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On the Analytic Continuation of Lauricella–Saran Hypergeometric Function $F_K(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z})$

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Abstract: The paper establishes an analytical extension of two ratios of Lauricella–Saran hypergeometric functions F_K with some parameter values to the corresponding branched continued fractions in their domain of convergence. The PC method used here is based on the correspondence between a formal triple power series and a branched continued fraction. As additional results, analytical extensions of the Lauricella–Saran hypergeometric functions $F_K(a_1, a_2, 1, b_2; a_1, b_2, c_3; \mathbf{z})$ and $F_K(a_1, 1, b_1, b_2; a_1, b_2, c_3; \mathbf{z})$ to the corresponding branched continued fractions were obtained. To illustrate this, we provide some numerical experiments at the end.

Keywords: Lauricella–Saran hypergeometric function; branched continued fraction; holomorphic functions of several complex variables; analytic continuation; convergence

MSC: 33C65; 32A17; 32A10; 30B40; 40A99

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1. Introduction

Hypergeometric functions of one and several variables occur naturally in a variety of applied mathematics, statistics and other decision sciences, chemistry and biology, mathematical physics, and engineering sciences. Their investigation has a very long history and a large bibliography (see, for example, [1–5]).

In 1893, G. Lauricella defined and studied four hypergeometric series F_A, F_B, F_C, and F_D of three variables [6]. He also indicated the existence of ten other hypergeometric functions of three variables $F_E, F_F, \ldots, F_T$, which were studied by Sh. Saran in 1954 [7].

Lauricella–Saran hypergeometric function F_K is defined by triple power series

$$F_K(a_1, a_2, b_1, b_2; c_1, c_2, c_3; \mathbf{z}) = \sum_{p,q,r=0}^{+\infty} \frac{(a_1)_p (a_2)_{q+r} (b_1)_{p+r} (b_2)_q}{(c_1)_p (c_2)_q (c_3)_r} \frac{z_1^p z_2^q z_3^r}{p!q!r!}, \tag{1}$$

where a_1, a_2, b_1, b_2, c_1, c_2, and c_3 are complex constants, $c_1, c_2, c_3 \notin \{0, -1, -2, \ldots\}$, $\mathbf{z} = (z_1, z_2, z_3) \in \mathrm{D}_{F_K}$,

$$\mathrm{D}_{F_K} = \{\mathbf{z} \in \mathbb{C}^3 : |z_k| < 1, \ k = 1, 2, \ |z_3| < (1 - |z_1|)(1 - |z_2|)\};$$

$(\cdot)_k$ is the Pochhammer symbol, defined as follows: $(\alpha)_0 = 1$, $(\alpha)_k = \alpha(\alpha + 1)_{k-1}$, $k \geq 1$. Applications and recent studies of these functions can be found, for instance, in [8–11] (see also [12–17]).

In this paper, we study the analytic continuation of the Lauricella–Saran hypergeometric function F_K with some parameter values into a branched continued fraction of the form

$$v_0(\mathbf{z})+\sum_{i_1=1}^{3}\frac{u_{i(1)}(\mathbf{z})}{v_{i(1)}(\mathbf{z})+\sum\limits_{i_2=1}^{3}\dfrac{u_{i(2)}(\mathbf{z})}{v_{i(2)}(\mathbf{z})+\ddots}}, \tag{2}$$

where the $v_0(\mathbf{z})$ and the elements $u_{i(k)}(\mathbf{z})$ and $v_{i(k)}(\mathbf{z})$, $i(k)\in\mathcal{I}$,

$$\mathcal{I}=\{i(k)=(i_1,i_2,\ldots,i_k):\ 1\le i_r\le 3,\ 1\le r\le k,\ k\ge 1\},$$

are functions of three variables in the certain domain D, $\mathrm{D}\subset\mathbb{C}^3$, (for more details on the branched continued fractions, see, for example, [18]).

The problem of the analytical continuation of the ratio of the Lauricella hypergeometric functions F_D with some real parameters to its branched continued fraction expansion were considered in [19,20]. In particular, it was proved in [19] that the expansion of the ratio is its analytic continuation in the domain

$$\mathrm{K}_{F_D}=\left\{\mathbf{z}\in\mathbb{C}^3:\ |z_k|<1,\ \mathrm{Re}(z_k)<\frac{1}{2},\ 1\le k\le 3\right\}.$$

In [21], it was established that the branched continued fraction expansion of the ratio of the Lauricella–Saran hypergeometric functions F_S with some real parameters is its analytic continuation in the domain

$$\mathrm{K}_{F_S}=\{\mathbf{z}\in\mathbb{C}^3:\ |z_k|+\mathrm{Re}(z_k)<1,\ 1\le k\le 3\}.$$

The paper is organized as follows. In Section 2, we give two methods for analytically extending a hypergeometric function (or ratio of hypergeometric functions) to a branched continued fraction in its domain of convergence. In Section 3, we derive two three-term recurrence relations for Lauricella–Saran hypergeometric functions F_K and construct the formal branched continued fraction expansions for two ratios of Lauricella–Saran hypergeometric functions F_K. Here, it is also proved that the branched continued fraction, which is an expansion of each ratio, uniformly converges to a holomorphic function of three variables on every compact subset of some domain of $\mathbb{C}^3$, and that this function is an analytic continuation of such a ratio in this domain.

2. Methods of Analytic Continuation

In the analytical theory of branched continued fractions, two methods are used to prove that the branched continued fraction expansion is an analytic continuation of a hypergeometric function (or ratio of hypergeometric functions) in some domain.

2.1. PC Method

The first method—let us call it the "PC method"—uses the so-called "principle of correspondence" (see, [22,23]). Its application requires that the branched continued fraction expansion corresponds at $\mathbf{z}=\mathbf{0}$ to a hypergeometric function (or ratio of hypergeometric functions) and that the sequence of its approximants converges uniformly on each compact subset of some neighborhood of the origin ($\mathbf{z}=\mathbf{0}$) to a function that is holomorphic in this neighborhood. Then, it remains to consistently apply the well-known Weierstrass' theorem ([24], p. 23) and the principle of analytic continuation ([25], p. 39).

Let us recall the necessary concepts.

An expression of the form

$$f_n(\mathbf{z}) = v_0(\mathbf{z}) + \sum_{i_1=1}^{3} \cfrac{u_{i(1)}(\mathbf{z})}{v_{i(1)}(\mathbf{z}) + \sum_{i_2=1}^{3} \cfrac{u_{i(2)}(\mathbf{z})}{v_{i(2)}(\mathbf{z}) + \ddots + \sum_{i_n=1}^{3} \cfrac{u_{i(n)}(\mathbf{z})}{v_{i(n)}(\mathbf{z})}}}$$

is called an nth approximant of (2) ([18], pp. 15–16).

A branched continued fraction (2) is called convergent at the point $\mathbf{z} = \mathbf{z}^0$, if at most a finite number of its approximants do not make sense, and if the limit of its sequence of approximants

$$\lim_{n \to +\infty} f_n(\mathbf{z}^0)$$

exists and is finite (see, [26] and ([27], p. 16)).

A branched continued fraction (2) is called uniformly convergent on subset E of D if its sequence $\{f_n(\mathbf{z})\}$ converges uniformly on E. If, moreover, this occurs for an arbitrary subset E such that $\overline{E} \subset D$ (here, $\overline{E}$ is the closure of the subset E), then (2) converges uniformly on each compact subset in D (see, [26] and ([27], p. 16)).

The concept of correspondence at $\mathbf{z} = \mathbf{0}$ (see, [28] and ([29], pp. 30–32)). Let $\mathbb{L}$ be a set of all formal triple power series of the form

$$L(\mathbf{z}) = \sum_{p,q,r=0}^{+\infty} d_{p,q,r} z_1^p z_2^q z_3^r, \tag{3}$$

where $d_{p,q,r} \in \mathbb{C}$, $p \geq 0$, $q \geq 0$, $r \geq 0$, $\mathbf{z} \in \mathbb{C}^3$. Let $f(\mathbf{z})$ be a function of three variables holomorphic in a neighborhood of the origin and let $\Lambda : f(\mathbf{z}) \to \Lambda(f)$ be a mapping associate with $f(\mathbf{z})$ its Taylor expansion in a neighborhood of the point $\mathbf{0}$.

