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Properties of Multivariable Hermite Polynomials in Correlation with Frobenius–Genocchi Polynomials

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Abstract: The evolution of a physical system occurs through a set of variables, and the problems can be treated based on an approach employing multivariable Hermite polynomials. These polynomials possess beneficial properties exhibited in functional and differential equations, recurring and explicit relations as well as symmetric identities, and summation formulae, among other examples. In view of these points, comprehensive schemes have been developed to apply the principle of monomiality from mathematical physics to various mathematical concepts of special functions, the development of which has encompassed generalizations, extensions, and combinations of other functions. Accordingly, this paper presents research on a novel family of multivariable Hermite polynomials associated with Frobenius–Genocchi polynomials, deriving the generating expression, operational rule, differential equation, and other defining characteristics for these polynomials. Additionally, the monomiality principle for these polynomials is verified, as well as establishing the series representations, summation formulae, operational and symmetric identities, and recurrence relations satisfied by these polynomials. This proposed scheme aims to provide deeper insights into the behavior of these polynomials and to uncover new connections between these polynomials, to enhance understanding of their properties.

Keywords: multivariable special polynomials; monomiality principle; explicit form; operational connection; symmetric identities; summation formulae; conjugates

MSC: 33E20; 33C45; 33B10; 33E30; 11T23

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Citation: Wani, S.A.; Oros, G.I.; Mahnashi, A.M.; Hamali, W. Properties of Multivariable Hermite Polynomials in Correlation with Frobenius–Genocchi Polynomials. *Mathematics* **2023**, *11*, 4523. https://doi.org/10.3390/math11214523

Academic Editors: Luca Gemignani and Francesco Aldo Costabile

Received: 30 August 2023
Revised: 26 October 2023
Accepted: 31 October 2023
Published: 2 November 2023

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1. Introduction

Multivariable special polynomials are mathematical expressions involving multiple variables, possessing distinctive properties, and serving crucial roles across a diverse array of mathematical and scientific domains. These polynomials are instrumental in describing intricate relationships among multiple variables, and they are extensively utilized in a wide range of mathematical fields. Notably, they play a fundamental role in algebraic geometry, a discipline dedicated to the examination of the geometric attributes of algebraic varieties. Thanks to multivariable special polynomials, significant geometric entities like algebraic curves, surfaces, and higher-dimensional varieties can be defined and explored. They prove invaluable in characterizing surface and curve intersections, identifying singularities within algebraic varieties, and gaining deeper insights into the properties of their coordinate rings. Furthermore, these polynomials have applications in various domains of theoretical physics, including quantum field theory and quantum mechanics. In particular, they have emerged as solutions to differential equations in the context of mathematical physics,

especially in scenarios involving eigenvalue problems, boundary value challenges, and symmetry analysis. Multiple fields, including quantum field theory, statistical mechanics, and the study of integrable systems, are profoundly influenced by the existence of these specialized polynomials.

In the realm of mathematics and other areas of knowledge, the development of multivariable special functions, achieved through generalizations, extensions, and amalgamations of various special functions, has cultivated a thriving domain, where the well-established identities and properties of special functions open doors to concrete applications. Accordingly, ongoing research is focused on exploring convolutions involving multiple polynomials, which holds promise for diverse applications across various research domains. This methodology introduces novel multivariable generalized polynomials, prized for their advantageous characteristics. These attributes encompass symmetric and convolutional identities, functional and differential equations, summation formulae, recurrent and explicit relationships, determinant expressions, and more.

Multiple disciplines, such as applied mathematics, theoretical physics, and approximation theory, among others within the mathematical sciences, greatly value polynomial sequences. A fundamental framework for the polynomial space is furnished by Bernstein polynomials of order n. Through diligent exploration of Bernstein polynomials, Dattoli and colleagues, as described in their work [1], employed operational techniques. Their investigations revolved around an extensive class of polynomial sequences, encompassing well-known sequences like the Miller–Lee, Bernoulli, and Euler polynomials. Given their significance, numerous scholars have introduced multivariable Hermite and other distinctive polynomials. For example, Dattoli and associates, in their study found in [2], introduced a generating function,

$$e^{r_1 t + r_2 t^2 + r_3 t^3} = \sum_{n=0}^{\infty} \mathfrak{H}_n(r_1, r_2, r_3) \frac{t^n}{n!}, \tag{1}$$

that represents 3-variable Hermite polynomials (3VHP) $\mathfrak{H}_n(r_1, r_2, r_3)$.

Moreover, when setting $r_3 = 0$, the 3VHP simplifies to the polynomials $\mathfrak{H}_n(r_1, r_2)$, renowned as the 2-variable Hermite Kamp'e de F'eriet polynomials (2VHKdFP) [3]. Furthermore, in the specific cases of $r_3 = 0$, $r_1 = 2r_1$, and $r_2 = -1$, the 3VHP transforms into the classical Hermite polynomials $\mathfrak{H}_n(r_1)$, as outlined in [4].

In recent times, a set of polynomials denoted as $\mathcal{F}_n^{[m]}(r_1, r_2, \cdots, r_m)$, labeled as multivariable Hermite polynomials (MHP), was introduced in publication [5]. These polynomials are defined by the following generating relation:

$$\exp(r_1\xi + r_2\xi^2 + \cdots + r_m\xi^m) = \sum_{n=0}^{\infty} \mathcal{F}_n^{[m]}(r_1, r_2, \cdots, r_m) \frac{\xi^n}{n!}, \tag{2}$$

and they possess the operational rule

$$\exp\left(r_2 \frac{\partial^2}{\partial {r_1}^2} + r_3 \frac{\partial^3}{\partial {r_1}^3} + \cdots + r_m \frac{\partial^m}{\partial {r_1}^m}\right) r_1^n = \mathcal{F}_n^{[m]}(r_1, r_2, \cdots, r_m), \tag{3}$$

with series representation

$$\mathcal{F}_n^{[m]}(r_1, r_2, \cdots r_m) = n! \sum_{s=0}^{[n/m]} \frac{u_m^s \, \mathcal{F}_{n-ms}^{[m]}(r_1, r_2, \cdots, u_{m-1})}{s! \, (n - ms)!}. \tag{4}$$

Several mathematicians have taken the opportunity to develop multiple special polynomial variations. The comprehensive structures of Apostol-type polynomials are presented in the investigation conducted by [6] and referred to as Apostol-type Frobenius–Genocchi polynomials. These special polynomials are symbolically denoted as $\mathbf{G}_n(r_1; \lambda; u)$ [7]. Furthermore, taking $\lambda = 1$, these polynomials reduce to the Frobenius–Genocchi polynomials

(FGN) $\mathbf{G}_n(r_1;u)$. Let us now revisit the generating expression for these polynomials, which, for $u \in \mathbb{C}$, $u \neq 1$, is articulated as follows:

$$\left(\frac{(1-u)\xi}{e^{\xi}-u}\right)e^{r_1\xi} = \sum_{n=0}^{\infty} \mathbf{G}_n(r_1;u)\frac{\xi^n}{n!}. \tag{5}$$

Consequently, when setting $r_1 = 0$, the earlier expression (5) yields the Frobenius–Genocchi numbers (FGN) denoted as $\mathbf{G}_n(u)$, which are defined as follows:

$$\left(\frac{(1-u)\xi}{e^{\xi}-u}\right) = \sum_{n=0}^{\infty} \mathbf{G}_n(u)\frac{\xi^n}{n!}. \tag{6}$$

Furthermore, on taking $u = -1$, the FGP becomes Genocchi polynomials (GP) $\mathbf{J}_n(r_1)$ [8].

