equations*

$$G_1'(w_1) = e^{w_1}, \quad G_2'(w_2) = (e-1)w_2 + 1, \quad G_3'(w_3) = 1$$

subject to $G_1(0) = G_2(0) = G_3(0) = 0$. *We consider* $G = (G_1, G_2, G_3)$ *and* $\Lambda = U[0, 1]$. *The* $\xi = (0, 0, 0)^T$ *is a solution. Let us assume that function G is defined on* Λ *with* $w = (w_1, w_2, w_3)^T$ *by*

$$G(w) = \begin{bmatrix} e^{w_1} - 1 \\ \frac{e-1}{2} w_2^2 + w_2 \\ w_3 \end{bmatrix}. \tag{36}$$

This definition gives

$$G'(w) = \begin{bmatrix} e^{w_1} & 0 & 0 \\ 0 & (e-1)w_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, by the definition of G, it follows that $G'(\xi) = 1$. *Let* $[x, y; G] = \int_0^1 G'\Big(x + \theta(y - x)\Big)d\theta$. *Then, hypotheses* (H_1)–(H_4) *are verified for*

$$\begin{aligned}
\phi_0(t_1, t_2) &= \frac{e}{2}(t_1 + t_2),\\
\phi(t_1, t_2) &= \frac{e-1}{2}t_2,\\
\phi_1(t_1, t_2, t_3) &= \frac{e-1}{2}(t_1 + t_2),\\
\Delta(t) &= \Big(2 + v(t)\Big)t,\\
v(t) &= \frac{e-1}{2}t,\\
\phi_2(t_1, t_2, t_3) &= \frac{e}{2}(t_1 + t_2) + \frac{e-1}{2}t_3,\\
&\textit{and}\\
\phi_3(t_1, t_2, t_3) &= \frac{e}{2}(t_1 + t_2).
\end{aligned}$$

The computational results are shown in Table 1.

Table 1. Numerical results of solver (2) for Example 1.

s_0		s_1		s_2		s
0.18365		0.19369		0.13771		0.13771
Cases	x_0	$\Big\|G(x_\sigma)\Big\|$	$\Big\|x_{\sigma+1} - x_\sigma\Big\|$	σ	κ	CPU Timing
Solver (3)	$(0.3, 0.3, 0.3)^{tr}$	9.1×10^{-132}	9.1×10^{-132}	4	4.0000	0.0237102
Solver (4)	$(0.3, 0.3, 0.3)^{tr}$	1.3×10^{-350}	1.3×10^{-350}	5	4.0000	0.0362553
Solver (5)	$(0.3, 0.3, 0.3)^{tr}$	6.6×10^{-114}	6.6×10^{-114}	4	4.0000	0.0251659

Example 2. *Let* $\mathbb{D} = U[0, 1]$ *and* $\mathbb{A}_1 = \mathbb{A}_2 = C[0, 1]$. *Consider the nonlinear integral equation of the first kind Hammerstein operator H defined by*

$$G(\Upsilon)(x) = \Upsilon(x) - 3\int_0^1 x\zeta\Upsilon(\zeta)^3 d\zeta.$$

The calculation for the derivative gives

$$G'\Big(\Upsilon(q)\Big)(x) = q(x) - 9\int_0^1 x\zeta\Upsilon(\zeta)^2 q(\zeta)d\zeta,$$

for $q \in C[0,1]$. By this value of the operator F', conditions (C_1)–(C_4) are verified; we choose

$$\begin{aligned}
\phi_0(t_1,t_2) &= \frac{9}{4}(t_1+t_2),\\
\phi(t_1,t_2) &= \frac{9}{4}(t_1+t_2),\\
\phi_1(t_1,t_2,t_3) &= \frac{9}{4}(t_1+t_2),\\
\phi_2(t_1,t_2,t_3) &= \frac{9}{4}(t_1+t_2+t_3),\\
\phi_3(t_1,t_2,t_3) &= \frac{9}{4}(t_1+t_2),\\
v(t) &= \frac{9}{4}t,\\
\Delta(t) &= \Big(2+v(t)\Big)t,
\end{aligned}$$

with $P = G'(x^*) = I$, where I is an identity matrix. By adopting the above functions, we obtain the radii for compositions (2) of Example 2 in Table 2.

