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Rolling Stiefel Manifolds Equipped with α-Metrics

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Abstract: We discuss the rolling, without slipping and without twisting, of Stiefel manifolds equipped with α-metrics, from an intrinsic and an extrinsic point of view. We, however, start with a more general perspective, namely, by investigating the intrinsic rolling of normal naturally reductive homogeneous spaces. This gives evidence as to why a seemingly straightforward generalization of the intrinsic rolling of symmetric spaces to normal naturally reductive homogeneous spaces is not possible, in general. For a given control curve, we derive a system of explicit time-variant ODEs whose solutions describe the desired rolling. These findings are applied to obtain the intrinsic rolling of Stiefel manifolds, which is then extended to an extrinsic one. Moreover, explicit solutions of the kinematic equations are obtained, provided that the development curve is the projection of a not necessarily horizontal one-parameter subgroup. In addition, our results are put into perspective with examples of the rolling Stiefel manifolds known from the literature.

Keywords: intrinsic rolling; extrinsic rolling; Stiefel manifolds; normal naturally reductive homogeneous spaces; covariant derivatives; parallel vector fields; kinematic equations

MSC: 53B21; 53C30; 53C25; 37J60; 58D19

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1. Introduction

In recent years, there has been increasing interest in the so-called rolling maps of differentiable manifolds. Researchers have taken different points of view to study the differential geometry behind these constructions. From our point of view, it seems to be natural to distinguish between two approaches, the intrinsic one and the extrinsic one. The first viewpoint does not require any embedding space to study rolling maps, whereas the second needs one. At first glance, the intrinsic approach seems to be of a more pure mathematical flavor, simply because intrinsic properties stay in the foreground and any influence of an embedding space, which might a priori not be known or even considered to be artificial, will be ignored. In some sense, in that framework, choosing coordinates is a no-go. On the other hand, however, the extrinsic approach might be considered to be of more applied character, mainly because some of the related applications actually stem from rolling rigid or convex bodies in the geometric mechanic sense and/or from closely related questions of geometric control. Although there is an overlap of both approaches, i.e., interpretations of the mathematical results of rolling without slipping or twisting have partially been discussed from both sides, the definitions usually differ, including assumptions and consequences. We want to emphasize that by extrinsic, we do not mean working with coordinates in the sense of charts. The access to an embedding vector space often nevertheless opens the path to a coordinate-free approach, similar to treating the standard sphere S^n embedded into $\mathbb{R}^{n+1}$.

The purpose of this paper is at least threefold. Firstly, we put both approaches, intrinsic and extrinsic, into perspective, clarifying the sometimes subtle differences and discussing their consequences. In particular, we claim that the role of the no-twist conditions become more clarified. Secondly, we study a sufficiently rich class of manifolds, namely, the rolling of normal natural reductive homogeneous spaces. An essentially constructive procedure to generalize the rolling of symmetric spaces is presented here for the first time. Thirdly, the rolling Stiefel manifold serves as our role model, as it is well known that although spheres and orthogonal groups within the set of real Stiefel manifolds are symmetric spaces, all the others are not. We also put all our results into perspective by comparing them to the partial results scattered in the literature.

Central to our treatise is the derivation of the so-called kinematic equations, i.e., a set of ODEs to be considered under certain nonholonomic constraints. Certainly, the rich theory behind differential geometric distributions, fiber bundle constructions, and differential systems can be applied here. For many examples, however, this theory often does not support explicit solutions for the nonholonomic problem of rolling with no slipping and no twisting. Here, we present explicit solutions for rolling Stiefel manifolds, even for a huge class of a one-parameter family of pseudo-Riemannian metrics for Stiefel manifolds. This class includes many of the known examples scattered through the literature.

We strongly believe that our work will influence future research, in particular, when rolling motions are driven by engineering applications. To be more specific, having solutions of the kinematic equations of rolling at hand is helpful in deriving explicit or closed formulas for differential geometric concepts, such as parallel transport and covariant derivatives, or even to tackle control theoretic questions. These in turn will facilitate finding solutions for interpolation, optimization, and path planning or other related engineering-type problems.

This paper is structured as follows. After introducing the necessary notations, we recall some facts on homogeneous spaces, with emphasis on normal naturally reductive homogeneous spaces. The Levi-Civita connection on a normal naturally reductive homogeneous space G/H is expressed in terms of vector fields on the Lie group G, which have been horizontally lifted from G/H in Section 3.1. This leads, in Section 3.2, to a characterization of parallel vector fields along curves, which is important for our further investigation of rolling.

We then come to Section 4, where three different notions of rolling a pseudo-Riemannian manifold over another one of equal dimension are introduced. Starting with one definition of intrinsic rolling, we continue with two different definitions of extrinsic rolling, the latter being closely related.

Although these definitions apply to general pseudo-Riemannian manifolds, we turn our attention to normal naturally reductive homogeneous spaces in Section 5. The rather simple form of rolling intrinsically pseudo-Riemannian symmetric spaces from [1] motivates an Ansatz, which is an obvious generalization of this rolling. Unfortunately, this does not yield the desired result, in general. This discussion is summarized in Lemma 5. In addition, it is illustrated by the example of Stiefel manifolds equipped with α-metrics in Section 5.2.

Afterward, we derive the so-called kinematic equations for rolling intrinsically normal naturally reductive homogeneous spaces. Their solutions describe the desired rolling explicitly if a control curve was given a priori.

In Section 6, our findings from Section 5.3 are applied to Stiefel manifolds. First, we recall some facts on Stiefel manifolds endowed with α-metrics from the literature. Afterward, the intrinsic rolling of Stiefel manifolds equipped with α-metrics is discussed by applying results from Section 5.3. For a specific choice of the parameter α, the α-metric on the Stiefel manifold $\mathrm{St}_{n,k}$ coincides with the metric induced by the Euclidean metric on the embedding space $\mathbb{R}^{n\times k}$. Using this fact, the extrinsic rolling of Stiefel manifolds is treated in Section 6.3 by extending the intrinsic rolling from Section 6.2.

In Section 6.4, the kinematic equations describing the rolling of Stiefel manifolds are solved explicitly where an additional assumption is imposed on the development curve. More precisely, an explicit formula for the extrinsic rolling of a tangent space of $\mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$ is obtained, provided that the development curve is the projection of a one-parameter subgroup in $O(n) \times O(k)$, which is not necessarily horizontal.

Finally, in Section 6.5, we relate our results about the extrinsic rolling of Stiefel manifolds to those derived in [2].

2. Notations and Terminology

These are some of the notations used throughout the paper:

M, N	smooth manifolds
T_pM	tangent space at $p \in M$
$d_pf\colon T_pM \to T_{f(p)}N$	tangent map of $f\colon M \to N$ at $p \in M$
N_pM	normal space at $p \in M$
NM	normal bundle of M
$\Gamma^\infty(TM)$	smooth vector fields on M
G	Lie group
H	closed subgroup of G
$\mathfrak{g}$	Lie algebra of G
$\pi\colon G \to G/H$	canonical projection
$\mathcal{H}$	horizontal bundle of $\pi\colon G \to G/H$
$\mathcal{V}$	vertical bundle, i.e., $\mathcal{V} = \ker(d\pi)$
$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$	reductive decomposition
$\mathrm{pr}_{\mathfrak{p}}\colon \mathfrak{g} \to \mathfrak{p}$	projection onto $\mathfrak{p}$ along $\mathfrak{h}$
$X\big\rvert_{\mathfrak{p}}$	$X\big\rvert_{\mathfrak{p}} = \mathrm{pr}_{\mathfrak{p}}(X)$ for $X \in \mathfrak{g}$
X, Y	smooth vector fields
$\nabla_X Y$	covariant derivative of Y in direction X
$\nabla_{\dot\alpha(t)} Y$	covariant derivative of Y along curve α
V	finite-dimensional (pseudo-Euclidean) Vector space
$\mathrm{End}(V)$	algebra of $\mathbb{R}$-linear endomorphism of V
$GL(V)$	general linear group of V
$O(V)$	pseudo-orthogonal group of V
$\mathfrak{so}(V)$	Lie algebra of $O(V)$
$SO(n)$	special orthogonal group, $SO(n) = \{R \in \mathbb{R}^{n\times n} \mid R^\top R = I_n, \det(R) = 1\}$
$\mathfrak{so}(n)$	Lie algebra of $SO(n)$, $\mathfrak{so}(n) = \{\Omega \in \mathbb{R}^{n\times n} \mid \Omega^\top = -\Omega\}$
$\otimes$	Kronecker product
$\mathrm{vec}\colon \mathbb{R}^{n\times k} \to \mathbb{R}^{nk}$	vec operator, $\mathrm{vec}(A) = \begin{bmatrix} A_1 \\ \vdots \\ A_k \end{bmatrix}$ for $A = (A_1, \dots, A_k) \in \mathbb{R}^{n\times k}$
$\oplus_\perp$	direct sum of vector spaces orthogonal with respect to scalar product
$\ltimes$	semi-direct product of groups

3. Normal Naturally Reductive Homogeneous Spaces

Lowercase Latin letters for the elements in a Lie group and uppercase Latin letters for the elements in the corresponding Lie algebra are used. For curves in the Lie algebra, it will be more convenient to use lowercase Latin letters as well.

Assume that a Lie group G acts transitively from the left on a smooth manifold M, with action

$$\tau\colon G \times M \to M, \quad (g, p) \mapsto \tau(g, p) = g.p.$$

Then, $\tau_g\colon M \to M$, defined by

$$\tau_g(p) = \tau(g, p), \quad p \in M,$$

is a diffeomorphism for any $g \in G$.

Let $\mathrm{Stab}(o) \subset G$ be the isotropy subgroup of a point $o \in M$, that is, $\mathrm{Stab}(o) = \{g \in G : g.o = \tau(g, o) = \tau_g(o) = o\}$. The isotropy subgroup of a point in M is a closed subgroup of G and any two isotropy subgroups are conjugate. To simplify notations, we may denote $\mathrm{Stab}(o)$ simply by H. The coset manifold G/H is diffeomorphic to M via $g.H \mapsto g.o$, where $g.H \in G/H$ denotes the coset defined by $g \in G$, and we can write $M = G/H$. The manifold $M = G/H$ is called a *homogeneous manifold*. We denote the corresponding Lie algebras of G and H by $\mathfrak{g}$ and $\mathfrak{h}$, respectively.

The coset manifold is said to be *reductive*, see, e.g., [3] (Chap. 11, Def. 21) or [4] (Def. 23.8), if there exists a subspace $\mathfrak{p} \subset \mathfrak{g}$, such that $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$ and $\mathrm{Ad}_h(X)$ for all $X \in \mathfrak{p}$ and $h \in H$. This Ad_H-invariance of $\mathfrak{p}$ implies $[\mathfrak{p}, \mathfrak{h}] \subset \mathfrak{p}$.

Let π denote the projection of G on the coset manifold, i.e.,

$$\pi \colon G \to G/H, \quad g \mapsto \pi(g) = g.H.$$

If e is the identity element in G, then the map π and its differential

$$d_e\pi \colon T_eG = \mathfrak{g} \to T_oM \tag{1}$$

have the following properties.

Proposition 1.

1. *π is a submersion;*
2. *$d_e\pi(\mathfrak{h}) = \{0\} \subset T_oM$;*
3. *$d_e\pi\big|_{\mathfrak{p}} \colon \mathfrak{p} \to T_oM$ is an isomorphism.*

Consider now M endowed with a pseudo-Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$. We write $\langle\langle \cdot, \cdot \rangle\rangle_p$ if we want to emphasize the value of the metric at the point $p \in M$. A metric tensor $\langle\langle \cdot, \cdot \rangle\rangle$ on M is said to be G-invariant if

$$\langle\langle X, Y \rangle\rangle_p = \langle\langle d_p\tau_g(X), d_p\tau_g(Y) \rangle\rangle_{\tau_g(p)},$$

for all $X, Y \in T_pM$. In other words, the diffeomorphism $\tau_g \colon M \to M$ is an isometry.

Next, we recall the definition of a pseudo-Riemannian submersion from [3] (Chap. 7, Def. 44).

Definition 1. *Let $\left(M, \langle\langle \cdot, \cdot \rangle\rangle^M\right)$ and $\left(N, \langle\langle \cdot, \cdot \rangle\rangle^N\right)$ be two pseudo-Riemannian manifolds and $\pi \colon N \to M$ be a submersion. Denote by $\mathcal{V}_n = \ker(d_n\pi)$ the vertical space at $n \in N$. Then, π is called a pseudo-Riemannian submersion if the fibers $\pi^{-1}(p)$ are pseudo-Riemannian submanifolds of N for all $p \in M$ and the maps $d_n\pi\big|_{\mathcal{H}_n} \colon \mathcal{H}_n \to T_{\pi(n)}M$ are isometries for all $n \in N$, where $\mathcal{H}_n = \mathcal{V}_n^\perp$.*

A scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{p}$ is said to be Ad_H-invariant if

$$\langle \mathrm{Ad}_h(X), \mathrm{Ad}_h(Y) \rangle = \langle X, Y \rangle, \text{ for all } h \in H \text{ and for all } X, Y \in \mathfrak{p}.$$

Next, we recall [3] (Chap. 11, Prop. 22).

Proposition 2. *By declaring the map $d_e\pi$ an isometry, there is one-to-one correspondence between the Ad_H-invariant scalar products on $\mathfrak{p}$ and the G-invariant metrics on G/H.*

Definition 2. *A coset manifold $M = G/H$ is called a naturally reductive space if the following:*

1. *$M = G/H$ is reductive;*
2. *M carries a G-invariant metric;*

3. *If $\langle \cdot, \cdot \rangle$ denotes the Ad_H-invariant scalar product on $\mathfrak{p}$ corresponding to the G-invariant metric (described in Proposition 2), then it has to satisfy*

$$\big\langle [X,Y]\big|_{\mathfrak{p}}, Z\big\rangle = \big\langle X, [Y,Z]\big|_{\mathfrak{p}}\big\rangle, \textit{ for all } X,Y,Z \in \mathfrak{p}.$$

Naturally reductive homogeneous spaces are complete, see [3] (Chap. 11, p. 313). Next, we introduce the notion of (pseudo-Riemannian) normal naturally reductive homogeneous space. This definition is a slight generalization of the homogeneous spaces that are considered in [4] (Prop. 23.29).

Definition 3. (Normal Naturally Reductive Spaces.) *Let G be a Lie group equipped with a bi-invariant metric and denote by $\langle \cdot, \cdot \rangle$ the corresponding Ad_G-invariant scalar product on its Lie algebra $\mathfrak{g}$. Moreover, let $H \subset G$ be a closed subgroup and denote its Lie algebra by $\mathfrak{h} \subset \mathfrak{g}$. If the orthogonal complement $\mathfrak{p} = \mathfrak{h}^{\perp}$ with respect to $\langle \cdot, \cdot \rangle$ is non-degenerated, we call G/H equipped with the G-invariant metric that turns $\pi\colon G \to G/H$ into a pseudo-Riemannian submersion a (pseudo-Riemannian) normal naturally reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$.*

By a trivial adaptation of the proof of [4] (Prop. 23.29), we show that normal naturally reductive spaces are naturally reductive.

Lemma 1. *Let G/H be normal naturally reductive. Then, G/H is naturally reductive.*

Proof. Let $X \in \mathfrak{p} = \mathfrak{h}^{\perp}$. Then, $\langle Y, X \rangle = 0$ for all $Y \in \mathfrak{h}$. The Ad_G invariance of $\langle \cdot, \cdot \rangle$ implies that

$$\big\langle \mathrm{Ad}_h(X), \mathrm{Ad}_h(Y)\big\rangle = 0, \quad h \in H. \tag{2}$$

Since $\mathrm{Ad}_h\colon \mathfrak{h} \to \mathfrak{h}$ is an isomorphism, this implies $\langle \mathrm{Ad}_h(X), \widehat{Y}\rangle = 0$ for $h \in H$ and all $\widehat{Y} \in \mathfrak{h}$, proving $\mathrm{Ad}_h(X) \in \mathfrak{p}$ for $h \in H$, i.e., $\mathrm{Ad}_h(\mathfrak{p}) \subset \mathfrak{p}$ for $h \in H$. In addition, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp} = \mathfrak{h} \oplus \mathfrak{p}$ is fulfilled because $\mathfrak{h}^{\perp}$ is assumed to be non-degenerated. Thus, G/H is a reductive homogeneous space.

In order to show that G/H is naturally reductive, we compute for $X, Y, Z \in \mathfrak{p}$

$$\begin{aligned} 0 &= \tfrac{\mathrm{d}}{\mathrm{d}t}\langle X, Z\rangle\big|_{t=0} \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big\langle \mathrm{Ad}_{\exp(tY)}(X), \mathrm{Ad}_{\exp(tY)}(Z)\big\rangle\big|_{t=0} \\ &= \big\langle [Y,X], Z\big\rangle + \big\langle X, [Y,Z]\big\rangle \\ &= -\big\langle [X,Y], Z\big\rangle + \big\langle X, [Y,Z]\big\rangle, \end{aligned} \tag{3}$$

where we have used the Ad_G-invariance of $\langle \cdot, \cdot \rangle$. Finally, because $\mathfrak{p} = \mathfrak{h}^{\perp}$, the last identity implies that

$$\big\langle [X,Y]\big|_{\mathfrak{p}}, Z\big\rangle = \big\langle X, [Y,Z]\big|_{\mathfrak{p}}\big\rangle, \quad X,Y,Z \in \mathfrak{p}, \tag{4}$$

i.e., G/H is a naturally reductive homogeneous space. □

Let G/H be a normal naturally reductive space. Then, by definition, the map $\pi\colon G \to G/H$ is a pseudo-Riemannian submersion. Obviously, the vertical bundle and horizontal bundle are given by

$$\mathcal{V}_g = \ker(d_g\pi) = \big(d_e L_g\big)\mathfrak{h} \quad \text{and} \quad \mathcal{H}_g = \mathcal{V}_g^{\perp} = \big(d_e L_g\big)\mathfrak{p},$$

for $g \in G$, respectively. From an algebraic point of view, the reductive decomposition has the following properties:

$$\mathfrak{g} = \mathfrak{p} \oplus_{\perp} \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{p}, \mathfrak{h}] \subset \mathfrak{p}.$$

We end this preliminary section by commenting on the regularity of curves. Throughout this text, for simplicity, if not indicated otherwise, a curve $c\colon I \to M$ on a manifold M is assumed to be smooth. However, we point out that many results can be generalized by requiring less regularity.

3.1. Levi-Civita Connection and Covariant Derivative

We first set some notations. The Levi-Civita connections on $M = G/H$ and on G will be denoted by ∇^M and ∇^G, respectively. In cases when it is clear from the context, we may use simply ∇ for both. If Y is a vector field on $M = G/H$, we denote by $\widetilde{Y} \in \Gamma^\infty(TG)$ its horizontal lift to G. Correspondingly, if $\alpha\colon I \to M$ is a curve in M and $r\colon I \to G$ is a lift of α to G, we write $\nabla_{\dot\alpha(t)} Y \in T_{\alpha(t)} M$ for the covariant derivative of Y along α and $\widetilde{\nabla_{\dot\alpha(t)} Y}\big|_{r(t)}$ for the horizontal lift of $\nabla_{\dot\alpha(t)} Y$ to $\mathcal{H}_{r(t)} \subset T_{r(t)} G$.

In the sequel, the lift of α to G will be denoted by q instead of r if it is considered to be horizontal. For $g \in G$, denote by $\mathrm{pr}_{\mathcal{H}_g}\colon T_g G \to \mathcal{H}_g$ the projection onto the horizontal bundle, explicitly given by

$$\mathrm{pr}_{\mathcal{H}_g} = \big(d_e L_g\big) \circ \mathrm{pr}_{\mathfrak{p}} \circ \big(d_e L_g\big)^{-1}. \tag{5}$$

Lemma 2. *Let G/H be a normal naturally reductive homogeneous space and let X, Y be vector fields on G/H. Denote by $\widetilde{X}$ and $\widetilde{Y}$ the horizontal lifts of X and Y, respectively. Moreover, let $\{A_1, \dots, A_k \mid i = 1, \dots, k\}$ be a basis of $\mathfrak{p}$ and denote by $\overline{A}_1, \dots, \overline{A}_k$ the corresponding left-invariant vector fields defined by $\overline{A}_i(g) = d_e L_g A_i$ for $g \in G$. Expanding $\widetilde{X} = \sum_{i=1}^k x_i \overline{A}_i$ and $\widetilde{Y} = \sum_{j=1}^k y_j \overline{A}_j$ with smooth functions $x_i, y_j\colon G \to \mathbb{R}$, we obtain for the Levi-Civita covariant derivative on G/H, for $g \in G$,*

$$\begin{aligned}(\nabla^M_X Y)(\pi(g)) = d_g\pi\Big(&\sum_{j=1}^k \big(\widetilde{X}(y_j)\big)(g)\overline{A}_j(g) \\ &+ \mathrm{pr}_{\mathcal{H}_g} \tfrac{1}{2} \sum_{i,j=1}^k x_i(g) y_j(g) \Big[\overline{A}_i, \overline{A}_j\Big](g)\Big),\end{aligned} \tag{6}$$

or, equivalently,

$$\widetilde{\nabla^M_X Y}\big|_g = \sum_{j=1}^k \big(\widetilde{X}(y_j)\big)(g)\overline{A}_j(g) + \tfrac{1}{2} \sum_{i,j=1}^k x_i(g) y_j(g) \overline{[A_i, A_j]|_{\mathfrak{p}}}(g). \tag{7}$$

Proof. Because the metric is bi-invariant, it follows that for left-invariant vector fields V, W on G, see [3] (p. 304) ,

$$\nabla^G_V W = \tfrac{1}{2}[V, W] \tag{8}$$

holds. Because G/H is a normal naturally reductive space, the map $\pi\colon G \to G/H$ is a pseudo-Riemannian submersion. Let X, Y be vector fields on M and $\widetilde{X}$, $\widetilde{Y}$ their horizontal lifts to G. We recall that the Levi-Civita connections on M and on G are related by, see [3] (Lemma 45, Chapter 7),

$$\nabla^M_X Y = d_g\pi\Big(\mathrm{pr}_{\mathcal{H}_g} \nabla^G_{\widetilde{X}} \widetilde{Y}\Big). \tag{9}$$

Expanding the horizontal lifts $\widetilde{X}$ and $\widetilde{Y}$ in terms of the left-invariant frame field $\{\overline{A}_1, \dots, \overline{A}_k\}$, i.e.,

$$\widetilde{X} = \sum_{i=1}^k x_i \overline{A}_i, \quad \widetilde{Y} = \sum_{j=1}^k y_j \overline{A}_j, \tag{10}$$

we have

$$\nabla^G_{\widetilde{X}}\widetilde{Y} = \nabla^G_{\widetilde{X}}\Big(\sum_{j=1}^{k} y_j \overline{A}_j\Big) = \sum_{j=1}^{k} (\widetilde{X}(y_j))\overline{A}_j + \tfrac{1}{2}\sum_{i,j=1}^{k} x_i y_j \Big[\overline{A}_i, \overline{A}_j\Big]. \tag{11}$$

Projecting to $\mathcal{H}_g$, and taking into consideration that the first term in the last equality belongs to $\mathcal{H}_g$, we obtain

$$\mathrm{pr}_{\mathcal{H}_g} \nabla^G_{\widetilde{X}}\widetilde{Y} = \sum_{j=1}^{k} (\widetilde{X}(y_j))\overline{A}_j + \mathrm{pr}_{\mathcal{H}_g} \tfrac{1}{2}\sum_{i,j=1}^{k} x_i y_j \Big[\overline{A}_i, \overline{A}_j\Big]. \tag{12}$$

Combining this dentity with (9), gives (6). Clearly, by using (5), one has $\mathrm{pr}_{\mathcal{H}_g}([\overline{A}_i, \overline{A}_j])(g) = \overline{[A_i, A_j]\big|_{\mathfrak{p}}}(g)$. Hence, (6) is equivalent to (7), as the vector field from (11) on G is horizontal and π-related to $\nabla^M_X Y$ by (6). $\square$

Lemma 2 yields an expression for the Levi-Civita covariant derivative on G/H in terms of horizontally lifted vector fields on G. This expression allows for determining the covariant derivative of vector fields along a curve in G/H in terms of horizontally lifted vector fields along a horizontal lift of the curve, as well. As preparation, we comment on the domain of horizontal lifts.