It has required a lot of work to improve and unify the operating rules, the monomiality principle, and other characteristics in the context of hybrid special polynomials. This research has its roots in 1941, when Steffenson originally proposed the idea of poweroids as a means of understanding monomiality [9]. Building on Steffenson's groundwork, Dattoli further developed the theory, providing insightful enhancements and refinements [10]. The foundation for a more thorough understanding of the monomiality principle and its use in the context of hybrid special polynomials was laid by their combined contributions. According to studies like [10–17], unique categories of hybrid special polynomials connected to Appell sequences have been created and thoroughly investigated, endowing these polynomials with significant utility in a variety of domains. These areas cover engineering, biological sciences, medicine, and physical sciences, where these polynomials have important applications. These hybrid polynomials' significant properties, such as differential equations, generating functions, series definitions, integral representations, and more, highlight their critical relevance. Because differential equations commonly model problems in a variety of scientific and technical domains, solutions frequently take the form of special functions. Therefore, by using the differential equations that these hybrid special polynomials fulfill, the challenges encountered while advancing in scientific areas can be successfully addressed.

A new composite polynomial ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)$ is introduced in this paper by combining the multivariable Hermite polynomials $\mathcal{F}_n^{[m]}(r_1, r_2, \cdots, r_m)$, as specified by (2), with the Apostol-type Frobenius–Genocchi polynomials [18,19] given by (5), while complying to the monomiality principle and operational rules. The following generating relation governs this newly introduced class of polynomials, known as multivariable Hermite–Frobenius–Genocchi polynomials:

$$\left(\frac{(1-u)\xi}{e^{\xi}-u}\right)\exp(r_1\xi + r_2\xi^2 + \cdots + r_m\xi^m) = \sum_{n=0}^{\infty} {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)\frac{\xi^n}{n!}. \tag{7}$$

The remainder of this article is divided into the following sections:

In Section 2, the multivariable Genocchi polynomials based on Hermite–Frobenius ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)$ are explored and analyzed. The development of operational formulae for these polynomials is also covered in this section.

In Section 3 of the paper, the validation of the monomiality principle and the development of the related differential equation are covered for the new class of polynomials ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)$. A number of identities met by these multivariable Hermite-based Frobenius–Genocchi polynomials are also demonstrated in this section, using operational formalism.

The focus of Section 4 is on the determining the symmetric identities and summation formulae applicable to the polynomials contained in ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)$. Several

specific examples of these polynomials are also explored in this section and implications regarding them are derived as well.

The last section of the paper contains a series of concluding remarks that highlight the main findings and impacts of this research.

2. Multivariable Hermite–Frobenius–Genocchi Polynomials

In this section, an original and comprehensive approach for establishing multivariable Hermite-based Frobenius–Genocchi polynomials (MHFGPs), denoted as ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)$, is presented. The new technique described here brings a fresh perspective and methods that distinguish it from previous approaches. The goal is to improve the understanding and investigation of these polynomial sequences through the use of this creative method, offering a new viewpoint on their characteristics and prospective uses. As a result, the study offers a fresh perspective that improves the understanding of these polynomials and the ability to use them for applications. The results of the investigation are the following stated and proved theorems.

Theorem 1. *Given that Formulation* (7) *serves as the defining expression for MHFGPs denoted as* ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)$*, the ensuing deductions can be made:*

$$\begin{aligned}
\frac{\partial}{\partial r_1}[{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)] &= n\ {}_{\mathcal{F}}\mathbf{G}_{n-1}^{[m]}(r_1, r_2, \cdots, r_m; u)\\
\frac{\partial}{\partial r_2}[{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)] &= n(n-1)\ {}_{\mathcal{F}}\mathbf{G}_{n-2}^{[m]}(r_1, r_2, \cdots, r_m; u)\\
\frac{\partial}{\partial u_3}[{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)] &= n(n-1)(n-2)\ {}_{\mathcal{F}}\mathbf{G}_{n-3}^{[m]}(r_1, r_2, \cdots, r_m; u)\\
\vdots &\qquad \vdots\\
\frac{\partial}{\partial r_m}[{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)] &= n(n-1)\cdots(n-m+1)\ {}_{\mathcal{F}}\mathbf{G}_{n-m}^{[m]}(r_1, r_2, \cdots, r_m; u). \qquad (8)
\end{aligned}$$

Proof. Upon differentiating expression (7), with respect to r_1, we deduce that:

$$\frac{\partial}{\partial r_1}\left[\left(\frac{(1-u)\xi}{e^{\xi}-u}\right)\exp(r_1\xi + r_2\xi^2 + \cdots + r_m\xi^m)\right] = \xi\left[\left(\frac{(1-u)\xi}{e^{\xi}-u}\right)\exp(r_1\xi + r_2\xi^2 + \cdots + r_m\xi^m)\right]. \qquad (9)$$

By substituting the right-hand side of expression (7) into the preceding Equation (9), we establish

$$\frac{\partial}{\partial r_1}\left[\sum_{n=0}^{\infty} {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)\frac{\xi^n}{n!}\right] = \sum_{n=0}^{\infty} {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)\frac{\xi^{n+1}}{n!}. \qquad (10)$$

After substituting $n \to n-1$ in the right-hand side of the preceding equation and subsequently equating the coefficients of similar exponents of ξ, we arrive at the derivation of the initial expression within the system of Equation (8).

Going forward, when differentiating expression (7) with respect to r_2, we observe

$$\frac{\partial}{\partial r_2}\left[\left(\frac{(1-u)\xi}{e^{\xi}-u}\right)\exp(r_1\xi + r_2\xi^2 + \cdots + r_m\xi^m)\right] = \xi^2\left[\left(\frac{(1-u)\xi}{e^{\xi}-u}\right)\exp(r_1\xi + r_2\xi^2 + \cdots + r_m\xi^m)\right]. \qquad (11)$$

Upon introducing the right-hand side of expression (7) into the equation preceding (11), we confirm:

$$\frac{\partial}{\partial r_2}\left[\sum_{n=0}^{\infty} {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)\frac{\xi^n}{n!}\right] = \sum_{n=0}^{\infty} {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)\frac{\xi^{n+2}}{n!}. \qquad (12)$$

By substituting $n \to n-2$ in the right-hand side of the preceding equation and subsequently equating the coefficients of similar exponents of ξ, we arrive at the derivation of the second expression within the system of Equation (8).

Likewise, following a similar approach, we derive the remaining expressions within the system (8). □

Theorem 2. *The generating expression for MHFGPs denoted as* ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)$ *is expressed as follows:*

$$\left(\frac{(1-u)\xi}{e^{\xi}-u}\right)\exp(r_1\xi + r_2\xi^2 + \cdots + r_m\xi^m) = \sum_{n=0}^{\infty} {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)\frac{\xi^n}{n!}. \tag{13}$$

Proof. We establish the validity of this result through two distinct approaches:

(i) By expanding the product of terms $\left(\frac{(1-u)\xi}{e^{\xi}-u}\right)$ and $\exp(r_1\xi + r_2\xi^2 + \cdots + r_m\xi^m)$ using the Newton series and organizing the product of function developments $\left(\frac{(1-u)\xi}{e^{\xi}-u}\right)$ and $\exp(r_1\xi + r_2\xi^2 + \cdots + r_m\xi^m)$ in terms of powers of ξ, we discern the emergence of the polynomials ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)$ expressed in Equation (7) as coefficients of $\frac{\xi^n}{n!}$. This realization establishes the generating function of the MHFGPs.