A sequence $\{f_n(\mathbf{z})\}$ of the functions of three variables holomorphic at the origin is said to correspond at $\mathbf{z} = \mathbf{0}$ to a formal triple power series (3) if

$$\lim_{n \to +\infty} \lambda\big(L - \Lambda(f_n)\big) = +\infty,$$

where λ is defined to be: $\lambda : \mathbb{L} \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$; if $L(\mathbf{z}) \equiv 0$, then $\lambda(L) = +\infty$; if $L(\mathbf{z}) \not\equiv 0$ then $\lambda(L) = k$, where k is the smallest degree of homogeneous terms for which $d_{p,q,r} \neq 0$, that is, $k = p + q + r$.

A branched continued fraction (2) is said to correspond at $\mathbf{z} = \mathbf{0}$ to a formal triple power series (3) (or a function $f(\mathbf{z})$ holomorphic at the origin) if its sequence $\{f_n(\mathbf{z})\}$ corresponds to $L(\mathbf{z})$ (or a formal triple power series $\Lambda(f)$).

Theorem 1 (Weierstrass' Theorem)**.** *Let a sequence $\{g_n(\mathbf{z})\}$ of holomorphic functions in a domain* D, $\mathrm{D} \subset \mathbb{C}^3$, *converge to a function $g(\mathbf{z})$ uniformly on each compact subset in* D, *then $f(\mathbf{z})$ is holomorphic in* D, *and for any $p \geq 0$, $q \geq 0$, $r \geq 0$,*

$$\frac{\partial^{p+q+r} g_n(\mathbf{z})}{\partial z_1^p \partial z_2^q \partial z_3^r} \to \frac{\partial^{p+q+r} g(\mathbf{z})}{\partial z_1^p \partial z_2^q \partial z_3^r} \quad as \quad n \to +\infty$$

on each compact subset in D.

Theorem 2 (The Principle of Analytic Continuation)**.** *Let the functions $g_1(\mathbf{z})$ and $g_2(\mathbf{z})$ be holomorphic in the domains* D_1, $\mathrm{D}_1 \subset \mathbb{C}^3$, *and* D_2, $\mathrm{D}_2 \subset \mathbb{C}^3$, *respectively, and let* $\mathrm{D}_1 \cap \mathrm{D}_2$ *be the domain. Let, further, in a real neighborhood of the point $\mathbf{z}^0$ from* $\mathrm{D}_1 \cap \mathrm{D}_2$ *the functions $g_1(\mathbf{z})$ and $g_2(\mathbf{z})$ coincide. Then these functions are an analytic continuation of one another, i.e., there is a unique function $g(\mathbf{z})$ that is holomorphic in* $\mathrm{D}_1 \cup \mathrm{D}_2$ *and coincides with $g_1(\mathbf{z})$ in* D_1 *and with $g_2(\mathbf{z})$ in* D_2.

2.2. PF Method

The second method, let us call it the "PF method", uses the so-called "property of fork" (see, [21,30,31]). This method is used when the hypergeometric function (or the ratio of hypergeometric functions) and the elements of the branched continued fraction expansion are positive-valued functions in some domain D. If it holds, then its approximant satisfies the "property of fork": the sequence of even (odd) approximants increases (decreases) and is no greater (no less) than any odd (even) approximant. If, in addition, the branched continued fraction expansion converges, then it converges to the hypergeometric function (or the ratio of hypergeometric functions) in D. Finally, for the same restrictions on the parameters of the hypergeometric function, it remains to prove the convergence of the branched continued fraction expansion in a wider domain than D and to apply Theorem 2.

3. Lauricella–Saran Hypergeometric Function $F_K(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z})$

We set $c_1 = a_1$ and $c_2 = b_2$. Then, from (1), it follows

$$F_K(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z}) = \sum_{p,q,r=0}^{+\infty} \frac{(a_2)_{q+r}(b_1)_{p+r}}{(c_3)_r} \frac{z_1^p z_2^q z_3^r}{p!q!r!}. \tag{4}$$

3.1. Recurrence Relations

Remark 1. *In the process of constructing a branched continued fraction expansion of the ratio of hypergeometric functions, recurrent relations (for instance, three-term and/or four-term) play an important role. The problem is not only in the direct construction of such an expansion, but also in obtaining a branched continued fraction of the simplest structure. This, in turn, can provide more opportunities to investigate the convergence of the constructed expansion.*

Let us prove the three-term recurrence relations for Lauricella–Saran hypergeometric function (4).

Lemma 1. *The following relations hold true:*

$$\begin{aligned} F_K(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z}) &= (1 - z_1) F_K(a_1, a_2, b_1 + 1, b_2; a_1, b_2, c_3 + 1; \mathbf{z}) \\ &\quad - \frac{a_2(c_3 - b_1)}{c_3(c_3 + 1)} z_3 F_K(a_1, a_2 + 1, b_1 + 1, b_2; a_1, b_2, c_3 + 2; \mathbf{z}), \end{aligned} \tag{5}$$

$$\begin{aligned} F_K(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z}) &= (1 - z_2) F_K(a_1, a_2 + 1, b_1, b_2; a_1, b_2, c_3 + 1; \mathbf{z}) \\ &\quad - \frac{b_1(c_3 - a_2)}{c_3(c_3 + 1)} z_3 F_K(a_1, a_2 + 1, b_1 + 1, b_2; a_1, b_2, c_3 + 2; \mathbf{z}). \end{aligned} \tag{6}$$

Proof. We have

$$\begin{aligned} &F_K(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z}) - F_K(a_1, a_2, b_1 + 1, b_2; a_1, b_2, c_3 + 1; \mathbf{z}) \\ &\quad = \sum_{p,q,r \geq 0} \frac{(a_2)_{q+r}(b_1)_{p+r}}{(c_3)_r} \frac{z_1^p z_2^q z_3^r}{p!q!r!} - \sum_{p,q,r \geq 0} \frac{(a_2)_{q+r}(b_1 + 1)_{p+r}}{(c_3 + 1)_r} \frac{z_1^p z_2^q z_3^r}{p!q!r!} \\ &\quad = \sum_{p+q+r \geq 1} \frac{(a_2)_{q+r}(b_1)_{p+r}}{(c_3)_r} \frac{z_1^p z_2^q z_3^r}{p!q!r!} - \sum_{p+q+r \geq 1} \frac{(a_2)_{q+r}(b_1 + 1)_{p+r}}{(c_3 + 1)_r} \frac{z_1^p z_2^q z_3^r}{p!q!r!} \\ &\quad = \sum_{q \geq 0,\, p+r \geq 1} (a_2)_{q+r} \left(\frac{(b_1)_{p+r}}{(c_3)_r} - \frac{(b_1 + 1)_{p+r}}{(c_3 + 1)_r} \right) \frac{z_1^p z_2^q z_3^r}{p!q!r!} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{q\geq 0,\ p=0,\ r\geq 1} (a_2)_{q+r} \frac{(b_1+1)_{r-1}}{(c_3+1)_{r-1}} \left(\frac{b_1}{c_3} - \frac{b_1+r}{c_3+r} \right) \frac{z_1^p z_2^q z_3^r}{p!q!r!} \\
&\quad + \sum_{q\geq 0,\ p\geq 1,\ r=0} (a_2)_{q+r} (b_1 - b_1 - p) \frac{z_1^p z_2^q z_3^r}{p!q!r!} \\
&\quad + \sum_{q\geq 0,\ p\geq 1,\ r\geq 1} \frac{(a_2)_{q+r}(b_1+1)_{p+r-1}}{(c_3+1)_{r-1}} \left(\frac{b_1}{c_3} - \frac{b_1+p+r}{c_3+r} \right) \frac{z_1^p z_2^q z_3^r}{p!q!r!} \\
&= - \sum_{q\geq 0,\ p=0,\ r\geq 1} \frac{(c_3-b_1)(a_2)_{q+r}(b_1+1)_{p+r-1}}{c_3(c_3+1)(c_3+2)_{r-1}} \frac{z_1^p z_2^q z_3^r}{p!q!(r-1)!} \\
&\quad - \sum_{q\geq 0,\ p\geq 1,\ r=0} \frac{(a_2)_{q+r}(b_1+1)_{p+r-1}}{(c_3)_r} \frac{z_1^p z_2^q z_3^r}{(p-1)!q!r!} \\
&\quad - \sum_{q\geq 0,\ p\geq 1,\ r\geq 1} \frac{(a_2)_{q+r}(b_1+1)_{p+r-1}(c_3-b_1)}{c_3(c_3+1)(c_3+2)_{r-1}} \frac{z_1^p z_2^q z_3^r}{p!q!(r-1)!} \\
&\quad - \sum_{q\geq 0,\ p\geq 1,\ r\geq 1} \frac{(a_2)_{q+r}(b_1+1)_{p+r-1}}{(c_3+1)_r} \frac{z_1^p z_2^q z_3^r}{(p-1)!q!r!} \\
&= -\frac{a_2(c_3-b_1)}{c_3(c_3+1)} z_3 \sum_{q\geq 0, p\geq 0, r\geq 1} \frac{(a_2+1)_{q+r-1}(b_1+1)_{p+r-1}}{(c_3+2)_{r-1}} \frac{z_1^p z_2^q z_3^{r-1}}{p!q!(r-1)!} \\
&\quad - z_1 \sum_{q\geq 0,\ p\geq 1,\ r\geq 0} \frac{(a_2)_{q+r}(b_1+1)_{p+r-1}}{(c_3+1)_r} \frac{z_1^{p-1} z_2^q z_3^r}{(p-1)!q!r!}, \\
&= -\frac{a_2(c_3-b_1)}{c_3(c_3+1)} z_3 F_K(a_1, a_2+1, b_1+1, b_2; a_1, b_2, c_3+2; \mathbf{z}) \\
&\quad - z_1 F_K(a_1, a_2, b_1+1, b_2; a_1, b_2, c_3+1; \mathbf{z}),
\end{aligned}
$$