Table 2. Radii of solver (2) of Example 2.

s_0	s_1	s_2	s
0.078031	0.070361	0.052654	0.052654

Example 3. *The system of nonlinear equations is a powerful tool for solving boundary value problems in many fields, such as physics, engineering, and finance. It allows for the modeling of complex systems with multiple variables and nonlinear relationships. By solving the system of equations, we can obtain solutions that satisfy the boundary conditions and accurately represent the behavior of the system. Therefore, we consider a boundary value problem (see [8]), which is given by*

$$u'' + u^3 = 0 \tag{37}$$

with $u(0) = u(1) = 0$. The interval $[0,1]$ is divided into 1006 sections to yield

$$\gamma_0 = 0 < \gamma_1 < \gamma_2 < \cdots < \gamma_{1006} = 1, \quad \gamma_{k+1} = \gamma_k + h, h = \frac{1}{1006}.$$

Then, we can choose $u_0 = u(\gamma_0) = 0, u_1 = u(\gamma_1), \ldots, u_{1006} = u(\gamma_{1006}) = 0$. We have

$$u''_\theta = \frac{u_{\theta-1} - 2u_\theta + u_{\theta+1}}{h^2}, \quad \theta = 1,2,3,\ldots,1005,$$

acquired by the use of the discretization approach. The following nonlinear system of equations is obtained as 1005×1005.

$$u_{\theta-1} + u_{\theta+1} - 2u_\theta + h^2 u_\theta^3 = 0. \tag{38}$$

We present iterations and the COC of Example 4 in Table 3. Expression (38) converges to the following resulting column vector (not a matrix):

$$x_* = \begin{pmatrix}
0.00105\ldots, 0.00210\ldots, 0.00315\ldots, 0.00420\ldots, 0.00525\ldots, 0.00630\ldots, 0.00735\ldots,\\
0.00840\ldots, 0.00945\ldots, 0.0105\ldots, 0.0115\ldots, 0.0126\ldots, 0.0136\ldots, 0.0147\ldots,\\
\vdots\\
0.991\ldots, 0.992\ldots, 0.992\ldots, 0.993\ldots, 0.994\ldots, 0.995\ldots, 0.996\ldots,\\
0.996\ldots, 0.997\ldots, 0.998\ldots, 0.999\ldots
\end{pmatrix}^{tr}.$$

Table 3. Numerical results for Example 3.

Cases	x_0	$\|G(x_\sigma)\|$	$\|x_{\sigma+1} - x_\sigma\|$	σ	κ	CPU Timing
Solver (3)	$(0.4, 0.4, \overset{1005}{\cdots}, 0.4)^{tr}$	2.0×10^{-290}	2.2×10^{-285}	4	4.0001	8395.78
Solver (4)	$(0.4, 0.4, \overset{1005}{\cdots}, 0.4)^{tr}$	1.8×10^{-275}	4.6×10^{-271}	4	4.0002	13475.8
Solver (5)	$(0.4, 0.4, \overset{1005}{\cdots}, 0.4)^{tr}$	5.5×10^{-286}	5.8×10^{-281}	4	4.0002	29148.4

Example 4. *We examine one of the most well-known applied science problems, the Hammerstein integral equation (see pp. 19–20 in [8] to compare the effectiveness and applicability of our suggested methods to those of others), which is given below:*

$$x(s) = 1 + \frac{1}{5}\int_0^1 G(s,t)x(t)^3 dt$$

where $x \in C[0,1]$ $s, t \in [0,1]$ *and kernel* G *is*

$$G(s,t) = \begin{cases} (1-s)t, t \leq s, \\ s(1-t), s \leq t. \end{cases}$$

To convert the aforementioned equation into a problem with finite dimensions, use the Gauss–Legendre quadrature formula, which is $\int_0^1 g(t)dt \simeq \sum_{j=1}^{10} w_j g(t_j)$, *where the abscissas* t_j *and the weights* w_j *are determined for* $j = 10$ *by the Gauss–Legendre quadrature formula. Denoting the approximations of* $x(t_i)$ *by* $x_i (i = 1, 2, ..., 10)$, *one obtains the system of nonlinear equations* $5x_i - 5 - \sum_{j=1}^{10} a_{ij} x_j^3 = 0$, *where* $i = 1, 2, ..., 10$

$$a_{ij} = \begin{cases} w_j t_j (1 - t_i), j \leq i, \\ w_j t_i (1 - t_j), i < j. \end{cases}$$