(ii) By replacing r_1 on both sides of expression (5) with the multiplicative operator $\hat{M} = r_1 + 2r_2\partial_{r_1} + 3u_3\partial_{r_1}^2 + \cdots + mr_m\partial_{r_1}^{m-1}$ of the MHFGPs as defined in [5], we arrive at

$$\left(\frac{(1-u)\xi}{e^{\xi}-u}\right)e^{(r_1+2r_2\partial_{r_1}+3u_3\partial_{r_1}^2+\cdots+mr_m\partial_{r_1}^{m-1})\xi} = \sum_{n=0}^{\infty}\mathbf{G}_n(r_1 + 2r_2\partial_{r_1} + 3u_3\partial_{r_1}^2 + \cdots + mr_m\partial_{r_1}^{m-1}; u)\frac{\xi^n}{n!}. \tag{14}$$

In light of the identity presented in expression (7) of [15], we obtain the left-hand side of Equation (13). By denoting the right-hand side as ${}_{\mathcal{F}}\mathbf{G}_n(r_1 + 2r_2\partial_{r_1} + 3u_3\partial_{r_1}^2 + \cdots + mr_m\partial_{r_1}^{m-1}; u)$ by ${}_{\mathcal{F}}\mathbf{G}_n(r_1, r_2, \cdots, r_m; u)$, we arrive at the deduction of assertion (13). □

Subsequently, we proceed to establish the operational formulae for MHFGPs denoted as ${}_{\mathcal{F}}\mathbf{G}_n(r_1, r_2, \cdots, r_m; u)$ through the verification of the following outcome:

Theorem 3. *For MHFGP* ${}_{\mathcal{F}}\mathbf{G}_n(r_1, r_2, \cdots, r_m; u)$*, the operational rule is as follows:*

$$\exp\left(r_2\frac{\partial^2}{\partial {r_1}^2} + u_3\frac{\partial^3}{\partial {r_1}^3} + \cdots + r_m\frac{\partial^m}{\partial {r_1}^m}\right)\left\{\mathbf{G}_n(r_1; u)\right\} = {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u). \tag{15}$$

Proof. To establish the validity of the result (15), we move forward by differentiating expression (7) in the following manner:

$$\begin{aligned}
\frac{\partial}{\partial r_1}[{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)] &= n\ {}_{\mathcal{F}}\mathbf{G}_{n-1}^{[m]}(r_1, r_2, \cdots, r_m; u)\\
\frac{\partial^2}{\partial {r_1}^2}[{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)] &= n(n-1)\ {}_{\mathcal{F}}\mathbf{G}_{n-2}^{[m]}(r_1, r_2, \cdots, r_m; u)\\
\frac{\partial^3}{\partial {r_1}^3}[{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)] &= n(n-1)(n-2)\ {}_{\mathcal{F}}\mathbf{G}_{n-3}^{[m]}(r_1, r_2, \cdots, r_m; u)\\
&\vdots \qquad\qquad \vdots\\
\frac{\partial^m}{\partial {r_1}^m}[{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)] &= n(n-1)(n-2)\cdots(n-m+1)\ {}_{\mathcal{F}}\mathbf{G}_{n-m}^{[m]}(r_1, r_2, \cdots, r_m; u)
\end{aligned} \tag{16}$$

and

$$\frac{\partial}{\partial r_2}[{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)] = n(n-1)\ {}_{\mathcal{F}}\mathbf{G}_{n-2}^{[m]}(r_1, r_2, \cdots, r_m; u)$$

$$\frac{\partial}{\partial u_3}[{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)] = n(n-1)(n-2)\ {}_{\mathcal{F}}\mathbf{G}_{n-3}^{[m]}(r_1, r_2, \cdots, r_m; u)$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{\partial}{\partial r_m}[{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)] = n(n-1)\cdots(n-m+1)\ {}_{\mathcal{F}}\mathbf{G}_{n-m}^{[m]}(r_1, r_2, \cdots, r_m; u). \quad (17)$$

Upon examining the system of Equations (16) and (17), it becomes evident that MHFGPs serve as solutions to the following equations:

$$\frac{\partial}{\partial r_2}[{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)] = \frac{\partial^2}{\partial {r_1}^2}[{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)]$$

$$\frac{\partial}{\partial u_3}[{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)] = \frac{\partial^3}{\partial {r_1}^3}[{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)]$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{\partial}{\partial r_m}[{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)] = \frac{\partial^m}{\partial {r_1}^m}[{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)], \quad (18)$$

subject to the following initial constraints:

$${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, 0, 0, \cdots, 0; u) = \mathbf{G}_n(r_1; u). \quad (19)$$

Hence, taking into consideration the earlier Equations (18) and (19), we arrive at the establishment of statement (15). □

Moving forward, our focus shifts to deriving the series representation of the MHFGPs ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, 0, 0, \cdots, 0; u) = \mathbf{G}_n(r_1; u)$ through the verification of the following outcomes:

Theorem 4. *For the MHFGPs* ${}_{\mathcal{F}}\mathbf{G}_n(r_1, r_2, \cdots, r_m; u)$*, the subsequent series representations are illustrated:*

$${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u) = \sum_{s=0}^{n} \binom{n}{s} \mathbf{G}_s(u) \mathcal{F}_{n-s}^{[m]}(r_1, r_2, \cdots, r_m) \quad (20)$$

and

$${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u) = \sum_{s=0}^{n} \binom{n}{s} \mathbf{G}_s(r_1; u) \mathcal{F}_{n-s}^{[m]}(r_2, u_3, \cdots, r_m). \quad (21)$$

Proof. Upon substituting formulations (2) and (6) into the left-hand side of Equation (7), our findings reveal

$$\sum_{s=0}^{\infty} \mathbf{G}_s(u) \frac{\xi^s}{s!} \sum_{n=0}^{\infty} \mathcal{F}_n^{[m]}(r_1, r_2, \cdots, r_m) \frac{\xi^n}{n!} = \sum_{n=0}^{\infty} {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u) \frac{\xi^n}{n!}. \quad (22)$$

Interchanging the expressions and performing the substitution $n \to n - s$ within the resultant expression, guided by the Cauchy product rule, gives rise to

$$\sum_{n=0}^{\infty} {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u) \frac{\xi^n}{n!} = \sum_{n=0}^{\infty} \sum_{s=0}^{n} \mathbf{G}_s(u) \mathcal{F}_n^{[m]}(r_1, r_2, \cdots, r_m) \frac{\xi^n}{(n-s)!\, s!}. \quad (23)$$

By multiplying and then dividing the right-hand side of the preceding expression by $n!$, and subsequently equating the coefficients of the matching powers of ξ on both sides, we arrive at the conclusion presented in (20).

Using a similar approach, by incorporating expressions (2) and (5) (with $r_1 = 0$) into the left-hand side of Equation (7), we ascertain

$$\sum_{s=0}^{\infty} \mathbf{G}_s(r_1;u)\frac{\xi^s}{s!}\sum_{n=0}^{\infty}\mathcal{F}_n^{[m]}(r_2,u_3,\cdots,r_m)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty}{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,\cdots,r_m;u)\frac{\xi^n}{n!}. \tag{24}$$

Through the interchange of expressions and the subsequent substitution of $n \to n-s$ within the resultant expression, guided by the Cauchy product rule, we attain

$$\sum_{n=0}^{\infty}{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,\cdots,r_m;u)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty}\sum_{s=0}^{n}\mathbf{G}_s(r_1;u)\mathcal{F}_n^{[m]}(r_2,u_3,\cdots,r_m)\frac{\xi^n}{(n-s)!\,s!}. \tag{25}$$

Upon multiplying and dividing the right-hand side of the preceding expression by $n!$, followed by equating the coefficients of matching powers of ξ on both sides, we reach the deduction presented in Equation (21). □

3. Monomiality Principle and Properties of Hybrid Special Polynomials

The idea of expressing certain functions or polynomials in terms of monomials is known as the monomiality principle, and it is a foundational idea in mathematics. According to the monomiality principle, complex functions or expressions can be represented as linear combinations of these monomials. The monomiality concept is fundamental to many areas of mathematics, particularly the study of special functions and polynomials. By disassembling complex expressions into their simpler component pieces, it enables their simplification and analysis. Researchers can frequently deduce features, correlations, and operational principles that are simpler to manipulate and comprehend by modeling functions as linear combinations of monomials. By matching the coefficients of monomials in the approximation and the target function, the monomiality principle can be utilized to build polynomial approximations to functions. The monomial basis is frequently used to define interpolating polynomials in numerical methods like polynomial interpolation, enabling quick calculations. Insights into convergence, completeness, and other analytical characteristics can be gained by investigating function spaces and their properties, using the monomial basis. The monomiality principle emphasizes the idea that many intricate mathematical constructions can be disassembled into less complicated components, allowing for a deeper comprehension and more efficient computations.

Hybrid special polynomials have undergone a thorough examination and integration of the monomiality principle, operational principles, and other characteristics. Steffenson first proposed the idea of monomiality in 1941, expanding on the idea of poweroids [9], and Dattoli later improved it [10]. The operators $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ play a crucial role in this particular context as multiplicative and derivative operators for a polynomial set described by $j_k(r_1)_{k\in\mathbb{N}}$.