from which follows the correctness of relation (5).

Similarly, we will prove the relation (6). By definition (4), we get

$$
\begin{aligned}
&F_K(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z}) - F_K(a_1, a_2+1, b_1, b_2; a_1, b_2, c_3+1; \mathbf{z}) \\
&\quad = \sum_{p,q,r\geq 0} \frac{(a_2)_{q+r}(b_1)_{p+r}}{(c_3)_r} \frac{z_1^p z_2^q z_3^r}{p!q!r!} - \sum_{p,q,r\geq 0} \frac{(a_2+1)_{q+r}(b_1)_{p+r}}{(c_3+1)_r} \frac{z_1^p z_2^q z_3^r}{p!q!r!} \\
&\quad = \sum_{p+q+r\geq 1} \frac{(a_2)_{q+r}(b_1)_{p+r}}{(c_3)_r} \frac{z_1^p z_2^q z_3^r}{p!q!r!} - \sum_{p+q+r\geq 1} \frac{(a_2+1)_{q+r}(b_1)_{p+r}}{(c_3+1)_r} \frac{z_1^p z_2^q z_3^r}{p!q!r!} \\
&\quad = \sum_{p\geq 0,\ q+r\geq 1} (b_1)_{p+r} \left(\frac{(a_2)_{q+r}}{(c_3)_r} - \frac{(a_2+1)_{q+r}}{(c_3+1)_r} \right) \frac{z_1^p z_2^q z_3^r}{p!q!r!} \\
&\quad = \sum_{p\geq 0,\ q=0,\ r\geq 1} (b_1)_{p+r} \frac{(a_2+1)_{r-1}}{(c_3+1)_{r-1}} \left(\frac{a_2}{c_3} - \frac{a_2+r}{c_3+r} \right) \frac{z_1^p z_2^q z_3^r}{p!q!r!} \\
&\qquad + \sum_{p\geq 0,\ q\geq 1,\ r=0} (b_1)_{p+r} (a_2 - a_2 - q) \frac{z_1^p z_2^q z_3^r}{p!q!r!} \\
&\qquad + \sum_{p\geq 0,\ q\geq 1,\ r\geq 1} \frac{(b_1)_{p+r}(a_2+1)_{q+r-1}}{(c_3+1)_{r-1}} \left(\frac{a_2}{c_3} - \frac{a_2+q+r}{c_3+r} \right) \frac{z_1^p z_2^q z_3^r}{p!q!r!}
\end{aligned}
$$

$$
\begin{aligned}
&= -\sum_{p\geq 0,\, q=0,\, r\geq 1} \frac{(c_3-a_2)(b_1)_{p+r}(a_2+1)_{q+r-1}}{c_3(c_3+1)(c_3+2)_{r-1}} \frac{z_1^p z_2^q z_3^r}{p!q!(r-1)!} \\
&\quad - \sum_{p\geq 0,\, q\geq 1,\, r=0} \frac{(b_1)_{p+r}(a_2+1)_{q+r-1}}{(c_3)_r} \frac{z_1^p z_2^q z_3^r}{p!(q-1)!r!} \\
&\quad - \sum_{p\geq 0,\, q\geq 1,\, r\geq 1} \frac{(a_2+1)_{q+r-1}(b_1)_{p+r}(c_3-a_2)}{c_3(c_3+1)(c_3+2)_{r-1}} \frac{z_1^p z_2^q z_3^r}{p!q!(r-1)!} \\
&\quad - \sum_{p\geq 0,\, q\geq 1,\, r\geq 1} \frac{(a_2)_{q+r-1}(b_1)_{p+r}}{(c_3+1)_r} \frac{z_1^p z_2^q z_3^r}{p!(q-1)!r!}. \\
&= -\frac{b_1(c_3-a_2)}{c_3(c_3+1)} z_3 \sum_{p\geq 0,\, q\geq 0,\, r\geq 1} \frac{(a_2+1)_{q+r-1}(b_1+1)_{p+r-1}}{(c_3+2)_{r-1}} \frac{z_1^p z_2^q z_3^{r-1}}{p!q!(r-1)!} \\
&\quad - z_2 \sum_{p\geq 0,\, q\geq 1,\, r\geq 0} \frac{(a_2+1)_{q+r-1}(b_1)_{p+r}}{(c_3+1)_r} \frac{z_1^p z_2^{q-1} z_3^r}{p!(q-1)!r!} \\
&= -\frac{b_1(c_3-a_2)}{c_3(c_3+1)} z_3 F_K(a_1, a_2+1, b_1+1, b_2; a_1, b_2, c_3+2; \mathbf{z}) \\
&\quad - z_2 F_K(a_1, a_2+1, b_1, b_2; a_1, b_2, c_3+1; \mathbf{z}),
\end{aligned}
$$

which had to be proved. □

3.2. Expansions

We set

$$
R_K^{(1)}(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z}) = \frac{F_K(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z})}{F_K(a_1, a_2, b_1+1, b_2; a_1, b_2, c_3+1; \mathbf{z})}, \tag{7}
$$

$$
R_K^{(2)}(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z}) = \frac{F_K(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z})}{F_K(a_1, a_2+1, b_1, b_2; a_1, b_2, c_3+1; \mathbf{z})}. \tag{8}
$$

The following theorem is true.

Theorem 3. *A ratio (7) has a formal branched continued fraction of the form*

$$
1 - z_1 - \cfrac{d_1 z_3}{1 - z_2 - \cfrac{d_2 z_3}{1 - z_1 - \cfrac{d_3 z_3}{1 - z_2 - \cfrac{d_4 z_3}{1 - \ddots}}}}, \tag{9}
$$

where, for all $k \geq 1$,

$$
d_{2k-1} = \frac{(a_2+k-1)(c_3+k-1-b_1)}{(c_3+2k-2)(c_3+2k-1)}, \quad d_{2k} = \frac{(b_1+k)(c_3+k-a_2)}{(c_3+2k-1)(c_3+2k)}. \tag{10}
$$

Proof. Dividing (5) and (6) by

$$
F_K(a_1, a_2, b_1+1, b_2; a_1, b_2, c_3+1; \mathbf{z}) \quad \text{and} \quad F_K(a_1, a_2+1, b_1, b_2; a_1, b_2, c_3+1; \mathbf{z}),
$$

respectively, we obtain

$$R_K^{(1)}(a_1,a_2,b_1,b_2;a_1,b_2,c_3;\mathbf{z}) = 1 - z_1 - \cfrac{\frac{a_2(c_3-b_1)}{c_3(c_3+1)}z_3}{R_K^{(2)}(a_1,a_2,b_1+1,b_2;a_1,b_2,c_3+1;\mathbf{z})}, \tag{11}$$

$$R_K^{(2)}(a_1,a_2,b_1,b_2;a_1,b_2,c_3;\mathbf{z}) = 1 - z_2 - \cfrac{\frac{b_1(c_3-a_2)}{c_3(c_3+1)}z_3}{R_K^{(1)}(a_1,a_2+1,b_1,b_2;a_1,b_2,c_3+1;\mathbf{z})}. \tag{12}$$