Remark 1. *Let $\alpha\colon I \to G/H$ be a curve on a normal naturally reductive space. The horizontal lift $q\colon I \to G$ is indeed defined on the same interval as α. This can be shown by exploiting that $\mathcal{H} \subset TG$ defines a principal connection that is known to be complete.*

Lemma 3. *Let $M = G/H$ be a normal naturally reductive homogeneous space, $\alpha\colon I \to M$ a curve, and Y a vector field along α. Let $q\colon I \to G$ be a horizontal lift of α and $\widetilde{Y}$ a horizontal lift of Y along q. Then,*

$$\begin{aligned}\nabla^M_{\dot{\alpha}(t)} Y(t) \quad &= \quad d_{q(t)}\pi\Big(\sum_{j=1}^{k} \tfrac{\mathrm{d}\, y_j(t)}{\mathrm{d}\, t}\overline{A}_j(t)\Big) \\ &+ \quad d_{q(t)}\pi\Big(\mathrm{pr}_{\mathcal{H}_{q(t)}} \tfrac{1}{2}\sum_{i,j=1}^{k} x_i(t) y_j(t) \Big[\overline{A}_i(t), \overline{A}_j(t)\Big]\Big),\end{aligned} \tag{13}$$

or, equivalently,

$$\widetilde{\nabla^M_{\dot{\alpha}(t)} Y}\big|_{q(t)} = \sum_{j=1}^{k} \tfrac{\mathrm{d}\, y_j(t)}{\mathrm{d}\, t}\overline{A}_j(t) + \tfrac{1}{2}\sum_{i,j=1}^{k} x_i(t) y_j(t) \overline{[A_i, A_j]\big|_{\mathfrak{p}}}(t), \tag{14}$$

where $\{A_1, \ldots, A_k\}$ is a basis of $\mathfrak{p}$, $\overline{A}_i$ denotes the left-invariant vector field corresponding to A_i for $i = 1, \ldots k$, and we write $\overline{A}_i(t) = \overline{A}_i(q(t))$ for short. The functions $x_i, y_j\colon I \to \mathbb{R}$ are defined by $\dot{q}(t) = \sum_{i=1}^{k} x_i(t)\overline{A}_i(t)$ and $\widehat{Y}(t) = \sum_{j=1}^{k} y_j \overline{A}_j(t)$.

Proof. Let $t \in I$. We extend the vector fields $\dot{\alpha}(t)$ and $Y(t)$ locally to vector fields $\widehat{X}$ and $\widehat{Y}$, respectively, defined on an open neighborhood of $\alpha(t)$ in G/H. The proof of [5] (Thm. 4.24) shows that such an extension is always possible. Moreover, we denote by $\widetilde{\widehat{X}}$ and $\widetilde{\widehat{Y}}$ the horizontal lifts of $\widehat{X}$ and $\widehat{Y}$, respectively. These vector fields are expanded as $\widetilde{\widehat{X}} = \sum_{i=1}^{k} \widehat{x}_i \overline{A}_i$ and $\widetilde{\widehat{Y}} = \sum_{j=1}^{k} \widehat{y}_j \overline{A}_j$ with uniquely locally defined functions $\widehat{x}_i, \widehat{y}_j$ on G. Clearly, these functions fulfill $\widehat{x}_i(q(t)) = x_i(t)$ and $\widehat{y}_j(q(t)) = y_j(t)$ whenever both sides are defined. In addition, $\widetilde{\widehat{X}}(q(t)) = \dot{q}(t)$ and $\widetilde{\widehat{Y}}(q(t)) = \widetilde{Y}(t)$ hold. By using Lemma 2, we compute

$$
\begin{aligned}
(\nabla^M_X Y)(\pi(q(t)) &= d_{q(t)}\pi\Big(\sum_{j=1}^{k}\Big(\widetilde{\widetilde{X}}(\widehat{y}_j)\Big)(q(t))\overline{A}_j(q(t))\Big) \\
&\quad + d_{q(t)}\pi\Big(\mathrm{pr}_{\mathcal{H}_{q(t)}}\tfrac{1}{2}\sum_{i,j=1}^{k}\widehat{x}_i(q(t))\widehat{y}_j(q(t))\Big[\overline{A}_i,\overline{A}_j\Big](q(t))\Big) \\
&= d_{q(t)}\pi\Big(\sum_{j=1}^{k}\tfrac{\mathrm{d}\, y_j(t)}{\mathrm{d}\, t}\overline{A}_j(t)\Big) \\
&\quad + d_{q(t)}\pi\Big(\mathrm{pr}_{\mathcal{H}_{q(t)}}\tfrac{1}{2}\sum_{i,j=1}^{k}x_i(t)y_j(t)\Big[\overline{A}_i,\overline{A}_j\Big](t)\Big),
\end{aligned}
$$

which proves (13). Clearly, this is equivalent to (14) by Lemma 2. □

Remark 2. *If $M = G/H$ is a symmetric space, then $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{h}$, and consequently the last summand in formula (13) vanishes. So, taking into consideration that, in this case, $\nabla^G_{\dot{q}(t)}\widetilde{Y}(t) = \sum_{j=1}^{k}\frac{\mathrm{d}\, y_j(t)}{\mathrm{d}\, t}\overline{A}_j$, the identity (13) reduces to*

$$\nabla^M_{\dot{\alpha}(t)}Y(t) = d_{q(t)}\pi\Big(\nabla^G_{\dot{q}(t)}\widetilde{Y}(t)\Big),$$

which shows that, in the case of a symmetric space, if Y is a parallel vector field along $\alpha(t) \in M$, its horizontal lift $\widetilde{Y}$ is actually a parallel vector field along the horizontal lift $q(t) \in G$ of $\alpha(t)$.

As we will see below, for nonsymmetric spaces, the presence of the second term in (13) reveals that the horizontal lift $q(t) \in G$ is not a good candidate for the property of preserving parallel vector fields. In the next section, we modify the "horizontal lift" in order to overcome this problem.

3.2. Parallel Vector Fields

Lemma 4. *Let $M = G/H$ be a normal naturally reductive homogeneous space, $\alpha\colon I \to M$ a curve, and $q\colon [0,T] \to G$ a horizontal lift of α. Moreover, let $s\colon I \to H$ and define the curve $r\colon I \to G$ by $r(t) = q(t)s(t)$. Let $Z\colon I \to TM$ be a vector field along α and denote by $\widetilde{Z}\colon I \to \mathcal{H}$ its horizontal lift along r. Then, the horizontal lift of $\nabla_{\dot{\alpha}(t)}Z\colon I \to TM$ along $r(t)$ is given by*

$$
\begin{aligned}
\widetilde{\nabla_{\dot{\alpha}(t)}Z}\big|_{r(t)} &= \sum_{j=1}^{k}\dot{z}_j(t)\overline{A}_j(r(t)) \\
&\quad + \sum_{i,j=1}^{k}\tfrac{1}{2}x_i(t)z_j(t)\mathrm{pr}_{\mathcal{H}_{r(t)}}\Big(\overline{[\mathrm{Ad}_{s(t)^{-1}}(A_i),A_j]}(r(t))\Big).
\end{aligned}
\tag{15}
$$

Here, we expanded $x(t) = \big(d_eL_{q(t)}\big)^{-1}\dot{q}(t) = \sum_{i=1}^{k}x_i(t)A_i \in \mathfrak{p}$ and $z(t) = \big(d_eL_{r(t)}\big)^{-1}\widetilde{Z}(t) = \sum_{i=1}^{k}z_i(t)A_i \in \mathfrak{p}$.

Proof. Let $X, Z \in \Gamma^\infty\big(T(G/H)\big)$ be vector fields with horizontal lifts $\widetilde{X}, \widetilde{Z} \in \Gamma^\infty(TG)$ and expand them by a left-invariant frame $\overline{A}_1,\ldots,\overline{A}_k$ of the horizontal bundle of $G \to G/H$, i.e., $\widetilde{X} = \sum_{i=1}^{k}x_i\overline{A}_i$ and $\widetilde{Z} = \sum_{j=1}^{k}z_j\overline{A}_j$. Then, by Lemma 2, the Levi-Civita connection on G/H can be expressed in terms of horizontal lifts by

$$\widetilde{\nabla_X Z} = \sum_{j=1}^{k}\widetilde{X}(z_j)\overline{A}_j + \frac{1}{2}\sum_{i,j=1}^{k}x_iz_j\overline{[A_i,A_j]\big|_{\mathfrak{p}}}. \tag{16}$$

Now, consider the curve $r(t) = q(t)s(t)$ being a lift of $\alpha(t)$. A simple computation shows that

$$\big(d_e L_{r(t)}\big)^{-1}\dot{r}(t) = \mathrm{Ad}_{s(t)^{-1}}\big(x(t)\big) + y(t), \tag{17}$$

where $y(t) := \big(d_e L_{s(t)}\big)^{-1}\dot{s}(t) \in \mathfrak{h}$. Thus, using (17) and $\pi\big(r(t)\big) = \alpha(t)$, we have

$$\begin{aligned}\dot{\alpha}(t) &= d_{r(t)}\pi\dot{r}(t)\\ &= \big(d_{r(t)}\pi \circ d_e L_{r(t)}\big)\big(\mathrm{Ad}_{s(t)^{-1}}\big(x(t)\big) + y(t)\big)\\ &= \big(d_{r(t)}\pi \circ d_e L_{r(t)}\big)\Big(\mathrm{Ad}_{s(t)^{-1}}\big(x(t)\big)\Big).\end{aligned} \tag{18}$$

Here, the last equality follows from the definition of the horizontal bundle. By extending $\dot{\alpha}(t)$ locally to a vector field X on G/H, the horizontal lift $\widetilde{X}$ of X satisfies $\widetilde{X}\big(r(t)\big) = d_e L_{r(t)}\Big(\mathrm{Ad}_{s(t)^{-1}}\big(x(t)\big)\Big)$ by (18). Moreover, the vector field Z along α can be extended locally to a vector field $\widehat{Z}$ on G/H, defined on an open neighborhood of α. Denote by $\widetilde{\widehat{Z}}$ the horizontal lift of $\widehat{Z}$. Then, $\widetilde{\widehat{Z}}\big(r(t)\big) = \widetilde{Z}(t)$ is fulfilled. By [5] (Thm. 4.24), we have

$$\widetilde{\nabla_{\dot{\alpha}(t)}Z}\big|_{r(t)} = \widetilde{\nabla_{\widetilde{X}}\widetilde{\widehat{Z}}}\big|_{r(t)}. \tag{19}$$

The desired result follows by exploiting (16), similarly to what was performed in the proof of Lemma 3. □

Corollary 1. *The vector field $Z\colon I \to T(G/H)$ along $\alpha\colon I \to G/H$ is parallel along α iff its horizontal lift $\widetilde{Z}$ along $r(t) = q(t)s(t) \in G$, defined as in Lemma 4 by $z(t) = (d_e L_{r(t)})^{-1}\widetilde{Z}(t) = \sum_{i=1}^{k} z_i(t)A_i \in \mathfrak{p}$, satisfies*

$$\dot{z}(t) = -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}}\Big(\big[\mathrm{Ad}_{s(t)^{-1}}\big(x(t)\big), z(t)\big]\Big) \tag{20}$$

for all $t \in I$, where $x(t) = \big(d_e L_{q(t)}\big)^{-1}\dot{q}(t) = \sum_{i=1}^{k} x_i(t)A_i \in \mathfrak{p}$.

Proof. Lemma 4 already implies the statement by applying the linear isomorphism $\big(d_{r(t)}\pi \circ d_e L_{r(t)}\big)^{-1}$ to both sides of $0 = \widetilde{\nabla_{\dot{\alpha}(t)}Z}\big|_{r(t)}$. □

When $s(t) = e$, for $t \in I$, Corollary 1 also gives the following characterization of parallel vector fields.

Corollary 2. *The vector field $Z\colon I \to T(G/H)$ along $\alpha\colon I \to G/H$ with a horizontal lift $q\colon I \to G$ is parallel along α iff its horizontal lift $\widetilde{Z}$ along q fulfills the ODE*

$$\dot{z}(t) = -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}}\big([x(t), z(t)]\big), \tag{21}$$

for all $t \in I$, where $x(t) = \big(d_e L_{q(t)}\big)^{-1}\dot{q}(t) \in \mathfrak{p}$ and

$$z(t) = \big(d_e L_{q(t)}\big)^{-1} \circ \big(d_{q(t)}\pi\big|_{\mathcal{H}_{q(t)}}\big)^{-1} Z(t) \in \mathfrak{p}. \tag{22}$$

4. Intrinsic and Extrinsic Formulation of Rolling

The goal of this section is to introduce the notation of rolling a pseudo-Riemannian manifold over another one.

In the following definitions, it is assumed that the pseudo-Riemannian manifolds (M, g) and $(\widehat{M}, \widehat{g})$ are of equal dimension and g and $\widehat{g}$ have the same signature.

Definition 4. (Intrinsic rolling.) *A curve $\alpha(t)$ on M is said to roll on a curve $\widehat{\alpha}(t)$ on $\widehat{M}$ intrinsically if there exists an isometry $A(t) : T_{\alpha(t)}M \to T_{\widehat{\alpha}(t)}\widehat{M}$ satisfying the following conditions:*

1. *No-slip condition: $\dot{\widehat{\alpha}}(t) = A(t)\dot{\alpha}(t)$;*
2. *No-twist condition: $A(t)X(t)$ is a parallel vector field in $\widehat{M}$ along $\widehat{\alpha}(t)$ iff $X(t)$ is a parallel vector field in M along $\alpha(t)$.*

The triple $\big(\alpha(t), \widehat{\alpha}(t), A(t)\big)$ is called a rolling (of M over $\widehat{M}$). The curve α is called a rolling curve, while $\widehat{\alpha}$ is called a development curve.

The next definition of extrinsic rolling is motivated by the description of extrinsic rolling in terms of bundles, see [6] (Def. 2) and [7] (Def. 3).

Definition 5. (Extrinsic rolling (I).) *Let M and $\widehat{M}$ be isometrically embedded into the same pseudo-Euclidean vector space V. A quadruple $\big(\alpha(t), \widehat{\alpha}(t), A(t), C(t)\big)$ is called an extrinsic rolling (of M over $\widehat{M}$), where $\alpha\colon I \to M$ and $\widehat{\alpha}\colon I \to \widehat{M}$ are curves, and $A(t)\colon T_{\alpha(t)}M \to T_{\widehat{\alpha}(t)}\widehat{M}$ and $C(t)\colon N_{\alpha(t)}M \to N_{\widehat{\alpha}(t)}\widehat{M}$ are isometries of the tangent and normal spaces, if the following conditions hold:*

1. *No-slip condition: $\dot{\widehat{\alpha}}(t) = A(t)\dot{\alpha}(t)$;*
2. *No-twist condition (tangential part): $A(t)X(t)$ is a parallel vector field in $\widehat{M}$ along $\widehat{\alpha}(t)$ if and only if $X(t)$ is a parallel vector field in M along $\alpha(t)$;*
3. *No-twist condition (normal part): $C(t)Z(t)$ is a normal parallel vector field in $\widehat{M}$ along $\widehat{\alpha}(t)$ iff $Z(t)$ is a normal parallel vector field in M along $\alpha(t)$.*

As for the intrinsic case, the curve α is called a rolling curve, while $\widehat{\alpha}$ is a called development curve.

Alternatively, we define extrinsic rolling as a reformulation of a slightly generalized version of [7] (Def. 1).

Definition 6. (Extrinsic rolling (II).) *Let M and $\widehat{M}$ be isometrically embedded into the same pseudo-Euclidean vector space V. A curve $(\alpha, E)\colon I \to M \times E(V)$, where $E(V) = O(V) \ltimes V$ denotes the pseudo-Euclidean group of V, is said to be an extrinsic rolling if the following conditions are satisfied:*

1. *$\widehat{\alpha}(t) := E(t)\alpha(t) \in \widehat{M}$;*
2. *$d_{\alpha(t)}E(t)(T_{\alpha(t)}M) = T_{\widehat{\alpha}(t)}\widehat{M}$;*
3. *No-slip condition: $\dot{\widehat{\alpha}}(t) = d_{\alpha(t)}E(t)\dot{\alpha}(t)$;*
4. *No-twist condition (tangential part): $d_{\alpha(t)}E(t)X(t)$ is parallel along $\widehat{\alpha}$ iff X is parallel along α;*
5. *No-twist condition (normal part): $d_{\alpha(t)}E(t)Z(t)$ is normal parallel along $\widehat{\alpha}$ iff Z is normal parallel along α.*

The curve α is called a rolling curve and the $\widehat{\alpha}$ is the development curve.

Remark 3. *The discussion in [1] (Sec. 3) reveals that a rolling in the sense of Definition 6 is closely related to the classical definition of rolling in [8] (Ap. B, Def. 1.1) . Indeed, the conditions Definition 6 and Claims 1–5 are equivalent to the conditions from [8] (Def. 1.1). Thus, the essential difference between Definition 6 and [8] (Def. 1.1) is that the rolling curve is already part of the Definition. This is motivated by [6] (Ex. 1).*

Motivated by [1] (Prop. 3), we relate the two different notions of extrinsic rolling from Definitions 5 and 6.

Proposition 3. *Let $\big(\alpha(t), \widehat{\alpha}(t), A(t), C(t)\big)$ be an extrinsic rolling in the sense of Definition 5. Then, the curve $g(t) = \big(\alpha(t), \big(R(t), s(t)\big)\big) \in M \times E(V)$, where*

$$\begin{aligned} R(t)\big|_{T_{\alpha(t)}M} &= A(t), \\ R(t)\big|_{N_{\alpha(t)}M} &= C(t), \\ s(t) &= \widehat{\alpha}(t) - R(t)\alpha(t), \end{aligned} \tag{23}$$

is an extrinsic rolling in the sense of Definition 6.

Conversely, given an extrinsic rolling $\big(\alpha(t), (R(t), s(t))\big)$ in the sense of Definition 6, $\big(\alpha(t), \widehat{\alpha}(t), A(t), C(t)\big)$ defines an extrinsic rolling in the sense of Definition 5, where

$$\begin{aligned} A(t) &= R(t)\big|_{T_{\alpha(t)}M}, \\ C(t) &= R(t)\big|_{N_{\alpha(t)}M}, \\ \widehat{\alpha}(t) &= s(t) + R(t)\alpha(t). \end{aligned} \tag{24}$$

Proof. Because this proposition follows analogously to [1] (Prop. 3), we only sketch the proof. Let $\big(\alpha(t), \widehat{\alpha}(t), A(t), C(t)\big)$ be an extrinsic rolling in the sense of Definition 6 and define $I \ni t \mapsto \big(\alpha(t), (R(t), s(t))\big) \in M \times E(V)$ by (23). We obtain

$$\begin{aligned} E(t)\alpha(t) &= R(t)\alpha(t) + s(t) \\ &= R(t)\alpha(t) + \big(\widehat{\alpha}(t) - R(t)\alpha(t)\big) \\ &= \widehat{\alpha}(t) \in \widehat{M}, \end{aligned} \tag{25}$$

which proves Claim 1 of Definition 6. Let $\gamma \colon (-\epsilon, \epsilon) \to M$ be a curve with $\gamma(0) = \alpha(t)$ and $\dot{\gamma}(0) = Z \in V$. Then,

$$d_{\alpha(t)}E(t)Z = \tfrac{\mathrm{d}}{\mathrm{d}\tau}\big(R(t)\gamma(\tau) + s(t)\big)\big|_{\tau=0} = R(t)Z \tag{26}$$

holds. Using (26), it is straightforward to verify that Definition (6) and Claims 2–5 are fulfilled.