The comprehensive exploration and integration of the monomiality principle, operational rules, and other attributes within hybrid special polynomials have undergone rigorous scrutiny. The notion of monomiality was initially introduced by Steffenson in 1941, building upon the concept of poweroids [9], and was later refined by Dattoli [10]. Within this particular context, the operators $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ play a central role as multiplicative and derivative operators for a polynomial set denoted by $j_k(r_1)_{k\in\mathbb{N}}$. The following expressions define these operators:

$$j_{k+1}(r_1) = \hat{\mathcal{M}}\{j_k(r_1)\} \tag{26}$$

and

$$k\, j_{k-1}(r_1) = \hat{\mathcal{D}}\{j_k(r_1)\}. \tag{27}$$

Consequently, the sequence of polynomials $j_k(r_1)_{m\in\mathbb{N}}$, which lends itself to manipulation through multiplicative and derivative operators, is recognized as a quasi-monomial. It is of paramount importance that this quasi-monomial conforms to the subsequent formulation:

$$\hat{\mathcal{D}}\hat{\mathcal{M}} - \hat{\mathcal{M}}\hat{\mathcal{D}} = \hat{1} = [\hat{\mathcal{D}}, \hat{\mathcal{M}}], \tag{28}$$

thus showing a Weyl group structure.

These $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ operators can be harnessed to uncover the importance of the set $j_k(r_1)_{k\in\mathbb{N}}$, assuming it meets the criteria of a quasi-monomial. Consequently, the following axioms come into play:

(i) The differential equation is exhibited by $j_k(r_1)$, by taking into account the expression

$$\hat{\mathcal{M}}\hat{\mathcal{D}}\{j_k(r_1)\} = k\, j_k(r_1); \tag{29}$$

(ii) With $j_0(r_1) = 1$, the expression

$$j_k(r_1) = \hat{\mathcal{M}}^k\,\{1\} \tag{30}$$

produces the explicit series representation;

(iii) Furthermore, through the utilization of identity (30), the generating expression is expressed in the form

$$e^{w\hat{\mathcal{M}}}\{1\} = \sum_{k=0}^{\infty} j_k(r_1)\frac{w^k}{k!}, \quad |w| < \infty. \tag{31}$$

These techniques are still used today in many areas of mathematical physics, quantum mechanics, and classical optics. This suggests that these techniques provide effective and efficient research tools. We demonstrate the validity of the monomiality concept for MHFGPs by highlighting the significance of this method. Here, we endorse the monomiality principle for MHFGPs ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)$ through showing the subsequent outcomes:

Theorem 5. *The MHFGPs* ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)$ *exhibit the subsequent multiplicative and derivative operators,*

$$\hat{M_{\mathcal{F}\mathbf{G}}} = r_1 + 2r_2\partial_{r_1} + 3u_3\partial_{r_1}^2 + \cdots + mr_m\partial_{r_1}^{m-1} - \frac{1-u-\partial_{r_1}\,e^{\partial_{r_1}}}{e^{\partial_{r_1}}-u} \tag{32}$$

and

$$\hat{D_{\mathcal{F}\mathbf{G}}} = \partial_{r_1}, \tag{33}$$

where $\partial_{r_1} = \frac{\partial}{\partial r_1}$.

Proof. Upon differentiating expression (7), with respect to ξ on both sides, we discover

$$\left(r_1 + 2r_2\xi + 3u_3\xi^2 + \cdots + mr_m\xi^{m-1} - \frac{1-u-\xi\,e^{\xi}}{e^{\xi}-u}\right)\left(\frac{(1-u)\xi}{e^{\xi}-u}\exp(r_1\xi + r_2\xi^2 + \cdots + r_m\xi^m)\right) = \sum_{n=0}^{\infty} n\ {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)\frac{\xi^{n-1}}{n!}. \tag{34}$$

This can be further expressed as

$$\left(r_1 + 2r_2\xi + 3u_3\xi^2 + \cdots + mr_m\xi^{m-1} - \frac{e^{\xi}}{e^{\xi}-u}\right)\left(\sum_{n=0}^{\infty}{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)\frac{\xi^{n-1}}{n!}\right) = \sum_{n=0}^{\infty} n\ {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, \cdots, r_m; u)\frac{\xi^{n-1}}{n!}. \tag{35}$$

Similarly, upon differentiating (7) with respect to r_1, we deduce the following identity:

$$\frac{\partial}{\partial r_1}\left(\frac{(1-u)\xi}{e^{\xi}-u}\exp(r_1\xi+r_2\xi^2+\cdots+r_m\xi^m)\right)=\xi\left(\frac{(1-u)\xi}{e^{\xi}-u}\exp(r_1\xi+r_2\xi^2+\cdots+r_m\xi^m)\right)$$

$$\frac{\partial}{\partial r_1}\left(\sum_{n=0}^{\infty}{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,\cdots,r_m;u)\frac{\xi^{n-1}}{n!}\right)=\xi\left(\sum_{n=0}^{\infty}{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,\cdots,r_m;u)\frac{\xi^{n-1}}{n!}\right). \quad (36)$$

By substituting $n\to n+1$ in the right-hand side of (35) and subsequently equating the coefficients of the matching exponents of ξ, utilizing expressions (26) and (37), within the resulting equation, we establish the validity of assertion (32).

Furthermore, the latter component of Equation (36) can be expressed as follows:

$$\frac{\partial}{\partial r_1}\left(\sum_{n=0}^{\infty}{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,\cdots,r_m;u)\frac{\xi^{n-1}}{n!}\right)=\left(\sum_{n=0}^{\infty}{}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,\cdots,r_m;u)\frac{\xi^{n+1}}{n!}\right). \quad (37)$$

Upon substituting $n\to n-1$ in the right-hand side of (37) and subsequently equating the coefficients of the identical exponents of ξ, utilizing expressions (27), within the resulting equation, we establish the validation of assertion (33). □

Subsequently, we derive the differential equation governing the MHFGPs ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,\cdots,r_m;u)$ by presenting the following outcome:

Theorem 6. *The MHFGPs ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,\cdots,r_m;u)$ adhere to the following differential equation:*

$$\left(r_1\partial_{r_1}+2r_2\partial_{r_1}^2+3u_3\partial_{r_1}^3+\cdots+mr_m\partial_{r_1}^m-\frac{1-u-\partial_{r_1}e^{\partial_{r_1}}}{e^{\partial_{r_1}}-u}\partial_{r_1}-n\right){}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,\cdots,r_m;u)=0. \quad (38)$$

Proof. By substituting expressions (32) and (33) into Equation (29), we substantiate the validity of assertion (38). □

4. Summation Formulae and Symmetric Identities

In the area of special functions, summation formulae and symmetric identities are essential concepts. These mathematical tools assist in establishing connections, streamlining computations, and revealing patterns among diverse classes of functions, particularly those that are frequently employed in mathematics, physics, engineering, and other fields of study. Summation formulae help with calculations and analyses and are frequently used to more easily determine the convergence or divergence of infinite series, offering us insights into the behavior of functions at various points, while symmetric identities help us simplify complex expressions that involve symmetric functions by substituting corresponding terms, resulting in a more elegant and manageable form. These identities are essential because they frequently offer additional ways to prove equality and equivalence, which is why they are commonly used to prove various features of functions. Furthermore, by highlighting the fundamental symmetry characteristics of functions, these identities make it easier to analyze and comprehend their behavior and characteristics. A number of special functions also have symmetry features, including trigonometric, Bessel, and Legendre functions. These functions can be connected to one another and have attributes derived from them, thanks to symmetric identities.