In fact, in (11), we have Step 1.1 of constructing a branched continued fraction. At Step 1.2, replacing b_1, c_3 by b_1+1 and c_3+1, respectively, in (12), we get

$$\begin{aligned} &R_K^{(1)}(a_1,a_2,b_1,b_2;a_1,b_2,c_3;\mathbf{z}) \\ &= 1 - z_1 - \cfrac{\frac{a_2(c_3-b_1)}{c_3(c_3+1)}z_3}{1 - z_2 - \cfrac{\frac{(b_1+1)(c_3+1-a_2)}{(c_3+1)(c_3+2)}z_3}{R_K^{(1)}(a_1,a_2+1,b_1+1,b_2;a_1,b_2,c_3+2;\mathbf{z})}}. \end{aligned} \tag{13}$$

Let us continue the next construction of the branched continued fraction in the same way as in steps 1.1–1.2. It is clear that the following relation holds, for all $k \geq 1$,

$$\begin{aligned} &R_K^{(1)}(a_1,a_2+k-1,b_1+k-1,b_2;a_1,b_2,c_3+2k-2;\mathbf{z}) \\ &= 1 - z_1 - \cfrac{\frac{(a_2+k-1)(c_3+k-1-b_1)}{(c_3+2k-2)(c_3+2k-1)}z_3}{1 - z_2 - \cfrac{\frac{(b_1+k)(c_3+k-a_2)}{(c_3+2k-1)(c_3+2k)}z_3}{R_K^{(1)}(a_1,a_2+k,b_1+k,b_2;a_1,b_2,c_3+2k;\mathbf{z})}}. \end{aligned} \tag{14}$$

At Steps 2.1–2.2, substituting (14) when $k=2$ in (13), we obtain

$$\begin{aligned} &R_K^{(1)}(a_1,a_2,b_1,b_2;a_1,b_2,c_3;\mathbf{z}) \\ &= 1 - z_1 - \cfrac{\frac{a_2(c_3-b_1)}{c_3(c_3+1)}z_3}{1 - z_2 - \cfrac{\frac{(b_1+1)(c_3+1-a_2)}{(c_3+1)(c_3+2)}z_3}{1 - z_1 - \cfrac{\frac{(a_2+1)(c_3+1-b_1)}{(c_3+2)(c_3+3)}z_3}{1 - z_2 - \cfrac{\frac{(b_1+2)(c_3+2-a_2)}{(c_3+3)(c_3+4)}z_3}{R_K^{(1)}(a_1,a_2+2,b_1+2,b_2;a_1,b_2,c_3+4;\mathbf{z})}}}}. \end{aligned}$$

Next, by (14) after the Steps n.1–n.2, we have

$$
\begin{aligned}
R_K^{(1)}(a_1,a_2,b_1,b_2;a_1,b_2,c_3;\mathbf{z}) &= 1 - z_1 \\
&- \cfrac{\frac{a_2(c_3-b_1)}{c_3(c_3+1)}z_3}{1 - z_2 - \cfrac{\frac{(b_1+1)(c_3+1-a_2)}{(c_3+1)(c_3+2)}z_3}{1 - \ddots - z_1 - \cfrac{\frac{(a_2+n-1)(c_3+n-1-b_1)}{(c_3+2n-2)(c_3+2n-1)}z_3}{1 - z_2 - \cfrac{\frac{(b_1+n)(c_3+n-a_2)}{(c_3+2n-1)(c_3+2n)}z_3}{R_K^{(1)}(a_1,a_2+n,b_1+n,b_2;a_1,b_2,c_3+2n;\mathbf{z})}}}}.
\end{aligned}
$$

Finally, as $n \to +\infty$, we obtain the formal expansion of (7) into branched continued fraction (9). □

The following theorem can be proved in much the same way as Theorem 3.

Theorem 4. *A ratio (8) has a formal branched continued fraction of the form*

$$
1 - z_2 - \cfrac{h_1 z_3}{1 - z_1 - \cfrac{h_2 z_3}{1 - z_2 - \cfrac{h_3 z_3}{1 - z_1 - \cfrac{h_4 z_3}{1 - \ddots}}}}, \tag{15}
$$

where, for all $k \geq 1$,

$$
h_{2k-1} = \frac{(b_1+k-1)(c_3+k-1-a_2)}{(c_3+2k-2)(c_3+2k-1)}, \quad h_{2k} = \frac{(a_2+k)(c_3+k-b_1)}{(c_3+2k-1)(c_3+2k)}. \tag{16}
$$

3.3. Analytic Continuation

We will apply the PC method to prove that expansion (9) is an analytic continuation of ratio (7) in some domain.

The following corollary follows directly from Theorem 1 [26].

Corollary 1. *Let $g_{0,0,k}$, $k \geq 1$, be real numbers such that, for all $k \geq 1$,*

$$
0 < g_{0,0,k} \leq 1.
$$

Then, the branched continued fraction,

$$
1 - z_{1,0,0} - \cfrac{g_{0,0,1} z_{0,0,1}}{1 - (1-g_{0,0,1})z_{0,1,1} - \cfrac{g_{0,0,2}(1-g_{0,0,1})z_{0,0,2}}{1 - (1-g_{0,0,2})z_{1,0,2} - \cfrac{g_{0,0,3}(1-g_{0,0,2})z_{0,0,3}}{1 - \ddots}}},
$$

converges if, for all $k \geq 0$,

$$
|z_{1,0,2k}| \leq \frac{1}{2}, \quad |z_{0,1,2k+1}| \leq \frac{1}{2}, \quad |z_{0,0,k+1}| \leq \frac{1}{2}.
$$

From the proof of Lemma 4.41 [32], we have following corollary.

Corollary 2. *If $x \geq c > 0$ and $v^2 \leq 4u + 4$, where $u, v \in \mathbb{R}$, then*

$$\min_{-\infty<y<+\infty} \mathrm{Re}\left(\frac{u+iv}{x+iy}\right) = -\frac{\sqrt{u^2+v^2}-u}{2x}.$$

Moreover, the following theorem clearly follows from Theorem 2.17 [18] (see also ([27], Theorem 24.2)).

Theorem 5. *Let a sequence of holomorphic functions $\{g_n(\mathbf{z})\}$ on the domain D, $\mathrm{D} \subset \mathbb{C}^3$, is uniform bounded on every compact subset of D. If, moreover, the sequence $\{g_n(\mathbf{z})\}$ converges at each point of the set E, $\mathrm{E} \subset \mathrm{D}$, which is the real neighborhood of the point $\mathbf{z}^0$ in D, then its converges uniformly on every compact subset of D to a holomorphic function in D.*

We will prove the following theorem.

Theorem 6. *Let a_2, b_1, and c_3 be constants such that, for all $k \geq 1$,*

$$0 < d_k \leq r, \tag{17}$$

where d_k, $k \geq 1$, are defined by (10), r is a positive number. Then:

(A) The branched continued fraction (9) converges uniformly on every compact subset of

$$\mathrm{H}_{r,r^*} = \bigcup_{-\pi/2<\alpha<\pi/2} \mathrm{H}_{r,r^*,\alpha}, \tag{18}$$

where $0 < r^* < 1$ and

$$\mathrm{H}_{r,r^*,\alpha} = \left\{ \mathbf{z} \in \mathbb{C}^3 : \ \frac{|z_k| + \mathrm{Re}(z_k e^{-2i\alpha})}{2(1-r^*)\cos^2\alpha} < 1,\ k = 1,2,\ \frac{|z_3| + \mathrm{Re}(z_3 e^{-2i\alpha})}{r^* \cos^2\alpha} < \frac{1}{2r} \right\}, \tag{19}$$

to a holomorphic function $f(\mathbf{z})$ in H_{r,r^*};

(B) The function $f(\mathbf{z})$ is an analytic continuation of (7) in the domain (18).