Conversely, assume that $I \ni t \mapsto M \times E(V)$ is a rolling in the sense of Definition 6. We now show that the quadruple $\big(\alpha(t), \widehat{\alpha}(t), A(t), C(t)\big)$, given by (24), is an extrinsic rolling in the sense of Definition 5. To this end, we note that $\widehat{\alpha}(t) = s(t) + R(t)\alpha(t) = E(t)\alpha(t)$ holds by Definition 6, Claim 1. Hence, by Definition 6, Claim 2, the map

$$A(t) = R(t)\big|_{T_{\alpha(t)}M} = \big(d_{\alpha(t)}E(t)\big)\big|_{T_{\alpha(t)}M} \colon T_{\alpha(t)}M \to T_{\widehat{\alpha}(t)}M \tag{27}$$

is indeed a well-defined isometry. Obviously, this implies that $C(t) = R(t)\big|_{N_{\alpha(t)}M} = \big(d_{\alpha(t)}E(t)\big)\big|_{N_{\alpha(t)}M} \colon N_{\alpha(t)}M \to N_{\widehat{\alpha}(t)}M$ is a well-defined isometry, as well. Using Definition 6, Claims 3–5, it is straightforward to show that $\big(\alpha(t), \widehat{\alpha}(t), A(t), C(t)\big)$ is indeed a rolling in the sense of Definition 5. □

Below, in Section 6, we use Proposition 3 to relate the rolling of the Stiefel manifolds constructed in this paper to the rolling maps of the Stiefel manifolds known from the literature.

5. Rolling Normal Naturally Reductive Homogeneous Spaces Intrinsically

We first formulate an Ansatz for the rolling of normal naturally reductive homogeneous spaces, which is a generalization of the rolling of pseudo-Riemannian symmetric spaces. It turns out, however, that such an assumption does not work in general.

5.1. No-Go Lemma

Assume that G/H is a pseudo-Riemannian symmetric space. Then, by [1] (Sec. 4.2), a rolling of $\mathfrak{p}$ over G/H along a given rolling curve can be viewed as a triple $\big(\alpha(t), \widehat{\alpha}(t), A(t)\big)$, where

$$\begin{aligned} A(t)\colon T_{\alpha(t)}\mathfrak{p} \cong \mathfrak{p} \to T_{\widehat{\alpha}(t)}(G/H), \\ A(t) = d_{q(t)}\pi \circ d_e L_{q(t)}, \end{aligned} \tag{28}$$

and $q\colon I \to G$ is defined by the initial value problem

$$\dot{q}(t) = d_e L_{q(t)} \dot{\alpha}(t), \quad q(0) = e, \tag{29}$$

whose solution is the horizontal lift of the development curve $\widehat{\alpha}(t) = \pi(q(t))$ through $q(0) = e$.

Note that in [1], G/H is always rolled over $\mathfrak{p}$, while in our work we consider $\mathfrak{p}$ rolling over G/H. This choice is more convenient for us, because there is no need to invert $q(t)$, as in [1] (Eq. 26).

Motivated by this rather simple form of the intrinsic rolling for symmetric spaces, we make the following Ansatz for the rolling of $\mathfrak{p}$ over G/H, where $q(t)$ will be replaced by another lift of $\hat{\alpha}$, $r(t) := q(t)s(t)$, $s(t)$ being a correction term, still to be specified, see below.

Ansatz:

Given a rolling curve $\alpha\colon I \to \mathfrak{p}$, let $u\colon I \ni t \mapsto u(t) = \dot{\alpha}(t) \in \mathfrak{p}$, and define the development curve $\widehat{\alpha}\colon I \to G/H$ by $\widehat{\alpha}(t) = \pi\big(q(t)\big)$, with $q\colon I \to G$ being the horizontal curve defined by the initial value problem

$$\dot{q}(t) = d_e L_{q(t)}\big(\mathrm{Ad}_{s(t)}(u(t))\big), \quad q(0) = e. \tag{30}$$

Here, $s\colon I \to H$ is a smooth curve that still needs to be specified. The definition of q in (30) is chosen such that the no-slip condition is satisfied, as will become clear in the computation (32) below. As a candidate for the isometry $A(t)\colon T_{\alpha(t)}\mathfrak{p} \cong \mathfrak{p} \to T_{\widehat{\alpha}(t)}(G/H)$, we define

$$A(t)(Z) = \big(d_{r(t)}\pi \circ d_e L_{r(t)}\big)(Z), \quad Z \in T_{\alpha(t)}\mathfrak{p} \cong \mathfrak{p}, \tag{31}$$

where $r\colon I \ni t \mapsto q(t)s(t) \in G$, for some $s\colon I \to H$.

Remark 4. *If G/H is a symmetric space, this yields a rolling of $\mathfrak{p}$ over G/H for $s(t) = e$, see [1].*

The more general situation, where G/H is a naturally reductive homogeneous space, is considered in the following. Our Ansatz satisfies the no-slip condition due to

$$\begin{aligned} A(t)\dot{\alpha}(t) &= d_{r(t)}\pi \circ d_e L_{r(t)}\, u(t) \\ &= d_e\big(\pi \circ L_{q(t)} \circ L_{s(t)}\big)\, u(t) \\ &= d_e\big(\tau_{q(t)} \circ \pi \circ L_{s(t)}\big)\, u(t) \\ &= d_e\big(\tau_{q(t)} \circ \tau_{s(t)} \circ \pi\big)\, u(t) \\ &= d_{\pi(q(t))}\tau_{q(t)} \circ d_e\pi \circ \mathrm{Ad}_{s(t)}\, u(t) \\ &= d_{q(t)}\pi \circ d_e L_{q(t)} \mathrm{Ad}_{s(t)}\, u(t) \\ &= d_{q(t)}\pi\, \dot{q}(t) \\ &= \dot{\widehat{\alpha}}(t), \end{aligned} \tag{32}$$

where $\tau\colon G \times G/H \ni (g, g'H) \mapsto (gg')H \in G/H$ denotes the G-action on G/H from the left, which fulfills $\tau_g \circ \pi = \pi \circ L_g$, for $g \in G$. Moreover, we exploited that the isotropy representation of G/H and the representation $\mathrm{Ad}\colon H \to GL(\mathfrak{p})$ are equivalent; to be more precise, $d_{\pi(e)}\tau_h \circ d_e\pi = d_e\pi \circ \mathrm{Ad}_h$, for $h \in H$, see, e.g., [4] (Sec. 23.4, p. 692).

Next, we try to specify the curve $s\colon I \to H$ by imposing the no-twist condition. To this end, let $Z\colon I \ni t \mapsto \big(\alpha(t), Z_2(t)\big) \in \mathfrak{p} \times \mathfrak{p} \cong T\mathfrak{p}$ be a parallel vector field along α. By identifying Z with its second component Z_2, Z can be expressed by $Z(t) = z$ for some $z \in \mathfrak{p}$. We need to determine $s\colon I \to H$ such that the vector field $t \mapsto A(t)Z(t) = \big(d_{r(t)}\pi \circ d_e L_{r(t)}\big)z$ along $\widehat{\alpha}$ is parallel. Note that by using (30), the curve $x(t) = \big(d_e L_{q(t)}\big)^{-1}\dot{q}(t)$ from Corollary 1 corresponds to $x(t) = \mathrm{Ad}_{s(t)}(u(t))$. Moreover, also due to

$$\big(d_e L_{r(t)}\big)^{-1} \circ \big(d_{r(t)}\pi\big|_{\mathcal{H}_{r(t)}}\big)^{-1} A(t)(z) = z = constant, \quad t \in I, \tag{33}$$

the condition $A(t)Z(t)$ being parallel tells us that

$$\begin{aligned} 0 &= -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}}\big(\big[\mathrm{Ad}_{s(t)^{-1}}(\mathrm{Ad}_{s(t)}(u(t)), z\big]\big) \\ &= -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}}\big([u(t), z]\big) \\ &= -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}}\big([\dot{\alpha}(t), z]\big). \end{aligned} \tag{34}$$

Assuming that for a given $0 \neq \dot{\alpha}(t) \in \mathfrak{p}$ there is a $z \in \mathfrak{p}$ such that $0 \neq [\dot{\alpha}(t), z] \in \mathfrak{p}$ holds, (34) cannot be satisfied independently of the choice of $s\colon I \to H$. We summarize the above discussion in the following lemma.

Lemma 5. (No-Go.) *Let $\alpha\colon I \to \mathfrak{p}$ be a curve so that $0 \neq \mathrm{pr}_{\mathfrak{p}}\big([\dot{\alpha}(t), z]\big)$ holds for some $z \in \mathfrak{p}$ and some $t \in I$. Then, $\big(\alpha(t), \widehat{\alpha}(t), A(t)\big)$, as defined in the Ansatz at the beginning of this section, does not define a rolling of $\mathfrak{p}$ over G/H no matter how $s\colon I \to H$ is chosen. To be more precise, the no-twist condition will never be fulfilled.*

5.2. Example: Stiefel Manifolds

We now specialize the above discussion to the Stiefel manifold $\mathrm{St}_{n,k}$ (for the definition and more details, see Section 6.1), equipped with the α-metrics introduced in [9]. These metrics will be recalled in Section 6.1, below. However, we think that it is convenient to apply Lemma 5 to a non-trivial example here. According to [9] (Eq. (37)) , for $E = \begin{bmatrix} I_k \\ 0 \end{bmatrix}$ and $\alpha \neq -1$, the projection $\mathrm{pr}_{\mathfrak{p}}\colon \mathfrak{so}(n) \times \mathfrak{so}(k) \to \mathfrak{p}$ is given by

$$\mathrm{pr}_{\mathfrak{p}}\Big(\begin{bmatrix} A & -B^\top \\ B & C \end{bmatrix}, \Psi\Big) = \Big(\begin{bmatrix} \frac{A-\Psi}{\alpha+1} & -B^\top \\ B & 0 \end{bmatrix}, \frac{\alpha(\Psi - A)}{\alpha+1}\Big). \tag{35}$$

We first assume that $1 \leq k \leq n-1$. Setting $\Psi = A$, we obtain elements of the form $\Big(\begin{bmatrix} 0 & -B^\top \\ B & 0 \end{bmatrix}, 0\Big) \in \mathfrak{p}$, where $B \in \mathbb{R}^{(n-k)\times k}$. Using (35), we can write

$$\begin{aligned} &\mathrm{pr}_{\mathfrak{p}}\Big[\Big(\begin{bmatrix} 0 & -B_1^\top \\ B_1 & 0 \end{bmatrix}, 0\Big]\Big), \Big(\begin{bmatrix} 0 & -B_2^\top \\ B_2 & 0 \end{bmatrix}, 0\Big)\Big] \\ &= \mathrm{pr}_{\mathfrak{p}}\Big(\begin{bmatrix} -B_1^\top B_2 + B_2^\top B_1 & 0 \\ 0 & -B_1 B_2^\top + B_2 B_1^\top \end{bmatrix}, 0\Big) \\ &= \Big(\begin{bmatrix} \frac{-B_1^\top B_2 + B_2^\top B_1}{\alpha+1} & 0 \\ 0 & 0 \end{bmatrix}, \frac{\alpha}{\alpha+1}\big(B_1^\top B_2 - B_2^\top B_1\big)\Big). \end{aligned} \tag{36}$$

Obviously, for $k = 1$, i.e., $B_1, B_2 \in \mathbb{R}^{(n-1)\times 1}$, one has $B_2^\top B_1 = B_1^\top B_2$ implying that (36) is vanishing for $k = 1$. Thus, for $\mathrm{St}_{n,1} \cong S^{n-1}$, the Ansatz actually yields a rolling.

Next, assume $k > 1$. Then, there are $B_1, B_2 \in \mathbb{R}^{(n-k)\times k}$ such that $B_2^\top B_1 - B_1^\top B_2 \neq 0$ holds. Indeed, choosing $B_2 = E_{12}$ given by $(E_{12})_{ij} = \delta_{1i}\delta_{2j}$, where δ_{1i} and δ_{2j} are Kronecker deltas, and $B_1 \in \mathbb{R}^{(n-k)\times k}$ with $(B_1)_{12} \neq 0$, we obtain

$$\begin{aligned}(B_2^\top B_1 - B_1^\top B_2)_{22} &= \sum_{\ell=1}^{n-k} \big((E_{12})_{k2}(B_1)_{k2} - (B_1)_{k2}(E_{12})_{k2}\big) \\ &= \sum_{\ell=1}^{n-k} \big(\delta_{1k}\delta_{22}(B_1)_{k2} - (B_1)_{k2}\delta_{2k}\delta_{12}\big) \\ &= (B_1)_{12} \neq 0.\end{aligned} \tag{37}$$

Consequently, the projection in (36) does not vanish identically for $1 < k < n$. It remains to consider the case $k = n$. This yields $\mathrm{St}_{n,n} \cong (O(n) \times O(n))/O(n)$, and for $(A, \Psi) \in \mathfrak{so}(n) \times \mathfrak{so}(n)$ the projection (35) reduces to

$$\mathrm{pr}_{\mathfrak{p}}(A, \Psi) = \Big(\tfrac{A-\Psi}{\alpha+1}, \tfrac{\alpha(\Psi-A)}{\alpha+1}\Big). \tag{38}$$

Parameterizing $\mathfrak{p}$ by

$$\mathfrak{p} = \Big\{\big(\tfrac{A}{\alpha+1}, -\tfrac{\alpha A}{\alpha+1}\big) \mid A \in \mathfrak{so}(n)\Big\}, \tag{39}$$

we obtain, for $A_1, A_2 \in \mathfrak{so}(k)$,

$$\begin{aligned}\mathrm{pr}_{\mathfrak{p}}\Big[\big(\tfrac{A_1}{\alpha+1}, -\tfrac{\alpha A_1}{\alpha+1}\big), \big(\tfrac{A_2}{\alpha+1}, -\tfrac{\alpha A_2}{\alpha+1}\big)\Big] &= \mathrm{pr}_{\mathfrak{p}}\Big(\tfrac{[A_1,A_2]}{(\alpha+1)^2}, \tfrac{\alpha^2[A_1,A_2]}{(\alpha+1)^2}\Big) \\ &= \Big(\tfrac{[A_1,A_2]-\alpha^2[A_1,A_2]}{(\alpha+1)^3}, \tfrac{\alpha(\alpha^2[A_1,A_2]-[A_1,A_2]}{(\alpha+1)^3}\Big).\end{aligned} \tag{40}$$

Clearly, the last equation vanishes for $k = n = 1$ and all $\alpha \in \mathbb{R} \setminus \{-1\}$. Moreover, it vanishes for $k = n > 1$ and all $A_1, A_2 \in \mathfrak{so}(n)$ iff $\alpha = 1$ holds. (Note that $\alpha = -1$ is excluded by the definition of the α-metrics in [9] (Def. 3.1).) We summarize these computations in the next corollary.

Corollary 3. *Let $1 < k < n$ and let $\alpha \in \mathbb{R} \setminus \{-1, 0\}$. Then, the Ansatz from Section 5.1 does not yield an intrinsic rolling, with respect to any α-metric, of a tangent space of the Stiefel manifold over the Stiefel manifold $\mathrm{St}_{n,k}$. However, for the case $k = n > 1$, the Ansatz yields only a rolling for $\alpha = 1$.*

5.3. Kinematic Equations for Intrinsic Rolling

Our aim is to find the triple $\big(\alpha(t), \widehat{\alpha}(t), A(t)\big)$ satisfying Definition 4 for a rolling of $\mathfrak{p}$ over the normal naturally reductive homogeneous space G/H.

More precisely, our goal is to find a system of ODEs, the so-called kinematic equations, which, for a prescribed rolling curve $\alpha\colon I \to \mathfrak{p}$, determines the development curve $\widehat{\alpha}\colon I \to G/H$ as well as the curve of isometries $A(t)\colon T_{\alpha(t)}\mathfrak{p} \cong \mathfrak{p} \to T_{\widehat{\alpha}(t)}(G/H)$.

The new terminology in the next definition is motivated by the theory of control, because the kinematic equations can be written as a control system whose control function is precisely $\dot{\alpha}(t)$.

Definition 7. *Given a rolling curve $\alpha\colon I \to \mathfrak{p}$, we call the curve $u\colon I \to \mathfrak{p}$, defined by $u(t) = \dot{\alpha}(t)$, the associated control curve.*

Note that a prescribed control curve $u\colon I \to \mathfrak{p}$ determines uniquely the rolling curve $\alpha\colon I \to \mathfrak{p}$ up to the initial condition $\alpha(0) = \alpha_0 \in \mathfrak{p}$.

In order to derive the kinematic equations, we start with the following remark.

Remark 5. *Let V and W be finite-dimensional pseudo-Euclidean vector spaces whose scalar products have the same signature and let $\phi\colon V \to W$ be an isometry. Then, the set of isometries between V and W is given by $\{\phi \circ S\colon V \to W \mid S \in O(V)\}$. Indeed, for $S \in O(V)$, $\phi \circ S$ is a* *composition of isometries, so it is an isometry, as well. Conversely, given an isometry $\psi\colon V \to W$, define the isometry $S = \phi^{-1} \circ \psi\colon V \to V$, which is an element of $O(V)$, and clearly $\psi = \phi \circ S$.*

In view of Remark 5, a possible candidate for the curve of isometries $A(t)\colon T_{\alpha(t)}\mathfrak{p} \cong \mathfrak{p} \to T_{\widehat{\alpha}(t)}(G/H)$ that is required for an intrinsic rolling is of the form

$$A(t) = \big(d_{q(t)}\pi\big) \circ \big(d_e L_{q(t)}\big) \circ S(t), \tag{41}$$

where $q\colon I \to G$ is the horizontal lift of the development curve $\widehat{\alpha}\colon I \to G/H$ through $q(0) = e$ and $S\colon I \to O(\mathfrak{p})$ is a curve in the orthogonal group of $\mathfrak{p}$ through $S(0) = \mathrm{id}_{\mathfrak{p}}$.

In the next theorem, we reproduce from [10] the kinematic equations for the rolling of $\mathfrak{p}$ over G/H. This statement holds for general normal naturally reductive homogeneous spaces, and the proof is provided to keep this paper as self-contained as possible.

Theorem 1. *Let G/H be a normal naturally reductive homogeneous space, $\alpha\colon I \to \mathfrak{p}$ a given curve, and $u\colon I \to \mathfrak{p}$ defined by $u(t) = \dot{\alpha}(t)$ the associated control curve. Moreover, let $S\colon I \to O(\mathfrak{p})$ and $q\colon I \to G$ be determined by the initial value problem*

$$\begin{aligned} \dot{S}(t) &= -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{S(t)u(t)} \circ S(t), \quad S(0) = \mathrm{id}_{\mathfrak{p}}, \\ \dot{q}(t) &= \big((d_e L_{q(t)}) \circ S(t)\big)u(t), \quad q(0) = e. \end{aligned} \tag{42}$$

Then, the triple $\big(\alpha(t), \widehat{\alpha}(t), A(t)\big)$, where

$$\widehat{\alpha}\colon I \to G/H, \quad t \mapsto \widehat{\alpha}(t) = (\pi \circ q)(t) \tag{43}$$

and

$$t \mapsto A(t) = (d_{q(t)}\pi) \circ (d_e L_{q(t)}) \circ S(t)\colon T_{\alpha(t)}\mathfrak{p} \cong \mathfrak{p} \to T_{\widehat{\alpha}(t)}(G/H), \tag{44}$$

is an intrinsic rolling of $\mathfrak{p}$ over G/H.

Proof. We show that $(\alpha(t), \widehat{\alpha}(t), A(t))$ satisfies the conditions of Definition 4. The solution S of the first equation in (42) is indeed a curve in $O(\mathfrak{p})$ because $-\frac{1}{2}\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{Su}\colon \mathfrak{p} \to \mathfrak{p}$ is skew-adjoint for all $S \in O(\mathfrak{p})$ and $u \in \mathfrak{p}$ with respect to the scalar product on $\mathfrak{p}$ defined by means of the bi-invariant metric on G. In fact, by exploiting that G/H is naturally reductive, using Definition 2, we obtain for $X, Y \in \mathfrak{p}$.

$$\begin{aligned} \big\langle -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{Su}(X), Y\big\rangle &= \big\langle -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}}\big([Su, X]\big), Y\big\rangle \\ &= \big\langle X, \tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{Su}(Y)\big\rangle, \end{aligned} \tag{45}$$

showing that $-\frac{1}{2}\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{Su} \in \mathfrak{so}(\mathfrak{p})$. Thus, $S(t) \in O(\mathfrak{p})$ because it is the integral curve of the time-variant vector field $-\frac{1}{2}\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{Su(t)} \circ S$ on $O(\mathfrak{p})$.

Next, we set $\widehat{\alpha}(t) = (\pi \circ q)(t)$. Obviously, the ODE for q in (42) implies that $q\colon I \to G$ is the horizontal lift of $\widehat{\alpha}$ through $q(0) = e$. Moreover, the map $A(t)\colon T_{\alpha(t)}\mathfrak{p} \cong \mathfrak{p} \to T_{\widehat{\alpha}(t)}(G/H)$ is well defined and an isometry because it is a composition of isometries.