For $i = j = 10$, *the abscissas* t_j *and weights* w_j *are known and shown in Table 4.*

Table 4. The abscissas t_j and weights w_j by Gauss–Legendre quadrature formula.

j	t_j	w_j
1	0.013046735741414139961017 99...	0.033335672154344068796784 40...
2	0.067468316655507744633951 65...	0.074725674575290296572888 16...
3	0.160295215850487796882836 32...	0.109543181257991021997767 46...
4	0.283302302935376404600367 03...	0.134633359654998177545613 46...
5	0.425562830509184394557587 00...	0.147762112357376435086946 49...
6	0.574437169490815605442413 00...	0.147762112357376435086946 49...
7	0.716697697064623595399632 97...	0.134633359654998177545613 46...
8	0.839704784149512203117163 68...	0.109543181257991021997767 46...
9	0.932531683344492255366048 34...	0.074725674575290296572888 16...
10	0.986953264258585860038982 01...	0.033335672154344068796784 40...

The convergence approaches towards the root $x_* = (1.001\ldots, 1.006\ldots, 1.014\ldots, 1.021\ldots, 1.026\ldots, 1.026\ldots, 1.021\ldots, 1.014\ldots, 1.006\ldots, 1.001\ldots)^{tr}$. *This is tested in Table 5.*

Table 5. Numerical results for Example 4.

Cases	x_0	$\|G(x_\sigma)\|$	$\|x_{\sigma+1} - x_\sigma\|$	σ	κ	CPU Timing
Solver (3)	$(1.1, 1.1, \overset{10}{\cdots}, 1.1)^{tr}$	5.8×10^{-379}	1.2×10^{-379}	4	3.9999	0.893317
Solver (4)	$(1.1, 1.1, \overset{10}{\cdots}, 1.1)^{tr}$	1.9×10^{-342}	4.1×10^{-343}	4	4.0000	0.796277
Solver (5)	$(1.1, 1.1, \overset{10}{\cdots}, 1.1)^{tr}$	8.4×10^{-372}	1.8×10^{-372}	4	3.9999	0.760152

Example 5. *Here, we solve the nonlinear, nondifferentiable system given as*

$$\begin{aligned} 3t_1^2 t_2 + t_2^2 - 1 + |t_1 - 1| &= 0 \\ t_1^4 + t_1 t_2^3 - 1 + |t_2| &= 0. \end{aligned}$$

Then, we set $G = (G_1, G_2)$, *and* $G_1 = Q_1 + Q_2$, $G_2 = Q_3 + Q_4$ *where*

$$\begin{aligned} Q_1(t_1, t_2) &= 3t_1^2 t_2 + t_2^2 - 1, \\ Q_2(t_1, t_2) &= |t_1 - 1|, \\ Q_3(t_1, t_2) &= t_1^4 + t_1 t_2^3 - 1, \\ &\textit{and} \\ Q_3(t_1, t_2) &= |t_2|. \end{aligned} \tag{39}$$

Notice that Q_2 *and* Q_4 *constitute the nondifferentiable part of the equation. The convergence towards the root* $(0.8946554, 0.3278265)^{tr}$ *is tested in Table 6.*

Table 6. Numerical results for Example 5.

Cases	x_0	$\|G(x_\sigma)\|$	$\|x_{\sigma+1} - x_\sigma\|$	σ	κ	CPU Timing
Solver (3)	$(0.9, .03)^{tr}$	1.9×10^{-157}	7.5×10^{-158}	5	3.0000	0.0449242
Solver (4)	$(0.9, .03)^{tr}$	5.8×10^{-174}	2.2×10^{-174}	5	3.0000	0.0528398
Solver (5)	$(0.9, .03)^{tr}$	1.7×10^{-187}	6.5×10^{-188}	5	3.0000	0.0525841

Example 6 (Bratu 2D Problem)**.** *The widely recognized two-dimensional Bratu problem, as described in [11,12], is defined as follows:*

$$\begin{aligned} &u_{xx} + u_{tt} + Ce^u = 0, on \\ &with\ boundary\ conditions\ u = 0\ on\ \Omega,\ \ where\ \ \Omega = \{(x,t) : 0 \le x \le 1, 0 \le t \le 1\}. \end{aligned} \tag{40}$$