Proof. We set, for $n \geq 1$,

$$G_n^{(n)}(\mathbf{z}) = 1 \tag{20}$$

and, for $n \geq 1$ and $1 \leq k \leq n$,

$$G_{2k-1}^{(2n)}(\mathbf{z}) = 1 - z_2 - \cfrac{d_{2k}z_3}{1 - z_1 - \cfrac{d_{2k+1}z_3}{1 - \ddots - z_1 - \cfrac{d_{2n-1}z_3}{1 - z_2 - d_{2n}z_3}}},$$

$$G_{2k}^{(2n)}(\mathbf{z}) = 1 - z_1 - \cfrac{d_{2k+1}z_3}{1 - z_2 - \cfrac{d_{2k+2}z_3}{1 - \ddots - z_1 - \cfrac{d_{2n-1}z_3}{1 - z_2 - d_{2n}z_3}}},$$

$$G_{2k-1}^{(2n+1)}(\mathbf{z}) = 1 - z_2 - \cfrac{d_{2k}z_3}{1 - z_1 - \cfrac{d_{2k+1}z_3}{1 - \ddots - z_2 - \cfrac{d_{2n}z_3}{1 - z_1 - d_{2n+1}z_3}}},$$

$$G_{2k}^{(2n+1)}(\mathbf{z}) = 1 - z_1 - \cfrac{d_{2k+1}z_3}{1 - z_2 - \cfrac{d_{2k+2}z_3}{1 - \ddots - z_2 - \cfrac{d_{2n}z_3}{1 - z_1 - d_{2n+1}z_3}}},$$

which gives us, for $n \geq 1$ and $1 \leq k \leq n$,

$$G_{2k-1}^{(2n)}(\mathbf{z}) = 1 - z_2 - \frac{d_{2k}z_3}{G_{2k}^{(2n)}(\mathbf{z})}, \quad G_{2k}^{(2n)}(\mathbf{z}) = 1 - z_1 - \frac{d_{2k+1}z_3}{G_{2k+1}^{(2n)}(\mathbf{z})}, \tag{21}$$

and

$$G_{2k-1}^{(2n+1)}(\mathbf{z}) = 1 - z_2 - \frac{d_{2k}z_3}{G_{2k}^{(2n+1)}(\mathbf{z})}, \quad G_{2k}^{(2n+1)}(\mathbf{z}) = 1 - z_1 - \frac{d_{2k+1}z_3}{G_{2k+1}^{(2n+1)}(\mathbf{z})}. \tag{22}$$

Thus, we write the nth approximants of (9) in the form

$$f_n(\mathbf{z}) = 1 - z_1 - \frac{d_1z_3}{G_1^{(n)}(\mathbf{z})}. \tag{23}$$

Let n be an arbitrary natural number, let α be an arbitrary real from $(-\pi/2, \pi/2)$, and let $\mathbf{z}$ be an arbitrary fixed point from (19). Then, the following inequalities are held, for all $1 \leq k \leq n$,

$$\mathrm{Re}(G_{2k-1}^{(2n)}(\mathbf{z})e^{-i\alpha}) > \frac{r^* \cos\alpha}{2} > 0 \tag{24}$$

and

$$\mathrm{Re}(G_{2k-1}^{(2n+1)}(\mathbf{z})e^{-i\alpha}) > \frac{r^* \cos\alpha}{2} > 0. \tag{25}$$

Let us prove that (24) is true. In view of (20), it is obvious that (24) holds for $k = n$. Assuming, by the induction, that (24) holds for $k = p+1$, $p+1 \leq n$, from (21) one obtains, for $k = p$,

$$G_{2p-1}^{(2n)}(\mathbf{z})e^{-i\alpha} = e^{-i\alpha} - \frac{z_2e^{-2i\alpha}}{e^{-i\alpha}} - \frac{d_{2p}z_3e^{-2i\alpha}}{G_{2p}^{(2n)}(\mathbf{z})e^{-i\alpha}}$$

and

$$G_{2p}^{(2n)}(\mathbf{z})e^{-i\alpha} = e^{-i\alpha} - \frac{z_1e^{-2i\alpha}}{e^{-i\alpha}} - \frac{d_{2p+1}z_3e^{-2i\alpha}}{G_{2p+1}^{(2n)}(\mathbf{z})e^{-i\alpha}}.$$

Then, using (17), (19), Corollary 2 , and the induction hypothesis, we have

$$\begin{aligned}
\mathrm{Re}(G_{2p}^{(2n)}(\mathbf{z})e^{-i\alpha}) &\geq \cos\alpha - \frac{|z_1e^{-2i\alpha}| + \mathrm{Re}(z_1e^{-2i\alpha})}{2\,\mathrm{Re}(e^{-i\alpha})} - \frac{d_{2p+1}(|z_3e^{-2i\alpha}| + \mathrm{Re}(z_3e^{-2i\alpha}))}{2\,\mathrm{Re}(G_{2p+1}^{(2n)}(\mathbf{z})e^{-i\alpha})} \\
&> \cos\alpha - (1-r^*)\cos\alpha - \frac{r^*\cos\alpha}{2} \\
&= \frac{r^*\cos\alpha}{2} > 0
\end{aligned}$$

and

$$\operatorname{Re}(G_{2p-1}^{(2n)}(\mathbf{z})e^{-i\alpha}) \geq \cos\alpha - \frac{|z_2e^{-2i\alpha}| + \operatorname{Re}(z_2e^{-2i\alpha})}{2\operatorname{Re}(e^{-i\alpha})} - \frac{d_{2p}(|z_3e^{-2i\alpha}| + \operatorname{Re}(z_3e^{-2i\alpha}))}{2\operatorname{Re}(G_{2p}^{(2n)}(\mathbf{z})e^{-i\alpha})}$$
$$> \cos\alpha - (1-r^*)\cos\alpha - \frac{r^*\cos\alpha}{2}$$
$$= \frac{r^*\cos\alpha}{2} > 0.$$

In the same way, we obtain the inequalities (25).

Thus, for all $n \geq 1$ and $\mathbf{z} \in \mathrm{H}_{r,r^*,\alpha}$,

$$G_1^{(n)}(\mathbf{z}) \neq 0.$$

This means that the sequence $\{f_n(\mathbf{z})\}$ is a sequence of holomorphic functions in (19), and, therefore, in domain H_{r,r^*} due to the arbitrariness α.

Let K be an arbitrary compact subset of H_{r,r^*}. Then, there exists an open triple-disk

$$\mathrm{H}_l = \{\mathbf{z} \in \mathbb{C}^3 : \; |z_k| < l, \; k = 1, 2.3\}, \quad l > 0,$$

such that $\mathrm{K} \subset \mathrm{H}_l$. Now, cover K by domains of the form

$$\mathrm{H}_{r,r^*,l,\alpha} = \mathrm{H}_{r,r^*,\alpha} \bigcap \mathrm{H}_l$$

and choose from this cover a finite subcover,

$$\mathrm{H}_{r,r^*,l,\alpha_1}, \; \mathrm{H}_{r,r^*,l,\alpha_2}, \; \dots, \; \mathrm{H}_{r,r^*,l,\alpha_k}.$$

Using (23)–(25), for any $n \geq 1$, $p \in \{1, \; 2, \; \dots, \; k\}$ and $\mathbf{z} \in \mathrm{H}_{r,r^*,l,\alpha_p}$, we have

$$|f_n(\mathbf{z})| \leq 1 + |z_1| + \frac{d_1|z_3|}{\operatorname{Re}(G_1^{(n)}(\mathbf{z})e^{-i\alpha_p})}$$
$$< 1 + l + \frac{2rl}{\cos\alpha_p}$$
$$= C(\mathrm{H}_{r,r^*,l,\alpha_p}).$$

Setting

$$C(\mathrm{K}) = \max_{1 \leq p \leq k} C(\mathrm{H}_{r,r^*,l,\alpha_p}),$$

for any $n \geq 1$ and $\mathbf{z} \in \mathrm{K}$, we obtain

$$|f_n(\mathbf{z})| \leq C(\mathrm{K}).$$

This means that the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded on every compact subset of the domain H_{r,r^*}.

It is clear that, for each real l^* such that

$$0 < l^* < \min\left\{\frac{1}{4}, \; \frac{1}{8r}\right\},$$

the domain

$$\mathrm{H}_{l^*} = \left\{\mathbf{z} \in \mathbb{R}^3 : \; -l^* < z_k < 0, \; 1 \leq k \leq 3\right\}$$

is contained in H_{r,r^*}, in particular, $\mathrm{H}_{l^*/2} \subset \mathrm{H}_{r,r^*}$.