We now show the no-slip condition. Indeed, by the chain-rule,

$$\begin{aligned} \dot{\widehat{\alpha}}(t) &= \tfrac{\mathrm{d}}{\mathrm{d}t}(\pi \circ q(t)) \\ &= (d_{q(t)}\pi)\dot{q}(t) \\ &= \big(d_{q(t)}\pi\big)\big(d_e L_{q(t)} \circ S(t)\big)u(t) \\ &= A(t)\dot{\alpha}(t). \end{aligned} \tag{46}$$

It remains to show the no-twist condition. Let $Z\colon I \to \mathfrak{p}$ be a parallel vector field along $\alpha\colon I \to \mathfrak{p}$, i.e., Z can be viewed as a constant function $Z(t) = Z_0$ for all $t \in I$ and some $Z_0 \in \mathfrak{p}$. We prove that the vector field $\widehat{Z}(t) = A(t)Z_0$ is parallel along the curve $\widehat{\alpha}$, by exploiting the result in Corollary 2. The curve $z \colon I \to \mathfrak{p}$ defined by

$$z(t) = (d_e L_{q(t)})^{-1} \circ (d_{q(t)}\pi)^{-1} A(t) Z_0 = S(t) Z_0 \tag{47}$$

fulfills

$$\begin{aligned} \dot{z}(t) &= \dot{S}(t) Z_0 \\ &= -\tfrac{1}{2} \circ \mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{S(t)u(t)} \circ S(t)(Z_0) \\ &= -\tfrac{1}{2} \big[S(t)u(t), S(t)(Z_0) \big] \Big|_{\mathfrak{p}} \\ &= -\tfrac{1}{2} \big[S(t)u(t), z(t) \big] \Big|_{\mathfrak{p}}. \end{aligned} \tag{48}$$

Thus, $Z(t) = A(t)Z_0$ is parallel along $\widehat{\alpha}(t) = (\pi \circ q)(t)$ by Corollary 2, due to the identity $\big(d_e L_{q(t)}\big)^{-1} \dot{q}(t) = S(t)u(t)$.

Conversely, assume that $A(t)Z(t)$ is parallel along $\widehat{\alpha}$ for some vector field $Z(t)$ along α. We define the parallel frame $A_i(t) = A(t)A_i$, where $\{A_1, \cdots, A_k\}$ forms a basis of $\mathfrak{p}$, and expand $A(t)Z(t)$ in this basis to obtain $A(t)Z(t) = \sum_{i=1}^k z_i A_i(t)$, where the coefficients $z_i \in \mathbb{R}$ are constant, because $A(t)Z(t)$ is assumed to be parallel, see [5] (Chap. 4, p.109). By the linearity of $A(t)$, we obtain

$$A(t)Z(t) = \sum_{i=1}^{k} z_i A_i(t) = A(t) \left(\sum_{i=1}^{k} z_i A_i \right) = A(t) Z_0, \tag{49}$$

for $Z_0 = \sum_{i=1}^k z_i A_i \in \mathfrak{p}$, i.e., $Z(t) = Z_0$ is constant. Thus, $Z(t)$ is a parallel vector field along α, as desired. □

Remark 6. *It is not clear whether the curve $S \colon I \to O(\mathfrak{p})$ from Theorem 1 is defined on the same interval I as the control curve $u \colon I \to \mathfrak{p}$ due to the nonlinearity of (42). We cannot rule out that S is defined only on a proper subinterval $I' \subsetneq I$ with $0 \in I'$. By abuse of notation, we write $S \colon I \to O(\mathfrak{p})$ nevertheless, even if S was defined on a proper subinterval. However, we are not aware of an example.*

If G/H is a Riemannian normal naturally reductive space, i.e., if the metric is positive definite, and the control defined on $\mathbb{R}$ is bounded, following [10], we can prove that S is defined on the whole interval $\mathbb{R}$. This is the next lemma.

Lemma 6. *Let $u \colon \mathbb{R} \to \mathfrak{p}$ be bounded and let G/H be a Riemannian normal naturally reductive homogeneous space. Then, the vector field given by*

$$X(t, S) = \big(1, -\tfrac{1}{2} \mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{S(t)u(t)} \circ S(t)\big) \tag{50}$$

on $\mathbb{R} \times O(\mathfrak{p})$ is complete.

Proof. We will show that this vector field is bounded in a complete Riemannian metric on $\mathbb{R} \times O(\mathfrak{p})$. Completeness then follows by [11] (Prop. 23.9). To this end, we view $O(\mathfrak{p})$ as a subset of $\mathrm{End}(\mathfrak{p})$. Because G/H is Riemannian, the corresponding scalar product on $\mathfrak{p}$ denoted by $\langle \cdot, \cdot \rangle$ is positive definite, i.e., an inner product. The norm on $\mathfrak{p}$ induced by this inner product is denoted by $\| \cdot \|$. We denote an extension of $\langle \cdot, \cdot \rangle$ to an inner product on $\mathfrak{g}$ by $\langle \cdot, \cdot \rangle$, too. The corresponding norm is denoted by $\| \cdot \|$, as well. We now endow $\mathrm{End}(\mathfrak{p})$ with the Frobenius scalar product given by $\langle S, T \rangle_F = \mathrm{trace}(S^\top T)$, where $S^\top$ is the adjoint of S with respect to $\langle \cdot, \cdot \rangle$. Then, $\langle \cdot, \cdot \rangle_F$ induces a bi-invariant and hence a complete metric on $O(\mathfrak{p})$. Moreover, the norm $\| \cdot \|_F$ defined by the Frobenius scalar product is equivalent to the operator norm $\| \cdot \|_2$. In particular, there is a $C > 0$ such that $\|S\|_F \leq C \|S\|_2$ holds for all $S \in \operatorname{End}(\mathfrak{p})$. In addition, on the $\mathbb{R}$-component, define the metric to be the Euclidean metric. In other words, the Riemannian metric on $\mathbb{R} \times O(\mathfrak{p})$ is given by

$$\langle (v, V), (w, W) \rangle_{(s,S)} = vw + \operatorname{trace}(V^\top W), \tag{51}$$

for all $(s, S) \in \mathbb{R} \times O(\mathfrak{p})$ and $(v, V), (w, W) \in T_{(s,S)}(\mathbb{R} \times O(\mathfrak{p}))$. Moreover, $\operatorname{ad}\colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{g}$ is bounded because $\mathfrak{p}$ is finite dimensional. Hence, there exists a $C' \geq 0$ with $\|\operatorname{ad}(X, Y)\| \leq C'\|X\|\|Y\|$. Consequently, for fixed $X \in \mathfrak{p}$, the operator norm of $\operatorname{ad}_X\colon \mathfrak{p} \to \mathfrak{g}$ can be estimated by $\|\operatorname{ad}(X, \cdot)\|_2 \leq C'\|X\|$. By this notation, we compute

$$\begin{aligned}
\|X(t, S)\|^2_{\mathbb{R}\times O(\mathfrak{p})} &= 1 + \|\tfrac{1}{2}\operatorname{pr}_{\mathfrak{p}} \circ \operatorname{ad}_{Su(t)} \circ S\|^2_F \\
&\leq 1 + \tfrac{C^2}{4}\|\operatorname{pr}_{\mathfrak{p}} \circ \operatorname{ad}_{Su(t)} \circ S\|^2_2 \\
&\leq 1 + \tfrac{C^2}{4}\|\operatorname{pr}_{\mathfrak{p}}\|^2_2\|\operatorname{ad}_{Su(t)}\|^2_2\|S\|^2_2 \\
&\leq 1 + \tfrac{(CC')^2}{4}\|S\|^2_2\|u(t)\|^2 \\
&\leq 1 + \tfrac{(CC')^2}{4}\|u\|^2_\infty < \infty,
\end{aligned} \tag{52}$$

where $\|u\|_\infty$ denotes the supremum norm of u and we exploited $\|S\|_2 = 1$ due to $S \in O(\mathfrak{p})$ and $\|\operatorname{pr}_{\mathfrak{p}}\|_2 \leq 1$, showing that X is bounded in a complete Riemannian metric. □

6. Rolling Stiefel Manifolds

A first attempt to generalize the rolling for pseudo-Riemannian symmetric spaces, as discussed in Section 5, does not work for Stiefel manifolds by Section 5.2. However, rolling maps for Stiefel manifolds have already appeared in [2] and more recently also in [1] (Sec. 5).

In this section, we reformulate the most recent results in [10], without using fiber-bundle techniques, to describe the intrinsic rolling of Stiefel manifolds equipped with the so-called α-metrics defined in [9]. Although, up to now, we have used the Greek letter α for rolling curves, in the first part of this section we will use the same letter α for the real parameter that defines a family of metrics on Stiefel manifolds. This will not create difficulties, because it will be clear from the context. In order to reach the above-mentioned objective, we specialize Theorem 1 to Stiefel manifolds. Eventually, this rolling is extended to an extrinsic rolling for the Euclidean metric. Finally, we show that our findings coincide with the rolling results from [2].

6.1. Stiefel Manifolds Equipped with α-Metrics as Normal Naturally Reductive Homogeneous Spaces

The Stiefel manifold $\operatorname{St}_{n,k}$ can be viewed as the embedded submanifold

$$\operatorname{St}_{n,k} = \{X \in \mathbb{R}^{n\times k} \mid X^\top X = I_k\}, \quad 1 \leq k \leq n \tag{53}$$

of $\mathbb{R}^{n\times k}$. In the sequel, we recall the so-called α-metrics on $\operatorname{St}_{n,k}$ introduced in [9] and show that $\operatorname{St}_{n,k}$ equipped with an α-metric can be viewed as a normal naturally reductive homogeneous space. The $(O(n) \times O(k))$-left action

$$\Phi\colon \big(O(n) \times O(k)\big) \times \mathbb{R}^{n\times k} \to \mathbb{R}^{n\times k}, \quad \big((R, \theta), X\big) \to RX\theta^\top, \tag{54}$$

by linear isomorphisms restricts to a transitive action

$$\big(O(n) \times O(k)\big) \times \operatorname{St}_{n,k} \to \operatorname{St}_{n,k}, \quad \big((R, \theta), X\big) \to RX\theta^\top \tag{55}$$

on $\operatorname{St}_{n,k}$, also denoted by Φ, which coincides with the action from in [9] (Eq. 12). Next, let $X \in \operatorname{St}_{n,k}$ be fixed, and denote by $H = \operatorname{Stab}(X) \subset O(n) \times O(k)$ the isotropy subgroup of X under the action Φ. Moreover, we write $G = O(n) \times O(k)$. Then, the Stiefel manifold $\operatorname{St}_{n,k}$ is diffeomorphic to the homogeneous space G/H. Moreover, the map

$$\iota_X \colon G/H \ni (R,\theta)\cdot H \mapsto RX\theta^\top \in \mathrm{St}_{n,k} \subset \mathbb{R}^{n\times k} \tag{56}$$

is a G-equivariant embedding, where $(R,\theta)\cdot H$ denotes the coset in G/H represented by $(R,\theta) \in G$.

In order to construct the α-metrics, the map

$$\langle\cdot,\cdot\rangle^{\alpha}_{\mathfrak{so}(n)\times\mathfrak{so}(k)} \colon \mathfrak{so}(n)\times\mathfrak{so}(k) \to \mathbb{R} \tag{57}$$

is defined on $\mathfrak{so}(n)\times\mathfrak{so}(k)$, for $\alpha \in \mathbb{R}\setminus\{0\}$, by

$$\big\langle(\Omega_1,\Psi_1),(\Omega_2,\Psi_2)\big\rangle^{\alpha}_{\mathfrak{so}(n)\times\mathfrak{so}(k)} = -\operatorname{trace}(\Omega_1\Omega_2) - \tfrac{1}{\alpha}\operatorname{trace}(\Psi_1\Psi_2), \tag{58}$$

see [9] (Eq. (21)).

Obviously, $\langle\cdot,\cdot\rangle^{\alpha}_{\mathfrak{so}(n)\times\mathfrak{so}(k)}$ yields a symmetric bilinear form on $\mathfrak{g} = \mathfrak{so}(n)\times\mathfrak{so}(k)$, which is Ad_G-invariant. Moreover, by [9] (Prop. 2), the subspace $\mathfrak{h} \subset \mathfrak{g}$ being the Lie algebra of $H = \mathrm{Stab}(X)$ for $X \in \mathrm{St}_{n,k}$ is non-degenerated for all $\alpha \in \mathbb{R}\setminus\{-1,0\}$.

After this preparation, we are in the position to reformulate [9] (Def. 3.3).

Definition 8. *Let $\alpha \in \mathbb{R}\setminus\{-1,0\}$. The α-metric on $\mathrm{St}_{n,k} \cong G/H$ is defined as the $G = O(n)\times O(k)$-invariant metric on G/H that turns the canonical projection $\pi\colon G \to G/H$ into a pseudo-Riemannian submersion, where G is equipped with the bi-invariant metric defined by means of the scalar product from* (58).

This definition turns G/H into a normal naturally reductive homogeneous space.

Lemma 7. *Let $\alpha \in \mathbb{R}\setminus\{-1,0\}$. Then, $G/H \cong \mathrm{St}_{n,k}$ equipped with an α-metric is a normal naturally reductive space. In particular, it is a naturally reductive homogeneous space.*

Proof. Obviously, $\mathrm{St}_{n,k} \cong G/H$ is a normal naturally reductive homogeneous space because the metric on G is bi-invariant and $\mathfrak{h} \subset \mathfrak{g}$ is a non-degenerated subspace. Hence, it is naturally reductive by Lemma 1. □

By requiring that $\iota_X \colon G/H \to \mathrm{St}_{n,k}$ from (56) is an isometry, the α-metric on $\mathrm{St}_{n,k}$ for $\alpha \in \mathbb{R}\setminus\{-1,0\}$, viewed as an embedded submanifold of $\mathbb{R}^{n\times k}$, is given by

$$\langle V,W\rangle^{(\alpha)}_X = 2\operatorname{trace}(V^\top W) + \tfrac{2\alpha+1}{\alpha+1}\operatorname{trace}(V^\top XX^\top W), \tag{59}$$

where $X \in \mathrm{St}_{n,k}$ and $V,W \in T_X\mathrm{St}_{n,k}$ by [9] (Cor. 2). In addition, if $\mathrm{St}_{n,k}$ is equipped with an α-metric, and $O(n)\times O(k)$ is equipped with the corresponding bi-invariant metric defined by the scalar product from (58), the map

$$\Phi_X = \iota_X \circ \pi \colon O(n)\times O(k) \to \mathrm{St}_{n,k}, \quad (R,\theta) \mapsto RX\theta^\top \tag{60}$$

is a pseudo-Riemannian submersion, where $X \in \mathrm{St}_{n,k}$ is arbitrary but fixed.

For considering the intrinsic rolling of $\mathrm{St}_{n,k} \cong G/H$, we need a formula for the orthogonal projection $\mathrm{pr}_{\mathfrak{p}} \colon \mathfrak{so}(n)\times\mathfrak{so}(k) \to \mathfrak{p}$ with respect to the metric defined in (58), where $\mathfrak{p} = \mathfrak{h}^\perp$, $\mathfrak{h}$ is the Lie algebra of $H = \mathrm{Stab}(X) \subset G$ for a fixed $X \in \mathrm{St}_{n,k}$. This is the next lemma, which is taken from [9] (Lem. 3.2).

Lemma 8. *Let $\alpha \in \mathbb{R}\setminus\{-1,0\}$. The orthogonal projection*

$$\mathrm{pr}_{\mathfrak{p}} \colon \mathfrak{so}(n)\times\mathfrak{so}(k) \to \mathfrak{p}, \quad (\Omega,\eta) \mapsto (\Omega^{\perp_X},\eta^{\perp_X}), \tag{61}$$

is given by

$$\begin{aligned} \Omega^{\perp_X} &= XX^\top\Omega + \Omega XX^\top - \tfrac{2\alpha+1}{\alpha+1}XX^\top\Omega XX^\top - \tfrac{1}{\alpha+1}X\eta X^\top, \\ \eta^{\perp_X} &= \tfrac{\alpha}{\alpha+1}(\eta - X^\top\Omega X). \end{aligned} \tag{62}$$

Proof. This is just a reformulation of [9] (Lem. 3.2). □

Because $\pi\colon G \to G/H$ is a pseudo-Riemannian submersion whose horizontal bundle is defined point-wise by $\mathcal{H}_g = (d_{(I_n,I_k)}L_g)(\mathfrak{p}) \subset T_gG$ and $\iota_X\colon G/H \to \mathrm{St}_{n,k}$ is an isometry, the map

$$d_{(I_n,I_k)}(\iota_X \circ \pi)\big|_{\mathfrak{p}}\colon \mathfrak{p} \to T_X\mathrm{St}_{n,k}, \quad (\Omega,\eta) \mapsto \Omega X - X\eta, \tag{63}$$

as well as its inverse are linear isometries. For the discussion of rolling Stiefel manifolds, we need an explicit formula for

$$\big(d_{(I_n,I_k)}(\iota_X \circ \pi)\big|_{\mathfrak{p}}\big)^{-1}\colon T_X\mathrm{St}_{n,k} \to \mathfrak{p}. \tag{64}$$

Such a formula is given in the next lemma, which is a trivial reformulation of [9] (Prop. 3).

Lemma 9. *Let $\alpha \in \mathbb{R} \setminus \{-1,0\}$ and $X \in \mathrm{St}_{n,k}$. The map*

$$\big(d_{(I_n,I_k)}(\iota_X \circ \pi)\big|_{\mathfrak{p}}\big)^{-1}\colon T_X\mathrm{St}_{n,k} \to \mathfrak{p}, \qquad V \mapsto \big(\Omega(V)^{\perp_X}, \eta(V)^{\perp_X}\big), \tag{65}$$

is given by

$$\begin{aligned} \Omega(V)^{\perp_X} &= VX^\top - XV^\top + \tfrac{2\alpha+1}{\alpha+1}XV^\top XX^\top, \\ \eta(V)^{\perp_X} &= -\tfrac{\alpha}{\alpha+1}X^\top V. \end{aligned} \tag{66}$$

Proof. This is a consequence of [9] (Prop. 3). □

Finally, we specialize the previous two lemmas for $\alpha = -\frac{1}{2}$. For this choice, the α-metric coincides with the Euclidean metric, scaled by the factor 2, see [9] (Sec. 4.2). Therefore, this special case will be important for discussing the extrinsic rolling of Stiefel manifolds equipped with the Euclidean metric.

Corollary 4. *Let $\alpha = -\frac{1}{2}$. Using the notation of Lemma 9, the following assertions are fulfilled:*

1. *The projection $\mathrm{pr}_{\mathfrak{p}}\colon \mathfrak{so}(n) \times \mathfrak{so}(k) \to \mathfrak{p}$ is given by*

$$\begin{aligned} \Omega^{\perp_X} &= XX^\top\Omega + \Omega XX^\top - 2X\eta X^\top, \\ \eta^{\perp_X} &= -(\eta - X^\top\Omega X). \end{aligned} \tag{67}$$

2. *The map $\big(d_{(I_n,I_k)}(\iota_X \circ \pi)\big|_{\mathfrak{p}}\big)^{-1}\colon T_X\mathrm{St}_{n,k} \to \mathfrak{p}$ is given by*

$$V \mapsto \big(\Omega(V)^{\perp_X}, \eta(V)^{\perp_X}\big) = \big(VX^\top - XV^\top, X^\top V\big). \tag{68}$$

Proof. This is a consequence of Lemmas 8 and 9. □

6.2. Intrinsic Rolling

In this section, using ideas from [10], we apply Theorem 1 to $\mathrm{St}_{n,k}$ equipped with an α-metric. More precisely, we use the isometry

$$\iota_X\colon G/H \to \mathrm{St}_{n,k} \tag{69}$$

to identify $\mathrm{St}_{n,k} \cong G/H$ as a normal naturally reductive homogeneous space, as well as the linear isometry

$$\left(d_{(I_n,I_k)}(\iota_X\circ\pi)\big|_{\mathfrak{p}}\right)^{-1}\colon T_X\mathrm{St}_{n,k}\to\mathfrak{p}, \tag{70}$$

identifying $T_X\mathrm{St}_{n,k}\cong\mathfrak{p}$ as vector spaces equipped with the scalar product from Section 6.1.

Throughout this section, if not indicated otherwise, we always assume that the maps from (69) and (70) are used to identify $G/H\cong\mathrm{St}_{n,k}$ and $\mathfrak{p}\cong T_X\mathrm{St}_{n,k}$, respectively.

These identifications allow for the construction of an intrinsic rolling of $T_X\mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$, where both manifolds are considered as embedded into $\mathbb{R}^{n\times k}$. We state the next definition in order to make this notion more precise.

Although, in the first part of this section, we have used the Greek letter α for the real parameter that defines a family of metrics on Stiefel, the same letter will be used later for rolling curves. This will not create difficulties, because it will be clear from the context.

Definition 9. *Consider the Stiefel manifold $\mathrm{St}_{n,k}\subset\mathbb{R}^{n\times k}$, equipped with an α-metric, as a submanifold of $\mathbb{R}^{n\times k}$. Moreover, let $X\in\mathrm{St}_{n,k}$ be fixed. Consider the triple $(\beta(t),\widehat{\beta(t)},B(t))$, where $\beta\colon I\to T_X\mathrm{St}_{n,k}\subset\mathbb{R}^{n\times k}$ and $\widehat{\beta}\colon I\to\mathrm{St}_{n,k}\subset\mathbb{R}^{n\times k}$ are curves and $B(t)\colon T_{\beta(t)}(T_X\mathrm{St}_{n,k})\cong T_X\mathrm{St}_{n,k}\to T_{\widehat{\beta}(t)}\mathrm{St}_{n,k}$ is a linear isometry. This triple is called an intrinsic rolling of $T_X\mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$, with both manifolds embedded into $\mathbb{R}^{n\times k}$, if the following conditions hold:*

1. *No-slip condition: $\dot{\widehat{\beta}}(t)=B(t)\dot{\beta}(t)$;*
2. *No-twist condition: $B(t)Z(t)$ is a parallel vector field along $\widehat{\beta}(t)$ iff $Z(t)$ is a parallel vector field along $\beta(t)$.*

The curve β is called a rolling curve and $\widehat{\beta}$ is called a development curve.

The next lemma uses Theorem 1 to obtain a rolling of $T_X\mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$ in the sense of Definition 9.