Further clarifying the method, in order to develop the summation formulae for the polynomials in ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,r_3,\cdots,r_m;u)$, we proceed to present the following outcomes:

Theorem 7. *For the MHFGPs ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,u_3,\cdots,r_m;u)$, the succeeding implicit summation formula holds true:*

$${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1+w,r_2,u_3,\cdots,r_m;u)=\sum_{k=0}^{n}\binom{n}{k}{}_{\mathcal{F}}\mathbf{G}_k^{[m]}(r_1,r_2,u_3,\cdots,r_m;u)w^{n-k}. \quad (39)$$

Proof. On taking $r_1 \to r_1 + w$ in expression (7), we find

$$\left(\frac{(1-u)\xi}{e^{\xi}-u}\right)\exp((r_1+w)\xi + r_2\xi^2 + \cdots + r_m\xi^m) = \sum_{n=0}^{\infty} {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1+w, r_2, \cdots, r_m; u)\frac{\xi^n}{n!}, \tag{40}$$

which can be expressed as

$$\left(\frac{(1-u)\xi}{e^{\xi}-u}\right)\exp(r_1\xi + r_2\xi^2 + \cdots + r_m\xi^m)\exp(w\xi) = \sum_{n=0}^{\infty} {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1+w, r_2, \cdots, r_m; u)\frac{\xi^n}{n!}. \tag{41}$$

Utilizing the series expansion of $\exp(w\xi)$ in the left-hand side of the preceding expression, we obtain

$$\sum_{k=0}^{\infty} {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1+w, r_2, \cdots, r_m; u)w^n\frac{\xi^{n+k}}{n!k!} = \sum_{n=0}^{\infty} {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1+w, r_2, \cdots, r_m; u)\frac{\xi^n}{n!}. \tag{42}$$

By following this procedure, we arrive at the derivation of statement (39) through the substitution of n with $n-k$ in the right-hand side of the ensuing expression. Subsequently, we equate the coefficients of the identical powers of ξ in the resulting equation. □

Corollary 1. *Substituting $w = 1$ into Equation (39), we obtain*

$${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1+1, r_2, u_3, \cdots, r_m; u) = \sum_{k=0}^{n}\binom{n}{k}{}_{\mathcal{F}}\mathbf{G}_k^{[m]}(r_1, r_2, u_3, \cdots, r_m; u). \tag{43}$$

Theorem 8. *For the MHFGPs ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, u_3, \cdots, r_m; u)$, the succeeding implicit summation formula holds true:*

$${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1+x, r_2+y, u_3+z, \cdots, r_m; u) = \sum_{k=0}^{n}\binom{n}{k}{}_{\mathcal{F}}\mathbf{G}_{n-k}^{[m]}(r_1, r_2, u_3, \cdots, r_m; u)\mathcal{F}_k(x, y, z). \tag{44}$$

Proof. On taking $r_1 \to r_1 + x$, $r_2 \to r_1 + y$ and $r_3 \to r_3 + z$ in expression (7), it follows that

$$\left(\frac{(1-u)\xi}{e^{\xi}-u}\right)\exp((r_1+x)\xi + (r_2+y)\xi^2 + (u_3+z)\xi^3 + \cdots + r_m\xi^m) = \sum_{n=0}^{\infty} {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1+x, r_2+y, u_3+z, \cdots, r_m; u)\frac{\xi^n}{n!}, \tag{45}$$

which can be expressed as

$$\left(\frac{(1-u)\xi}{e^{\xi}-u}\right)\exp(r_1\xi + r_2\xi^2 + \cdots + r_m\xi^m)\exp(x\xi + y\xi^2 + z\xi^3) = \sum_{n=0}^{\infty} {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1+x, r_2+y, u_3+z, \cdots, r_m; u)\frac{\xi^n}{n!}. \tag{46}$$

Utilizing the series expansion of $\exp(x\xi + y\xi^2 + z\xi^3)$ on the left-hand side of the preceding expression, we obtain

$$\sum_{n=0}^{\infty} {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1, r_2, u_3, \cdots, r_m; u)\mathcal{F}_k(x, y, z)\frac{\xi^{n+k}}{n!k!} = \sum_{n=0}^{\infty} {}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1+x, r_2+y, u_3+z, \cdots, r_m; u)\frac{\xi^n}{n!}. \tag{47}$$

The derivation of statement (44) is accomplished by substituting $n \to n-k$ into the left-hand side of the following expression and subsequently equating the coefficients of the corresponding powers of ξ in the resultant equation. □

Corollary 2. *Substituting $z = 0$ in expression (44), we obtain*

$$\mathcal{F}\mathbf{G}_{n}^{[m]}(r_1 + x, r_2 + y, u_3, \cdots, r_m; u) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{F}\mathbf{G}_{n-k}^{[m]}(r_1, r_2, u_3, \cdots, r_m; u)\mathcal{F}_k(x, y). \quad (48)$$

Theorem 9. *For the MHFGPs $\mathcal{F}\mathbf{G}_{n}^{[m]}(r_1, r_2, u_3, \cdots, r_m; u)$, the succeeding implicit summation formula holds true:*

$$\mathcal{F}\mathbf{G}_{n+s}^{[m]}(q, r_2, u_3, \cdots, r_m; u) = \sum_{l,m=0}^{n,s} \binom{n}{l} \binom{s}{m} (q - r_1)^{l+m} \mathcal{F}\mathbf{G}_{n+s-l-m}^{[m]}(r_1, r_2, u_3, \cdots, r_m; u). \quad (49)$$

Proof. By substituting $\xi \to \xi + \eta$, considering the expression

$$\sum_{M=0}^{\infty} g(M) \frac{(r_1 + r_2)^M}{M!} = \sum_{l,m=0}^{\infty} g(l + m) \frac{r_1^l \, r_2^m}{l! \, m!} \quad (50)$$

in (7), and subsequently simplifying the resulting expression, we obtain

$$e^{-r_1(\xi+\eta)} \sum_{n,s=0}^{\infty} \mathcal{F}\mathbf{G}_{n+s}^{[m]}(r_1, r_2, u_3, \cdots, r_m; u) \frac{\xi^n \, \eta^s}{n! \, s!} = \left(\frac{(1 - u)(\xi + \eta)}{e^{\xi+\eta} - u} \right) \exp(r_2(\xi + \eta)^2 + \cdots + r_m(\xi + \eta)^m). \quad (51)$$

Upon substituting $r_1 \to q$ into the preceding Equation (51) and comparing the resulting expression to the previous one, followed by an expansion of the exponential function, we arrive at

$$\sum_{n,s=0}^{\infty} \mathcal{F}\mathbf{G}_{n+s}^{[m]}(r_1, r_2, u_3, \cdots, r_m; u) \frac{\xi^n \, \eta^s}{n! \, s!} = \sum_{M=0}^{\infty} (q - r_1)^M \frac{(\xi + \eta)^M}{M!} \times \sum_{n,s=0}^{\infty} \mathcal{F}\mathbf{G}_{n+s}^{[m]}(r_1, r_2, u_3, \cdots, r_m; u) \frac{\xi^n \, \eta^s}{n! \, s!}. \quad (52)$$

Hence, considering expression (50) from the previous Equation (52) and subsequently substituting $n \to n - l$ and $s \to s - m$ in the resultant expression, we ascertain

$$\sum_{n,s=0}^{\infty} \mathcal{F}\mathbf{G}_{n+s}^{[m]}(r_1, r_2, u_3, \cdots, r_m; u) \frac{\xi^n \, \eta^s}{n! \, s!} = \sum_{n,s=0}^{\infty} \sum_{l,m=0}^{n,s} \frac{(q - r_1)^{l+m}}{l! \, m!} \times \mathcal{F}\mathbf{G}_{n+s-l-m}^{[m]}(r_1, r_2, u_3, \cdots, r_m; u) \frac{\xi^n \, \eta^s}{(n - l)! \, (s - m)!}. \quad (53)$$

By comparing the coefficients of similar exponents of ξ and η on both sides of the preceding equation, we validate assertion (49). □

Corollary 3. *On substitution of $n = 0$ in expression (49), we find*

$$\mathcal{F}\mathbf{G}_{s}^{[m]}(q, r_2, u_3, \cdots, r_m; u) = \sum_{m=0}^{s} \binom{s}{m} (q - r_1)^m \mathcal{F}\mathbf{G}_{s-m}^{[m]}(r_1, r_2, u_3, \cdots, r_m; u). \quad (54)$$