Taking into account (17), it is easy to show that, for any $\mathbf{z} \in \mathrm{H}_{l^*}$, $\mathrm{H}_{l^*} \subset \mathrm{H}_{r,r^*}$, the following inequalities hold, for all $k \geq 1$,

$$|z_k| < \frac{1}{4} \quad k = 1, 2, \quad |d_k z_3| < \frac{1}{8}.$$

This means that the elements of branched continued fraction (9) satisfy the conditions of Corollary 1, with $g_{0,0,k} = 1/2$ for all $k \geq 1$. By this corollary the branched continued fraction (9) converges in H_{l^*}, $\mathrm{H}_{l^*} \subset \mathrm{H}_{r,r^*}$. It follows from Theorem 5 that the convergence is uniform on compact subsets of H_{r,r^*} to a holomorphic function $f(\mathbf{z})$ in H_{r,r^*}. This proves *(A)*.

Now, we prove *(B)*. Setting, for $n \geq 1$,

$$F_{2n}^{(2n)}(\mathbf{z}) = R_K^{(1)}(a_1, a_2 + n, b_1 + n, b_2; a_1, b_2, c_3 + 2n; \mathbf{z}),$$
$$F_{2n+1}^{(2n+1)}(\mathbf{z}) = R_K^{(2)}(a_1, a_2 + n, b_1 + n + 1, b_2; a_1, b_2, c_3 + 2n + 1; \mathbf{z}),$$

and, for $n \geq 1$ and $1 \leq k \leq n$,

$$F_{2k-1}^{(2n)}(\mathbf{z}) = 1 - z_2 - \cfrac{d_{2k} z_3}{1 - z_1 - \cfrac{d_{2k+1} z_3}{1 - \ddots - z_1 - \cfrac{d_{2n-1} z_3}{1 - z_2 - \cfrac{d_{2n} z_3}{F_{i(2n)}^{(2n)}(\mathbf{z})}}}},$$

$$F_{2k}^{(2n)}(\mathbf{z}) = 1 - z_1 - \cfrac{d_{2k+1} z_3}{1 - z_2 - \cfrac{d_{2k+2} z_3}{1 - \ddots - z_1 - \cfrac{d_{2n-1} z_3}{1 - z_2 - \cfrac{d_{2n} z_3}{F_{i(2n)}^{(2n)}(\mathbf{z})}}}},$$

$$F_{2k-1}^{(2n+1)}(\mathbf{z}) = 1 - z_2 - \cfrac{d_{2k} z_3}{1 - z_1 - \cfrac{d_{2k+1} z_3}{1 - \ddots - z_2 - \cfrac{d_{2n} z_3}{1 - z_1 - \cfrac{d_{2n+1} z_3}{F_{i(2n+1)}^{(2n+1)}(\mathbf{z})}}}},$$

$$F_{2k}^{(2n+1)}(\mathbf{z}) = 1 - z_1 - \cfrac{d_{2k+1} z_3}{1 - z_2 - \cfrac{d_{2k+2} z_3}{1 - \ddots - z_2 - \cfrac{d_{2n} z_3}{1 - z_1 - \cfrac{d_{2n+1} z_3}{F_{i(2n+1)}^{(2n+1)}(\mathbf{z})}}}},$$

we have, for $n \geq 1$ and $1 \leq k \leq n$,

$$F_{2k-1}^{(2n)}(\mathbf{z}) = 1 - z_2 - \frac{d_{2k} z_3}{F_{2k}^{(2n)}(\mathbf{z})}, \quad F_{2k}^{(2n)}(\mathbf{z}) = 1 - z_1 - \frac{d_{2k+1} z_3}{F_{2k+1}^{(2n)}(\mathbf{z})}, \tag{26}$$

and

$$F_{2k-1}^{(2n+1)}(\mathbf{z}) = 1 - z_2 - \frac{d_{2k} z_3}{F_{2k}^{(2n+1)}(\mathbf{z})}, \quad F_{2k}^{(2n+1)}(\mathbf{z}) = 1 - z_1 - \frac{d_{2k+1} z_3}{F_{2k+1}^{(2n+1)}(\mathbf{z})}. \tag{27}$$

Hence, and from the proof of Theorem 3, it follows that for each $n \geq 1$,

$$R_K^{(1)}(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z}) = 1 - z_1 - \cfrac{d_1 z_3}{1 - z_2 - \cfrac{d_2 z_3}{1 - \ddots - z_1 - \cfrac{d_{2n-1} z_3}{1 - z_2 - \cfrac{d_{2n} z_3}{F_{2n}^{(2n)}(\mathbf{z})}}}}$$

$$= 1 - z_1 - \frac{d_1 z_3}{F_1^{(2n)}(\mathbf{z})}$$

and

$$R_K^{(1)}(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z}) = 1 - z_1 - \cfrac{d_1 z_3}{1 - z_2 - \cfrac{d_2 z_3}{1 - \ddots - z_2 - \cfrac{d_{2n} z_3}{1 - z_1 - \cfrac{d_{2n+1} z_3}{F_{2n+1}^{(2n+1)}(\mathbf{z})}}}}$$

$$= 1 - z_1 - \frac{d_1 z_3}{F_1^{(2n+1)}(\mathbf{z})}.$$

Since $F_k^{(n)}(\mathbf{0}) = 1$ and $G_k^{(n)}(\mathbf{0}) = 1$ for any $1 \leq k \leq n$, $n \geq 1$, then there exist $\Lambda(1/F_k^{(n)})$ and $\Lambda(1/G_k^{(n)})$, i.e., the $1/F_k^{(n)}$ and $1/G_k^{(n)}$ have Taylor expansions in a neighborhood of the origin. It is clear that $F_k^{(n)}(\mathbf{z}) \not\equiv 0$ and $G_k^{(n)}(\mathbf{z}) \not\equiv 0$ for all indices. Applying the method suggested in ([18], p. 28) and (20)–(22), (26), and (27), for each $n \geq 1$ one obtains

$$R_K^{(1)}(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z}) - f_{2n-1}(\mathbf{z})$$
$$= \frac{d_1 z_3}{F_1^{(2n)}(\mathbf{z}) G_1^{(2n-1)}(\mathbf{z})} \cdots \frac{d_{2n-1} z_3}{F_{2n-1}^{(2n)}(\mathbf{z}) G_{2n-1}^{(2n-1)}(\mathbf{z})} \left(-z_2 - \frac{d_{2n} z_3}{F_{2n}^{(2n)}(\mathbf{z})} \right)$$

and

$$R_K^{(1)}(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z}) - f_{2n}(\mathbf{z})$$
$$= \frac{d_1 z_3}{F_1^{(2n+1)}(\mathbf{z}) G_1^{(2n)}(\mathbf{z})} \cdots \frac{d_{2n} z_3}{F_{2n}^{(2n+1)}(\mathbf{z}) G_{2n}^{(2n)}(\mathbf{z})} \left(-z_1 - \frac{d_{2n+1} z_3}{F_{2n+1}^{(2n+1)}(\mathbf{z})} \right).$$

Hence, in a neighborhood of origin for any $n \geq 1$, we have

$$\Lambda(R_K^{(1)}) - \Lambda(f_n) = \sum_{\substack{p+q+r \geq n \\ p \geq 0,\, q \geq 0,\, r \geq 0}} d_{p,q,r}^{(n)} z_1^p z_2^q z_3^r,$$

where $d_{p,q,r}^{(n)}$, $p \geq 0$, $q \geq 0$, $r \geq 0$, $p + q + r \geq n$, are some coefficients. It follows that

$$\lambda(\Lambda(R_1) - \Lambda(f_n)) = n + 1$$

tends monotonically to $+\infty$ as $n \to +\infty$.

Thus, the branched continued fraction (9) corresponds at $\mathbf{z} = \mathbf{0}$ to a formal triple power series $\Lambda(R_K^{(1)})$.