The approximate solution for a nonlinear partial differential equation can be determined by employing finite difference discretization. This approach simplifies the problem into solving a system of nonlinear equations. Let us denote the approximate solution at the grid points of the mesh as $u_{i,j}$, *where* $u_{i,j}$ *represents the solution at position* x_i *and time* t_j. *Additionally, we define M and N as the number of steps in the x and t directions, and h and k as the corresponding step sizes. To tackle the provided partial differential equation, we will apply the central difference method to* u_{xx} *and* u_{tt}, *i.e.,*

$$u_{xx}(x_i, t_j) = (u_{i+1,j} - 2u_{i,j} + u_{i-1,j})/h^2,\ C = 0.1, t \in [0,1].$$

We seek the solution to a system with dimension 225×225 *by choosing* $M = 16$ *and* $N = 16$. *It converges to the following resulting column vector (not a matrix):*

$$x_* = \begin{pmatrix} 0.000619\ldots, 0.00104\ldots, 0.00135\ldots, 0.00157\ldots, 0.00173\ldots, 0.00184\ldots, 0.00190\ldots, \\ 0.00192\ldots, 0.00190\ldots, 0.00184\ldots, 0.00173\ldots, 0.00157\ldots, 0.00135\ldots, 0.00104\ldots, \\ 0.000619\ldots, 0.00104\ldots, 0.00181\ldots, 0.00239\ldots, 0.00281\ldots, 0.00312\ldots, 0.00333\ldots, \\ 0.00345\ldots, 0.00349\ldots, 0.00345\ldots, 0.00333\ldots, 0.00312\ldots, 0.00281\ldots, 0.00239\ldots, \\ 0.00181\ldots, 0.00104\ldots, 0.00135\ldots, 0.00239\ldots, 0.00318\ldots, 0.00379\ldots, 0.00423\ldots, \\ 0.00453\ldots, 0.0047\ldots, 0.00476\ldots, 0.00470\ldots, 0.00453\ldots, 0.00423\ldots, 0.00379\ldots, \\ 0.00318\ldots, 0.00239\ldots, 0.00135\ldots, 0.00157\ldots, 0.00281\ldots, 0.00379\ldots, 0.00453\ldots, \\ 0.00508\ldots, 0.00545\ldots, 0.00567\ldots, 0.00574\ldots, 0.00567\ldots, 0.00545\ldots, 0.00508\ldots, \\ 0.00453\ldots, 0.00379\ldots, 0.00281\ldots, 0.00157\ldots, 0.00173\ldots, 0.00312\ldots, 0.00423\ldots, \\ 0.00508\ldots, 0.00571\ldots, 0.00614\ldots, 0.00639\ldots, 0.00648\ldots, 0.00639\ldots, 0.00614\ldots, \\ 0.00571\ldots, 0.00508\ldots, 0.00423\ldots, 0.00312\ldots, 0.00173\ldots, 0.00184\ldots, 0.00333\ldots, \\ 0.00453\ldots, 0.00545\ldots, 0.00614\ldots, 0.00661\ldots, 0.00689\ldots, 0.00698\ldots, 0.00689\ldots, \\ 0.00661\ldots, 0.00614\ldots, 0.00545\ldots, 0.00453\ldots, 0.00333\ldots, 0.00184\ldots, 0.00190\ldots, \\ 0.00345\ldots, 0.00470\ldots, 0.00567\ldots, 0.00639\ldots, 0.00689\ldots, 0.00719\ldots, 0.00728\ldots, \\ 0.00719\ldots, 0.00689\ldots, 0.00639\ldots, 0.00567\ldots, 0.00470\ldots, 0.00345\ldots, 0.00190\ldots, \\ 0.00192\ldots, 0.00349\ldots, 0.00476\ldots, 0.00574\ldots, 0.00648\ldots, 0.00698\ldots, 0.00728\ldots, \\ 0.00738\ldots, 0.00728\ldots, 0.00698\ldots, 0.00648\ldots, 0.00574\ldots, 0.00476\ldots, 0.00349\ldots, \\ 0.00192\ldots, 0.00190\ldots, 0.00345\ldots, 0.00470\ldots, 0.00567\ldots, 0.00639\ldots, 0.00689\ldots, \\ 0.00719\ldots, 0.00728\ldots, 0.00719\ldots, 0.00689\ldots, 0.00639\ldots, 0.00567\ldots, 0.00470\ldots, \\ 0.00345\ldots, 0.00190\ldots, 0.00184\ldots, 0.00333\ldots, 0.00453\ldots, 0.00545\ldots, 0.00614\ldots, \\ 0.00661\ldots, 0.00689\ldots, 0.00698\ldots, 0.00689\ldots, 0.00661\ldots, 0.00614\ldots, 0.00545\ldots, \\ 0.00453\ldots, 0.00333\ldots, 0.00184\ldots, 0.00173\ldots, 0.00312\ldots, 0.00423\ldots, 0.00508\ldots, \\ 0.00571\ldots, 0.00614\ldots, 0.00639\ldots, 0.00648\ldots, 0.00639\ldots, 0.00614\ldots, 0.00571\ldots, \\ 0.00508\ldots, 0.00423\ldots, 0.00312\ldots, 0.00173\ldots, 0.00157\ldots, 0.00281\ldots, 0.00379\ldots, \\ 0.00453\ldots, 0.00508\ldots, 0.00545\ldots, 0.00567\ldots, 0.00574\ldots, 0.00567\ldots, 0.00545\ldots, \\ 0.00508\ldots, 0.00453\ldots, 0.00379\ldots, 0.00281\ldots, 0.00157\ldots, 0.00135\ldots, 0.00239\ldots, \\ 0.00318\ldots, 0.00379\ldots, 0.00423\ldots, 0.00453\ldots, 0.00470\ldots, 0.00476\ldots, 0.00470\ldots, \\ 0.00453\ldots, 0.00423\ldots, 0.00379\ldots, 0.00318\ldots, 0.00239\ldots, 0.00135\ldots, 0.00104\ldots, \\ 0.00181\ldots, 0.00239\ldots, 0.00281\ldots, 0.00312\ldots, 0.00333\ldots, 0.00345\ldots, 0.00349\ldots, \\ 0.00345\ldots, 0.00333\ldots, 0.00312\ldots, 0.00281\ldots, 0.00239\ldots, 0.00181\ldots, 0.00104\ldots, \\ 0.000619\ldots, 0.00104\ldots, 0.00135\ldots, 0.00157\ldots, 0.00173\ldots, 0.00184\ldots, 0.00190\ldots, \\ 0.00192\ldots, 0.00190\ldots, 0.00184\ldots, 0.00173\ldots, 0.00157\ldots, 0.00135\ldots, 0.00104\ldots, \\ 0.000619\ldots \end{pmatrix}^{tr}.$$