Corollary 4. *On replacement of q with $q + r_1$ and setting $m = 2$ in Equation (49), we obtain*

$$\mathcal{F}\mathbf{G}_{n+s}^{[m]}(q + r_1, r_2, u_3, \cdots, r_m; u) = \sum_{l,m=0}^{n,s} \binom{n}{l} \binom{s}{m} (q)^{l+m} \mathcal{F}\mathbf{G}_{n+s-l-m}^{[m]}(r_1, r_2, u_3, \cdots, r_m; u). \quad (55)$$

Corollary 5. *On substituting q with $q + r_1$ and setting $m = 1$ in Equation (49), we arrive at*

$$ {}_{\mathcal{F}}\mathbf{G}^{[m]}_{n+s}(q+r_1;u) = \sum_{l,m=0}^{n,s} \binom{n}{l}\binom{s}{m}(q)^{l+m}\, {}_{\mathcal{F}}\mathbf{G}^{[m]}_{n+s-l-m}(r_1;u). \tag{56} $$

Corollary 6. *By setting $q = 0$ in Equation (49), we obtain*

$$ {}_{\mathcal{F}}\mathbf{G}^{[m]}_{n+s}(r_2,u_3,\cdots,r_m;u) = \sum_{l,m=0}^{n,s} \binom{n}{l}\binom{s}{m}(-r_1)^{l+m}\, {}_{\mathcal{F}}\mathbf{G}^{[m]}_{n+s-l-m}(r_1,r_2,u_3,\cdots,r_m;u). \tag{57} $$

The assessment of infinite sums containing specialized functions is a common problem-solving strategy in the fields of applied mathematics and physics. Applications for generalized special functions can be found in many fields, including electromagnetics and combinatorics. Numerous academics have identified and carefully examined numerous identity categories connected to Apostol-type polynomials. Examples of research can be seen in [18–29]. These investigations are what motivate the MHAFEP's search for symmetry identities. Let us proceed now to consider the following definitions:

Definition 1. *The generalized summation of integer powers $\mathfrak{S}k(n;\lambda)$ is defined by the following generating function, valid for any real or complex input λ:*

$$ \sum j = 0^{\infty}\mathfrak{S}_j(n;\lambda)\frac{\xi^j}{j!} = \frac{\lambda e^{(n+1)\xi}-1}{\lambda e^{\xi}-1}. \tag{58} $$

Definition 2. *The multiple power sums $\mathfrak{S}k^{(l)}(m;\lambda)$ are defined using the following generating function, applicable for every real or complex input λ:*

$$ \frac{1}{\lambda^l}\sum n = 0^{\infty}\left(\sum_{q=0}^{n}\binom{n}{q}(-l)^{n-q}\mathfrak{S}_k^{(l)}(m;\lambda)\right)\frac{\xi^n}{n!} = \left(\frac{1-\lambda^m e^{m\xi}}{1-\lambda e^{\xi}}\right)^l. \tag{59} $$

We present justification for the next findings, in order to demonstrate the symmetry identities for MHFGP ${}_{\mathcal{F}}\mathbf{G}^{[m]}_{n}(r_1,r_2,r_3,\cdots,r_m;u)$.

Theorem 10. *The subsequent symmetry relationship connecting MHFGPs and generalized integer power sums holds true for any positive integers μ and η, alongside non-negative integer n, and complex parameter u in $\mathbb{C}$:*

$$ \begin{aligned} &\sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}\, {}_{\mathcal{F}}\mathbf{G}^{[m]}_{n-k}(\eta r_1,\eta^2 r_2,\eta^3 u_3,\cdots,\eta^m r_m;u)\sum_{l=0}^{k}\binom{k}{l}\eta^k\mathfrak{S}_l(\mu-1;\frac{1}{u}) \\ &\qquad\times {}_{\mathcal{F}}\mathbf{G}^{[m]}_{k-1}(\mu r_1,\mu^2 r_2,\mu^3 U_3,\cdots,\mu^m r_m;u) \\ &= \sum_{k=0}^{n}\binom{n}{k}\eta^{n-k}\, {}_{\mathcal{F}}\mathbf{G}^{[m]}_{n-k}(\mu r_1,\mu^2 r_2,\mu^3 u_3,\cdots,\mu^m r_m;u)\sum_{l=0}^{k}\binom{k}{l}\mu^k\mathfrak{S}_l(\eta-1;\frac{1}{u}) \\ &\qquad\times {}_{\mathcal{F}}\mathbf{G}^{[m]}_{k-1}(\eta r_1,\eta^2 r_2,\eta^3 U_3,\cdots,\eta^m r_m;u). \end{aligned} \tag{60} $$

Proof. Let

$$ \mathfrak{G}(\xi) := \frac{(1-u)\xi\; e^{\mu r_1\eta\xi + r_2(\mu\eta\xi)^2 + u_3(\mu\eta\xi)^3}(e^{\mu\eta\xi}-u)\; e^{\mu\eta\xi r_1 + r_2(\mu\eta\xi)^2 + U_3(\mu\eta\xi)^3}}{(e^{\mu\xi}-u)\;(e^{\eta\xi}-u)}, \tag{61} $$

which, when considering the Cauchy product rule, transforms into

$$ \begin{aligned} \mathfrak{G}(\xi) = \sum_{n=0}^{\infty}\Bigg(\sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}\, {}_{\mathcal{F}}\mathbf{G}^{[m]}_{n-k}(\eta r_1,\eta^2 r_2,\eta^3 u_3,\cdots,\eta^m r_m;u)\sum_{l=0}^{k}\binom{k}{l}\eta^k\mathfrak{S}_l(\mu-1;\frac{1}{u}) \\ \times {}_{\mathcal{F}}\mathbf{G}^{[m]}_{k-1}(\mu r_1,\mu^2 r_2,\mu^3 U_3,\cdots,\mu^m r_m;u)\frac{\xi^n}{n!}. \end{aligned} \tag{62} $$

Proceeding with a comparable approach, we deduce:

$$\mathfrak{G}(\xi)=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\binom{n}{k}\eta^{n-k}{}_{\mathcal{F}}\mathbf{G}_{n-k}^{[m]}(\mu r_1,\mu^2 r_2,\mu^3 u_3,\cdots,\mu^m r_m;u)\sum_{l=0}^{k}\binom{k}{l}\mu^k\mathfrak{S}_l(\eta-1;\frac{1}{u})$$
$$\times\,{}_{\mathcal{F}}\mathbf{G}_{k-1}^{[m]}(\eta r_1,\eta^2 r_2,\eta^3 U_3,\cdots,\eta^m r_m;u)\Big)\frac{\xi^n}{n!}. \quad (63)$$

By comparing the coefficients of the corresponding powers of ξ in expressions (62) and (63), we arrive at the conclusion stated in assertion (60). □

Theorem 11. *The subsequent symmetry relation concerning MHAGPs holds true for positive integers μ and η, non-negative integer n, and complex parameter u in $\mathbb{C}$:*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\mu-1}\sum_{j=0}^{\eta-1}u^{\mu+\eta-2}(\frac{1}{u})^{i+j}\mu^{n-k}\eta^k{}_{\mathcal{F}}\mathbf{G}_k^{[m]}(\mu r_1+\frac{\mu}{\eta}j,\mu^2 r_2,\mu^3 U_3,\cdots,\mu^m r_m;u)$$
$$\times\,{}_{\mathcal{F}}\mathbf{G}_{n-k}^{[m]}(\eta r_1+\frac{\eta}{\mu}i,\eta^2 r_2,\eta^3 u_3,\cdots,\eta^m r_m;u)$$
$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\eta-1}\sum_{j=0}^{\mu-1}u^{\mu+\eta-2}(\frac{1}{u})^{i+j}\eta^{n-k}\mu^k{}_{\mathcal{F}}\mathbf{G}_k^{[m]}(\eta r_1+\frac{\eta}{\mu}j,\mu^2 r_2,\mu^3 U_3,\cdots,\mu^m r_m;u)$$
$$\times\,{}_{\mathcal{F}}\mathbf{G}_{n-k}^{[m]}(\mu r_1+\frac{\mu}{\eta}i,\eta^2 r_2,\eta^3 u_3,\cdots,\eta^m r_m;u). \quad (64)$$