Lemma 10. *Let $\beta\colon I\to T_X\mathrm{St}_{n,k}\subset\mathbb{R}^{n\times k}$ be a curve and define the curve $\alpha\colon I\to\mathfrak{p}$ by $\alpha(t)=\left(d_{(I_n,I_k)}(\iota_X\circ\pi)\big|_{\mathfrak{p}}\right)^{-1}(\beta(t))$ for $t\in I$. Let $(\alpha(t),\widehat{\alpha}(t),A(t))$ be the triple obtained in Theorem 1 for the rolling along α of $T_X\mathrm{St}_{n,k}$ (identified with $\mathfrak{p}$), over G/H (identified with $\mathrm{St}_{n,k}$). Moreover, define the curve*

$$\widehat{\beta}\colon I\to\mathrm{St}_{n,k},\quad t\mapsto\widehat{\beta}(t)=\iota_X(\widehat{\alpha}(t)) \tag{71}$$

and the isometry $B(t)\colon T_{\beta(t)}(T_X\mathrm{St}_{n,k})\cong T_X\mathrm{St}_{n,k}\to T_{\widehat{\beta}(t)}\mathrm{St}_{n,k}$ by

$$B(t)=\left(d_{\widehat{\alpha}(t)}\iota_X\right)\circ A(t)\circ\left(d_{(I_n,I_k)}(\iota_X\circ\pi)^{-1}\right). \tag{72}$$

Then, the triple $(\beta(t),\widehat{\beta}(t),B(t))$ defines an intrinsic rolling of $T_X\mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$ in the sense of Definition 9.

Proof. The proof follows by applying Theorem 1 because G/H can be isometrically and G-equivariantly identified with $\mathrm{St}_{n,k}$ via $\iota_X\colon G/H\to\mathrm{St}_{n,k}$. Moreover, parallel vector fields are mapped to parallel vector fields by isometries.

In more detail, the no-slip condition holds as

$$\begin{aligned}\dot{\widehat{\beta}}(t)&=\tfrac{\mathrm{d}}{\mathrm{d}t}(\iota_X\circ\widehat{\alpha})(t)\\&=(d_{\widehat{\alpha}(t)}\iota_X)\dot{\widehat{\alpha}}(t)\\&=(d_{\widehat{\alpha}(t)}\iota_X)\big(A(t)\dot{\alpha}(t)\big)\\&=(d_{\widehat{\alpha}(t)}\iota_X)\circ A(t)\circ\left(d_{(I_n,I_k)}(\iota_X\circ\pi)\big|_{\mathfrak{p}}\right)^{-1}\big(\dot{\beta}(t)\big)\\&=B(t)\dot{\beta}(t).\end{aligned} \tag{73}$$

Next, we consider a parallel vector field $V\colon I\to T(T_X\mathrm{St}_{n,k})$ along β, i.e., V can be viewed as the constant map $V(t)=V_0$ for $t\in I$ and some $V_0\in T_X\mathrm{St}_{n,k}$. Clearly,

$Z(t) = \big(d_{(I_n,I_k)}(\iota_X \circ \pi)^{-1}\big)(V(t)) = Z_0$ is constant, with $Z_0 = \big(d_{(I_n,I_k)}(\iota_X \circ \pi)^{-1}\big)(V_0)$, i.e., $Z(t)$ is a parallel vector field along the curve α. Thus, by Theorem 1, the vector field $A(t)Z(t)$ is parallel along $\widehat{\alpha}$. Because $\iota_X \colon G/H \to \mathrm{St}_{n,k}$ is an isometry, this parallel vector field is mapped to the parallel vector field $d_{\widehat{\alpha}(t)}\iota_X\big(A(t)Z(t)\big)$ along the curve $\widehat{\beta}(t) = \iota_X(\widehat{\alpha}(t))$.

Conversely, assuming that $d_{\widehat{\alpha}(t)}\iota_X(A(t)Z(t))$ is parallel along $\widehat{\beta}$, one shows by exploiting Theorem 1 that $Z(t)$ is parallel along $\widehat{\alpha}$ because $\iota_X^{-1} \colon \mathrm{St}_{n,k} \to G/H$ is an isometry. Hence, $V(t) = T_{(I_n,I_k)}(\iota_X \circ \pi)(Z(t))$ is parallel along β. □

As a corollary, we reformulate the kinematic equations for the intrinsic rolling of Stiefel manifolds in the sense of Definition 9.

Corollary 5. *Let $\beta \colon I \to \mathrm{St}_{n,k}$ be a curve and let $u \colon I \to \mathfrak{p}$ be the associated control curve, so that $u(t) = \big(d_{(I_n,I_k)}(\iota_X \circ \pi)\big|_{\mathfrak{p}}\big)^{-1}\big(\dot{\beta}(t)\big)$ for $t \in I$. Consider the curves $S \colon I \to O(\mathfrak{p})$ as well as $q \colon I \ni t \mapsto q(t) = (R(t), \theta(t)) \in O(n) \times O(k)$ defined by the initial value problems*

$$\begin{aligned} \dot{S}(t) &= -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{S(t)u(t)} \circ S(t), \quad S(0) = \mathrm{id}_{\mathfrak{p}} \\ \dot{q}(t) &= \big(d_{(I_n,I_k)}L_{q(t)}\big)S(t)u(t), \quad q(0) = (I_n, I_k). \end{aligned} \tag{74}$$

Then, the triple $(\beta(t), \widehat{\beta}(t), B(t))$ defines an intrinsic rolling of $T_X\mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$, where

$$\widehat{\beta} \colon I \to \mathrm{St}_{n,k}, \quad t \mapsto (\iota_X \circ \pi)(q(t)) = R(t)X\theta(t)^\top \tag{75}$$

and

$$B(t) = d_{q(t)}(\iota_X \circ \pi) \circ d_e L_{q(t)} \circ S(t) \circ d_{(I_n,I_k)}(\iota_X \circ \pi)^{-1}. \tag{76}$$

Proof. This is a consequence of Lemma 10 combined with Theorem 1. □

6.3. Extrinsic Rolling

We now consider $\mathrm{St}_{n,k}$ embedded into $\mathbb{R}^{n\times k}$, equipped with the metric induced by the Frobenius scalar product scaled by the factor of two, i.e., the metric on $\mathrm{St}_{n,k}$ is given by

$$\langle V, W\rangle_X = 2\,\mathrm{trace}\,\big(V^\top W\big), \quad X \in \mathrm{St}_{n,k},\ V, W \in T_X\mathrm{St}_{n,k}. \tag{77}$$

This metric corresponds to the α-metric, when $\alpha = -\frac{1}{2}$. In the sequel, we will refer to this metric as the Euclidean metric.

We now construct a quadruple $(\beta(t), \widehat{\beta}(t), B(t), C(t))$, which satisfies Definition 5.

To this end, we first recall that a vector field $\widehat{Z} \colon I \to N\mathrm{St}_{n,k}$ along a curve $\widehat{\beta} \colon I \to \mathrm{St}_{n,k}$ is normal parallel if

$$\nabla_{\dot{\widehat{\beta}}(t)}\widehat{Z}(t) = P^{\perp}_{\widehat{\beta}(t)}\big(\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{Z}(t)\big) = 0, \quad t \in I, \tag{78}$$

holds, where $P^{\perp}_X \colon \mathbb{R}^{n\times k} \to N_X\mathrm{St}_{n,k}$ denotes the orthogonal projection onto the normal space $N_X\mathrm{St}_{n,k} = (T_X\mathrm{St}_{n,k})^{\perp}$ of $\mathrm{St}_{n,k}$ at the point X with respect to the Euclidean metric. This projection is given by

$$P^{\perp}_X(V) = \tfrac{1}{2}X(X^\top V + V^\top X), \quad V \in \mathbb{R}^{n\times k}, \tag{79}$$

see, e.g., [12].

In order to determine the curve $T \colon I \to O(N_X\mathrm{St}_{n,k})$, we derive an ODE that is satisfied by a curve associated to a normal vector field iff the vector field is parallel. To this end, we first recall that $\Phi_X = \iota_X \circ \pi \colon O(n) \times O(k) \to \mathrm{St}_{n,k}$ from (60) is a pseudo-Riemannian submersion. Hence, it makes sense to consider the horizontal lift of a curve $\widehat{\beta} \colon I \to \mathrm{St}_{n,k}$. In addtion, for fixed $(\xi_1, \xi_2) \in \mathfrak{so}(n) \times \mathfrak{so}(k)$, we define the linear map:

$$f_{(\xi_1,\xi_2)} \colon \mathbb{R}^{n\times k} \to \mathbb{R}^{n\times k}, \quad V \mapsto \xi_1 V - V\xi_2. \tag{80}$$

Lemma 11. *Let $X \in \mathrm{St}_{n,k}$ be fixed, $\widehat{\beta}\colon I \to \mathrm{St}_{n,k}$ a curve, and $\widehat{Z}\colon I \to N\mathrm{St}_{n,k}$ be a normal vector field along $\widehat{\beta}$. Moreover, let $q\colon I \ni t \mapsto q(t) = (R(t), \theta(t)) \in O(n) \times O(k)$ be a horizontal lift of $\widehat{\beta}$. Then, $\widehat{Z}$ is parallel along $\widehat{\beta}$ iff the curve*

$$z^\perp\colon I \to N_X\mathrm{St}_{n,k}, \quad t \mapsto z^\perp(t) = \Phi_{q(t)^{-1}}\big(\widehat{Z}(t)\big) = R(t)^\top \widehat{Z}(t)\theta(t), \tag{81}$$

satisfies the ODE

$$\dot{z}^\perp(t) = -\big(P_X^\perp \circ f_{(\xi_1(t), \xi_2(t))}\big)\big(z^\perp(t)\big), \quad t \in I, \tag{82}$$

where $(\xi_1(t), \xi_2(t)) = \big(R(t)^\top \dot{R}(t), \theta(t)^\top \dot{\theta}(t)\big) \in \mathfrak{so}(n) \times \mathfrak{so}(k)$.

Proof. Let $(R, \theta) \in O(n) \times O(k)$ and $X \in \mathrm{St}_{n,k}$. Then,

$$P^\perp_{\Phi_{(R,\theta)}(X)}(V) = \Phi_{(R,\theta)} \circ P_X^\perp \circ \Phi_{(R^\top, \theta^\top)}(V) \tag{83}$$

holds for $V \in \mathbb{R}^{n\times k}$ by the Φ-invariance of the Euclidean metric. Because $q(t) = (R(t), \theta(t))$ is a horizontal lift of $\widehat{\beta}$, i.e., $\widehat{\beta}(t) = (\iota_X \circ \pi)(q(t)) = R(t)X\theta(t)^\top$, (83) implies that

$$P^\perp_{\widehat{\beta}(t)}(V) = \Phi_{(R(t),\theta(t))} \circ P_X^\perp\big(R(t)^\top V \theta(t)\big). \tag{84}$$

Moreover, the condition $P^\perp_{\widehat{\beta}(t)}\big(\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{Z}(t)\big) = 0$ is equivalent to

$$P_X^\perp\big(R(t)^\top \big(\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{Z}(t)\big)\theta(t)\big) = 0 \tag{85}$$

by (84), because $\Phi_{(R(t),\theta(t))}\colon \mathbb{R}^{n\times k} \to \mathbb{R}^{n\times k}$ is a linear isomorphism. Obviously, by the definition of $z^\perp$, we have

$$\widehat{Z}(t) = R(t) z^\perp(t) \theta(t)^\top. \tag{86}$$

Plugging (86) into (85) yields

$$\begin{aligned}
0 &= P_X^\perp\big(R(t)^\top \big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big(R(t)z^\perp(t)\theta(t)^\top\big)\big)\theta(t)\big) \\
&= P_X^\perp\big(R(t)^\top \big(\dot{R}(t)z^\perp(t)\theta(t)^\top + R(t)\dot{z}^\perp(t)\theta(t)^\top + R(t)z^\perp(t)\dot{\theta}(t)^\top\big)\theta(t)\big) \\
&= P_X^\perp\big(R(t)^\top \dot{R}(t) z^\perp(t) + \dot{z}^\perp(t) + z^\perp(t)\dot{\theta}(t)^\top\theta(t)\big).
\end{aligned} \tag{87}$$

Using $(\xi_1(t), \xi_2(t)) = \big(R(t)^\top \dot{R}(t), \theta(t)^\top \dot{\theta}(t)\big)$ and $\theta(t)^\top\dot{\theta}(t) = -\dot{\theta}(t)^\top \theta(t)$, as well as $P_X^\perp\big(\dot{z}^\perp(t)\big) = \dot{z}^\perp(t)$ due to $z^\perp(t) \in N_X\mathrm{St}_{n,k}$, we can equivalently rewrite (87) by

$$\begin{aligned}
0 &= P_X^\perp\big(\xi_1(t)z^\perp(t) + \dot{z}^\perp(t) - z^\perp(t)\xi_2(t)\big) \\
&= \dot{z}^\perp(t) + \big(P_X^\perp \circ f_{(\xi_1(t),\xi_2(t))}\big)\big(z^\perp(t)\big).
\end{aligned} \tag{88}$$

This yields the desired result. □

After this preparation, we are in the position to determine the extrinsic rolling of $T_X\mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$ with respect to the Euclidean metric in the sense of Definition 5.

Theorem 2. *Let $X \in \mathrm{St}_{n,k}$ be fixed and let $\beta\colon I \to T_X\mathrm{St}_{n,k}$ be a curve. Moreover, let $(\beta(t), \widehat{\beta}(t), B(t))$ denote the intrinsic rolling of $T_X\mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$ from Lemma 10 for $\alpha = -\frac{1}{2}$. Furthermore, let $q\colon I \ni t \mapsto q(t) = (R(t), \theta(t)) \in O(n) \times O(k)$ be the horizontal lift of $\widehat{\beta}\colon I \to \mathrm{St}_{n,k}$ through $q(0) = (I_n, I_k)$ and define $(\xi_1, \xi_2)\colon I \to \mathfrak{so}(n) \times \mathfrak{so}(k)$ by*

$$(\xi_1(t), \xi_2(t)) = \big(d_{(I_n,I_k)}L_{q(t)}\big)^{-1}\dot{q}(t) = \big(R(t)^\top \dot{R}(t), \theta(t)^\top\dot{\theta}(t)\big), \tag{89}$$

for $t \in I$. Let $T\colon I \to O(N_X\mathrm{St}_{n,k})$ be the solution of the initial value problem

$$\dot{T}(t) = -P_X^{\perp} \circ f_{(\xi_1(t), \xi_2(t))} \circ T(t), \quad T(0) = \mathrm{id}_{N_X \mathrm{St}_{n,k}}. \tag{90}$$

Then, the quadruple $(\beta(t), \widehat{\beta}(t), B(t), C(t))$*, with*

$$C(t) \colon N_{\beta(t)}(T_X \mathrm{St}_{n,k}) \cong N_X \mathrm{St}_{n,k} \to N_{\widehat{\beta}(t)} \mathrm{St}_{n,k}, \tag{91}$$

defined by

$$C(t) = \Phi_{(R(t), \theta(t))} \circ T(t), \tag{92}$$

is an extrinsic rolling of $T_X \mathrm{St}_{n,k}$ *over* $\mathrm{St}_{n,k}$ *with respect to the Euclidean metric.*

Proof. We only need to show the normal no-twist condition because the tangential no-twist condition and the no-slip condition are fulfilled by Lemma 10. We start with proving that $T(t) \in O(N_X \mathrm{St}_{n,k})$, for $t \in I$. For that, we compute

$$\begin{aligned} \big\langle (-P_X^{\perp} \circ f_{(\xi_1(t), \xi_2(t))})(Y), Z \big\rangle_X &= -\big\langle f_{(\xi_1(t), \xi_2(t))}(Y), Z \big\rangle_X \\ &= -2 \operatorname{trace}\big((\xi_1(t) Y - Y \xi_2(t))^\top Z \big) \\ &= 2 \operatorname{trace}\big(Y^\top \xi_1(t) Z - \xi_2(t) Y^\top Z \big) \\ &= 2 \operatorname{trace}\big(Y^\top (\xi_1(t) Z - Z \xi_2(t)) \big) \\ &= \big\langle Y, (P_X^{\perp} \circ f_{(\xi_1(t), \xi_2(t))})(Z) \big\rangle, \end{aligned} \tag{93}$$

for $Y, Z \in N_X \mathrm{St}_{n,k}$, by exploiting $(\xi_1(t), \xi_2(t)) \in \mathfrak{so}(n) \times \mathfrak{so}(k)$. Thus, $-P_X^{\perp} \circ f_{(\xi_1(t), \xi_2(t))} \colon N_X \mathrm{St}_{n,k} \to N_X \mathrm{St}_{n,k}$ is skew-adjoint with respect to the Euclidean metric, implying that $-P_X^{\perp} \circ f_{(\xi_1(t), \xi_2(t))}) \circ T$, for $T \in O(N_X \mathrm{St}_{n,k})$, can be viewed as a time-variant vector field on $O(N_X \mathrm{St}_{n,k})$.

Next, we note that $C(t) \colon N_{\beta(t)}(T_X \mathrm{St}_{n,k}) \cong N_X \mathrm{St}_{n,k} \to N_{\widehat{\beta}(t)} \mathrm{St}_{n,k}$ is an isometry (as the composition of isometries). Now, let $Z^{\perp} \colon I \to N(T_X \mathrm{St}_{n,k})$ be a normal parallel vector field along $\beta \colon I \to T_X \mathrm{St}_{n,k}$. Then, $Z^{\perp}$ can be viewed as the constant curve $Z^{\perp}(t) = Z_0^{\perp}$, for $t \in I$ and some $Z_0^{\perp} \in N_X \mathrm{St}_{n,k}$. Obviously, $\widehat{Z}^{\perp} \colon I \to N\mathrm{St}_{n,k}$ given by

$$\widehat{Z}^{\perp}(t) = C(t) Z^{\perp}(t) = \big(\Phi_{(R(t), \theta(t))} \circ T(t) \big)(Z_0^{\perp}), \quad t \in I, \tag{94}$$

is a normal vector field along the curve $\widehat{\beta}$. It remains to show that $\widehat{Z}^{\perp}$ is parallel along $\widehat{\beta}$. To this end, we exploit Lemma 11. We consider the curve $z^{\perp} \colon I \to N_X \mathrm{St}_{n,k}$ given by

$$z^{\perp}(t) = \Phi_{(R(t)^\top, \theta(t)^\top)}\big(\widehat{Z}^{\perp}(t) \big) = T(t)(Z_0^{\perp}) \tag{95}$$

and obtain

$$\begin{aligned} \dot{z}^{\perp}(t) &= \dot{T}(t)(Z_0^{\perp}) \\ &= -\big(P_X^{\perp} \circ f_{(\xi_1(t), \xi_2(t))} \circ T(t) \big)(Z_0^{\perp}) \\ &= -\big(P_X^{\perp} \circ f_{(\xi_1(t), \xi_2(t))} \big)\big(z^{\perp}(t) \big) \end{aligned} \tag{96}$$

due to (90). Thus, $\widehat{Z}^{\perp}$ is parallel along $\widehat{\beta}$ by Lemma 11.

Conversely, assume that $\widehat{Z}^{\perp} \colon I \to N\mathrm{St}_{n,k}$ given by $Z^{\perp}(t) = C(t) Z(t)^{\perp}$ for some $Z^{\perp} \colon I \to N_X \mathrm{St}_{n,k}$ is normal parallel along $\widehat{\beta}$. We define the normal parallel frame along $\widehat{\beta}$ by $A_i^{\perp}(t) = C(t) A_i$, where the vectors $A_i^{\perp} \in N_X \mathrm{St}_{n,k}$ for $i \in \{1, \ldots, \ell_n\}$ with $\ell_n = \dim(N_X \mathrm{St}_{n,k})$ form a basis. Then, analogously to [5] (Chap. 4, p. 106), one shows that $\widehat{Z}^{\perp}$ is normal parallel along $\widehat{\beta}$ iff the coefficient functions $z^i \colon I \to \mathbb{R}$ defined by $\widehat{Z}^{\perp}(t) = \sum_{i=1}^{\ell_n} z_i(t) A_i^{\perp}(t)$ are constant. Because $\widehat{Z}^{\perp}$ is assumed to be normal parallel, there exists a uniquely determined $z_i \in \mathbb{R}$ such that $Z^{\perp}(t) = \sum_{i=1}^{\ell_n} z_i A_i^{\perp}(t)$ is fulfilled. Hence, by the linearity of $C(t) \colon N_{\beta(t)}(T_X \mathrm{St}_{n,k}) \cong N_X \mathrm{St}_{n,k} \to N_{\widehat{\beta}(t)} \mathrm{St}_{n,k}$, we obtain

$$\widehat{Z}^{\perp}(t) = \sum_{i=1}^{\ell_n} z_i A_i^{\perp}(t) = \sum_{i=1}^{\ell_n} z_i C(t) A_i^{\perp} = \sum_{i=1}^{\ell_n} C(t)(z^i A_i^{\perp}) = C(t) Z^{\perp}, \tag{97}$$

where $Z^{\perp} = \sum_{i=1}^{\ell_n} z_i A_i^{\perp}$ is viewed as a normal vector field along β, which is clearly normal parallel. This yields the desired result. □

As a corollary of Theorem 2, we obtain the kinematic equations for the extrinsic rolling of $T_X \mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$ with respect to the Euclidean metric.