Let Δ be the neighborhood of the origin which contained (18), and in which

$$\Lambda(R_1) = \sum_{p,q,r=0}^{+\infty} d_{p,q,r} z_1^p z_2^q z_3^r. \tag{28}$$

From part *(A)*, it follows that the sequence $\{f_n(\mathbf{z})\}$ converges uniformly on each compact subset of the domain Δ to function $f(\mathbf{z})$, which is holomorphic in Δ. Then, according to Theorem 1 for arbitrary $k+l,\ k \geq 0,\ l \geq 0$, we have

$$\frac{\partial^{p+q+r} f_n(\mathbf{z})}{\partial z_1^p \partial z_2^q \partial z_3^r} \to \frac{\partial^{p+q+r} f(\mathbf{z})}{\partial z_1^p \partial z_2^q \partial z_3^r} \quad \text{as} \quad n \to +\infty$$

on each compact subset of the domain Δ. And now, according to the above proven, the expansion of each approximant $f_n(\mathbf{z}),\ n \geq 1$, into formal triple power series and series (28) agree for all homogeneous terms up to and including degree $(n-1)$. Then, for arbitrary $p+q+r$, $p \geq 0$, $q \geq 0$, $r \geq 0$, we obtain

$$\lim_{n \to +\infty} \left(\frac{\partial^{p+q+r} f_n}{\partial z_1^p \partial z_2^q \partial z_3^r}(\mathbf{0}) \right) = \frac{\partial^{p+q+r} f}{\partial z_1^p \partial z_2^q \partial z_3^r}(\mathbf{0}) = p!q!r! d_{p,q,r}.$$

Hence,

$$f(\mathbf{z}) = \sum_{p,q,r=0}^{+\infty} \frac{1}{p!q!r!} \left(\frac{\partial^{p+q+r} f}{\partial z_1^p \partial z_2^q \partial z_3^r}(\mathbf{0}) \right) z_1^p z_2^q z_3^r = \sum_{p,q,r=0}^{+\infty} \alpha_{p,q,r} z_1^p z_2^q z_3^r$$

for all $\mathbf{z} \in \Delta$.

Finally, Theorem 2 follows part *(B)*. $\square$

Setting $b_1 = 0$ and replacing c_3 by $c_3 - 1$ in Theorem 6, we have the following result.

Corollary 3. *Let a_2 and c_3 be constants such that, for all $k \geq 1$,*

$$0 < \frac{(a_2+k-1)(c_3+k-2)}{(c_3+2k-3)(c_3+2k-2)} \leq r, \quad 0 < \frac{k(c_3+k-1-a_2)}{(c_3+2k-2)(c_3+2k-1)} \leq r,$$

where r is a positive number. Then:

(A) The branched continued fraction

$$\cfrac{1}{1 - z_1 - \cfrac{d_1 z_3}{1 - z_2 - \cfrac{d_2 z_3}{1 - z_1 - \cfrac{d_3 z_3}{1 - z_2 - \cfrac{d_4 z_3}{1 - \ddots}}}}}, \tag{29}$$

where, for all $k \geq 1$,

$$d_{2k-1} = \frac{(a_2+k-1)(c_3+k-2)}{(c_3+2k-3)(c_3+2k-2)}, \quad d_{2k} = \frac{k(c_3+k-1-a_2)}{(c_3+2k-2)(c_3+2k-1)},$$

converges uniformly on every compact subset of the domain (18) to a function $f(\mathbf{z})$ holomorphic in H_{r,r^*};

(B) The function $f(\mathbf{z})$ is an analytic continuation of $F_K(a_1, a_2, 1, b_2; a_1, b_2, c_3; \mathbf{z})$ in H_{r,r^*}.

The following theorem can be proved in much the same way as Theorem 6 using Theorems 1, 2, and 5 and Corollaries 1 and 2.

Theorem 7. *Let a_2, b_1, and c_3 be constants such that, for all $k \geq 1$,*

$$0 < h_k \leq r,$$

where h_k, $k \geq 1$, are defined by (16), r is a positive number. Then:

(A) The branched continued fraction (15) converges uniformly on every compact subset of the domain (18) to a function $f(\mathbf{z})$ holomorphic in H_{r,r^*};
(B) The function $f(\mathbf{z})$ is an analytic continuation of (8) in the domain (18).

Corollary 4. *Let b_1, and c_3 be constants such that, for all $k \geq 1$,*

$$0 < \frac{(b_1+k-1)(c_3+k-2)}{(c_3+2k-3)(c_3+2k-2)} \leq r, \quad 0 < \frac{k(c_3+k-1-b_1)}{(c_3+2k-2)(c_3+2k-1)} \leq r,$$

where r is a positive number. Then:

(A) The branched continued fraction

$$\cfrac{1}{1-z_2-\cfrac{d_1 z_3}{1-z_1-\cfrac{d_2 z_3}{1-z_2-\cfrac{d_3 z_3}{1-z_1-\cfrac{d_4 z_3}{1-\ddots}}}}}, \tag{30}$$

where, for all $k \geq 1$,

$$d_{2k-1} = \frac{(b_1+k-1)(c_3+k-2)}{(c_3+2k-3)(c_3+2k-2)}, \quad d_{2k} = \frac{k(c_3+k-1-b_1)}{(c_3+2k-2)(c_3+2k-1)},$$

converges uniformly on every compact subset of the domain (18) to a holomorphic function $f(\mathbf{z})$ in H_{r,r^*};
(B) The function $f(\mathbf{z})$ is an analytic continuation of $F_K(a_1, 1, b_1, b_2; a_1, b_2, c_3; \mathbf{z})$ in H_{r,r^*}.

Remark 2. *Theorems 6 and 7, as and Corollaries 3 and 4, establish the convergence criteria for the constructed branched continued fraction expansions for real parameter values of the Lauricella–Saran hypergeometric function (4). The method used for this also allows us to obtain the convergence criteria for complex parameter values, sacrificing the domain for variable z_3.*

Remark 3. *Estimates of the rate of convergence for the branched continued fractions (9), (15), (29), and (30) can be established in the same way as in [33].*

4. Numerical Experiments

By Corollary 3, we have

$$\begin{aligned}\ln\left(1+\frac{z_3}{(1+z_1)(1+z_2)}\right) &= z_3\, F_K(a_1, 1, 1, b_2; a_1, b_2, 2; -z_1, -z_2, -z_3) \\ &= \cfrac{z_3}{1+z_1+\cfrac{d_1 z_3}{1+z_2+\cfrac{d_2 z_3}{1+z_1+\cfrac{d_3 z_3}{1+z_2+\cfrac{d_4 z_3}{1+\ddots}}}}},\end{aligned} \tag{31}$$

where, for $k \geq 1$,

$$d_{2k-1} = \frac{k}{2(2k-1)}, \quad d_{2k} = \frac{k}{2(2k+1)}.$$

The branched continued fraction in (31) converges and represents a single-valued branch of the function,

$$\left(1 + \frac{z_3}{(1+z_1)(1+z_2)}\right), \tag{32}$$

in the domain

$$\mathrm{H}_{r^*} = \left\{\mathbf{z} \in \mathbb{C}^3 : \ |\arg(z_k + 1 - r^*)| < \pi, \ k = 1, 2, \ \left|\arg\left(z_3 + \frac{r^*}{2}\right)\right| < \pi\right\}, \quad 0 < r^* < 1.$$

The numerical illustration of triple power series

$$\begin{aligned} \ln\left(1 + \frac{z_3}{(1+z_1)(1+z_2)}\right) &= z_3\, F_K(a_1, 1, 1, b_2; a_1, b_2, 2; -z_1, -z_2, -z_3) \\ &= z_3 - z_1 z_3 - z_2 z_3 - z_3^2 + \ldots, \end{aligned} \tag{33}$$

and the branched continued fraction (31) is given in Table 1.

Table 1. Relative error of 5th partial sum and 5th approximant for (32).

$\mathbf{z}$	**(32)**	**(33)**	**(31)**
$(0.1, 0.1, -0.4)$	-0.40134	2.96×10^{-1}	1.00×10^{-1}
$(0.4, 0.4, -0.4)$	-0.22826	4.92×10^{-1}	4.00×10^{-1}
$(0.5, 0.5, 0.5)$	0.20067	8.22×10^{-1}	5.00×10^{-1}
$(0.9, 0.9, 0.9)$	0.22259	2.22×10^{1}	9.00×10^{-1}
$(0.1, 0.1, 10)$	2.22619	9.10×10^{3}	1.89×10^{-1}
$(-0.01, -0.01, 10)$	2.41619	7.54×10^{3}	9.03×10^{-2}
$(0.1, 0.1, 50)$	3.74531	1.66×10^{7}	9.32×10^{-1}
$(-0.9, -0.9, 9)$	6.80351	3.32×10^{2}	7.14×10^{-1}

Calculations were performed using Wolfram Mathematica software 13.1.0.0 for Linux.