Proof. Let

$$\mathfrak{H}(\xi):=\Big((1-u)\xi\Big)^2 e^{\mu\eta\xi r_1+r_2(\mu\eta\xi)^2+u_3(\mu\eta\xi)^3+\cdots+r_m(\mu\eta\xi)^m}$$
$$\times\;\frac{(e^{\mu\eta\xi}-u^{\mu})\,(e^{\mu\eta\xi}-u^{\eta})e^{\mu\eta\xi r_1+r_2(\mu\eta\xi)^2+U_3(\mu\eta\xi)^3+\cdots+r_m(\mu\eta\xi)^m}}{(e^{\mu\xi}-u)\,(e^{\eta\xi}-u)}, \quad (65)$$

which, in light of the series representations of $\frac{(e^{\mu\eta\xi}-u^{\mu})}{(e^{\eta\xi}-u)}$ and $\frac{(e^{\mu\eta\xi}-u^{\eta})}{(e^{\mu\xi}-u)}$ in the final expression gives

$$\mathfrak{H}(\xi)=\Big(\frac{(1-u)\xi}{e^{\mu\xi}-u}\Big)e^{\eta r_1(\mu\xi)+\eta^2 r_2(\mu\xi)^2+\eta^3 u_3(\mu\xi)^3+\cdots+\eta^m r_m(\mu\xi)^m}u^{\mu-1}\sum_{i=0}^{\mu-1}\Big(\frac{1}{u}\Big)^i e^{\eta\xi i}$$
$$\times\Big(\frac{(1-u)\xi}{e^{\eta\xi}-u}\Big)e^{\mu r_1(\eta\xi)+\mu^2 r_2(\eta\xi)^2+\mu^3 U_3(\eta\xi)^3+\cdots+\mu^m r_m(\eta\xi)^m}u^{\eta-1}\sum_{j=0}^{\eta-1}\Big(\frac{1}{u}\Big)^j e^{\mu\xi\, j}. \quad (66)$$

Therefore, taking into account (7) and employing the Cauchy product rule as applied in the preceding Equation (66), we ascertain that

$$\mathfrak{H}(\xi):=\sum_{n=0}^{\infty}\Bigg[\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\mu-1}\sum_{j=0}^{\eta-1}u^{\mu+\eta-2}(\frac{1}{u})^{i+j}\mu^{n-k}\eta^k{}_{\mathcal{F}}\mathbf{G}_k^{[m]}(\mu r_1+\frac{\mu}{\eta}j,\mu^2 r_2,\mu^3 U_3,\cdots,\mu^m r_m;u)$$
$$\times\,{}_{\mathcal{F}}\mathbf{G}_{n-k}^{[m]}(\eta r_1+\frac{\eta}{\mu}i,\eta^2 r_2,\eta^3 u_3,\cdots,\eta^m r_m;u)\Bigg]. \quad (67)$$

Proceeding in a comparable manner, we establish an additional identity:

$$\mathfrak{H}(\xi):=\sum_{n=0}^{\infty}\Bigg[\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\eta-1}\sum_{j=0}^{\mu-1}u^{\mu+\eta-2}(\frac{1}{u})^{i+j}\eta^{n-k}\mu^k{}_{\mathcal{F}}\mathbf{G}_k^{[m]}(\eta r_1+\frac{\eta}{\mu}j,\mu^2 r_2,\mu^3 U_3,\cdots,\mu^m r_m;u)$$
$$\times\,{}_{\mathcal{F}}\mathbf{G}_{n-k}^{[m]}(\mu r_1+\frac{\mu}{\eta}i,\eta^2 r_2,\eta^3 u_3,\cdots,\eta^m r_m;u)\Bigg]. \quad (68)$$

By analyzing the coefficients of similar powers of ξ in expressions (67) and (68), we arrive at the conclusion stated in assertion (64). □

Theorem 12. *The subsequent symmetry relationship concerning MHFGPs holds true for positive integers μ and η, non-negative integer n, and complex parameter u in $\mathbb{C}$:*

$$\sum_{k=0}^{\eta-1} u^{\eta-1}(\frac{1}{u})^k \sum_{i=0}^{n} \binom{n}{i} \mathcal{F}\mathbf{G}_{n-i}^{[m]}(\mu r_1, \mu^2 r_2, \mu^3 u_3, \cdots, \mu^m r_m; u)\eta^{n-i}(\mu k)^i$$
$$= \sum_{k=0}^{\mu-1} u^{\mu-1}(\frac{1}{u})^k \sum_{i=0}^{n} \binom{n}{i} \mathcal{F}\mathbf{G}_{n-i}^{[m]}(\eta r_1, \eta^2 r_2, \eta^3 u_3, \cdots, \eta^m r_m; u)\mu^{n-i}(\eta k)^i. \quad (69)$$

Proof. Let

$$\mathfrak{N}(\xi) := \frac{(1-u)\xi\, e^{\mu\eta\xi r_1 + r_2(\mu\eta\xi)^2 + u_3(\mu\eta\xi)^3 + \cdots + r_m(\mu\eta\xi)^m}}{(e^{\mu\eta\xi} - u^\eta)\,(e^{\mu\xi} - u)\,(e^{\eta\xi} - u)}. \quad (70)$$

By following a similar approach, as demonstrated in the preceding theorem, we derive the conclusion stated in assertion (70). □

Theorem 13. *The subsequent symmetry relationship connecting MHFGPs and multiple power sums holds true for positive integers μ and η, non-negative integer n, and complex parameter u in $\mathbb{C}$:*

$$\sum_{k=0}^{n} \binom{n}{k} \mathcal{F}\mathbf{G}_{n-k}^{[m]}(\eta r_1, \eta^2 r_2, \eta^3 u_3, \cdots, \eta^m r_m; u)u^\eta \sum_{l=0}^{k} \binom{k}{l} \sum_{r=0}^{l} \binom{l}{r} (-1)^{l-r} \mathcal{S}_k(\eta; \frac{1}{u})$$
$$\times \mathcal{F}\mathbf{G}_{k-l}^{[m+1]}(\mu r_1, \mu^2 r_2, \mu^3 U_3, \cdots, \mu^m r_m; u)\mu^{n-k+l}\eta^{k-l}$$
$$= \sum_{k=0}^{n} \binom{n}{k} \mathcal{F}\mathbf{G}_{n-k}^{[m]}(\mu r_1, \mu^2 r_2, \mu^3 u_3, \cdots, \mu^m r_m; u)u^\mu \sum_{l=0}^{k} \binom{k}{l} \sum_{r=0}^{l} \binom{l}{r} (-1)^{l-r} \mathcal{S}_k(\mu; \frac{1}{u})$$
$$\times \mathcal{F}\mathbf{G}_{k-l}^{[m+1]}(\eta r_1, \eta^2 r_2, \eta^3 U_3, \cdots, \eta^m r_m; u)\eta^{n-k+l}\mu^{k-l}. \quad (71)$$

Proof. Let

$$\mathfrak{F}(\xi) := ((\frac{(1-u)\xi}{e^\xi - u})\xi)^2\, e^{\mu r_1(\eta\xi) + \mu^2 r_2(\eta\xi)^2 + \mu^3 u_3(\eta\xi)^3 + \cdots + \mu^m r_m(\eta\xi)^m}$$
$$\times \frac{(e^{\mu\eta\xi} - u^\eta)\, e^{\mu r_1(\eta\xi) + \mu^2 r_2(\eta\xi)^2 + \mu^3 U_3(\eta\xi)^3 + \cdots + \mu^m r_m(\eta\xi)^m}}{(e^{\eta\xi} - u)\,(e^{\mu\xi} - u)}. \quad (72)$$