Corollary 6. *Let $X \in \mathrm{St}_{n,k}$ be fixed and let $\beta \colon I \to T_X \mathrm{St}_{n,k}$ be a prescribed rolling curve with an associated control curve*

$$u \colon I \ni t \mapsto \left(d_{(I_n, I_k)}(\iota_X \circ \pi)\big|_{\mathfrak{p}}\right)^{-1}(\dot{\beta}(t)) \in \mathfrak{p} \tag{98}$$

viewed as a curve in $\mathfrak{p}$, where

$$\left(d_{(I_n, I_k)}(\iota_X \circ \pi)\big|_{\mathfrak{p}}\right)^{-1} \colon T_X \mathrm{St}_{n,k} \to \mathfrak{p} \tag{99}$$

is given by Corollary 4. Moreover, let the curves $S \colon I \to O(\mathfrak{p})$ and $q \colon I \to O(n) \times O(k)$, as well as $T \colon I \to O(N_X \mathrm{St}_{n,k})$, be defined by the initial value problem

$$\begin{aligned}
\dot{S}(t) &= -\tfrac{1}{2} \mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{S(t)u(t)} \circ S(t), \quad S(0) = \mathrm{id}_{\mathfrak{p}}, \\
\dot{q}(t) &= \left(d_{(I_n, I_k)} L_{q(t)}\right) S(t) u(t), \quad q(0) = (I_n, I_k), \\
\dot{T}(t) &= -P_X^{\perp} \circ f_{(\xi_1(t), \xi_2(t))} \circ T(t), \quad T(0) = \mathrm{id}_{N_X \mathrm{St}_{n,k}},
\end{aligned} \tag{100}$$

where $f_{(\xi_1, \xi_2)} \colon \mathbb{R}^{n \times k} \to \mathbb{R}^{n \times k}$ is given by (80) and $\mathrm{pr}_{\mathfrak{p}} \colon \mathfrak{so}(n) \times \mathfrak{so}(k) \to \mathfrak{p}$ is determined in Corollary 4. Then, $(\beta(t), \widehat{\beta}(t), B(t), C(t))$ defines an extrinsic rolling of $T_X \mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$ with respect to the Euclidean metric, where

$$\widehat{\beta} \colon I \to \mathrm{St}_{n,k}, \quad t \mapsto (\iota_X \circ \pi)(q(t)) = R(t) X \theta(t)^{\top}, \tag{101}$$

$$B(t) = d_{(q(t))}(\iota_X \circ \pi) \circ \left((d_e L_{q(t)}) \circ S(t)\right) \circ d_{(I_n, I_k)}(\iota_X \circ \pi)^{-1}, \tag{102}$$

and

$$C(t) = \Phi_{(R(t), \theta(t))} \circ T(t). \tag{103}$$

We call the Equation (100) kinematic equations for the extrinsic rolling of $T_X \mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$ with respect to the Euclidean metric.

6.4. Rolling Along Special Curves

In this subsection, we consider a rolling of $T_X \mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$ such that its development curve $\widehat{\beta} \colon I \to \mathrm{St}_{n,k}$ is the projection of a *not* necessarily horizontal one-parameter subgroup, i.e., a curve

$$\widehat{\beta} \colon I \to \mathrm{St}_{n,k}, \quad t \mapsto (\iota_X \circ \pi)(\exp(t\xi)) = \mathrm{e}^{t\xi_1} X \mathrm{e}^{-t\xi_2}, \tag{104}$$

for some $(\xi_1, \xi_2) \in \mathfrak{so}(n) \times \mathfrak{so}(k)$, where $X \in \mathrm{St}_{n,k}$ is fixed. For this special case, which includes the curves considered in [13], we determine an extrinsic rolling $(\beta(t), \widehat{\beta}(t), B(t), C(t))$ explicitly. To this end, we proceed as in [10], where the intrinsic rolling of general reductive spaces along such a curve are determined explicitly. However, for the following discussion, we will restrict to the study of Stiefel manifolds equipped with the Euclidean metric, as it allows for simplifying some arguments.

Before we continue, we fix some notations. Let $\xi = (\xi_1, \xi_2) \in \mathfrak{so}(n) \times \mathfrak{so}(k)$. Let $\xi_{\mathfrak{h}} = (\xi_{1,\mathfrak{h}}, \xi_{2,\mathfrak{h}})$ and $\xi_{\mathfrak{p}} = (\xi_{1,\mathfrak{p}}, \xi_{2,\mathfrak{p}})$ denote the projections of ξ onto $\mathfrak{h}$ and onto $\mathfrak{p}$, respectively. Here, the reductive decomposition is always understood to be taken with respect to the α-

metric, where $\alpha = -\frac{1}{2}$. In particular, the subspaces $\mathfrak{h}$ and $\mathfrak{p}$ of $\mathfrak{so}(n) \times \mathfrak{so}(k)$ are orthogonal with respect to the scalar product $\langle \cdot, \cdot \rangle^{\alpha}_{\mathfrak{so}(n)\times\mathfrak{so}(k)}$ defined in (58).

We first consider the horizontal lift of a curve given by (104).

Lemma 12. *Let $X \in \mathrm{St}_{n,k}$ and $\xi = (\xi_1, \xi_2) \in \mathfrak{so}(n) \times \mathfrak{so}(k)$. The horizontal lift of*

$$\widehat{\beta}\colon I \to \mathrm{St}_{n,k}, \quad t \mapsto \widehat{\beta}(t) = (\iota_X \circ \pi)\big(\exp(t\xi)\big) = \mathrm{e}^{t\xi_1} X \mathrm{e}^{-t\xi_2}, \tag{105}$$

through $q(0) = (I_n, I_k)$ is given by

$$\begin{aligned} q\colon I &\to O(n) \times O(k) \\ t &\mapsto \exp(t\xi)\exp(-t\xi_{\mathfrak{h}}) = \big(\mathrm{e}^{t\xi_1} e^{-t\xi_{1,\mathfrak{h}}}, \mathrm{e}^{t\xi_2}\mathrm{e}^{-t\xi_{2,\mathfrak{h}}}\big). \end{aligned} \tag{106}$$

Moreover, it is the solution of the initial value problem

$$\dot{q}(t) = \big(d_{(I_n,I_k)} L_{q(t)}\big) \mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{p}}), \quad q(0) = (I_n, I_k). \tag{107}$$

Proof. Obviously, $q(0) = (I_n, I_k)$ holds and

$$\widehat{\beta}(t) = (\iota_X \circ \pi)(\exp(t\xi)) = (\iota_X \circ \pi)\big(\exp(t\xi)\exp(-t\xi_{\mathfrak{h}})\big) \tag{108}$$

is fulfilled because $t \mapsto \exp(-t\xi_{\mathfrak{h}})$ is a curve in $H \subset O(n) \times O(k)$.

We claim that q is horizontal. Indeed, by using the well-known properties of the matrix exponential

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\exp(t\xi) = \exp(t\xi)\xi \quad \text{and} \quad \tfrac{\mathrm{d}}{\mathrm{d}t}\exp(t\xi_{\mathfrak{h}}) = \xi_{\mathfrak{h}}\exp(t\xi_{\mathfrak{h}}), \tag{109}$$

we compute

$$\dot{q}(t) = \exp(t\xi)\xi\exp(-t\xi_{\mathfrak{h}}) - \exp(t\xi)\xi_{\mathfrak{h}}\exp(-t\xi_{\mathfrak{h}}) \tag{110}$$

yielding

$$\begin{aligned} (d_{(I_n,I_k)} L_{q(t)})^{-1}\dot{q}(t) &= \exp(t\xi_{\mathfrak{h}})\exp(-t\xi)\dot{q}(t) \\ &= \exp(t\xi_{\mathfrak{h}})\exp(-t\xi)\big(\exp(t\xi)\xi\exp(-t\xi_{\mathfrak{h}}) \\ &\quad - \exp(t\xi)\xi_{\mathfrak{h}}\exp(-t\xi_{\mathfrak{h}})\big) \\ &= \exp(t\xi_{\mathfrak{h}})\xi\exp(-t\xi_{\mathfrak{h}}) \\ &\quad - \exp(t\xi_{\mathfrak{h}})\xi_{\mathfrak{h}}\exp(-t\xi_{\mathfrak{h}}) \\ &= \exp(t\xi_{\mathfrak{h}})\xi_{\mathfrak{p}}\exp(-t\xi_{\mathfrak{h}}) \\ &= \mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{p}}). \end{aligned} \tag{111}$$

Here, we exploited the fact that that $O(n) \times O(k)$ can be viewed as a matrix Lie group. Hence, $q\colon I \to O(n) \times O(k)$ is horizontal due to $\mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{p}}) \in \mathfrak{p}$ because $\mathfrak{so}(n) \times \mathfrak{so}(k) = \mathfrak{h} \oplus \mathfrak{p}$ is a reductive decomposition. In addition, (111) implies that q is the solution of (107), as desired. □

Next, we determine the intrinsic rolling $(\alpha(t), \widehat{\alpha}(t), A(t))$ of $T_X\mathrm{St}_{n,k} \cong \mathfrak{p}$ over $\mathrm{St}_{n,k} \cong (O(n) \times O(k))/H$ viewed as a normal naturally reductive homogeneous space, where $\widehat{\alpha}(t) = \pi(\exp(t\xi))$ for some $\xi \in \mathfrak{so}(n) \times \mathfrak{so}(k)$.

To this end, we recall the kinematic equations from Theorem 1. They are given by

$$\begin{aligned} \dot{S}(t) &= -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{S(t)u(t)} \circ S(t), \quad S(0) = \mathrm{id}_{\mathfrak{p}}, \\ \dot{q}(t) &= \big(d_{(I_n,I_k)} L_{q(t)}\big) S(t)u(t), \quad q(0) = (I_n, I_k), \end{aligned} \tag{112}$$

where

$$S(t)u(t) = \mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{p}}), \tag{113}$$

for $t \in I$, by the definition of $\widehat{\alpha}(t) = \pi(\exp(t\xi)) = (\iota_X)^{-1}(\widehat{\beta}(t))$ and Lemma 12. Thus, the ODE for $S\colon I \to O(\mathfrak{p})$ in (112) becomes

$$\dot{S}(t) = -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{\mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{p}})} \circ S(t), \quad S(0) = \mathrm{id}_{\mathfrak{p}}. \tag{114}$$

In order to determine the intrinsic rolling explicitly, we need to solve this equation. As a preparation, we state a lemma on time-variant linear ODEs, which is inspired by [14] (p. 48).

Lemma 13. *Let V be a finite-dimensional real vector space and let $A, B \in \mathrm{End}(V)$ be linear maps on V. Consider the curve $S\colon I \to GL(V)$ defined by the initial value problem*

$$\dot{S}(t) = \exp(tA) \circ B \circ \exp(-tA) \circ S(t), \quad S(0) = S_0 \in GL(V). \tag{115}$$

Then, S is given by

$$S(t) = \exp(tA) \circ \exp(t(B - A)) \circ S_0. \tag{116}$$

Proof. Define $\widetilde{S}\colon I \to GL(V)$ by $\widetilde{S}(t) = \exp(-tA) \circ S(t)$. Then,

$$\begin{aligned} \dot{\widetilde{S}}(t) &= -A \circ \exp(-tA) \circ S(t) + \exp(-tA) \circ \dot{S}(t) \\ &= -A \circ \widetilde{S}(t) + \exp(-tA) \circ \exp(tA) \circ B \circ \exp(-tA) \circ S(t) \\ &= (B - A) \circ \widetilde{S}(t), \end{aligned} \tag{117}$$

for $t \in I$, implying that $\widetilde{S}(t) = \exp(t(B - A)) \circ S_0$. Consequently, by the definition of $\widetilde{S}$, we obtain

$$S(t) = \exp(tA) \circ \widetilde{S}(t) = \exp(tA) \circ \exp(t(B - A)) \circ S_0, \quad t \in I. \tag{118}$$

□

Lemma 14. *Let $\xi \in \mathfrak{so}(n) \times \mathfrak{so}(k)$. The solution of the initial value problem*

$$\dot{S}(t) = -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{\mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{p}})} \circ S(t), \quad S(0) = \mathrm{id}_{\mathfrak{p}}, \tag{119}$$

is given by

$$S\colon I \to O(\mathfrak{p}), \quad t \mapsto \mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})} \circ \exp\big(-t\big(\mathrm{ad}_{\xi_{\mathfrak{h}}} + \tfrac{1}{2}(\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{\xi_{\mathfrak{p}}})\big)\big). \tag{120}$$

Proof. Rewrite (119) such that Lemma 13 can be applied. We compute

$$\begin{aligned} \dot{S}(t) &= -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{\mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{p}})} \circ S(t) \\ &= -\tfrac{1}{2}\mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})} \circ \mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{\xi_{\mathfrak{p}}} \circ \mathrm{Ad}_{\exp(-t\xi_{\mathfrak{h}})} \circ S(t) \\ &= -\tfrac{1}{2}\exp(t\mathrm{ad}_{\xi_{\mathfrak{h}}}) \circ \mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{\xi_{\mathfrak{p}}} \circ \exp(-t\mathrm{ad}_{\xi_{\mathfrak{h}}}) \circ S(t), \end{aligned} \tag{121}$$

where in the first equality we used the fact that $\mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}\colon \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra morphism and, moreover, $\mathrm{Ad}_h \circ \mathrm{pr}_{\mathfrak{p}} = \mathrm{pr}_{\mathfrak{p}} \circ \mathrm{Ad}_h$ holds due to $\mathrm{Ad}_h(\mathfrak{p}) \subset \mathfrak{p}$ as well as $\mathrm{Ad}_h(\mathfrak{h}) \subset \mathfrak{h}$, for $h \in H$. For the second equality, $\mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})} = \exp(\mathrm{ad}_{t\xi_{\mathfrak{h}}})$ is used. Hence, we can apply Lemma 13 with $A = \mathrm{ad}_{\xi_{\mathfrak{h}}}$ and $B = -\frac{1}{2}\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{\xi_{\mathfrak{p}}}$. This yields

$$\begin{aligned} S(t) &= \exp(t\mathrm{ad}_{\xi_{\mathfrak{h}}}) \circ \exp\big(t\big(-\tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{\xi_{\mathfrak{p}}} - \mathrm{ad}_{\xi_{\mathfrak{h}}}\big)\big) \circ \mathrm{id}_{\mathfrak{p}} \\ &= \mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})} \circ \exp\big(-t\big(\mathrm{ad}_{\xi_{\mathfrak{h}}} + \tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{\xi_{\mathfrak{p}}}\big)\big), \end{aligned} \tag{122}$$

as desired. □

We proceed with determining the intrinsic rolling $(\alpha(t), \widehat{\alpha}(t), A(t))$. Recall that the control curve $u\colon I \to \mathfrak{p}$ is defined by $u(t) = \dot{\alpha}(t)$. Hence, (113) yields

$$\dot{\alpha}(t) = S(t)^{-1}\mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{p}}) = \exp\left(t\left(\mathrm{ad}_{\xi_{\mathfrak{h}}} + \tfrac{1}{2}(\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{\xi_{\mathfrak{p}}})\right)\right)(\xi_{\mathfrak{p}}), \tag{123}$$

where we used the formula for $S\colon I \to O(\mathfrak{p})$ from Lemma 14. Therefore,

$$\alpha(t) = \int_0^t \exp\left(s\left(\mathrm{ad}_{\xi_{\mathfrak{h}}} + \tfrac{1}{2}(\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{\xi_{\mathfrak{p}}})\right)\right)(\xi_{\mathfrak{p}})\,\mathrm{d}\,s \tag{124}$$

is the rolling curve $\alpha\colon I \to \mathfrak{p}$.

We summarize our findings for the intrinsic rolling of $T_X\mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$ in the next proposition.

Proposition 4. *Let $\xi = (\xi_1, \xi_2) \in \mathfrak{so}(n) \times \mathfrak{so}(k)$ and $X \in \mathrm{St}_{n,k}$. Then, the triple $(\alpha(t), \widehat{\alpha}(t), A(t))$ with*

$$\begin{aligned}
\alpha(t) &= \int_0^t \exp\left(s\left(\mathrm{ad}_{\xi_{\mathfrak{h}}} + \tfrac{1}{2}(\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{\xi_{\mathfrak{p}}})\right)\right)(\xi_{\mathfrak{p}})\,\mathrm{d}\,s,\\
\widehat{\alpha}(t) &= \pi(\exp(t\xi)),\\
A(t) &= (d_{q(t)}\pi) \circ d_{(I_n,I_k)}L_{q(t)} \circ S(t),
\end{aligned} \tag{125}$$

for $t \in I$, where $q\colon I \ni t \mapsto \exp(t\xi)\exp(-t\xi_{\mathfrak{h}}) \in O(n) \times O(k)$ and

$$S\colon I \to O(\mathfrak{p}), \quad t \mapsto \mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})} \circ \exp\left(-t\left(\mathrm{ad}_{\xi_{\mathfrak{h}}} + \tfrac{1}{2}(\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{\xi_{\mathfrak{p}}})\right)\right), \tag{126}$$

is an intrinsic rolling of $T_X\mathrm{St}_{n,k} \cong \mathfrak{p}$ over $\mathrm{St}_{n,k} \cong (O(n) \times O(k))/H$, viewed as normal naturally reductive homogeneous space.

Remark 7. *Obviously, proceeding analogously to the proof of Proposition 4, one derives an explicit expression for the intrinsic rolling $(\alpha(t), \widehat{\alpha}(t), A(t))$ of $T_X\mathrm{St}_{n,k} \cong \mathfrak{p}$ over $\mathrm{St}_{n,k}$, where $\widehat{\alpha}(t) = \pi(\exp(t\xi))$ for $\xi \in \mathfrak{so}(n) \times \mathfrak{so}(k)$ for any α-metric, where $\alpha \in \mathbb{R} \setminus \{-1, 0\}$. Indeed, an explicit expression for the rolling of general reductive homogeneous spaces G/H whose development curve is given by $t \mapsto \pi(\exp(t\xi))$ for $\xi \in \mathfrak{g}$ is known, see [10].*

From now on, whenever convenient, we may interchangeably use two different notations, e^A and $\exp(A)$, for the exponential of a matrix.

To determine an extrinsic rolling $(\beta(t), \widehat{\beta}(t), B(t), C(t))$ of $T_X\mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$, with respect to the Euclidean metric whose development curve is given by $\widehat{\beta}\colon I \ni t \mapsto (\iota_X \circ \pi)\big(\exp(t\xi)\big) \in \mathrm{St}_{n,k}$, we recall from Corollary 6 that the normal part $C(t)$ is given by

$$C(t) = \Phi_{(R(t),\theta(t))} \circ T(t), \quad t \in I. \tag{127}$$

Here, $T\colon I \to O(N_X\mathrm{St}_{n,k})$ is the solution of the initial value problem

$$\dot{T}(t) = -P_X^{\perp} \circ f_{(\xi_1(t),\xi_2(t))} \circ T(t), \quad T(0) = \mathrm{id}_{N_X\mathrm{St}_{n,k}}, \tag{128}$$

and the horizontal lift $q\colon I \to O(n) \times O(k)$ of $\widehat{\beta}$ and $S(t)u(t)$ are, as in the intrinsic case, given by (106) and (113), respectively. That is,

$$q(t) = \exp(t\xi)\exp(-t\xi_{\mathfrak{h}}) \in O(n) \times O(k),$$

$$
\begin{aligned}
(\xi_1(t), \xi_2(t)) = S(t)u(t) =& \mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{p}}) \\
=& \big(\mathrm{e}^{t\xi_{1,\mathfrak{h}}}\xi_{1,\mathfrak{p}}\mathrm{e}^{-t\xi_{1,\mathfrak{h}}}, \mathrm{e}^{t\xi_{2,\mathfrak{h}}}\xi_{2,\mathfrak{p}}\mathrm{e}^{-t\xi_{2,\mathfrak{h}}}\big).
\end{aligned} \tag{129}
$$

In order to determine the normal part of the extrinsic rolling explicitly, we need to solve (128).