The computational results are given in Table 7.

Table 7. Numerical results for Example 6.

Cases	$\left\|G(x_\sigma)\right\|$	$\left\|x_{\sigma+1} - x_\sigma\right\|$	σ	κ	CPU Timing
Solver (3)	2.2×10^{-194}	2.5×10^{-193}	3	3.9997	1004.13
Solver (4)	4.1×10^{-194}	4.5×10^{-193}	3	3.9997	1640.15
Solver (5)	2.6×10^{-194}	2.9×10^{-193}	3	3.9997	1741.74

The initial estimation is $x_0 = \left(0.1\sin(i\pi h)\sin(j\pi h), \cdots, 0.1\sin(i\pi h)\sin(j\pi h)\right)^{tr}$, $i, j = 1, 2, 3, \cdots, 15$.

6. Conclusions

A process is introduced that shows convergence for a family of Steffensen-like methods. The advantage of this process over earlier ones is that the condition about the existence of $F'(x^*)^{-1}$ is not required. Consequently, the methods can be applied to solve nondifferentiable equations with a convergence theory to back them up. Moreover, the process is applicable to other methods involving the inverses of linear operators such as [1–10,13–18]. It is worth noting that in earlier work the local convergence analysis is shown using Taylor series expansions and assumptions on derivatives of high order not present in the method. Furthermore, no computable error estimates or uniqueness of the solution results are provided. Finally, the more interesting semi-local convergence analysis not studied in [1,19–29] is also developed in this paper. All these concerns are positively addressed in this paper and in the more general setting of a Banach space. Therefore, the applicability of such methods is extended to the local as well as the semi-local convergence cases using only conditions on the operators appearing in the method. Finally, the numerical applications further complement the theoretical findings.

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