Upon simplifying the exponents and employing expressions (7) and (59) in the final equation, we obtain

$$\mathfrak{F}(\xi) := \sum_{n=0}^{\infty} \mathcal{F}\mathbf{G}_n^{[m]}(\eta r_1, \eta^2 r_2, \eta^3 u_3, \cdots, \eta^m r_m; u)\mu^n \frac{\xi^n}{n!} u^\eta \sum_{m=0}^{\infty} \sum_{r=0}^{m} \binom{m}{r} (-1)^{m-r} \mathcal{S}_k(\eta; \frac{1}{u})\mu^m \frac{\xi^m}{m!}$$
$$\times \mathcal{F}\mathbf{G}_{k-l}^{[m+1]}(\mu r_1, \mu^2 r_2, \mu^3 U_3, \cdots, \mu^m r_m; u)\eta^l \frac{\xi^l}{l!}. \quad (73)$$

Thus, considering the Cauchy product rule, we arrive at

$$\mathfrak{F}(\xi) := \sum_{n=0}^{\infty} \Bigg[\sum_{l=0}^{n} \binom{n}{l} \mathcal{F}\mathbf{G}_{n-l}^{[m]}(\eta r_1, \eta^2 r_2, \eta^3 u_3, \cdots, \eta^m r_m; u)\mu^{n-l}u^\eta \sum_{m=0}^{l} \binom{l}{m} \sum_{r=0}^{m} \binom{m}{r} (-1)^{m-r} \mathcal{S}_k(\eta; \frac{1}{u})$$
$$\times \mathcal{F}\mathbf{G}_{l-m}^{[m+1]}(\mu r_1, \mu^2 r_2, \mu^3 U_3, \cdots, \mu^m r_m; u)\mu^m \eta^{l-m} \Bigg] \frac{\xi^n}{n!}. \quad (74)$$

Proceeding in a comparable manner, we obtain

$$\mathfrak{F}(\xi) := \sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}\binom{n}{l}{}_{\mathcal{F}}\mathbf{G}_{n-l}^{[m]}(\mu r_1,\mu^2 r_2,\mu^3 u_3,\cdots,\mu^m r_m;u)\eta^{n-l}u^{\mu}\sum_{m=0}^{l}\binom{l}{m}\sum_{r=0}^{m}\binom{m}{r}(-1)^{m-r}\mathcal{S}_k(\mu;\frac{1}{u})\right.$$
$$\left.\times\,{}_{\mathcal{F}}\mathbf{G}_{l-m}^{[m+1]}(\eta r_1,\eta^2 r_2,\eta^3 U_3,\cdots,\eta^m r_m;u)\mu^m\mu^{l-m}\right]\frac{\xi^n}{n!}. \quad (75)$$

By comparing the coefficients of the corresponding powers of ξ in expressions (74) and (75), we establish the conclusion presented in assertion (71). □

Theorem 14. *The subsequent symmetry relationship linking MHFGPs and the generalized integer power sums holds true for positive integers μ and η, non-negative integer n, and complex parameter u in $\mathbb{C}$:*

$$\sum_{m=0}^{n}\binom{n}{m}{}_{\mathcal{F}}\mathbf{G}_{n-m}^{[m]}(\eta r_1,\eta^2 r_2,\eta^3 u_3,\cdots,\eta^m r_m;u)\mu^{n-m}u^{\mu}\sum_{r=0}^{m}\binom{m}{r}(-1)^{m-r}\mathcal{S}_k(\mu;\frac{1}{u})\eta^m$$
$$=\sum_{m=0}^{n}\binom{n}{k}{}_{\mathcal{F}}\mathbf{G}_{n-m}^{[m]}(\mu r_1,\mu^2 r_2,\mu^3 u_3,\cdots,\mu^m r_m;u)\eta^{n-m}u^{\eta}\sum_{r=0}^{m}\binom{m}{r}(-1)^{m-r}\mathcal{S}_k(\eta;\frac{1}{u})\mu^m. \quad (76)$$

Proof. Let

$$\mathfrak{M}(\xi) := \frac{(1-u)\xi\, e^{\eta r_1(\mu\xi)+\eta^2 r_2(\mu\xi)^2+\eta^3 u_3(\mu\xi)^3+\cdots+\eta^m r_m(\mu\xi)^m}\left(e^{\mu\eta\xi}-u^{\mu}\right)}{\left(e^{\eta\xi}-u\right)\left(e^{\mu\xi}-u\right)}. \quad (77)$$

By following a similar approach to that demonstrated in the preceding theorem, we deduce the conclusion stated in assertion (76). □

5. Conclusions

The multivariable Hermite-based Frobenius–Genocchi polynomials ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,\cdots,r_m;u)$ were introduced in this paper and thoroughly examined for establishing properties and potential uses. The polynomials contained in ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,\cdots,r_m;u)$ were given a thorough introduction in Section 2, and operational formulae that simplify their computation were also provided in this section. In Section 3, the inquiry was carried out further, and the behavior of the polynomials ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,\cdots,r_m;u)$ was clarified by thoroughly validating the monomiality principle and determining the corresponding differential equation. To further deepen the understanding of the properties of multivariable Hermite-based Frobenius–Genocchi polynomials ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,\cdots,r_m;u)$, a variety of significant identities governing them were established. Additional connections and interactions between the polynomials ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,\cdots,r_m;u)$ were highlighted in Section 3, using operational formalism, deepening the understanding of their nature. For the multivariable Hermite-based Frobenius–Genocchi polynomials ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,\cdots,r_m;u)$, summation equations and symmetric identities were deduced in Section 4. These results provide the ability to modify and streamline equations utilizing these polynomials, making them more adaptable for use in a variety of mathematical contexts. Additionally, this section's examination of particular polynomial instances offered helpful indications on the range of applications of multivariable Hermite-based Frobenius–Genocchi polynomials for answering mathematical queries.

In summary, by providing a thorough understanding of multivariable Hermite-based Frobenius–Genocchi polynomials, this research significantly contributes to the topic of polynomial theory. The summation formulae, identities, derived operational formulae, and analyses of particular cases all contribute to a better understanding of these polynomials. The goal of this investigation is to contribute to further exploration for finding applications of the polynomials ${}_{\mathcal{F}}\mathbf{G}_n^{[m]}(r_1,r_2,\cdots,r_m;u)$ in related domains.

Investigating these polynomials' characteristics and prospective uses in physics and related fields is an interesting area for potential future investigation. In order to investigate these polynomials' algebraic and analytical features, the derived generating function provides a succinct representation of these polynomials. The developed recurrence relations also facilitate the computation and study of polynomial values via recursive computational methods. Quantum mechanics, statistical physics, mathematical physics, and engineering are just several of the various domains where these polynomials are employed. These findings not only increase comprehension of multivariable Hermite-based Frobenius–Genocchi polynomials but also open up new research directions into their properties and prospective uses in physics and related fields. This could lead to generating their extended forms via fractional operators, investigating approximation and weighted approximation attributes, and analyzing hypergeometric representations.

Operational techniques are useful instruments for developing novel families of special functions and determining properties pertaining to both regular and extended special functions. These strategies enable the quick derivation of explicit solutions for families of partial differential equations, including the Heat and D'Alembert equations. We are able to examine solutions for a variety of physical issues involving several classes of partial differential equations using the methodology described in this work in conjunction with the monomiality principle.

Author Contributions: Conceptualization, S.A.W., G.I.O., A.M.M. and W.H.; data curation, S.A.W. and G.I.O.; formal analysis, S.A.W. and G.I.O.; funding acquisition, G.I.O.; investigation, S.A.W., G.I.O., A.M.M. and W.H.; methodology, S.A.W.; project administration, S.A.W. and G.I.O.; resources, S.A.W., G.I.O., A.M.M. and W.H.; software, S.A.W. and G.I.O.; supervision, S.A.W. and G.I.O.; validation, S.A.W., G.I.O., A.M.M. and W.H.; visualization, S.A.W., G.I.O., A.M.M. and W.H.; writing—original draft, S.A.W., G.I.O. and A.M.M.; writing—review and editing, S.A.W., G.I.O., A.M.M. and W.H. All authors have read and agreed to the published version of the manuscript.

Funding: The APC was funded by University of Oradea, Romania.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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