5. Conclusions

In this paper, we constructed two formal branched continued fraction expansions for Lauricella–Saran hypergeometric function ratios defined by (7) and (8). Our method is based on the classical method of constructing a Gaussian continued fraction [34], which can be applied to other Lauricella–Saran functions. To prove the convergence of expansions to ratios, we used the PC method, which is described in Section 2.1. These branched continued fractions are fascinating in their forms and have good approximate properties (in particular, compared with triple power series under certain conditions, they have wider convergence domains and are endowed with the property of numerical stability). They can bring new insights into the study of the hypergeometric functions of several variables. Their potential wide domain convergence and estimates of the rate of convergence are an interesting direction worth exploring in the future. Along this path, ideas implemented in [35–39] can be used.

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References

1. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions, Volume 1*; McGraw-Hill Book Co.: New York, NY, USA, 1953.
2. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions, Volume 2*; McGraw-Hill Book Co.: New York, NY, USA, 1953.
3. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions, Volume 3*; McGraw-Hill Book Co.: New York, NY, USA, 1955.
4. Exton, H. *Multiple Hypergeometric Functions and Applications*; Horwood, E., Ed.; Halsted Press: Chichester, UK, 1976.
5. Srivastava, H.M.; Karlsson, P.W. *Multiple Gaussian Hypergeometric Series*; Halsted Press: New York, NY, USA, 1985.
6. Lauricella, G. Sulle funzioni ipergeometriche a più variabili. *Rend. Circ. Matem.* **1893**, *7*, 111–158. (In Italian) [CrossRef]
7. Saran, Sh. Hypergeometric functions of three variables. *Ganita* **1954**, *5*, 77–91.
8. Blümlein, J.; Saragnese, M.; Schneider, C. Hypergeometric structures in Feynman integrals. *Ann. Math. Artif. Intell.* **2023**, *91*, 591–649. [CrossRef]
9. Kol, B.; Shir, R. The propagator seagull: General evaluation of a two loop diagram. *J. High Energy Phys.* **2019**, *2019*, 83. [CrossRef]
10. Luo, M.-J.; Raina, R.K. On certain results related to the hypergeometric function F_K. *J. Math. Anal. Appl.* **2021**, *504*, 125439. [CrossRef]
11. Luo, M.-J.; Xu, M.-H.; Raina, R.K. On certain integrals related to Saran's hypergeometric function F_K. *Fractal Fract.* **2022**, *6*, 155. [CrossRef]
12. Hutchinson, T.P. Compound gamma bivariate distributions. *Metrika* **1981**, *28*, 263–271. [CrossRef]
13. Hutchinson, T.P. Four applications of a bivariate Pareto distribution. *Biom. J.* **1979**, *21*, 553–563. [CrossRef]
14. Lee, P.A. The correlated bivariate inverted beta distribution. *Biom. J.* **1981**, *23*, 693–703. [CrossRef]
15. Mazars, M. Statistical physics of the freely jointed chain. *Phys. Rev. E* **1996**, *53*, 6297–6319. [CrossRef]
16. Mazars, M. Canonical partition functions of freely jointed chains. *J. Phys. A Math. Gen.* **1998**, *31*, 1949–1964. [CrossRef]
17. Ong, S.H. Computation of bivariate gamma and inverted beta distribution functions. *J. Statist. Comput. Simul.* **1995**, *51*, 153–163. [CrossRef]
18. Bodnar, D.I. *Branched Continued Fractions*; Naukova Dumka: Kyiv, Ukraine, 1986. (In Russian)
19. Antonova, T.M.; Hoyenko, N.P. Approximation of Lauricella's functions F_D ratio by Nörlund's branched continued fraction in the complex domain. *Mat. Metody Fiz. Mekh. Polya* **2004**, *47*, 7–15. (In Ukrainian)
20. Bodnar, D.I.; Hoyenko, N.P. Approximation of the ratio of Lauricella functions by a branched continued fraction. *Mat. Studii* **2003**, *20*, 210–214. (In Ukrainian)
21. Hoyenko, N.; Antonova, T.; Rakintsev, S. Approximation for ratios of Lauricella–Saran fuctions F_S with real parameters by a branched continued fractions. *Math. Bul. Shevchenko Sci. Soc.* **2011**, *8*, 28–42. (In Ukrainian)
22. Antonova, T.; Dmytryshyn, R.; Kravtsiv, V. Branched continued fraction expansions of Horn's hypergeometric function H_3 ratios. *Mathematics* **2021**, *9*, 148. [CrossRef]
23. Antonova, T.; Dmytryshyn, R.; Lutsiv, I.-A.; Sharyn, S. On some branched continued fraction expansions for Horn's hypergeometric function $H_4(a,b;c,d;z_1,z_2)$ ratios. *Axioms* **2023**, *12*, 299. [CrossRef]
24. Shabat, B.V. *Introduce to Complex Analysis. Part II. Functions of Several Variables*; American Mathematical Society: Providence, RI, USA, 1992.
25. Vladimirov, V.S. *Methods of the Theory of Functions of Many Complex Variables*; The MIT Press: Cambridge, MA, USA, 1966.
26. Dmytryshyn, R.I. Convergence of multidimensional A- and J-fractions with independent variables. *Comput. Methods Funct. Theory* **2022**, *22*, 229–242. [CrossRef]
27. Wall, H.S. *Analytic Theory of Continued Fractions*; D. Van Nostrand Co.: New York, NY, USA, 1948.
28. Dmytryshyn, R.I.; Sharyn, S.V. Approximation of functions of several variables by multidimensional S-fractions with independent variables. *Carpathian Math. Publ.* **2021**, *13*, 592–607. [CrossRef]
29. Cuyt, A.A.M.; Petersen, V.; Verdonk, B.; Waadeland, H.; Jones, W.B. *Handbook of Continued Fractions for Special Functions*; Springer: Dordrecht, The Netherlands, 2008.
30. Antonova, T.; Dmytryshyn, R.; Kurka, R. Approximation for the ratios of the confluent hypergeometric function $\Phi_D^{(N)}$ by the branched continued fractions. *Axioms* **2023**, *11*, 426. [CrossRef]
31. Manzii, O.S. Investigation of expansion of the ratio of Appel hypergeometric functions F_3 into a branching continued fraction. In *Approximation Theory and Its Applications*; Stepanets, O.I., Ed.; Proceedings of Institute of Mathematics of NAS of Ukraine; Institute of Mathematics of NAS of Ukraine: Kyiv, Ukraine, 2000; Volume 31, pp. 344–353. (In Ukrainian)
32. Jones, W.B.; Thron, W.J. *Continued Fractions: Analytic Theory and Applications*; Addison-Wesley Pub. Co.: Reading, MA, USA, 1980.
33. Antonova, T.; Dmytryshyn, R.; Sharyn, S. Branched continued fraction representations of ratios of Horn's confluent function H_6. *Constr. Math. Anal.* **2023**, *6*, 22–37. [CrossRef]

34. Gauss, C.F. Disquisitiones generales circa seriem infinitam $1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+3)}x^3$+etc. In *Commentationes Societatis Regiae Scientiarum Gottingensis Recentiores*; Classis Mathematicae, 1812; H. Dieterich: Gottingae, Germany, 1813; Volume 2, pp. 3–46. (In Latin)
35. Bilanyk, I.B. A truncation error bound for some branched continued fractions of the special form. *Mat. Stud.* **2019**, *52*, 115–123. [CrossRef]
36. Bodnar, D.I.; Bilanyk, I.B. Estimation of the rates of pointwise and uniform convergence of branched continued fractions with inequivalent variables. *J. Math. Sci.* **2022**, *265*, 423–437. [CrossRef]
37. Bodnar, D.I.; Bilanyk, I.B. On the convergence of branched continued fractions of a special form in angular domains. *J. Math. Sci.* **2020**, *246*, 188–200. [CrossRef]
38. Bodnar, D.I.; Bilanyk, I.B. Parabolic convergence regions of branched continued fractions of the special form. *Carpathian Math. Publ.* **2021**, *13*, 619–630. [CrossRef]
39. Bodnar, D.I.; Bilanyk, I.B. Two-dimensional generalization of the Thron-Jones theorem on the parabolic domains of convergence of continued fractions. *Ukr. Math. J.* **2023**, *74*, 1317–1333. [CrossRef]