Lemma 15. *Let $X \in \mathrm{St}_{n,k}$ and $\xi = (\xi_1, \xi_2) \in \mathfrak{so}(n) \times \mathfrak{so}(k)$. Then, the initial value problem*

$$
\begin{aligned}
\dot{T}(t) &= -P_X^{\perp} \circ f_{(\mathrm{e}^{t\xi_{1,\mathfrak{h}}}\xi_{1,\mathfrak{p}}\mathrm{e}^{-t\xi_{1,\mathfrak{h}}}, \mathrm{e}^{t\xi_{2,\mathfrak{h}}}\xi_{2,\mathfrak{p}}\mathrm{e}^{-t\xi_{2,\mathfrak{h}}})} \circ T(t), \\
T(0) &= \mathrm{id}_{N_X\mathrm{St}_{n,k}},
\end{aligned} \tag{130}
$$

has the unique solution $T \colon I \to N_X\mathrm{St}_{n,k}$ given by

$$
T(t) = \Phi_{\exp(t\xi_{\mathfrak{h}})} \circ \exp\big(-t(P_X^{\perp} \circ f_{(\xi_1,\xi_2)})\big). \tag{131}
$$

Proof. By direct computation, we verify that T from (131) is indeed a solution. We first calculate two alternative formulas for $\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\exp(t\xi_{\mathfrak{h}})}(V)$, with $V \in \mathbb{R}^{n\times k}$, as follows:

$$
\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\Phi_{\exp(t\xi_{\mathfrak{h}})}(V) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{t\xi_{1,\mathfrak{h}}}V\mathrm{e}^{-t\xi_{2,\mathfrak{h}}}\big) \\
&= \mathrm{e}^{t\xi_{1,\mathfrak{h}}}\xi_{1,\mathfrak{h}}V\mathrm{e}^{-t\xi_{2,\mathfrak{h}}} - \mathrm{e}^{t\xi_{1,\mathfrak{h}}}V\xi_{2,\mathfrak{h}}\mathrm{e}^{-t\xi_{2,\mathfrak{h}}} \\
&= \mathrm{e}^{t\xi_{1,\mathfrak{h}}}\xi_{1,\mathfrak{h}}\mathrm{e}^{-t\xi_{1,\mathfrak{h}}}\mathrm{e}^{t\xi_{1,\mathfrak{h}}}V\mathrm{e}^{-t\xi_{2,\mathfrak{h}}} - \mathrm{e}^{t\xi_{1,\mathfrak{h}}}V\mathrm{e}^{-t\xi_{2,\mathfrak{h}}}\mathrm{e}^{t\xi_{2,\mathfrak{h}}}\xi_{2,\mathfrak{h}}\mathrm{e}^{-t\xi_{2,\mathfrak{h}}} \\
&= f_{(\mathrm{e}^{t\xi_{1,\mathfrak{h}}}\xi_{1,\mathfrak{h}}\mathrm{e}^{-t\xi_{1,\mathfrak{h}}}, \mathrm{e}^{t\xi_{2,\mathfrak{h}}}\xi_{2,\mathfrak{h}}\mathrm{e}^{-t\xi_{2,\mathfrak{h}}})} \circ \Phi_{\exp(t\xi_{\mathfrak{h}})}(V),
\end{aligned} \tag{132}
$$

and also

$$
\begin{aligned}
\Phi_{\exp(t\xi_{\mathfrak{h}})} \circ f_{(\xi_{1,\mathfrak{p}},\xi_{2,\mathfrak{p}})}(V) &= \mathrm{e}^{t\xi_{1,\mathfrak{h}}}\big(\xi_{1,\mathfrak{p}}V - V\xi_{2,\mathfrak{p}}\big)\mathrm{e}^{-t\xi_{2,\mathfrak{h}}} \\
&= \big(\mathrm{e}^{t\xi_{1,\mathfrak{h}}}\xi_{1,\mathfrak{p}}\mathrm{e}^{-t\xi_{1,\mathfrak{h}}}\big)\big(\mathrm{e}^{t\xi_{1,\mathfrak{h}}}V\mathrm{e}^{-t\xi_{2,\mathfrak{h}}}\big) \\
&\quad - \big(\mathrm{e}^{t\xi_{1,\mathfrak{h}}}V\mathrm{e}^{-t\xi_{2,\mathfrak{h}}}\big)\big(\mathrm{e}^{t\xi_{2,\mathfrak{h}}}\xi_{2,\mathfrak{p}}e^{-t\xi_{2,\mathfrak{h}}}\big) \\
&= f_{(\mathrm{e}^{t\xi_{1,\mathfrak{h}}}\xi_{1,\mathfrak{p}}\mathrm{e}^{-t\xi_{1,\mathfrak{h}}}, \mathrm{e}^{t\xi_{2,\mathfrak{h}}}\xi_{2,\mathfrak{p}}\mathrm{e}^{-t\xi_{2,\mathfrak{h}}})}\big(\mathrm{e}^{t\xi_{1,\mathfrak{h}}}V\mathrm{e}^{-t\xi_{2,\mathfrak{h}}}\big) \\
&= f_{(\mathrm{e}^{t\xi_{1,\mathfrak{h}}}\xi_{1,\mathfrak{p}}\mathrm{e}^{-t\xi_{1,\mathfrak{h}}}, \mathrm{e}^{t\xi_{2,\mathfrak{h}}}\xi_{2,\mathfrak{p}}\mathrm{e}^{-t\xi_{2,\mathfrak{h}}})} \circ \Phi_{\exp(t\xi_{\mathfrak{h}})}(V).
\end{aligned} \tag{133}
$$

Using (132) and (133), we can write:

$$
\begin{aligned}
\dot{T}(t) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\Phi_{\exp(t\xi_{\mathfrak{h}})} \circ \exp\big(-t(P_X^{\perp} \circ f_{(\xi_1,\xi_2)})\big)\Big) \\
&\overset{(132)}{=} f_{(\mathrm{e}^{t\xi_{1,\mathfrak{h}}}\xi_{1,\mathfrak{h}}\mathrm{e}^{-t\xi_{1,\mathfrak{h}}}, \mathrm{e}^{t\xi_{2,\mathfrak{h}}}\xi_{2,\mathfrak{h}}\mathrm{e}^{-t\xi_{2,\mathfrak{h}}})} \circ \Phi_{\exp(t\xi_{\mathfrak{h}})} \circ \exp\big(-t(P_X^{\perp} \circ f_{(\xi_1,\xi_2)})\big)\big) \\
&\quad - \Phi_{\exp(t\xi_{\mathfrak{h}})} \circ \big(P_X^{\perp} \circ f_{(\xi_1,\xi_2)}\big) \circ \exp\big(-t(P_X^{\perp} \circ f_{(\xi_1,\xi_2)})\big) \\
&= P_X^{\perp} \circ \Phi_{\exp(t\xi_{\mathfrak{h}})} \circ f_{(\xi_{1,\mathfrak{h}},\xi_{2,\mathfrak{h}})} \circ \exp\big(-t(P_X^{\perp} \circ f_{(\xi_1,\xi_2)})\big)\big) \\
&\quad - \big(P_X^{\perp} \circ \Phi_{\exp(t\xi_{\mathfrak{h}})} \circ f_{(\xi_1,\xi_2)}\big) \circ \exp\big(-t(P_X^{\perp} \circ f_{(\xi_1,\xi_2)})\big) \\
&= P_X^{\perp} \circ \Phi_{\exp(t\xi_{\mathfrak{h}})} \circ \big(f_{(\xi_{1,\mathfrak{h}},\xi_{2,\mathfrak{h}})} - f_{(\xi_1,\xi_2)}\big) \circ \exp\big(-t(P_X^{\perp} \circ f_{(\xi_1,\xi_2)})\big) \\
&= -P_X^{\perp} \circ \Phi_{\exp(t\xi_{\mathfrak{h}})} \circ f_{(\xi_{1,\mathfrak{p}},\xi_{2,\mathfrak{p}})} \circ \exp\big(-t(P_X^{\perp} \circ f_{(\xi_1,\xi_2)})\big) \\
&\overset{(133)}{=} -P_X^{\perp} \circ f_{(\mathrm{e}^{t\xi_{1,\mathfrak{h}}}\xi_{1,\mathfrak{p}}\mathrm{e}^{-t\xi_{1,\mathfrak{h}}}, \mathrm{e}^{t\xi_{2,\mathfrak{h}}}\xi_{2,\mathfrak{p}}\mathrm{e}^{-t\xi_{2,\mathfrak{h}}})} \circ \Phi_{\exp(t\xi_{\mathfrak{h}})} \circ \exp\big(-t(P_X^{\perp} \circ f_{(\xi_1,\xi_2)})\big) \\
&= -P_X^{\perp} \circ f_{(\mathrm{e}^{t\xi_{1,\mathfrak{h}}}\xi_{1,\mathfrak{p}}\mathrm{e}^{-t\xi_{1,\mathfrak{h}}}, \mathrm{e}^{t\xi_{2,\mathfrak{h}}}\xi_{2,\mathfrak{p}}\mathrm{e}^{-t\xi_{2,\mathfrak{h}}})} \circ T(t),
\end{aligned} \tag{134}
$$

where we have also used $P_X^\perp \circ f_{(\xi_{1,\mathfrak{h}},\xi_{2,\mathfrak{h}})} = f_{(\xi_{1,\mathfrak{h}},\xi_{2,\mathfrak{h}})} \circ P_X^\perp$. Together with the obvious observation that the initial condition $T(0) = \mathrm{id}_{N_X \mathrm{St}_{n,k}}$ is satisfied, this gives the desired result. □

Now, we are in the position to give an explicit expression for the extrinsic rolling of $T_X\mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$ with respect to the Euclidean metric whose development curve is of the desired form.

Proposition 5. *Let $\xi = (\xi_1, \xi_2) \in \mathfrak{so}(n) \times \mathfrak{so}(k)$ and $X \in \mathrm{St}_{n,k}$. Then, the quadruple $(\beta(t), \widehat{\beta}(t), B(t), C(t))$ is an extrinsic rolling of $T_X\mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$ with respect to the Euclidean metric, where*

$$\begin{aligned} \beta(t) &= (d_{(I_n,I_k)}\iota_X \circ \pi)(\alpha(t)), \\ \widehat{\beta}(t) &= (\iota_X \circ \pi)(\exp(t\xi)) = \mathrm{e}^{t\xi_1} X \mathrm{e}^{-t\xi_2}, \\ B(t) &= (d_{(I_n,I_k)}\iota_X) \circ A(t) \circ (d_{(I_n,I_k)}\iota_X \circ \pi)^{-1}, \\ C(t) &= \Phi_{q(t)} \circ T(t), \end{aligned} \tag{135}$$

for $t \in I$, and

$$\begin{aligned} \alpha(t) &= \int_0^t \exp\big(s\big(\mathrm{ad}_{\xi_\mathfrak{h}} + \tfrac{1}{2}(\mathrm{pr}_\mathfrak{p} \circ \mathrm{ad}_{\xi_\mathfrak{p}})\big)\big)(\xi_\mathfrak{p})\,\mathrm{d}s, \\ q(t) &= \exp(t\xi)\exp(-t\xi_\mathfrak{h}) = \big(\mathrm{e}^{t\xi_1}\mathrm{e}^{-t\xi_{1,\mathfrak{h}}}, \mathrm{e}^{t\xi_2}\mathrm{e}^{-t\xi_{2,\mathfrak{h}}}\big), \\ S(t) &= \mathrm{Ad}_{\exp(t\xi_\mathfrak{h})} \circ \exp\big(-t\big(\mathrm{ad}_{\xi_\mathfrak{h}} + \tfrac{1}{2}(\mathrm{pr}_\mathfrak{p} \circ \mathrm{ad}_{\xi_\mathfrak{p}})\big)\big), \\ A(t) &= (d_{q(t)}\pi) \circ (d_{(I_n,I_k)}L_{q(t)}) \circ S(t), \\ T(t) &= \Phi_{\exp(t\xi_\mathfrak{h})} \circ \exp\big(-t(P_X^\perp \circ f_{(\xi_1,\xi_2)})\big). \end{aligned} \tag{136}$$

Proof. This is a consequence of the above discussion. Essentially, the assertion follows by combining Proposition 4, Lemma 15, and Theorem 2. □

Proposition 5 implies an explicit expression for the rolling along geodesics. In fact, by exploiting that geodesics on naturally reductive homogeneous spaces are projections of horizontal one-parameter groups, we obtain the next corollary.

Corollary 7. *Let $\xi = (\xi_1, \xi_2) \in \mathfrak{p}$ and $X \in \mathrm{St}_{n,k}$. Then, the quadruple $(\beta(t), \widehat{\beta}(t), B(t), C(t))$ is an extrinsic rolling of $T_X\mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$ with respect to the Euclidean metric, where*

$$\begin{aligned} \beta(t) &= (d_{(I_n,I_k)}\iota_X \circ \pi)(t\xi_1, t\xi_2) = t(\xi_1 X - X\xi_2), \\ \widehat{\beta}(t) &= (\iota_X \circ \pi)(\exp(t\xi)) = \mathrm{e}^{t\xi_1} X \mathrm{e}^{-t\xi_2}, \\ B(t) &= (d_{(I_n,I_k)}\iota_X \circ \pi) \circ (d_{(I_n,I_k)}L_{(\mathrm{e}^{t\xi_1},\mathrm{e}^{t\xi_2})}), \\ &\quad \circ \exp\big(-\tfrac{1}{2}t(\mathrm{pr}_\mathfrak{p} \circ \mathrm{ad}_{\xi_\mathfrak{p}})\big) \circ (d_{(I_n,I_k)}\iota_X \circ \pi)^{-1}, \\ C(t) &= \Phi_{(\mathrm{e}^{t\xi_1},\mathrm{e}^{t\xi_2})} \circ \exp\big(-t(P_X^\perp \circ f_{(\xi_1,\xi_2)})\big), \end{aligned} \tag{137}$$

for $t \in I$, whose development curve is a geodesic.

Proof. Clearly, $\xi \in \mathfrak{p}$ implies $\xi_\mathfrak{h} = 0$. Thus, the assertion follows by Proposition 5. □

6.5. Comparison with Existing Literature

In this final section, we relate our results with the known rolling of Stiefel manifolds from [2].

We discuss how the rolling of $T_X\mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$ is related to the rolling obtained in [2]. As in [2], we specify $X = E = \begin{bmatrix} I_k \\ 0 \end{bmatrix}$. It is well known that

$$T_E\mathrm{St}_{n,k} = \left\{ \begin{bmatrix} \Omega \\ B \end{bmatrix} \mid \Omega \in \mathfrak{so}(k), B \in \mathbb{R}^{(n-k)\times k} \right\} = \mathfrak{so}(n)E \tag{138}$$

holds. We now recall the rolling map from [2], where trivial modifications concerning the terminology and notations were made in order to adapt it to our notation.

Let $\alpha\colon I \to \mathrm{St}_{n,k}$ be a rolling curve with $\alpha(0) = E$. Then, there exists a curve $U\colon I \to SO(n)$ such that $\alpha(t) = U(t)E$. Denote

$$G = \{W \in SO(nk) \mid W = V \otimes U, V \in SO(k), U \in SO(n)\} \subset SO(nk) \tag{139}$$

and

$$\mathcal{U}(t) = \{Q(t) \in G \mid Q(t)\,\mathrm{vec}(E) = (V(t) \otimes U(t))\,\mathrm{vec}(E) = \mathrm{vec}(\alpha(t))\}. \tag{140}$$

The rotational part, $R(t) \in SO(nk)$, describing the rolling of $T_E\mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$ is obtained in [2] by the following Ansatz:

$$R(t) = Q(t)\widetilde{S}(t), \tag{141}$$

where $Q(t) \in \mathcal{U}(t)$ and $\widetilde{S}(t)$ is a curve in the isotropy subgroup of E under the $SO(nk)$-action on $\mathbb{R}^{nk} = \mathrm{vec}(\mathbb{R}^{n\times k})$, i.e.,

$$\begin{aligned} \widetilde{S}(t) &\in \{R \in SO(nk) \mid R\,\mathrm{vec}(E) = \mathrm{vec}(E)\} \\ &= \mathrm{Stab}(\mathrm{vec}(E)) \cong SO(nk-1), \end{aligned} \tag{142}$$

where the isomorphism in the above equation is obtained by choosing an orthogonal transformation $P_0 \in O(nk)$ such that $P_0E \in \mathrm{span}\{e_{nk}\}$ holds, as well as

$$\begin{aligned} P_0(T_E\mathrm{St}_{n,k}) &= \mathrm{span}\{e_1, \dots e_{\ell_t}\}, \\ P_0(N_E\mathrm{St}_{n,k}) &= \mathrm{span}\{e_{\ell_t+1}, \dots e_{nk}\}, \end{aligned} \tag{143}$$

where $\ell_t = \dim(\mathrm{St}_{n,k})$ and $\ell_n = \dim(N_X\mathrm{St}_{n,k})$, yielding

$$P_0\big(\mathrm{Stab}(E)\big)P_0^\top = \left\{ \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \mid S \in SO(nk-1) \right\}. \tag{144}$$

Note that $\widetilde{S}$ in this text corresponds to S in [2]. By this notation, it is shown in [2] that $\widetilde{S}(t)$ needs to fulfill

$$\widetilde{S}(t) \in \begin{bmatrix} O(\ell_t) & 0 & 0 \\ 0 & O(\ell_n - 1) & 0 \\ & 0 & 1 \end{bmatrix} \cap SO(nk), \tag{145}$$

where $\ell_t = \dim(T_E\mathrm{St}_{n,k})$ and $\ell_n = \dim(N_E\mathrm{St}_{n,k})$.

The orthogonal projection of a matrix $A \in \mathbb{R}^{nk\times nk}$ onto a matrix with the structure given in the above equation is denoted by $A_{\mathrm{bl-diag}}$. Using this notation, we recall [2] (Lem. 3.2).

Lemma 16. *Let $h = (R, s)$ be a rolling map for the Stiefel manifold $\mathrm{St}_{n,k}$. If $Q(t) \in \mathcal{U}(t)$ and $R(t) = Q(t)\widetilde{S}(t)$ with $\widetilde{S}(t) \in \mathrm{Stab}(E)$, then $\widetilde{S}(t)$ obeys the ODE*

$$\begin{aligned} \dot{\widetilde{S}}(t) &= P_0^\top \left(P_0\dot{Q}(t)^\top Q(t)P_0^\top\right)_{\mathrm{bl-diag}} P_0\widetilde{S}(t) \\ &= -P_0^\top \left(P_0Q(t)^\top \dot{Q}(t)P_0^\top\right)_{\mathrm{bl-diag}} P_0\widetilde{S}(t), \end{aligned} \tag{146}$$

where $s\colon I \to \mathbb{R}^{nk}$ *fulfills the ODE*

$$\dot{s}(t) = -\widetilde{S}(t)\dot{Q}(t)^\top Q(t)\operatorname{vec}(E) = \widetilde{S}(t)Q(t)^\top \dot{Q}(t)\operatorname{vec}(E) \tag{147}$$

by [2] (Eq. (44)).

Note that the second equations in (146) and (146) of Lemma 16 are correct by $Q^\top \dot{Q} \in \mathfrak{so}(nk)$ because of $Q\colon I \to SO(nk)$.

The goal of the remaining part of this subsection is to show that the extrinsic rolling of the Stiefel manifold obtained in Section 6.3 fulfills Lemma 16. To this end, we recall that the extrinsic rolling $(\beta(t), \widehat{\beta}(t), B(t), C(t))$ from Section 6.3 is constructed by using the kinematic equations

$$\begin{aligned}
u(t) &= \big(d_{(I_n,I_k)}(\iota_E \circ \pi)\big|_{p}\big)^{-1}(\dot{\beta}(t)),\\
\dot{S}(t) &= -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{S(t)u(t)} \circ S(t), \quad S(0) = S_0 = \mathrm{id}_{\mathfrak{p}} \in O(\mathfrak{p}),\\
\dot{q}(t) &= (d_{(I_n,I_k)}L_{q(t)}) \circ S(t)u(t), \quad q(0) = (I_n, I_k) \in O(n) \times O(k),\\
\dot{T}(t) &= -P_E^{\perp} \circ f_{(\xi_1(t),\xi_2(t))} \circ T(t), \quad T(0) = \mathrm{id}_{N_E \mathrm{St}_{n,k}} \in O(N_E \mathrm{St}_{n,k}),
\end{aligned} \tag{148}$$

according to Corollary 6 for $X = E$. The development curve reads

$$\widehat{\beta}(t) = R(t)E\theta(t)^\top. \tag{149}$$

Hence, $q(t) = (R(t), \theta(t)) \in \mathcal{U}(t)$ is fulfilled by the definition of $\mathcal{U}(t)$, after identifying $q(t)$ with $Q(t) = \theta(t) \otimes R(t)$ by the map

$$O(n) \times O(k) \ni (R, \theta) \mapsto \theta \otimes R \in O(k) \otimes O(n), \tag{150}$$

which is an isomorphism of the Lie groups onto its images. Using this identification, we obtain that

$$\big(d_e L_{q(t)}\big)^{-1}\dot{q}(t) = S(t)u(t) = (\xi_1(t), \xi_2(t)) \tag{151}$$

corresponds to

$$Q^\top \dot{Q} = \xi_2(t) \otimes I_n + I_k \otimes \xi_1(t), \tag{152}$$

by using properties of the Kronecker product, see, e.g., [15] (Sec. 7.1).

It remains to relate the curves $S(t)$ and $T(t)$ from (148) to the curve $\widetilde{S}(t)$ considered in Lemma 16.

We first consider the normal part. We show that $E \in N_E \mathrm{St}_{n,k}$ is invariant under $T\colon I \to O(N_E \mathrm{St}_{n,k})$, where T is defined by the kinematic equation. We obtain, by the definition of $f_{(\xi_1,\xi_2)}$ for $X = E$,

$$f_{(\xi_1,\xi_2)}(E) = (\xi_1 E - E\xi_2) \tag{153}$$

implying that $f_{(\xi_1,\xi_2)}\big(\operatorname{span}\{E\}\big) \subset T_E \mathrm{St}_{n,k}$ by the linearity of $f_{(\xi_1,\xi_2)}\colon \mathbb{R}^{n\times k} \to \mathbb{R}^{n\times k}$. Next, we consider the curve $I \ni t \mapsto E(t) = T(t)(E)$, where $T\colon I \to O(N_X \mathrm{St}_{n,k})$ is given by the kinematic equation. We may view $E(t)$ as a solution of the initial value problem

$$\dot{E}(t) = -\big(P_E^{\perp} \circ f_{(\xi_1(t),\xi_2(t))}\big)(E(t)), \quad E(0) = E. \tag{154}$$

The unique solution of this ODE is given by $E(t) = E$, for $t \in I$, because $E(0) = E$ is clearly fulfilled and

$$-\big(P_E^{\perp} \circ f_{(\xi_1(t),\xi_2(t))}\big)(E) = 0 = \dot{E}(t) \tag{155}$$

holds due to $f_{(\xi_1,\xi_2)}\big(\operatorname{span}\{E\}\big) \subset T_E\mathrm{St}_{n,k}$. In other words, because $T(0) = \mathrm{id}_{N_E\mathrm{St}_{n,k}}$, one has

$$T(t)E = E, \quad t \in I. \tag{156}$$

Clearly, by choosing $P_0 \in O(nk)$ such that (143) holds, one obtains, for $v \in \mathbb{R}^{nk}$,

$$\begin{aligned} P_0 \operatorname{vec}(P_E^{\perp}(\operatorname{vec}^{-1}(v))) &= \begin{bmatrix} 0_{\ell_t} & 0 \\ 0 & I_{\ell_n} \end{bmatrix} P_0 v \in \mathbb{R}^{nk}, \\ P_0 \operatorname{vec}(P_E(\operatorname{vec}^{-1}(v))) &= \begin{bmatrix} I_{\ell_t} & 0 \\ 0 & 0_{\ell_n} \end{bmatrix} P_0 v \in \mathbb{R}^{nk}, \end{aligned} \tag{157}$$

which implies, for $v \in \mathbb{R}^{nk}$,

$$\begin{aligned} \operatorname{vec} \circ P_E \circ \operatorname{vec}^{-1}(v) &= P_0^{\top} \begin{bmatrix} I_{\ell_t} & 0 \\ 0 & 0_{\ell_n} \end{bmatrix} P_0 v, \\ \operatorname{vec} \circ P_E^{\perp} \circ \operatorname{vec}^{-1}(v) &= P_0^{\top} \begin{bmatrix} 0_{\ell_t} & 0 \\ 0 & I_{\ell_n} \end{bmatrix} P_0 v. \end{aligned} \tag{158}$$

We now identify the curve $S \colon I \to O(\mathfrak{p})$ with the curve $\widehat{S} \colon I \to O(T_E\mathrm{St}_{n,k})$ via

$$\widehat{S}(t) = \big(d_{(I_n,I_k)}(\iota_E \circ \pi)\big) \circ S(t) \circ \big(d_{(I_n,I_k)}(\iota_E \circ \pi)\big|_{\mathfrak{p}}\big)^{-1}. \tag{159}$$

In the sequel, we find a matrix representation for $\widehat{S}$, roughly speaking, by considering $\mathcal{S} = \operatorname{vec} \circ \widehat{S} \circ \operatorname{vec}^{-1}$.

We start with computing (159) explicitly. The ODE (148) for $S(t) \in O(\mathfrak{p})$ can be equivalently rewritten as

$$\dot{S}(t) \circ S(t)^{-1} = -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{(\xi_1(t),\xi_2(t))} \tag{160}$$

and, therefore,

$$\begin{aligned} &\dot{\widehat{S}}(t) \circ \widehat{S}(t)^{-1} \\ &= \big(d_{(I_n,I_k)}(\iota_E \circ \pi)\big) \circ \big(-\tfrac{1}{2}\mathrm{pr}_{\mathfrak{p}} \circ \mathrm{ad}_{(\xi_1(t),\xi_2(t))}\big) \circ \big(d_{(I_n,I_k)}(\iota_E \circ \pi)\big|_{\mathfrak{p}}\big)^{-1} \\ &= \big(d_{(I_n,I_k)}(\iota_E \circ \pi)\big) \circ \big(-\tfrac{1}{2}\mathrm{ad}_{(\xi_1(t),\xi_2(t))}\big) \circ \big(d_{(I_n,I_k)}(\iota_E \circ \pi)\big|_{\mathfrak{p}}\big)^{-1}, \end{aligned} \tag{161}$$

where, for the last equality, we use the fact that $\mathfrak{h}$ belongs to the kernel of $d_{(I_n,I_k)}(\iota_E \circ \pi)$.

We now compute the right-hand side of the above equation. To this end, we write

$$V = \begin{bmatrix} \Omega \\ C \end{bmatrix} \in T_E\mathrm{St}_{n,k} \quad \text{and} \quad (\xi_1,\xi_2) = \left(\begin{bmatrix} 2\Psi & -B^{\top} \\ B & 0 \end{bmatrix}, \Psi\right) \in \mathfrak{p}. \tag{162}$$

Taking into account that $\Omega^{\top} = -\Omega$, $\Psi^{\top} = -\Psi$, and

$$d_{(I_n,I_k)}(\iota_E \circ \pi)\left(\begin{bmatrix} 2\Psi & -B^{\top} \\ B & 0 \end{bmatrix}, \Psi\right) = \begin{bmatrix} 2\Psi & -B^{\top} \\ B & 0 \end{bmatrix}\begin{bmatrix} I_k \\ 0 \end{bmatrix} - \begin{bmatrix} I_k \\ 0 \end{bmatrix}\Psi = \begin{bmatrix} \Psi \\ B \end{bmatrix}, \tag{163}$$

we can write

$$
\begin{aligned}
&\left(d_{(I_n,I_k)}(\iota_E\circ\pi)\right)\circ\dot{S}(t)\circ S(t)^{-1}\circ\left(d_{(I_n,I_k)}(\iota_E\circ\pi)\big|_{\mathfrak{p}}\right)^{-1}(V)\\
&=-\tfrac{1}{2}d_{(I_n,I_k)}(\iota_X\circ\pi)\Big([\xi_1,(EV^\top-VE^\top)],[\xi_2,E^\top V]\Big)\\
&=-\tfrac{1}{2}d_{(I_n,I_k)}(\iota_X\circ\pi)\Big(\begin{bmatrix}2\Psi&-B^\top\\B&0\end{bmatrix}\begin{bmatrix}2\Omega&-C^\top\\C&0\end{bmatrix}-\begin{bmatrix}2\Omega&-C^\top\\C&0\end{bmatrix}\begin{bmatrix}2\Psi&-B^\top\\B&0\end{bmatrix},\Psi\Omega-\Omega\Psi\Big)\\
&=-\tfrac{1}{2}d_{(I_n,I_k)}(\iota_X\circ\pi)\Big(\begin{bmatrix}4\Psi\Omega-B^\top C&-2\Psi C^\top\\2B\Omega&-BC^\top\end{bmatrix}-\begin{bmatrix}4\Omega\Psi-C^\top B&-2\Omega B^\top\\2C\Psi&-CB^\top\end{bmatrix},\Psi\Omega-\Omega\Psi\Big)\\
&=-\tfrac{1}{2}d_{(I_n,I_k)}(\iota_X\circ\pi)\Big(\begin{bmatrix}4\Psi\Omega-B^\top C-4\Omega\Psi+C^\top B&-2\Psi C^\top+2\Omega B^\top\\2B\Omega-2C\Psi&-BC^\top+CB^\top\end{bmatrix},\Psi\Omega-\Omega\Psi\Big)\\
&=-\tfrac{1}{2}\Big(\begin{bmatrix}4\Psi\Omega-B^\top C-4\Omega\Psi+C^\top B&-2\Psi C^\top+2\Omega B^\top\\2B\Omega-2C\Psi&-BC^\top+CB^\top\end{bmatrix}\begin{bmatrix}I_k\\0\end{bmatrix}-\begin{bmatrix}I_k\\0\end{bmatrix}(\Psi\Omega-\Omega\Psi)\Big)\\
&=-\tfrac{1}{2}\begin{bmatrix}4\Psi\Omega-B^\top C-4\Omega\Psi+C^\top B-\Psi\Omega+\Omega\Psi\\2B\Omega-2C\Psi\end{bmatrix}\\
&=-\tfrac{1}{2}\begin{bmatrix}3\Psi\Omega-3\Omega\Psi-B^\top C+C^\top B\\2B\Omega-2C\Psi\end{bmatrix},
\end{aligned}
\tag{164}
$$

as well as

$$
\begin{aligned}
P_E\Big(\xi_1V-V\xi_2\Big)&=P_E\Big(\begin{bmatrix}2\Psi&-B^\top\\B&0\end{bmatrix}\begin{bmatrix}\Omega\\C\end{bmatrix}-\begin{bmatrix}\Omega\\C\end{bmatrix}\Psi\Big)\\
&=P_E\Big(\begin{bmatrix}2\Psi\Omega-B^\top C-\Omega\Psi\\B\Omega-C\Psi\end{bmatrix}\Big)\\
&=\begin{bmatrix}(\Psi\Omega-(\Psi\Omega)^\top)-\frac{1}{2}(B^\top C-(B^\top C)^\top)-\frac{1}{2}(\Omega\Psi-(\Psi\Omega)^\top)\\B\Omega-C\Psi\end{bmatrix}\\
&=\begin{bmatrix}\Psi\Omega-\Omega\Psi-\frac{1}{2}B^\top C+\frac{1}{2}C^\top B-\frac{1}{2}\Omega\Psi+\frac{1}{2}\Psi\Omega\\B\Omega-C\Psi\end{bmatrix}\\
&=\tfrac{1}{2}\begin{bmatrix}3\Psi\Omega-3\Omega\Psi-B^\top C+C^\top B\\2B\Omega-2C\Psi\end{bmatrix}.
\end{aligned}
\tag{165}
$$

By comparing (164) and (165), we obtain

$$
\begin{aligned}
&\left(d_{(I_n,I_k)}(\iota_E\circ\pi)\right)\circ\dot{S}(t)\circ S(t)^{-1}\circ\left(d_{(I_n,I_k)}(\iota_X\circ\pi)\big|_{\mathfrak{p}}\right)^{-1}(V)\\
&=-P_E\big(\xi_1(t)V-V\xi_2(t)\big).
\end{aligned}
\tag{166}
$$

Therefore, (161) can be written as

$$
\dot{\widehat{S}}(t)\circ\widehat{S}(t)^{-1}(V)=-P_E\big(\xi_1(t)V-V\xi_2(t)\big), \tag{167}
$$

for $V\in T_E\mathrm{St}_{n,k}$ or, equivalently, as

$$
\dot{\widehat{S}}(t)\circ\widehat{S}(t)^{-1}(V)=-P_E\circ f_{(\xi_1(t),\xi_2(t))}(V), \tag{168}
$$

for $V\in T_E\mathrm{St}_{n,k}$. Applying $\mathrm{vec}\colon\mathbb{R}^{n\times k}\to\mathbb{R}^{nk}$, we obtain for $\mathcal{S}=\mathrm{vec}\circ\widehat{S}(t)\circ\mathrm{vec}^{-1}$ the ODE

$$
\dot{\mathcal{S}}(t)=-\big(\,\mathrm{vec}\circ P_E\circ\mathrm{vec}^{-1}\,\big)\circ\big(\,\mathrm{vec}\circ f_{(\xi_1(t),\xi_2(t))}\circ\mathrm{vec}^{-1}\,\big)\circ\mathcal{S}(t). \tag{169}
$$

For $W\in\mathbb{R}^{n\times k}$, we have

$$
\begin{aligned}
\mathrm{vec}\,\big(f_{(\xi_1(t),\xi_2(t))}(W)\big)&=\mathrm{vec}(\xi_1(t)W-W\xi_2(t))\\
&=\big(I_k\otimes\xi_1(t)+\xi_2(t)\otimes I_n\big)\,\mathrm{vec}(W).
\end{aligned}
\tag{170}
$$

Denoting the representation matrix of $\mathcal{S}$ by $\mathcal{S}$, as well, and using the identity (170) with W replaced by $\mathrm{vec}^{-1}\circ\mathcal{S}(t)\circ\mathrm{vec}(V)$, we obtain

$$
\dot{\mathcal{S}}(t)\,\mathrm{vec}(V)=-\big(\,\mathrm{vec}\circ P_E\circ\mathrm{vec}^{-1}\,\big)(I_k\otimes\xi_1(t)+\xi_2(t)\otimes I_n)\mathcal{S}\,\mathrm{vec}(V), \tag{171}
$$

for $V \in T_E\mathrm{St}_{n,k}$.

Recalling the definition of $P_0 \in SO(nk)$ from (143), and using (158), we can rewrite (171) for $V \in T_E\mathrm{St}_{n,k}$ equivalently as

$$\begin{aligned}
&\dot{\mathcal{S}}(t) \circ \mathcal{S}(t)^{-1} \operatorname{vec}(V) \\
&= -P_0^\top \begin{bmatrix} I_{\ell_t} & 0 \\ 0 & 0 \end{bmatrix} P_0 (I_k \otimes \xi_1(t) + \xi_2(t) \otimes I_n) P_0^\top P_0 \operatorname{vec}(V) \\
&= -P_0^\top \begin{bmatrix} I_{\ell_t} & 0 \\ 0 & 0 \end{bmatrix} P_0 (I_k \otimes \xi_1(t) + \xi_2(t) \otimes I_n) P_0^\top \begin{bmatrix} I_{\ell_t} & 0 \\ 0 & 0 \end{bmatrix} P_0 \operatorname{vec}(V) \\
&= -P_0^\top \begin{bmatrix} I_{\ell_t} & 0 \\ 0 & 0 \end{bmatrix} P_0 Q(t)^\top \dot{Q}(t) P_0^\top \begin{bmatrix} I_{\ell_t} & 0 \\ 0 & 0 \end{bmatrix} P_0 \operatorname{vec}(V),
\end{aligned} \tag{172}$$

where the last equality holds due to (152).

Similarly, for $T\colon I \to O(N_E\mathrm{St}_{n,k})$, if we define $\mathcal{T}(t) = \operatorname{vec} \circ T(t) \circ \operatorname{vec}^{-1}$ and denote its representation matrix by the same symbol, we have, for $V \in N_X\mathrm{St}_{n,k}$,

$$\begin{aligned}
&\dot{\mathcal{T}}(t) \circ \mathcal{T}(t)^{-1} \operatorname{vec}(V) \\
&= -P_0^\top \begin{bmatrix} 0 & 0 \\ 0 & I_{\ell_n} \end{bmatrix} P_0 (I_k \otimes \xi_1(t) + \xi_2(t) \otimes I_n) P_0^\top \begin{bmatrix} 0 & 0 \\ 0 & I_{\ell_n} \end{bmatrix} P_0 \operatorname{vec}(V) \\
&= -P_0^\top \begin{bmatrix} 0 & 0 \\ 0 & I_{\ell_n} \end{bmatrix} P_0 Q(t)^\top \dot{Q}(t) P_0^\top \begin{bmatrix} 0 & 0 \\ 0 & I_{\ell_n} \end{bmatrix} P_0 \operatorname{vec}(V).
\end{aligned} \tag{173}$$

Next, we define $\widetilde{S}\colon I \ni t \mapsto \widetilde{S}(t) \in \mathbb{R}^{nk \times nk}$ and show that this curve $\widetilde{S}(t)$ is exactly the curve $\widetilde{S}(t)$ from Lemma 16. For that, let $v \in \mathbb{R}^{nk}$ and compute

$$\begin{aligned}
\dot{\widetilde{S}}(t)\widetilde{S}(t)^{-1} v &= \dot{\mathcal{S}}(t) \circ \mathcal{S}(t)^{-1} \circ (\operatorname{vec} \circ P_E \circ \operatorname{vec}^{-1})(v) \\
&\quad + \dot{\mathcal{T}}(t) \circ \mathcal{T}(t)^{-1} \circ (\operatorname{vec} \circ P_E^\perp \circ \operatorname{vec}^{-1})(v) \\
&= -P_0^\top \begin{bmatrix} I_{\ell_t} & 0 \\ 0 & 0 \end{bmatrix} P_0 Q(t)^\top \dot{Q}(t) P_0^\top \begin{bmatrix} I_{\ell_t} & 0 \\ 0 & 0 \end{bmatrix} P_0 v \\
&\quad - P_0^\top \begin{bmatrix} 0 & 0 \\ 0 & I_{\ell_n} \end{bmatrix} P_0 Q(t)^\top \dot{Q}(t) P_0^\top \begin{bmatrix} 0 & 0 \\ 0 & I_{\ell_n} \end{bmatrix} P_0 v \\
&= -P_0^\top \Bigg(\begin{bmatrix} I_{\ell_t} & 0 \\ 0 & 0 \end{bmatrix} P_0 Q(t)^\top \dot{Q}(t) P_0^\top \begin{bmatrix} I_{\ell_t} & 0 \\ 0 & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & 0 \\ 0 & I_{\ell_n} \end{bmatrix} P_0 Q(t)^\top \dot{Q}(t) P_0^\top \begin{bmatrix} 0 & 0 \\ 0 & I_{\ell_n} \end{bmatrix} \Bigg) P_0 v.
\end{aligned} \tag{174}$$

In order to show that $\widetilde{S}(t)$ indeed satisfies the ODE from Lemma 16, we state the following auxiliar result.

Lemma 17. *Let $\ell_t, \ell_n \in \mathbb{N}$ with $\ell_t + \ell_n = nk$, and consider the matrix $A \in \mathfrak{so}(nk)$ partitioned as*

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ -A_{12}^\top & A_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{where} \quad A_{11} \in \mathfrak{so}(\ell_t), \quad A_{22} \in \mathfrak{so}(\ell_n - 1).$$

Then, for $v \in \mathbb{R}^{nk}$,

$$A_{\mathrm{bl-diag}} v = \left(\begin{bmatrix} 0 & 0 \\ 0 & I_{\ell_n} \end{bmatrix} A \begin{bmatrix} 0 & 0 \\ 0 & I_{\ell_n} \end{bmatrix} + \begin{bmatrix} I_{\ell_t} & 0 \\ 0 & 0 \end{bmatrix} A \begin{bmatrix} I_{\ell_t} & 0 \\ 0 & 0 \end{bmatrix} \right) v \tag{175}$$

holds.

Proof. Writing $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, where $v_1 \in \mathbb{R}^{\ell_t}$, $v_2 \in \mathbb{R}^{\ell_n - 1}$ and $v_3 \in \mathbb{R}$, we compute

$$A_{\mathrm{bl-diag}} v = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} A_{11} v_1 \\ A_{22} v_2 \\ 0 \end{bmatrix}. \tag{176}$$

Moreover, we also have

$$\begin{aligned}
&\left(\begin{bmatrix} 0 & 0 \\ 0 & I_{\ell_n} \end{bmatrix} A \begin{bmatrix} 0 & 0 \\ 0 & I_{\ell_n} \end{bmatrix} + \begin{bmatrix} I_{\ell_t} & 0 \\ 0 & 0 \end{bmatrix} A \begin{bmatrix} I_{\ell_t} & 0 \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & I_{\ell_n} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} I_{\ell_t} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & I_{\ell_n} \end{bmatrix} \begin{bmatrix} A_{12}v_2 \\ A_{22}v_2 \\ 0 \end{bmatrix} + \begin{bmatrix} I_{\ell_t} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11}v_1 \\ A_{21}v_1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} A_{11}v_1 \\ A_{22}v_2 \\ 0 \end{bmatrix},
\end{aligned} \tag{177}$$

showing the desired result. □

Applying Lemma 17 to (174) yields

$$\dot{\widetilde{S}}(t)\widetilde{S}(t)^{-1} = -P_0^\top \left(P_0 Q(t)^\top \dot{Q}(t) P_0^\top\right)_{\mathrm{bl-diag}} P_0. \tag{178}$$

So, $\widetilde{S}$ defined in (174) fulfills the ODE from Lemma 16.

It remains to show that our approach also gives the curve $s\colon I \to \mathbb{R}^{nk}$ from Lemma 16. Recalling that

$$\left(d_e L_{q(t)}\right)^{-1} \dot{q}(t) = S(t)u(t) = (\xi_1(t), \xi_2(t)), \tag{179}$$

we write $S(t)^{-1}(\xi_1(t), \xi_2(t)) = (u_1(t), u_2(t))$, and

$$\dot{\beta}(t) = \left(d_{(I_n, I_k)}(\iota_E \circ \pi)\right)\left(u_1(t), u_2(t)\right) = u_1(t)E - Eu_2(t), \tag{180}$$

where β is the rolling curve for the rolling of $T_E\mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$. We now consider the curve $s\colon I \to \mathbb{R}^{nk}$ from Lemma 16 and perform the following computations:

$$\begin{aligned}
\dot{s}(t) &= \widetilde{S}(t)^\top Q(t)^\top \dot{Q}(t)\,\mathrm{vec}(E) \\
&= \widetilde{S}(t)^\top\,\mathrm{vec}(\xi_1(t)E - E\xi_2(t)) \\
&= \mathrm{vec} \circ \left(d_{(I_n, I_k)}(\iota_E \circ \pi)\right) \circ S(t)^{-1} \circ \left(d_{(I_n, I_k)}(\iota_E \circ \pi)\big|_{\mathfrak{p}}\right)^{-1} \\
&\quad \circ \mathrm{vec}^{-1} \circ \mathrm{vec}(\xi_1(t)E - E\xi_2(t)) \\
&= \mathrm{vec} \circ \left(d_{(I_n, I_k)}(\iota_E \circ \pi)\right) \circ S(t)^{-1}(\xi_1(t), \xi_2(t)) \\
&= \mathrm{vec}\,\left(d_{(I_n, I_k)}(\iota_E \circ \pi)(u_1(t), u_2(t))\right) \\
&= \mathrm{vec}(u_1(t)E - Eu_2(t)) \\
&= \mathrm{vec}(\dot{\beta}(t)).
\end{aligned} \tag{181}$$

By (181), $\mathrm{vec}(\beta(t)) + b_0 = s(t)$ holds for $t \in I$ and some $b_0 \in \mathbb{R}^{nk}$.

Recalling, from Lemma 16, that $(R(t)^\top, s(t))$ defines a rolling of $\mathrm{St}_{n,k}$ over $T_E\mathrm{St}_{n,k}$, the development curve is given by $Q(t)\,\mathrm{vec}(E) = \mathrm{vec}(\widehat{\beta}(t))$ and the rolling curve by $s(t) = \mathrm{vec}(\beta(t))$. Thus, $\widehat{\alpha}(t)$, $\alpha(t)$, and $R(t)$ from Proposition 3 correspond to $\mathrm{vec}(\beta(t))$, $Q(t)\,\mathrm{vec}(E)$, and $(Q(t)\widetilde{S}(t))^\top$, respectively. Therefore, we obtain

$$\begin{aligned}
s(t) &= \mathrm{vec}(\beta(t)) - (Q(t)\widetilde{S}(t))^\top Q(t)\,\mathrm{vec}(E) \\
&= \mathrm{vec}(\beta(t)) - \widetilde{S}(t)^\top Q(t)^\top Q(t)\,\mathrm{vec}(E) \\
&= \mathrm{vec}(\beta(t)) - \widetilde{S}(t)^\top Q(t)^\top Q(t)\widetilde{S}(t)\,\mathrm{vec}(E) \\
&= \mathrm{vec}(\beta(t)) - E,
\end{aligned} \tag{182}$$

by exploiting that $\widetilde{S}(t)\,\mathrm{vec}(E) = \mathrm{vec}(E)$. Obviously, using (181), we may conclude that $s(t)$ from (182) fulfills the ODE

$$\dot{s}(t) = \widetilde{S}(t)^\top Q(t)^\top \dot{Q}(t)\,\mathrm{vec}(E) \tag{183}$$

from Lemma 16.

In conclusion, after having developed the theoretical results for the rolling normal naturally reductive homogeneous spaces over their tangent spaces, we specialized this to the Stiefel manifold. The results presented here for rolling extrinsically the Stiefel manifold $\mathrm{St}_{n,k}$ over its tangent space $T_E\mathrm{St}_{n,k}$ coincide with those obtained previously in [2].

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